# Patterns in knot cohomology I

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#### Abstract

Cohomology theory of links, introduced in [Kh1], is combinatorial. Dror Bar-Natan recently wrote a program that found ranks of cohomology groups of all prime knots with up to 11 crossings [BN]. His surprising experimental data is discussed in this note.

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## 1 Notations

The Jones polynomial is determined by the skein relation

$$q^2J(L_1) - q^{-2}J(L_2) = (q - q^{-1})J(L_3),$$

where  $L_i$  are depicted in figure 1, and by the normalization J(unknot) = 1. This standard normalization is different from the one in [Kh1,2].

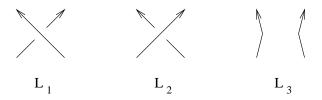


Figure 1:

Familiarity with [Kh1,2] or [BN] is assumed. Warning: we use the grading conventions of [Kh2], and the cohomology group that we denote by  $\mathcal{H}^{i,j}$  is denoted  $\mathcal{H}^{i,-j}$  in [BN] and [Kh1]. Let  $h^{i,j}(K)$  (or simply  $h^{i,j}$ ) be the rank of  $\mathcal{H}^{i,j}(K)$ . Ranks of cohomology groups satisfy (notice  $q^{-j}$ , rather than  $q^j$ )

$$(q+q^{-1})J(K) = \sum_{i,j} (-1)^i q^{-j} h^{i,j}(K).$$
(1)

We use the Rolfsen enumeration for knots with 10 or fewer crossings. Knots with more than 10 crossings are enumerated as in Knotscape, for instance,  $11_{77}^n$  denotes the 77th non-alternating 11-crossing knot.

### 2 Initial observations

There are 249 prime unoriented knots with at most 10 crossings (not counting mirror images). From Bar-Natan [BN] we learn that for all but 12 of these knots the nontrivial cohomology groups lie on two adjacent diagonals. Let us call such knots homologically thin, or H-thin, for short. We have no clue why nearly all small knots are H-thin. Figure 2 depicts  $10_{117}$ , an H-thin knot, and ranks of its cohomology groups.  $h^{i,j}$  is zero if the (i,j)-square is empty.

Squares with even j-coordinates are omitted from the picture, since cohomology groups  $\mathcal{H}^{i,2k}(K)$ , for a knot K, are always zero. By a diagonal we mean a line 2i+j=b, for some b, also referred to as the b-diagonal.

All H-thin knots with up to 10 crossings share the following properties

- (i) cohomology groups are supported on  $(\sigma \pm 1)$ -diagonals, where  $\sigma$  is the signature of the knot;
- (ii) after substracting 1 from  $h^{0,\sigma\pm 1}$ , the numbers on the upper diagonal coincide with numbers on the lower diagonal after the (1,-4) shift;
- (iii) the Jones polynomial is alternating:  $J(K) = \sum c_i q^{2i}$ , if  $c_i c_j > 0$  then  $j \equiv i \pmod{2}$ , if  $c_i c_j < 0$  then  $j \not\equiv i \pmod{2}$ . Unless the knot is a (2, n)-torus knot, for  $n \in \{3, 5, 7, 9\}$ , the Jones polynomial has no gaps, i.e.  $c_i \neq 0, c_{i+k} \neq 0$  implies  $c_{i+m} \neq 0$  for all m between 1 and k-1.
- (iv) The Alexander polynomial  $\Delta(K) = \sum a_i t^i$  is alternating and has no gaps.

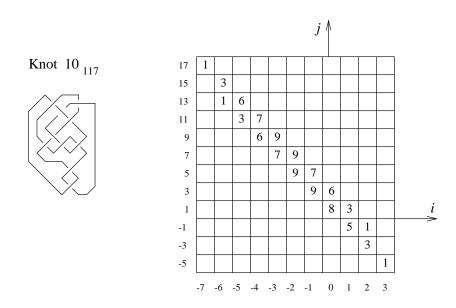


Figure 2:  $10_{117}$  and ranks of its cohomology groups

All alternating and the majority of non-alternating knots with up to 10 crossings are H-thin. Knots that are not H-thin will be called H-thick (homologically thick). The twelve H-thick knots with at most 10 crossings are

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8_{19}, 9_{42}, 10_{124}, 10_{128}, 10_{132}, 10_{136}, 10_{139}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, 10_{161}.
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Figure 3 shows the knot  $10_{132}$  and ranks of its cohomology groups.

Properties (i), (iii), and (iv) of H-thin knots (with at most 10 crossings) fail on many of these knots. The 12 H-thick knots satisfy

- (i') cohomology groups are supported on three adjacent diagonals. Discard the diagonal with the smallest total rank of cohomology groups supported on it. The two remaining ones are  $(\sigma \pm 1)$ -diagonals.
- (ii') if, for a suitable i, we substract 1 from  $h^{0,i}$  and  $h^{0,i+2}$ , the remaining numbers can be arranged into pairs with the (1, -4) difference in the bigrading (figure 4 does it for  $10_{132}$ ).
- (iii') The Jones polynomials of  $10_{124}$ ,  $10_{139}$ ,  $10_{145}$ ,  $10_{152}$ ,  $10_{153}$ ,  $10_{154}$ ,  $10_{161}$  are not alternating. The Jones polynomials of  $8_{19}$ ,  $10_{124}$ ,  $10_{132}$ ,  $10_{139}$ ,  $10_{145}$ ,  $10_{152}$ ,  $10_{153}$ ,  $10_{154}$ ,  $10_{161}$  have gaps.
- (iv') The Alexander polynomials of  $8_{19}$ ,  $10_{124}$ ,  $10_{128}$ ,  $10_{139}$ ,  $10_{145}$ ,  $10_{152}$ ,  $10_{153}$ ,  $10_{154}$ ,  $10_{161}$  are not alternating. The Alexander polynomials of  $8_{19}$ ,  $10_{124}$ ,  $10_{139}$ ,  $10_{154}$ ,  $10_{161}$  have gaps.

We verified (iii), (iii'), (iv), and (iv') using the tables in [St].

For any knot K the Alexander polynomial at -1 equals the Jones polynomial

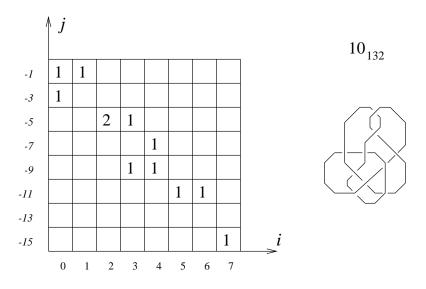


Figure 3:  $10_{132}$  and ranks of its cohomology groups

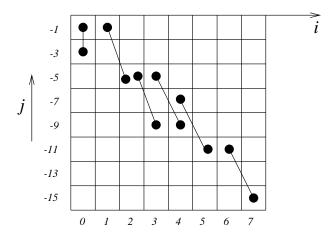


Figure 4: Cohomology of  $10_{132}$  arranged in pairs

at  $\sqrt{-1}$ :

$$\Delta_{-1}(K) = J_{\sqrt{-1}}(K) \tag{2}$$

(because of our choice of variable q, the R.H.S. is  $J_{\sqrt{-1}}(K)$  rather than the more common  $J_{-1}(K)$ ).

Coefficient-wise, with notations from (iii),(iv),

$$\sum_{i} (-1)^{i} a_{i} = \sum_{i} (-1)^{i} c_{i}.$$

Since Jones and Alexander polynomials of H-thin knots with at most 10 crossings are alternating, for these knots we obtain

$$\sum_{i} |a_i| = \sum_{i} |c_i|.$$

Properties (i) and (ii) imply that, in addition,

$$\operatorname{rank}\mathcal{H}(K) - 1 = \sum_{i} |c_{i}|$$

where  $\operatorname{rank} \mathcal{H}(K) = \sum_{i,j} h^{i,j}$  is the rank of total cohomology of the knot. To summarize, H-thin knots with at most 10 crossings satisfy

$$\operatorname{rank}\mathcal{H}(K) - 1 = \sum_{i} |c_{i}| = \sum_{i} |a_{i}|. \tag{3}$$

What about the twelve H-thick knots? For each of them the inequalities hold

$$\sum_{i} |a_i| \le \operatorname{rank} \mathcal{H}(K) - 3 \ge \sum_{i} |c_i|, \tag{4}$$

(Note that  $\sum_i |a_i|$  and  $\sum_i |c_i|$  are odd for any knot and rank( $\mathcal{H}$ ) is even.)

Alternating knots with at most 10 crossings are H-thin, and it was conjectured in [BN] and [G] that all alternating knots are H-thin. This conjecture is now a theorem, due to Eun Soo Lee [Lee]:

**Theorem 1** Non-split alternating links are H-thin.

We now look at the data for 11-crossing knots [BN]. There are 367 alternating and 185 non-alternating prime knots with 11 crossings. H-thick knots among them number 41.

Properties (i)-(iv) continue to hold for H-thin knots with 11 crossings. There are several 11-crossing knots with non-alternating Jones or Alexander polynomial. All of them are H-thick. Likewise, 11-crossing knots with a gap in the Alexander polynomial are H-thick.

**Problem:** Explain why so many non-alternating knots with 11 or fewer crossings are H-thin.

### 3 A-module structure of knot cohomology

In this section we work over  $\mathbb{Q}$  (rather than over  $\mathbb{Z}$ , as in [Kh1, Section 7]). In particular, the base ring is  $\mathcal{A} = \mathbb{Q}[X]/(X^2)$  and the chain complex  $\mathcal{C}(D)$  associated to a plane diagram D of a knot K is a complex of  $\mathbb{Q}$ -vector spaces. Cohomology groups  $\mathcal{H}^{i,j}(D)$  are finite-dimensional  $\mathbb{Q}$ -vector spaces, only finitely many of them are nontrivial. Dimensions  $h^{i,j}$  of these groups are invariants of K.

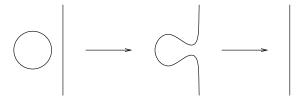


Figure 5: Cobordism between circle  $\cup D$  and D

Choose a segment I of D that does not contain crossings. Place an unknotted circle next to I and consider the cobordism that merges the circle and I (figure 5). This cobordism induces a map of complexes  $A \otimes C(D) \to C(D)$  and makes C(D) into a complex of graded A-modules. A Reidemeister move from D to D' that happens away from I induces a chain homotopy equivalence between complexes of A-modules C(D) and C(D'). Given two diagrams  $D_1$  and  $D_2$  of K and two segments  $I_1$  and  $I_2$  in them, there is a sequence of Reidemeister moves that takes  $(D_1, I_1)$  to  $(D_2, I_2)$  such that all moves happens away from  $I_1$ . Instead of moving an arc over or under  $I_1$  we can move it across the rest of the plane (or  $\mathbb{S}^2$ ). In other words, there are as many knots as one-component (1, 1)-tangles.

We obtain an invariant of K, the complex  $\mathcal{C}(D)$  of free A-modules up to chain homotopy equivalence. The Krull-Schmidt theorem, valid for bounded complexes of finite-dimensional modules over finite-dimensional algebras, tells us that  $\mathcal{C}(D)$  decomposes (uniquely up to an isomorphism) as direct sum of an acyclic complex and indecomposable complexes with nontrivial cohomology. The multiplicity of each indecomposable complex in this decomposition is an invariant of K. Denote by  $C_n$  the complex

$$0 \longrightarrow \mathcal{A} \xrightarrow{X} \mathcal{A}\{-2\} \xrightarrow{X} \cdots \xrightarrow{X} \mathcal{A}\{-2n+2\} \xrightarrow{X} \mathcal{A}\{-2n\} \longrightarrow 0, \quad (5)$$

where the leftmost  $\mathcal{A}$  is in cohomological degree 0.

**Proposition 1** A non-acyclic indecomposable complex of free graded A-modules is isomorphic to  $C_n[i]\{j\}$  for a unique triple  $(n, i, j), n \geq 0$ .

Example: If K is a (2, 2m+1)-torus knot, C(K) is a direct sum of  $C_0\{2m\}$  and  $C_1[2i+1]\{4i+2m+2\}, 1 \le i \le m$ .

**Proposition 2**  $C(K_1 \# K_2) \cong C(K_1) \otimes_{\mathcal{A}} C(K_2)$ .

*Proof:* Obvious.  $\square$ 

Define homological width of K, denoted hw(K), as the minimal number m such that cohomology of K lie on m adjacent diagonals. The homological width of a knot is at least 2, since cohomology groups of indecomposable complexes  $C_n$  lie on 2 adjacent diagonals, and any knot has nontrivial cohomology (since the Jones polynomial does not vanish). According to our definitions, a knot is H-thin if and only if it has homological width 2.

Proposition 2 implies

**Proposition 3**  $hw(K_1 \# K_2) = hw(K_1) + hw(K_2) - 2.$ 

Corollary 1  $K_1 \# K_2$  is H-thin if and only if both  $K_1$  and  $K_2$  are H-thin.

### Reduced cohomology

Let Q = A/XA be the one-dimensional representation of A. Define the reduced complex of D by

$$\widetilde{\mathcal{C}}(D) = \mathcal{C}(D) \otimes_{\mathcal{A}} Q.$$

This is a complex of graded  $\mathbb{Q}$ -vector spaces. We call its cohomology the *reduced* cohomology of D (and K) and denote by  $\widetilde{\mathcal{H}}(D)$  and  $\widetilde{\mathcal{H}}(K)$ , the latter are defined up to isomorphism. Ranks of cohomology groups  $\widetilde{\mathcal{H}}^{i,j}(D)$  are invariants of K. The Euler characteristic of  $\widetilde{\mathcal{H}}$  is the Jones polynomial (compare to (1)):

$$J(K) = \sum_{i,j} (-1)^i q^{-j} \operatorname{rank}(\widetilde{\mathcal{H}}^{i,j}(K)),$$

therefore,

$$\mathrm{rank}\widetilde{\mathcal{H}}(K) \geq |J_{\sqrt{-1}}(K)| = |\Delta_{-1}(K)|.$$

**Proposition 4** Reduced cohomology groups  $\widetilde{\mathcal{H}}^{i,j}(K)$  lie on one diagonal (2i+j) is constant) if and only if K is H-thin.

Corollary 2 The Jones polynomial of an H-thin knot is alternating. The absolute values of its coefficients are dimensions of reduced cohomology groups.

### H-restricted knots

Properties (ii),(ii') admit a homological interpretation. We say that a knot K is H-restricted if non-acyclic indecomposable summands of the A-module complex C(K) are one  $A\{i\}$ , for some i, and one or several  $C_1[j]\{k\}$ , for  $j,k \in \mathbb{Z}$ . Cohomology groups of a H-restricted knot can be paired up as in (ii'). Existence of such pairing, however, does not imply that a knot is H-restricted.

(2, 2m+1)-torus knots are H-restricted. The figure eight knot is H-restricted.

**Proposition 5** If  $K_1$  and  $K_2$  are H-restricted then  $K_1 \# K_2$  is H-restricted.

Conjecture 1 All knots are H-restricted.

This is a homological counterpart of Conjecture 1 in [BN] about  $Kh_{\mathbb{Q}}$ .

**Proposition 6** If K is H-restricted then  $\operatorname{rank} \mathcal{H}(K) = \operatorname{rank} \widetilde{\mathcal{H}}(K) - 1$ .

## 4 Cohomology with $\mathbb{Z}_2$ -coefficients

Let us now work over  $\mathbb{Z}$  rather that  $\mathbb{Q}$ , so that  $\mathcal{A} = \mathbb{Z}[X]/(X^2)$ . A computation in [Kh1, Section 6.2] implies that  $\mathcal{C}(K)$ , where K is a (2, 2m + 1)-torus knot, is isomorphic to the direct sum (modulo acyclic complexes) of the complex  $0 \longrightarrow \mathcal{A}\{2m\} \longrightarrow 0$  and m complexes  $C'_1$ 

$$0 \longrightarrow \mathcal{A} \xrightarrow{2X} \mathcal{A}\{-2\} \longrightarrow 0 \tag{6}$$

with various shifts.

Cohomology of  $C'_1 \otimes_{\mathbb{Z}} \mathbb{Q}$  is two-dimensional (over  $\mathbb{Q}$ ), and is a matching pair of cohomology groups in bidegrees that differ by (1, -4).

Now change the base field to  $\mathbb{Z}_2$ . In characteristic 2 the differential in (6) is 0, and the dimension of cohomology groups of  $C'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_2$  is 4 (as a  $\mathbb{Z}_2$ -vector space), see figure 6.

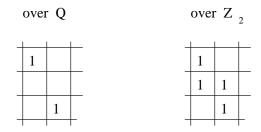


Figure 6: Dimensions of cohomology of  $C'_1$  over  $\mathbb{Q}$  and  $\mathbb{Z}_2$ 

According to the tables in Bar-Natan [BN], the same patterns relates rational and  $\mathbb{Z}_2$ -cohomology of any prime knot with at most seven crossings. Pair up the rational cohomology groups as in (ii), so that all but one pair look as on the left hand side of figure 6, and change each on them to the quadruple of 1's on the right hand side. We get ranks of  $\mathbb{Z}_2$ -cohomology groups.

It is likely that for any knot K with at most 7 crossings  $\mathcal{C}(K)$  decomposes as a direct sum of

• an acyclic complex,

- complex  $0 \longrightarrow \mathcal{A}\{i\} \longrightarrow 0$  for some  $i \in \mathbb{Z}$ ,
- complexes  $0 \longrightarrow \mathcal{A}\{j\} \xrightarrow{2kX} \mathcal{A}\{j-2\} \longrightarrow 0$  for  $j, k \in \mathbb{Z}$ .

This would explain the observed relation between rational and  $\mathbb{Z}_2$ -cohomology of these knots.

#### 5 Cohomology of adequate knots

For a link diagram D denote by  $s_{+}D$  (respectively  $s_{-}D$ ) the diagram obtained by taking 0-resolution (respectively 1-resolution) of each crossing of D, see figure 7.



Figure 7: Two resolutions of a crossing

We say that D is adequate if

- for any crossing of D the two segments of  $s_+D$  that replace this crossing belong to distinct components of  $s_+D$ ,
- for any crossing of D the two segments of  $s_D$  that replace this crossing belong to distinct components of  $s_{-}D$ .

A reduced alternating link diagram is adequate. A link admitting an adequate diagram is called adequate. For further information about adequate links see Thistlethwaite [Th] and Lickorish [Li, Chapter 5].

Proposition 7 Adequate non-alternating knots are H-thick.

*Proof:* Assume D is an adequate non-alternating diagram of a knot K. We continue to use cohomology with integer coefficients. Recall from [Kh1, Chapter 7] that

$$\overline{\mathcal{H}}^0(D) \neq 0 \neq \overline{\mathcal{H}}^n(D)$$

where n is the number of crossing of D. More precisely,

$$\overline{\mathcal{H}}^{0,|s_+D|}(D) \cong \mathbb{Z}, \qquad \overline{\mathcal{H}}^{0,i}(D) \cong 0, \text{ if } i > |s_+D|,$$
 (7)

$$\overline{\mathcal{H}}^{0,|s_{+}D|}(D) \cong \mathbb{Z}, \qquad \overline{\mathcal{H}}^{0,i}(D) \cong 0, \text{ if } i > |s_{+}D|, \qquad (7)$$

$$\overline{\mathcal{H}}^{n,-|s_{-}D|-n}(D) \cong \mathbb{Z}, \qquad \overline{\mathcal{H}}^{0,i}(D) \cong 0, \text{ if } i < -|s_{-}D|-n, \qquad (8)$$

where  $|s_+D|$  is the number of components of  $s_+D$ , etc.

From discussion is Section 3 we know that rational cohomology groups come in pairs (complex (5) contributes  $\mathbb{Q} \oplus \mathbb{Q}$  to cohomology, in two degrees that differ by (n, -2n-2)). The companion of  $\overline{\mathcal{H}}^{0,|s_+D|}(D) \otimes \mathbb{Q} \cong \mathbb{Q}$  will lie one diagonal below it, while the companion of  $\overline{\mathcal{H}}^{n,-|s_-D|-n} \otimes \mathbb{Q}$  will lie one diagonal above it. This is illustrated in figure 8, which unintentionally shows the case  $n = |s_+D| + |s_-D|$ .

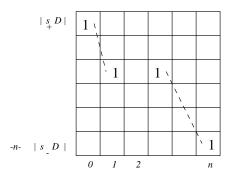


Figure 8:

If K is H-thin, these two pairs of cohomology groups must lie on two adjancent diagonals. This implies  $n + 2 = |s_+D| + |s_-D|$ .

**Lemma 1** If D is adequate, non-alternating and prime then  $n + 2 > |s_+D| + |s_-D|$ . If D is alternating then  $n + 2 = |s_+D| + |s_-D|$ .

This lemma is proved in [Li, Chapter 5].  $\square$ 

Therefore, if D is prime, K is H-thick. The case of composite D follows from Corollary 1.  $\square$ 

There are no adequate non-alternating knots with 9 or fewer crossings, 3 adequate non-alternating knots with 10 crossings:  $10_{152}$ ,  $10_{153}$ ,  $10_{154}$ , and 15 adequate non-alternating 11-crossing knots.

## 6 Cohomology of positive and braid positive knots

### Positive knots

We say that a knot is *positive* if it has a diagram with only positive crossings (figure 9).

**Proposition 8** If K is a positive knot then  $\mathcal{H}^{i,j}(K) = 0$  if i < 0,

$$\mathcal{H}^{0,j}(K) = \left\{ \begin{array}{ll} \mathbb{Z} & \textit{if } j = s - n - 1 \pm 1 \\ 0 & \textit{otherwise}, \end{array} \right.$$

and  $\mathcal{H}^{i,j} = 0$  if i > 0 and  $j \ge s - n$ , where s is the number of Seifert circles and n the number of crossings in a positive diagram of K.



Figure 9: A positive crossing

*Proof:* Left to the reader.  $\square$  Note that  $\frac{n-s+1}{2}$  is the genus of K.

### Braid positive knots

 $8_{19}$  is a (3,4)-torus knot,  $10_{124}$  is a (3,5)-torus knot. Both are H-thick. If n,m are odd, the (n,m)-torus knot is H-thick since its Jones polynomial is not alternating. We expect that (n,m)-torus knots, 2 < n < m, are H-thick.

Torus knots are examples of *braid positive* knots, i.e. knots that are closures of positive braids.

Braid positive prime knots with at most 10 crossings are (2, n)-torus knots, for  $n \in \{3, 5, 7, 9\}$ , and the four H-thick knots  $8_{19}$ ,  $10_{124}$ ,  $10_{139}$ ,  $10_{152}$ .

There are two braid positive prime 11-crossing knots: the (2,11)-torus knot and  $11_{77}^n$ , the closure of the braid  $\sigma_1^2 \sigma_2^2 \sigma_1 \sigma_3 \sigma_2^3 \sigma_3^2$ . The latter is H-thick [BN].

There are 7 braid positive prime knots with 12 crossings. All of them are H-thick, since their Jones polynomials are not alternating.

Not counting the (2,13)-torus knot, there are 12 braid positive prime 13-crossing knots. At least 10 are H-thick (the Jones polynomial is not alternating). We don't know if the remaining knots  $13_{4587}^n$  and  $13_{5016}^n$  are H-thick.

There are 17 braid positive prime knots with 14 crossings. All but 3 have non-alternating Jones polynomial.

**Problem:** Are all braid positive prime knots other than (2, n)-torus knots H-thick?

**Problem:** If K is braid positive, is  $\mathcal{H}^{1,j}(K) = 0$  for all j?

## 7 Alexander polynomial and cohomology

We say that a prime knot is Ap-special if its Alexander polynomial is not alternating or has a gap. A well-known theorem of Murasugi [Mu] can be restated as

**Proposition 9** Ap-special knots are not alternating.

Few small knots are Ap-special, and all or nearly all small Ap-special knots are H-thick:

- There are 9 Ap-special knots with at most 10 crossings. All of them are H-thick.
- There are 19 Ap-special knots with 11 crossings. All of them are H-thick.
- There are 104 Ap-special knots with 12 crossings. For all but 8 of them the Jones polynomial is not alternating, so that at least 96 of these knots are H-thick.
- There are 115 knots with 13 crossings and a gap in the Alexander polynomial. All but 13 have non-alternating Jones polynomial, thus, at least 102 of these knots are H-thick.

### **Problem:** Is any Ap-special knot H-thick?

Knots with non-alternating Jones polynomial are a minority among nonalternating knots with at most 14 crossings, as seen in the table below.

$\operatorname{crossings}$	$\leq 9$	10	11	12	13	14
non-alternating	11	42	185	888	5110	27110
Jones not alternating	0	7	26	169	1154	7075
H-thick	2	10	41	> 169	> 1154	> 7075

For instance, the fifth column says that there are 888 prime non-alternating knots with 12 crossings (not distinguishing mirror images); among them 169 have non-alternating Jones polynomial, and, therefore, at least 169 are H-thick. On the other hand, there is no doubt that for large n most n-crossing knots are H-thick.

The following examples provide another experimental relation between the Alexander polynomial and knot cohomology.

- 1. The only knot with the trivial Alexander polynomial and at most 10 crossings is the unknot. There are two 11-crossing, two 12-crossing, fifteen 13-crossing and thirty-six 14-crossing knots with the trivial Alexander polynomial. All of them are H-thick (since their Jones polynomials are not alternating).
- 2. The Alexander polynomial of the trefoil is  $t^{-1} 1 + t$ . There are no other knots with at most 12 crossings and this Alexander polynomial. There are eight 13-crossing knots and seventeen prime 14-crossing knots with this Alexander polynomial. All of them are H-thick (for the same reason).
- 3. The figure eight knot is the only one with less than 13 crossings and Alexander polynomial  $-t^{-1} + 3 t$ . There are two 13-crossing knots and fifteen 14-crossing knots with this Alexander polynomial. All are H-thick.

- 4.  $\Delta(5_2) = 2t^{-1} 3 + 2t$ . There are no other knots with this Alexander polynomial and less than 12 crossings. Four 12-crossing, three 13-crossing, and nine 14-crossing knots have Alexander polynomial  $2t^{-1} 3 + 2t$ . All of these knots are H-thick.
- 5. Consider knots with at most 14 crossings and Alexander polynomial  $-2t^{-1}+5-2t$ . Four of them:  $6_1$ ,  $9_{46}$ ,  $11_{139}^n$ , and  $13_{3523}^n$  are H-thin (these knots are examples of (n, -3, 3)-pretzel knots; any (n, -3, 3)-pretzel knot is slice, H-thin, and its cohomology has rank 10). The remaining two 11-crossing knots, four 12-crossing knots, eleven 13-crossing knots, and fifty 14-crossing knots with this polynomial are H-thick.
- 6.  $\Delta(5_1) = \Delta(10_{132}) = t^{-2} t^{-1} + 1 t + t^2$ .  $5_1$  is the (2,5)-torus knot and is H-thin.  $10_{132}$  is H-thick. There are no 11 and 12-crossing knots with this Alexander polynomial. Two 13-crossing knots and twelve 14-crossing knots have this Alexander polynomial. All are H-thick.
- 7.  $10_{153}$  is the only knot with at most 11 crossing and Alexander polynomial  $t^{-3} t^{-2} t^{-1} + 1 t t^2 + t^3$ . Four 12-crossing, seven 13-crossing and and nineteen 14-crossing knots have this Alexander polynomial. All are H-thick. Unlike other examples, this Alexander polynomial is not alternating.

These examples suggest that knots with small Alexander polynomial relative to the crossing number tend to be H-thick.

## 8 Volume and cohomology

H-thick knots with few crossings tend to have small hyperbolic volume or to be non-hyperbolic:

- 8<sub>19</sub> is the only H-thick knot with 8 crossings and the only non-hyperbolic knot with 8 crossings (it is the (3, 4)-torus knot).
- The H-thick knot 10<sub>124</sub> is the (3,5)-torus knot and the only non-hyperbolic 10-crossing knot.
- $9_{42}$ , the only H-thick 9-crossing knot, has the second smallest volume ( $\approx 4.05686$ ) among all 48 hyperbolic knots with 9 crossings (and the smallest determinant (= 7) among all 9-crossing knots).  $9_{42}$  has the same volume as  $10_{132}$ , another H-thick knot. The latter has the smallest volume among all hyperbolic knots with 10 crossings. Among known pairs of knots with the same volume,  $(9_{42}, 10_{132})$  is the pair with the second smallest volume. The pair with the smallest volume ( $\approx 2.8281$ ) consists of  $5_2$  and the famous (-2, 3, 7)-pretzel knot. Knot  $5_2$  is H-thin, while the (-2, 3, 7)-pretzel knot is H-thick, since its Jones polynomial is not alternating.

• Three out of the four hyperbolic 10-crossing knots with the smallest volumes are H-thick, even though among 164 hyperbolic 10-crossing knots only 9 are H-thick.

Determinant det(K) of a knot K is the determinant of the matrix  $M + M^T$  where M is a Seifert matrix of K. Determinant is a common specialization of the Alexander and Jones polynomials:

$$det(K) = \Delta_{-1}(K) = J_{\sqrt{-1}}(K)$$

|det(K)| is also the number of elements in the first homology group of the double cover of  $\mathbb{S}^3$  branched over K.

Nathan Dunfield documents a fascinating relation between determinants and volumes of hyperbolic knots [D]. First, he plots  $\log |det(K)|$  versus the volume of K for all alternating knots K with a fixed number of crossings. Amazingly, the points cluster around a straight line. Next, he combines the pictures into one by plotting  $\frac{\log |det(K)|}{\log(\deg J(K))}$  versus the volume of K for all alternating knots with at most 13 crossings and samples of 14-16 crossing alternating knots. Again, all points stay close to a straight line.

Dunfield comments: " $\log(J(-1))$  is one of the first terms in Kashaev's conjecture about the relationship between the colored Jones polynomial and hyperbolic volume. However, the above doesn't appear to simply be saying that you have fast convergence in Kashaev's conjecture as the slope of the line is not what you would expect."

When non-alternating knots are included, the plots become less impressive. The majority of points still lie close to the coveted straight line, but there are defections. For instance, there are hyperbolic knots with  $det(K) = \pm 1$ , and points assigned to them will lie on the x-axis, far away from where we would like them to. This is illustrated in figures 10, 11, where we plot  $(vol(K), \log |det(K)|)$  for all hyperbolic non-alternating knots with 10 and 11 crossings (for 12, 13 crossings consult [D]).

To save the day, we change from det(K) to the rank of the reduced cohomology group of K. The inequality

$$\mathrm{rank}\widetilde{\mathcal{H}}(K) \geq |det(K)|$$

is valid for all knots, and turns into equality for H-thin knots. If the knot is H-restricted (and we expect that all knots are),  $\operatorname{rank} \widetilde{\mathcal{H}}(K) = \operatorname{rank} \mathcal{H}(K) - 1$ . In figures 12, 13 we plot  $(\operatorname{vol}(K), \operatorname{log}(\operatorname{rank} \mathcal{H}(K) - 1))$  for all hyperbolic non-alternating knots with 10 and 11 crossings (there are 41, respectively 185, such knots).

Clearly, for non-alternating knots with 10 and 11 crossings the correlation between the volume and the rank of cohomology is even better than the one between the volume and the determinant. Somehow vol(K) and  $rank\mathcal{H}(K)$  are successful in spying on each other. We have no explanation for this behaviour.

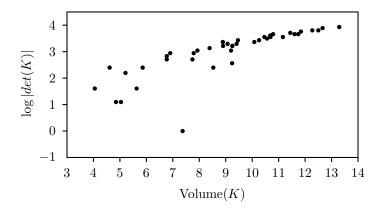


Figure 10: Volume versus  $\log |det(K)|$  for 10-crossing non-alternating knots

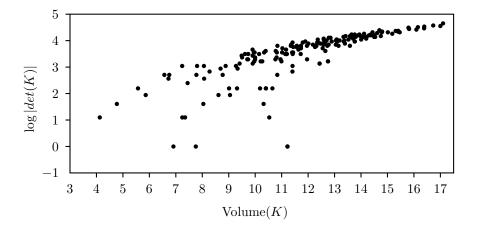


Figure 11: Volume versus  $\log |det(K)|$  for 11-crossing non-alternating knots

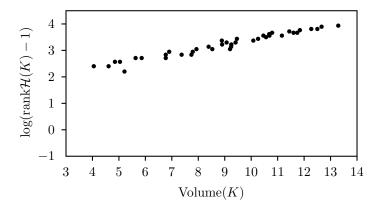


Figure 12: Volume versus  $\log(\mathrm{rank}\mathcal{H}(K)-1)$  for 10-crossing non-alternating knots

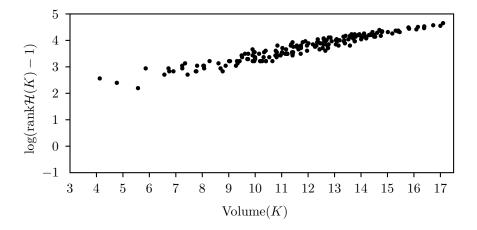


Figure 13: Volume versus  $\log(\mathrm{rank}\mathcal{H}(K)-1)$  for 11-crossing non-alternating knots

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