

# Construction techniques for cubical complexes, odd cubical 4-polytopes, and prescribed dual manifolds

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## Abstract

We provide a number of new construction techniques for cubical complexes and cubical polytopes, and thus for cubifications (hexahedral mesh generation).

Thus we obtain the first instance of a cubical 4-polytope that has a non-orientable dual manifold (a Klein bottle). This confirms the existence conjecture of Heteyi [17, Conj. 2, p. 325].

More systematically, we prove that every normal crossing codimension one immersion of a compact 2-manifold into  $\mathbb{R}^3$  is PL-equivalent to a dual manifold immersion of a cubical 4-polytope. As an instance we obtain a cubical 4-polytope with a cubation of Boy's surface as a dual manifold immersion, and with an odd number of facets. This solves problems of Eppstein, Thurston and others. Our explicit example has 19 520 vertices and 18 333 facets.

**Keywords:** Cubical complexes, cubical polytopes, regular subdivisions, normal crossing codimension one PL immersions, construction techniques, cubical meshes, Boy's surface

**MSC 2000 Subject Classification:** 52B12, 52B11, 52B05, 57Q05

## 1 Introduction

A  $d$ -polytope is *cubical* if all its proper faces are combinatorial cubes, that is, each  $k$ -face of the polytope,  $k \in \{0, \dots, d-1\}$  is combinatorially isomorphic to the  $k$ -dimensional standard cube. It known (cf. [3] [20]) that every cubical  $d$ -polytope  $P$  determines a PL immersion of an abstract cubical  $(d-2)$ -manifold into the polytope boundary  $\partial P \cong S^{d-1}$ . The immersed manifold is orientable if and only if the 2-skeleton of the cubical  $d$ -polytope ( $d \geq 3$ ) is “edge

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orientable” in the sense of Heteyi [17]. He conjecture that there are cubical 4-polytopes that are not edge-orientable.

In the more general setting of PL cubical  $(d - 1)$ -spheres, Babson and Chan [3] have observed that *every* type of normal crossing PL immersion of a PL  $(d - 2)$ -manifold into an  $(d - 1)$ -sphere appears among the dual manifolds of a cubical PL  $(d - 1)$ -sphere.

No similarly general result is available for cubical polytopes. The reason for this may be traced/blamed to a lack of flexible construction techniques for cubical polytopes, and more general, for cubical complexes (such as the “hexahedral meshes” that are of great interest in CAD and in Numerical Analysis).

In this paper, we develop a number of new and improved construction techniques for cubical polytopes. Here our point of view is that it always pays off to carry along convex lifting functions of high symmetry. The most complicated and subtle one of our constructions is the “generalized regular Hexhoop” construction of Section 6.2, which yields a cubification of a  $d$ -polytope with a hyperplane of symmetry, where a (suitable) lifting function may be specified on the boundary.

Our work is extended by the first author in [30], where additional construction techniques for *cubifications* (i.e. cubical subdivisions of  $d$ -polytopes with prescribed boundaries) are discussed.

Using the constructions developed here, we achieve the following constructions and results:

- A rather simple construction yields a cubical 4-polytope (with 72 vertices and 62 facets) for which the immersed dual 2-manifold is not orientable: One of its components is a Klein bottle. Apparently this is the first example of a cubical polytope with a non-orientable dual manifold. Thus its existence also confirms a conjecture of Heteyi (Section 5).
- More generally, all PL-types of normal crossing immersions of 2-manifolds appear as dual manifolds in the boundary complexes of cubical 4-polytopes (Section 7). In the case of non-orientable 2-manifolds of odd genus, this yields cubical 4-polytopes with an odd number of facets. From this, we also obtain a complete characterization of the lattice of  $f$ -vectors of cubical 4-polytopes (Section 9).
- In particular, we construct an explicit example with 19 520 vertices and 18 333 facets of a cubical 4-polytope with a cubation of Boy’s surface (projective plane with exactly one triple point) as a dual manifold immersion (Section 8).
- Via Schlegel diagrams, this implies that every 3-cube has a cubical subdivision into an even number of cubes that does not subdivide the boundary complex. Thus for every cubification of a 3-dimensional domain there is also a cubification of the opposite parity (Section 10). This answers questions by Eppstein and Thurston [5] [9] [34].

## 2 Basics

For the following we assume that the readers are familiar with the basic combinatorics and geometry of convex polytopes. In particular, we will be dealing with cubical polytopes (see Grünbaum [15, Sect. 4.6]), polytopal (e.g. cubical) complexes, regular subdivisions (see Ziegler [36, Sect. 5.1]), and Schlegel diagrams [15, Sect. 3.3] [36, Sect. 5.2]. For regular cell complexes, barycentric subdivision and related notions we refer to Munkres [25]. Suitable references for the basic concepts about PL manifolds, embeddings and (normal crossing) immersions include Hudson [19] and Rourke & Sanderson [29].

## 2.1 Almost cubical polytopes

All proper faces of a cubical  $d$ -polytope have to be combinatorial cubes. We define an *almost cubical*  $d$ -polytope as a pair  $(P, F)$ , where  $F$  is a specified facet of  $P$  such that all facets of  $P$  other than  $F$  are required to be combinatorial cubes. Thus,  $F$  need not be a cube, but it will be cubical.

By  $\mathcal{C}(P)$  we denote the polytopal complex given by a polytope  $P$  and all its faces. By  $\mathcal{C}(\partial P)$  we denote the *boundary complex* of  $P$ , consisting of all proper faces of  $P$ . If  $P$  is a cubical polytope, then  $\mathcal{C}(\partial P)$  is a cubical complex. If  $(P, F)$  is almost cubical, then the *Schlegel complex*  $\mathcal{C}(\partial P) \setminus \{F\}$  is a cubical complex that is combinatorially isomorphic to the Schlegel diagram  $\text{SCHLEGEL}(P, F)$  of  $P$  based on  $F$ .

## 2.2 Cubifications

A *cubification* of a cubical PL  $(d-1)$ -sphere  $\mathcal{S}^{d-1}$  is a cubical  $d$ -ball  $\mathcal{B}^d$  with boundary  $\mathcal{S}^{d-1}$ . A double counting argument shows that every cubical  $(d-1)$ -sphere that admits a cubification has an even number of facets. Whether this condition is sufficient is a challenging open problem, even for  $d=3$  (compare e.g. [5], [9]).

## 2.3 Dual manifolds

For every (pure) cubical  $d$ -dimensional complex  $\mathcal{C}$ ,  $d > 1$ , the *derivative complex* is an abstract cubical cell  $(d-1)$ -dimensional complex  $\mathcal{D}(\mathcal{C})$  whose vertices may be identified with the edge midpoints of the complex, while the facets “separate the opposite facets of a facet of  $\mathcal{C}$ ,” that is, they correspond to pairs  $(F, [e])$ , where  $F$  is a facet of  $\mathcal{C}$  and  $[e]$  denotes a “parallel class” of edges of  $F$ . Thus this is a cell complex with  $f_1(\mathcal{C})$  vertices and  $(d-1)f_{d-1}(\mathcal{C})$  cubical facets of dimension  $d-1$ ,  $d-1$  of them for each facet. Hence, the derivative complex  $\mathcal{D}(\mathcal{C})$  is a  $(d-1)$ -dimensional cubical complex.

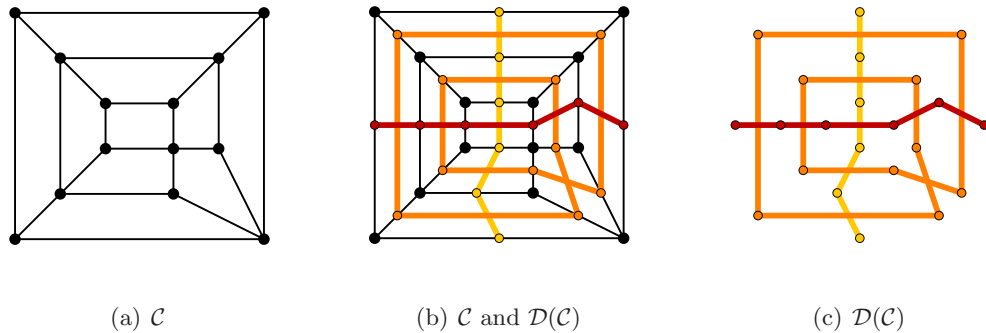


FIGURE 1: The derivative complex of a cubical 2-complex  $\mathcal{C}$ .

Most cubical complexes we consider in this paper are cubical spheres (for instance boundary complexes of cubical polytopes), or cubical balls. In this case, the derivate complex is a (not necessarily connected) manifold, and we call each connected component of the derivative complex  $\mathcal{D}(P)$  of a cubical complex  $\mathcal{C}$  a *dual manifold* of  $\mathcal{C}$ . If the cubical complex  $\mathcal{C}$  is a sphere than the dual manifolds of  $\mathcal{C}$  are manifolds without boundary. If  $\mathcal{C}$  is a cubical (PL)  $d$ -ball, then some

(possibly all) dual manifolds have non-empty boundary components, namely the dual manifolds of  $\partial\mathcal{C}$ .

The derivative complex, and thus each dual manifold, comes with a canonical immersion into the boundary of  $P$ . More precisely, the barycentric subdivision of  $\mathcal{D}(P)$  has a simplicial map to the barycentric subdivision of the boundary complex  $\partial P$ , which is a codimension one normal crossing immersion into the simplicial sphere  $\text{sd}(\mathcal{C}(\partial P))$ . (*Normal crossing* means that each multiple-intersection point is of degree  $k \leq d$  and there is a neighborhood of each multiple intersection point that is PL isomorphic to (a neighborhood of) a point which is contained in  $k$  pair-wise perpendicular hyperplanes.)

Restricted to a dual manifold, this immersion may be an embedding or not.

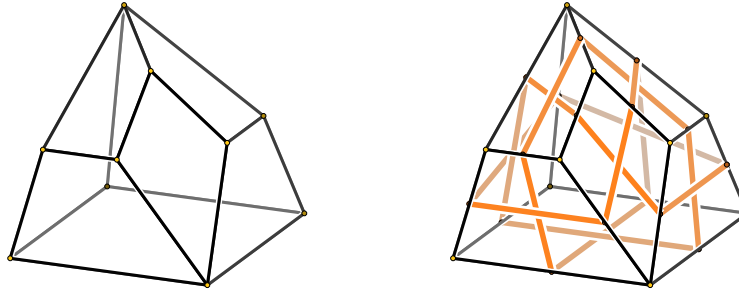


FIGURE 2: The cubical octahedron  $O_8$  (the only combinatorial type of a cubical 3-polytope with 8 facets), and its single immersed dual manifold.

In the case of cubical 3-polytopes, the derivative complex may consist of one or many 1-spheres. For example, for the 3-cube it consists of 3 1-spheres, while for the “cubical octahedron”  $O_8$  displayed in Figure 2 the dual manifold is a single immersed  $S^1$  (with 8 double points).

In the case of 4-polytopes, the dual manifolds are surfaces (compact 2-manifolds without boundary). As an example, we here display a Schlegel diagram of a “neighborly cubical” 4-polytope (with the graph of the 5-cube), with  $f$ -vector  $(32, 80, 96, 48)$ .

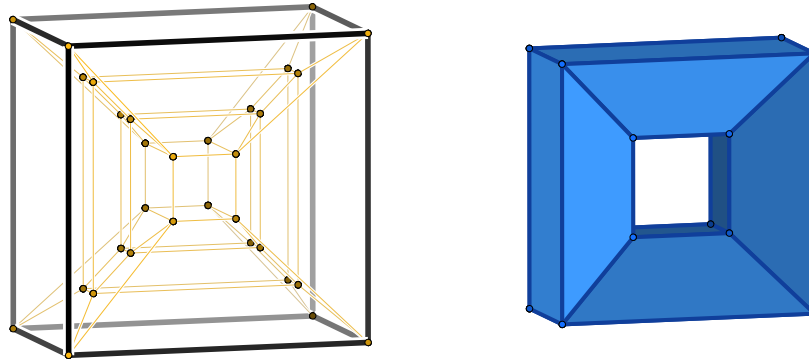


FIGURE 3: A Schlegel diagram of the “neighborly cubical” 4-polytope  $C_4^5$  with the graph of the 5-cube, and its dual torus. All other dual manifolds are embedded 2-spheres.

According to Joswig & Ziegler [21] this may be constructed as

$$C_4^5 := \text{conv}((Q \times 2Q) \cup (2Q \times Q)), \quad \text{where } Q = [-1, +1]^2.$$

Here the dual manifolds are four embedded cubical 2-spheres  $S^2$  with  $f$ -vector  $(16, 28, 14)$  — of two different combinatorial types — and one embedded torus  $T$  with  $f$ -vector  $(16, 32, 16)$ .

## 2.4 Orientability

Let  $P$  be a cubical  $d$ -polytope ( $d \geq 3$ ). The immersed dual manifolds in its boundary are crossed transversally by the edges of the polytope.

Thus we find that orientability of the dual manifolds is equivalent to the possibility to give *consistent* edge orientations to the edges of the  $P$ , that is, in each 2-face of  $P$  opposite edges get parallel (rather than antiparallel) orientations; compare Heteyi [17]. Figure 4 shows such an edge orientation for a cubical 3-polytope (whose derivative complex consists of three circles, so it has 8 consistent edge orientations in total).

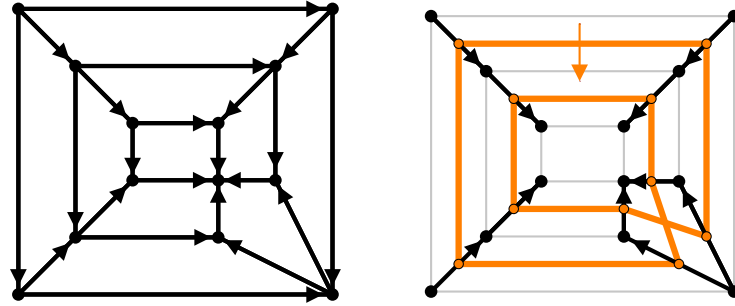


FIGURE 4: An edge-orientation of (the Schlegel diagram) of a cubical 3-polytope with 13 vertices and 3 dual circles. The second figure shows on class of edges that must be oriented consistently.

One can attempt to obtain such edge orientations by moving from edge to edge across 2-faces. The obstruction to this arises if on a path moving from edge to edge across quadrilateral 2-faces we return to an already visited edge, with reversed orientation, that is, if we close a *cubical Möbius strip* with parallel inner edges, as displayed in the figure. (Such an immersion would not necessarily be embedded, that is, some 2-face may be used twice for the Möbius strip.)

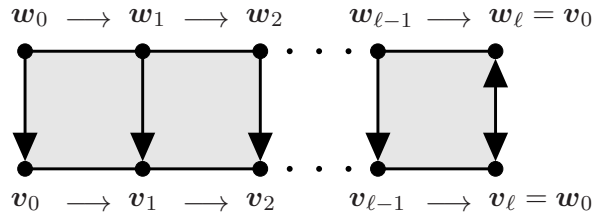


FIGURE 5: A cubical Möbius strip with parallel inner edges.

**Proposition 2.1.** *For every cubical  $d$ -polytope ( $d \geq 3$ ), the following are equivalent:*

- All dual manifolds of  $P$  are orientable.
- The 2-skeleton of  $P$  has a consistent edge orientation.
- The 2-skeleton of  $P$  does not admit an immersion of a cubical Möbius strip with parallel inner edges.

## 2.5 From PL immersions to cubical PL spheres

The emphasis in this paper is on cubical convex  $d$ -polytopes. In the more general setting of cubical PL  $(d-1)$ -spheres, one has more flexible tools available. In this setting, Babson & Chan [3] proved that “all PL codimension 1 normal crossing immersions appear.” The following sketch is meant to explain the Babson-Chan theorem geometrically (it is presented in a combinatorial framework and terminology in [3]), and to briefly indicate which parts of their construction are available in the polytope world.

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### Construction 1: BABSON-CHAN [3]

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**Input:** A normal crossing immersion  $j : \mathcal{M}^{d-2} \rightarrow \mathcal{S}^{d-1}$  of a triangulated PL manifold  $\mathcal{M}^{d-2}$  of dimension  $d-2$  into a PL simplicial  $(d-1)$ -sphere.

**Output:** A cubical PL  $(d-1)$ -sphere with a dual manifold immersion PL-equivalent to  $j$ .

- (1) Perform a barycentric subdivision on  $\mathcal{M}^{d-2}$  and  $\mathcal{S}^{d-1}$ .

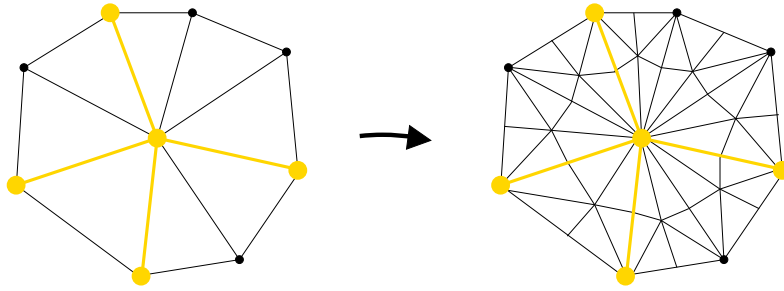


FIGURE 6: Step 1. Performing a barycentric subdivision. (We illustrate the impact of the construction on 2-ball, which might be part of the boundary of a 2-sphere. The immersion which is shown in bold has a single double-intersection point.)

(Here each  $i$ -simplex is replaced by  $(i+1)!$  new  $i$ -simplices, which is an even number for  $i > 0$ . This step is done only to ensure parity conditions on the  $f$ -vector, especially that the number of facets of the final cubical sphere is congruent to the Euler characteristic of  $\mathcal{M}^{d-2}$ . Barycentric subdivisions are easily performed in the polytopal category as well, see Ewald & Shephard [10].)

- (2) Perform a “cubical barycentric subdivision” on  $\mathcal{M}^{d-2}$  and  $\mathcal{S}^{d-1}$ .

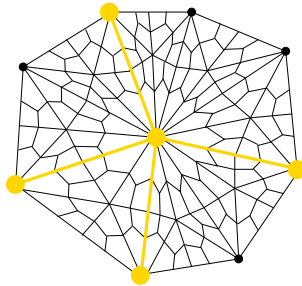


FIGURE 7: Step 2. Performing a cubical barycentric subdivision.

(This is the standard tool for passage from a simplicial complex to a PL-homeomorphic cubical complex; here every  $i$ -simplex is subdivided into  $i+1$  different  $i$ -cubes. Such

cubations can be performed in the polytopal category according to Shephard [31]: If the starting triangulation of  $\mathcal{S}^{d-1}$  was polytopal, the resulting cubation will be polytopal as well.)

- (3) “Thicken” the cubical  $(d-1)$ -sphere along the immersed  $(d-2)$ -manifold, to obtain the cubical  $(d-1)$ -sphere  $BC(\mathcal{S}^{d-1}, j(\mathcal{M}^{d-2}))$ .

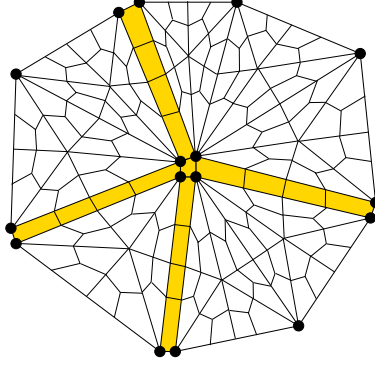


FIGURE 8: The outcome of the Babson-Chan construction: A cubical sphere with a dual manifold immersion that is PL-equivalent to the input immersion  $j$ .

(In this step, every  $(d-1-i)$ -cube in the  $i$ -fold multiple point locus results in a new  $(d-1)$ -cube. The original immersed manifold, in its cubified subdivided version, now appears as a dual manifold in the newly resulting  $(d-1)$ -cubes. This last step is the one that seems hard to perform for polytopes in any non-trivial instance.)

### 3 Lifting polytopal subdivisions

#### 3.1 Regular balls

In the following, the primary object we deal with is a *regular ball*: a regular polytopal subdivision  $\mathcal{B}$  of a convex polytope  $P = |\mathcal{B}|$ . A polytopal subdivision  $\mathcal{B}$  is *regular* (also known as *coherent* or *projective*) if it admits a *lifting function*, that is, a concave function  $f : |P| \rightarrow \mathbb{R}$  whose domains of linearity are the facets of the subdivision. (A function  $g : D \rightarrow \mathbb{R}$  is *concave* if for all  $\mathbf{x}, \mathbf{y} \in D$  and  $0 < \lambda < 1$  we have  $g(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda g(\mathbf{x}) + (1-\lambda)g(\mathbf{y})$ .)

In this definition, subdivisions of the boundary are allowed, that is, we do not necessarily require that the faces of  $P = |\mathcal{B}|$  are themselves faces in  $\mathcal{B}$ .

In the sequel we focus on regular *cubical* balls. Only in some cases we consider regular non-cubical balls.

**Example.** If  $(P, F)$  is an almost cubical polytope, then the Schlegel diagram based on  $F$ , which we denote by  $\text{SCHLEGEL}(P, F)$ , is a regular cubical ball (without subdivision of the boundary).

**Lemma 3.1.** *If  $\mathcal{B}$  is a regular cubical  $d$ -ball, then there is a regular cubical ball  $\mathcal{B}'$  without subdivision of the boundary, combinatorially isomorphic to  $\mathcal{B}$ .*

*Proof.* Using a positive lifting function  $f : |\mathcal{B}| \rightarrow \mathbb{R}$ , the  $d$ -ball  $\mathcal{B}$  may be lifted to  $\tilde{\mathcal{B}}$  in  $\mathbb{R}^{d+1}$ , by mapping each  $\mathbf{x} \in |\mathcal{B}|$  to  $(\mathbf{x}, f(\mathbf{x})) \in \mathbb{R}^{d+1}$ .

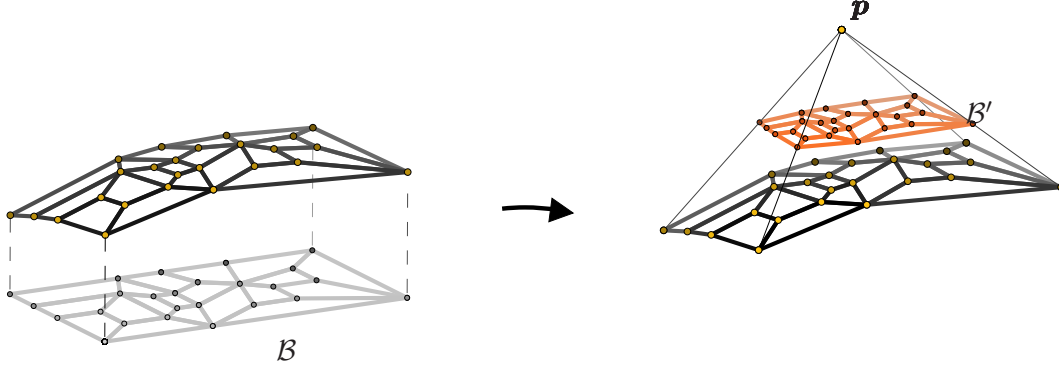


FIGURE 9: Illustration of the ‘convexification’ of a regular ball (Lemma 3.1).

Viewed from  $\mathbf{p} := \lambda \mathbf{e}_{d+1}$  for sufficiently large  $\lambda$ , this lifted ball will appear to be *strictly convex*, that is, its boundary is a convex polytope (rather than a boundary subdivision of a convex polytope). Thus one may look at the polytopal complex that consists of the cones spanned by faces of  $\tilde{\mathcal{B}}$  with apex  $\mathbf{p}$ . This polytopal complex is regular, since it appears convex when viewed from  $\mathbf{p}$ , which yields a lifting function for the restriction of  $\tilde{\mathcal{B}}$  to the hyperplane given by  $x_{d+1} = 0$ , which may be taken to be  $\mathcal{B}'$ .  $\square$

### 3.2 Lifted balls

When constructing cubical complexes we often deal with regular cubical balls which are equipped with a lifting function. A *lifted  $d$ -ball* is a pair  $(\mathcal{B}, h)$  consisting of a regular  $d$ -ball  $\mathcal{B}$  and a lifting function  $h$  of  $\mathcal{B}$ . The *lifted boundary* of a lifted ball  $(\mathcal{B}, h)$  is the pair  $(\partial\mathcal{B}, h|_{\partial\mathcal{B}})$ .

If  $(\mathcal{B}, h)$  is a lifted  $d$ -ball in  $\mathbb{R}^{d'}$  then  $\text{lift}(\mathcal{B}, h)$  denotes the copy of  $\mathcal{B}$  in  $\mathbb{R}^{d'+1}$  with vertices  $(\mathbf{v}, h(\mathbf{v})) \in \mathbb{R}^{d'+1}$ ,  $\mathbf{v} \in \text{vert}(\mathcal{B})$ . (In the sequel we sometimes do not distinguish between these two interpretations of a lifted ball.) We rely on Figure 10 for the illustration of this correspondence.

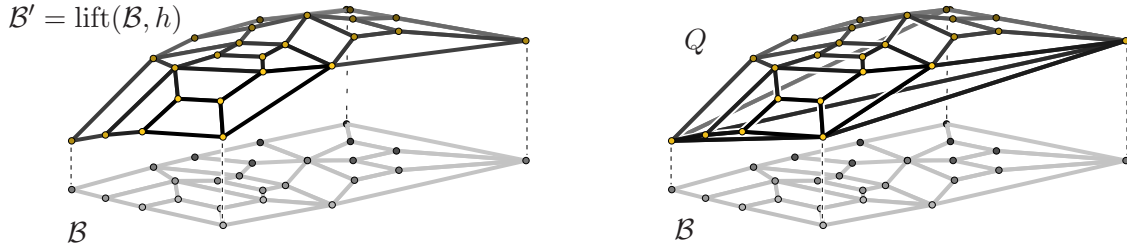


FIGURE 10: A lifted cubical ball  $(\mathcal{B}, h)$  and its lifted copy  $\text{lift}(\mathcal{B}, h)$ . The figure on the right shows the convex hull  $Q = \text{conv}(\text{lift}(\mathcal{B}, h))$ .

*Notation.* We identify  $\mathbb{R}^d$  with  $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$ , and decompose a point  $\mathbf{x} \in \mathbb{R}^{d+1}$  as  $\mathbf{x} = (\pi(\mathbf{x}), \gamma(\mathbf{x}))$ , where  $\gamma(\mathbf{x})$  is the last coordinate of  $\mathbf{x}$  and  $\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  is the projection that eliminates the last coordinate.

Often a lifted ball  $(\mathcal{B}, \psi)$  is constructed as follows: Let  $P$  be a  $d$ -polytope (in  $\mathbb{R}^d$ ) and  $Q \subset \mathbb{R}^{d+1}$  a polytope such that  $\pi(Q) = P$ . Then the complex  $\mathcal{B}'$  given as the set of upper faces of  $Q$



determines a lifted polytopal subdivision  $(\mathcal{B}, \psi)$  of  $P$  (where  $\mathcal{B} := \pi(\mathcal{B}')$  and  $\psi$  is determined the vertex heights  $\gamma(\mathbf{v})$ ,  $\mathbf{v} \in \text{vert}(\mathcal{B}')$ ). Hence  $\text{lift}(\mathcal{B}, \psi)$  equals  $\mathcal{B}'$ . Compare again Figure 10.

*lifted boundary subdivision* of a  $d$ -polytope  $P$  is a pair  $(\mathcal{S}^{d-1}, \psi)$  consisting of a polytopal subdivision  $\mathcal{S}^{d-1}$  of the boundary of  $P$  and a piece-wise linear function  $\psi : |\partial P| \rightarrow \mathbb{R}$  such that for each facet  $F$  of  $P$  the restriction of  $\psi$  on  $F$  is a lifting function of the induced subdivision  $\mathcal{S}^{d-1} \cap F$  of  $F$ .

### 3.3 Patching lemma

Often regular cubical balls are constructed from other regular balls. The following “patching lemma”, which appears frequently in the construction of regular subdivisions (see [22, Cor. 1.12] or [7, Lemma 3.2.2]), helps to produce regular balls.

*Notation.* For a  $d$ -polytope  $P \subset \mathbb{R}^d$ , a polytopal subdivision  $\mathcal{T}$  of  $P$  and a hyperplane  $H$  in  $\mathbb{R}^d$ , we denote by  $\mathcal{T} \cap H$  the *restriction* of  $\mathcal{T}$  to  $H$ , which is given by

$$\mathcal{T} \cap H := \{F \cap H : F \in \mathcal{T}\}.$$

For two  $d$ -polytopes  $P, Q$  with  $Q \subset P$  and a polytopal subdivision  $\mathcal{T}$  of  $P$  we denote by  $\mathcal{T} \cap Q$  the *restriction* of  $\mathcal{T}$  to  $Q$ , which is given by

$$\mathcal{T} \cap Q := \{F \cap Q : F \in \mathcal{T}\}.$$

**Lemma 3.2 (“Patching lemma”).** *Let  $Q$  be a  $d$ -polytope. Assume we are given the following data:*

- ▷ *A regular polytopal subdivision  $\mathcal{S}$  of  $Q$  (the “raw subdivision”).*
- ▷ *For each facet  $F$  of  $\mathcal{S}$ , a regular polytopal subdivision  $\mathcal{T}_F$  of  $F$ , such that  $\mathcal{T}_F \cap F' = \mathcal{T}_{F'} \cap F$  for all facets  $F, F'$  of  $\mathcal{S}$ .*
- ▷ *For each facet  $F$  of  $\mathcal{S}$ , a concave lifting function  $h_F$  of  $\mathcal{T}_F$ , such that  $h_F(\mathbf{x}) = h_{F'}(\mathbf{x})$  for all  $\mathbf{x} \in F \cap F'$ , where  $F, F'$  are facets of  $\mathcal{S}$ .*

*Then this uniquely determines a regular polytopal subdivision  $\mathcal{U} = \bigcup_F \mathcal{T}_F$  of  $Q$  (the “fine subdivision”). Furthermore, for every lifting function  $g$  of  $\mathcal{S}$  there exists a small  $\varepsilon_0 > 0$  such that for all  $\varepsilon_0 > \varepsilon > 0$  the function  $g + \varepsilon h$  is a lifting function of  $\mathcal{U}$ , where  $h$  is the piece-wise linear function  $h : |Q| \rightarrow \mathbb{R}$  which on each  $F \in \mathcal{S}$  is given by  $h_F$ .*

(By  $\text{fac}(\mathcal{S})$  we denote the set of facets of a complex  $\mathcal{S}$ .)

*Proof.* Let  $g$  be a lifting function of  $\mathcal{S}$ . For a parameter  $\varepsilon > 0$  we define a piece-wise linear function  $\phi_\varepsilon : |P| \rightarrow \mathbb{R}$  that on  $\mathbf{x} \in F \in \text{fac}(\mathcal{S})$  takes the value  $\phi_\varepsilon(\mathbf{x}) = g(\mathbf{x}) + \varepsilon h_F(\mathbf{x})$ . (It is well-defined since the  $h_F$  coincide on the ridges of  $\mathcal{S}$ .) The domains of linearity of  $\phi_\varepsilon$  are given by the facets of the “fine” subdivision  $\mathcal{U}$ . If  $\varepsilon$  tends to zero then  $\phi_\varepsilon$  tends to the concave function  $g$ . This implies that there exists a small  $\varepsilon_0 > 0$  such that  $\phi_\varepsilon$  is concave and thus a lifting function of  $\mathcal{U}$ , for all  $\varepsilon_0 > \varepsilon > 0$ .  $\square$

### 3.4 Products and prisms

**Lemma 3.3 (“Product lemma”).** *Let  $(\mathcal{B}_1, h_1)$  be a lifted cubical  $d_1$ -ball in  $\mathbb{R}^{d_1}$  and  $(\mathcal{B}_2, h_2)$  be a lifted cubical  $d_2$ -ball in  $\mathbb{R}^{d_2}$ .*

*Then the product  $\mathcal{B}_1 \times \mathcal{B}_2$  of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  is a regular cubical  $(d_1 + d_2)$ -ball in  $\mathbb{R}^{d_1 + d_2}$ .*

*Proof.* Each cell of  $\mathcal{B}_1 \times \mathcal{B}_2$  is a product of two cubes. Hence  $\mathcal{B}_1 \times \mathcal{B}_2$  is a cubical complex. A lifting function  $h$  of  $\mathcal{B}_1 \times \mathcal{B}_2$  is given by the sum of  $h_1$  and  $h_2$ , that is, by  $h((\mathbf{x}, \mathbf{y})) := h_1(\mathbf{x}) + h_2(\mathbf{y})$ , for  $\mathbf{x} \in |\mathcal{B}_1|, \mathbf{y} \in |\mathcal{B}_2|$ .  $\square$

As a consequence, the *prism*  $\text{prism}(\mathcal{C})$  over a cubical  $d$ -complex  $\mathcal{C}$  yields a cubical  $(d + 1)$ -dimensional complex. Furthermore, the prism over a regular cubical ball  $\mathcal{B}$  yields a regular cubical  $(d + 1)$ -ball.

### 3.5 Piles of cubes

For integers  $\ell_1, \dots, \ell_d \geq 1$ , the *pile of cubes*  $P_d(\ell_1, \dots, \ell_d)$  is the cubical  $d$ -ball formed by all unit cubes with integer vertices in the  $d$ -polytope

$$P := [0, \ell_1] \times \dots \times [0, \ell_d],$$

that is the cubical  $d$ -ball formed by the set of all  $d$ -cubes

$$C(k_1, \dots, k_d) := [k_1, k_1 + 1] \times \dots \times [k_d, k_d + 1]$$

for integers  $0 \leq k_i < \ell_i$  together with their faces (cf. [36, Sect. 5.1]).

The pile of cubes  $P_d(\ell_1, \dots, \ell_d)$  is a product of 1-dimensional subdivisions, which are regular. Hence the product lemma implies that  $P_d(\ell_1, \dots, \ell_d)$  is a regular cubical subdivision of the  $d$ -polytope  $P$ .

### 3.6 Connector polytope

The following construction yields a “connector” polytope that may be used to attach cubical 4-polytopes resp. regular cubical 4-balls without the requirement that the attaching facets are projectively equivalent.

**Lemma 3.4.** *For any combinatorial 3-cube  $F$  there is a combinatorial 4-cube  $C$  that has both (a projective copy of)  $F$  and a regular 3-cube  $F'$  as (adjacent) facets.*

*Proof.* After a suitable projective transformation, we may assume that  $F \subset \mathbb{R}^3$  has a unit square  $Q$  as a face. Now the prism  $F \times I$  over  $F$  has  $F$  and  $Q \times I$  as adjacent facets, where the latter is a unit cube.  $\square$

## 4 Basic construction techniques

### 4.1 Lifted prisms

While there appears to be no simple construction that would produce a cubical  $(d + 1)$ -polytope from a given cubical  $d$ -polytope, we do have a simple prism construction that produces regular cubical  $(d + 1)$ -balls from regular cubical  $d$ -balls.

---

**Construction 2: LIFTED PRISM**


---

**Input:** A lifted cubical  $d$ -ball  $(\mathcal{B}, h)$ .

**Output:** A lifted cubical  $(d + 1)$ -ball  $\text{LIFTEDPRISM}(\mathcal{B}, h)$   
which is combinatorially isomorphic to the prism over  $\mathcal{B}$ .

We may assume that the convex lifting function  $h$  defined on  $P := |\mathcal{B}|$  is strictly positive. (Otherwise substitute  $h$  by  $h - C + M$ , where  $M > 0$  is constant and  $C := \max\{0, \min\{h(\mathbf{x}) : \mathbf{x} \in |\mathcal{B}|\}\}$ .)

Then the lifted facets of  $\text{LIFTEDPRISM}(\mathcal{B}, h)$  may be taken to be the sets

$$\tilde{F} := \{(\mathbf{x}, t, h(\mathbf{x})) : \mathbf{x} \in F, -h(\mathbf{x}) \leq t \leq +h(\mathbf{x})\}, \quad F \in \text{fac}(\mathcal{B}).$$

---

If  $\mathcal{B}$  does not subdivide the boundary of  $P$ , then  $\text{LIFTEDPRISM}(\mathcal{B}, h)$  does not subdivide the boundary of  $|\text{LIFTEDPRISM}(\mathcal{B}, h)|$ . In this case  $\hat{P} := |\text{LIFTEDPRISM}(\mathcal{B}, h)|$  is a cubical  $(d + 1)$ -polytope whose boundary complex is combinatorially isomorphic to the boundary of the prism over  $\mathcal{B}$ . The  $f$ -vector of  $\hat{P}$  is then given by

$$f_k(\hat{P}) = \begin{cases} 2f_0(\mathcal{B}) & \text{for } k = 0, \\ 2f_k(\mathcal{B}) + f_{k-1}(\partial\mathcal{B}) & \text{for } 0 < k \leq d, \\ f_d(\partial\mathcal{B}) & \text{for } k = d + 1. \end{cases}$$

Figure 11 shows the lifted prism over a lifted cubical 2-ball.

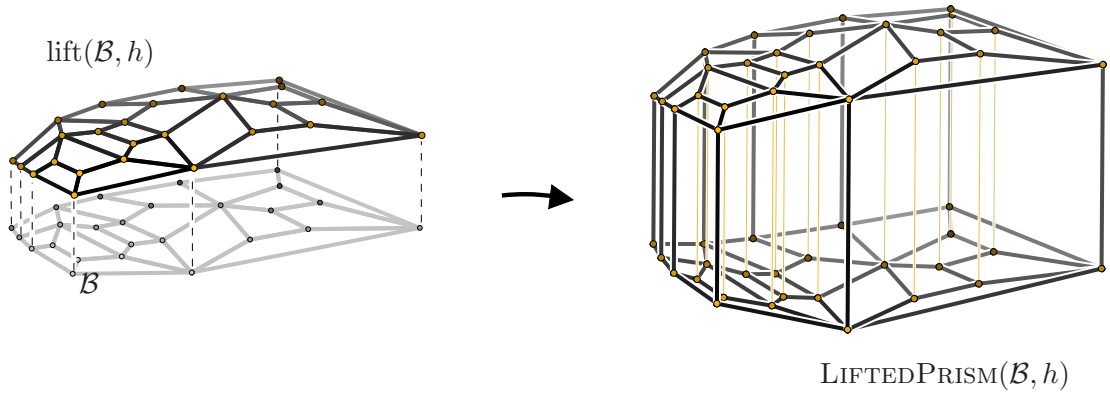


FIGURE 11: The lifted prism of a lifted cubical  $d$ -ball  $(\mathcal{B}, h)$ , displayed for  $d = 2$ . The outcome is a (regular) cubical  $(d + 1)$ -ball which is combinatorially isomorphic to the prism over  $\mathcal{B}$ .

**Remark 1 (Dual manifolds).** Up to PL-homeomorphism, the cubical ball  $\text{LIFTEDPRISM}(\mathcal{B}, h)$  has the following dual manifolds:

- $\mathcal{N} \times I$  for each dual manifold  $\mathcal{N}$  of  $\mathcal{B}$ ,
- one  $(d - 1)$ -sphere combinatorially isomorphic to  $\partial\mathcal{B}$ .

## 4.2 Lifted prisms over two balls

Another modification of this construction is to take two different lifted cubical balls  $(\mathcal{B}_1, h_1)$  and  $(\mathcal{B}_2, h_2)$  with the same lifted boundary complex (that is  $\partial\mathcal{B}_1 = \partial\mathcal{B}_2$  and  $h_1(\mathbf{x}) = h_2(\mathbf{x})$  for all  $\mathbf{x} \in \partial\mathcal{B}_1 = \partial\mathcal{B}_2$ ) as input. In this case the outcome is a cubical  $(d+1)$ -polytope which may not even have a cubification.

---

### Construction 3: LIFTED PRISM OVER TWO BALLS

---

**Input:** Two lifted cubical  $d$ -balls  $(\mathcal{B}_1, h_1)$  and  $(\mathcal{B}_2, h_2)$  with the same lifted boundary.

**Output:** A cubical  $(d+1)$ -polytope  $\text{LIFTEDPRISM}((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$  with lifted copies of  $\mathcal{B}_1$  and  $\mathcal{B}_2$  in its boundary.

If both balls do not subdivide their boundaries, we set  $\mathcal{B}'_k := \mathcal{B}_k$  and  $h'_k := h_k$  for  $k \in \{1, 2\}$ . Otherwise we apply the construction of the proof of Lemma 3.1 simultaneously to both lifted cubical balls  $(\mathcal{B}_1, h_1)$  and  $(\mathcal{B}_2, h_2)$  to obtain two lifted cubical  $d$ -balls  $(\mathcal{B}'_1, h'_1)$  and  $(\mathcal{B}'_2, h'_2)$  with the same support  $Q = |\mathcal{B}_1| = |\mathcal{B}_2|$  which do not subdivide the boundary of  $Q$ .

As in the previous construction we can assume that  $h'_1, h'_2$  are strictly positive. Then  $\widehat{Q} := \text{LIFTEDPRISM}((\mathcal{B}_1, h_1), (\mathcal{B}_2, h_2))$  is the convex hull of the points

$$\{(\mathbf{x}, +h'_1(\mathbf{x})) : \mathbf{x} \in |\mathcal{B}'_1|\} \cup \{(\mathbf{x}, -h'_2(\mathbf{x})) : \mathbf{x} \in |\mathcal{B}'_2|\},$$

Since both  $\mathcal{B}'_1$  and  $\mathcal{B}'_2$  do not subdivide their boundaries, each of their proper faces yields a face of  $\widehat{Q}$ . Furthermore,  $\widehat{Q}$  is a cubical  $(d+1)$ -polytope whose  $f$ -vector is given by

$$f_k(\widehat{Q}) = \begin{cases} f_0(\mathcal{B}_1) + f_0(\mathcal{B}_2) & \text{for } k = 0, \\ f_k(\mathcal{B}_1) + f_k(\mathcal{B}_2) + f_{k-1}(\partial\mathcal{B}_1) & \text{for } 0 < k \leq d, \\ f_d(\partial\mathcal{B}_1) & \text{for } k = d + 1. \end{cases}$$


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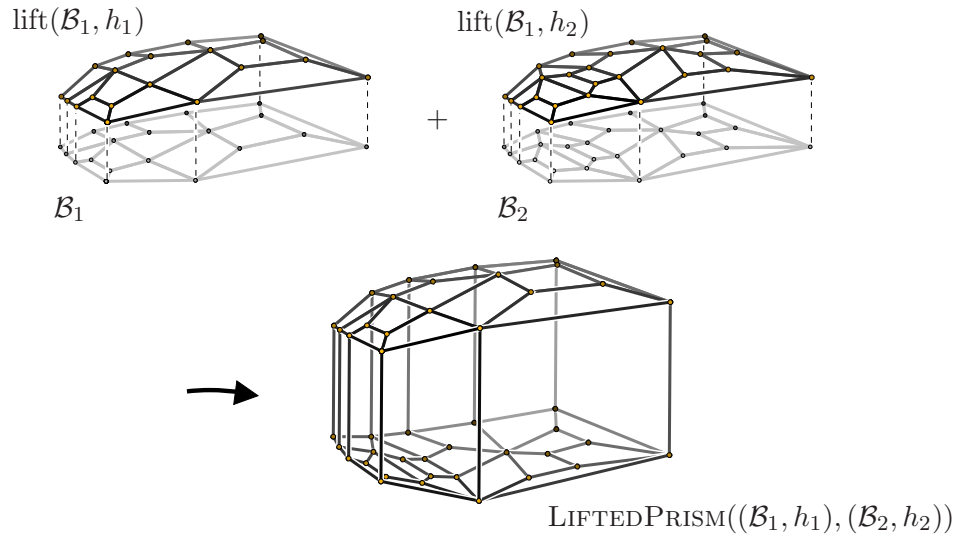


FIGURE 12: The lifted prism over two lifted cubical  $d$ -balls  $(\mathcal{B}_1, h_1)$  and  $(\mathcal{B}_2, h_2)$ , displayed for  $d = 2$ . The outcome is a cubical  $(d+1)$ -polytope.

The following is a projective variant of the prism construction, applied to a  $d$ -polytope  $P$ .

### 4.3 Schlegel caps

---

#### Construction 4: SCHLEGEL CAP

---

**Input:** An almost cubical  $d$ -polytope  $(P, F_0)$  and a point  $\mathbf{x}_0 \in \mathbb{R}^d$  beyond  $F_0$  (and beneath all other facets of  $P$ ; cf. [15, Sect. 5.2]).

**Output:** A regular cubical  $d$ -ball  $\text{SCHLEGELCAP}(P)$ , with  

$$P \subset |\text{SCHLEGELCAP}(P)| \subset \text{conv}(P \cup \mathbf{x}_0),$$
which is combinatorially isomorphic to the prism over  $\text{SCHLEGEL}(P, F)$ .

The construction of the Schlegel cap depends on a further piece of input data, namely a hyperplane  $H$  that separates  $\mathbf{x}_0$  from  $P$ . It is obtained as follows:

- (1) Apply a projective transformation that moves  $\mathbf{x}_0$  to infinity while fixing  $H$ . This projective transformation moves the Schlegel complex  $\mathcal{C}(\partial P) \setminus \{F_0\}$  to a new cubical complex  $\mathcal{E}$ .
- (2) Reflect the image  $\mathcal{E}$  of the Schlegel complex in  $H$ , and call its reflected copy  $\mathcal{E}'$ .
- (3) Build the polytope bounded by  $\mathcal{E}$  and  $\mathcal{E}'$ .
- (4) Reverse the projective transformation of (1).

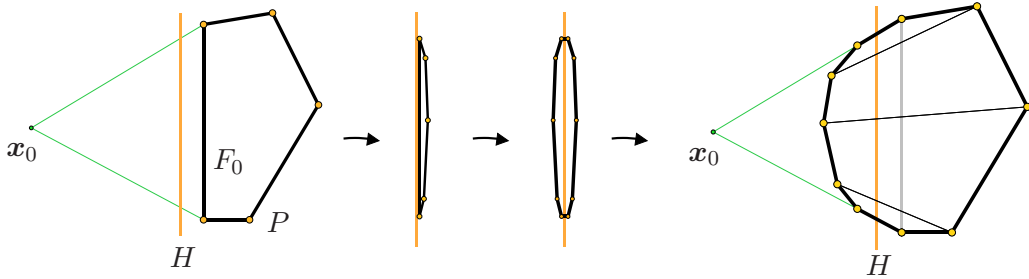


FIGURE 13: Construction steps of the Schlegel cap over an almost cubical polytope.

An alternative description, avoiding projective transformations, is as follows:

- (1) For each point  $\mathbf{x}$  in the Schlegel complex  $\mathcal{C}(\partial P) \setminus \{F_0\}$  let  $\bar{\mathbf{x}}$  be the intersection point of  $H$  and the segment  $[\mathbf{x}_0, \mathbf{x}]$ , and let  $\mathbf{x}'$  be the point on the segment  $[\mathbf{x}_0, \mathbf{x}]$  such that  $[\mathbf{x}_0, \bar{\mathbf{x}}; \mathbf{x}', \mathbf{x}]$  form a harmonic quadruple (cross ratio  $-1$ ).  
That is, if  $\vec{v}$  is a direction vector such that  $\mathbf{x} = \mathbf{x}_0 + t\vec{v}$  for some  $t > 1$  denotes the difference  $\mathbf{x} - \mathbf{x}_0$ , while  $\bar{\mathbf{x}} = \mathbf{x}_0 + \vec{v}$  lies on  $H$ , then  $\mathbf{x}' = \mathbf{x}_0 + \frac{t}{2t-1}\vec{v}$ .
  - (2) For each face  $G$  of the Schlegel complex,  $G' := \{\mathbf{x}' : \mathbf{x} \in G\}$  is the “projectively reflected” copy of  $G$  on the other side of  $H$ .
  - (3) Now the Schlegel cap in this model is the regular polytopal ball with faces  $G, G'$  and  $\text{conv}(G \cup G')$  for faces  $G$  in the Schlegel complex.
- 

Again we rely on our figures to illustrate this.

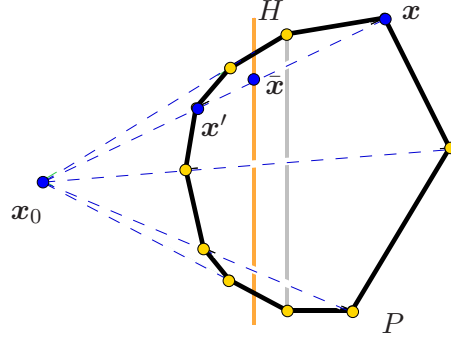


FIGURE 14: Constructing the Schlegel cap via cross ratios.

## 5 A small cubical 4-polytope with a dual Klein bottle

In this section we present the first instance of a cubical 4-polytope with a non-orientable dual manifold. By Proposition 2.1 this instance is not edge-orientable. Hence, its existence also confirms the conjecture of Heteyi [17, Conj. 2, p. 325]. Apparently this is the first example of a cubical polytope with a non-orientable dual manifold.

**Theorem 5.1.** *There is a cubical 4-polytope  $P_{72}$  with  $f$ -vector*

$$f(P_{72}) = (72, 196, 186, 62),$$

*one of whose dual manifolds is an immersed Klein bottle of  $f$ -vector  $(80, 160, 80)$ .*

**Step 1.** We start with a cubical octahedron  $O_8$ , the smallest cubical 3-polytope that is not a cube, with  $f$ -vector

$$f(O_8) = (10, 16, 8).$$

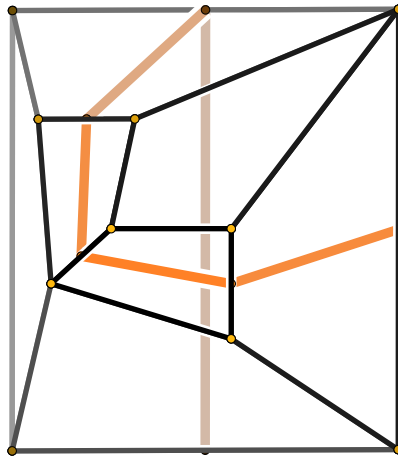


FIGURE 15: The cubical octahedron  $O_8$  positioned in  $\mathbb{R}^3$  with a regular square base facet  $Q$  and acute dihedral angles at this square base. Furthermore, a subcomplex of the dual manifold is highlighted.

The  $f$ -vector of any Schlegel diagram of this is

$$f(\text{SCHLEGEL}(O_8)) = (10, 16, 7).$$

We may assume that  $O_8$  is already positioned in  $\mathbb{R}^3$  with a regular square base facet  $Q$  and acute dihedral angles at this square base; compare the figure below.

Let  $O'_8$  be a congruent copy of  $O_8$ , obtained by reflection of  $O_8$  in its square base followed by a  $90^\circ$  rotation around the axis orthogonal to the base; compare the figure below. This results in a regular 3-ball with cubical 2-skeleton with  $f$ -vector

$$f(B_2) = (16, 28, 15, 2).$$

The special feature of this complex is that it contains a cubical Möbius strip with parallel inner edges of length 9 in its 2-skeleton, as is illustrated in the figure.

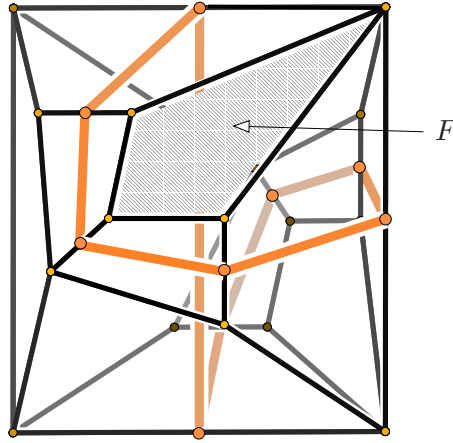


FIGURE 16: The outcome of step 1 of the construction: The 2-cubical convex 3-ball  $\mathcal{B}_2$  which contains a Möbius strip with parallel inner edges in the 2-skeleton. (The dual manifold of the Möbius strip (an embedded  $S^1$ ) is given by the old edges.)

**Step 2.** Now we perform a Schlegel cap construction on  $O_8$ , based on the (unique) facet  $F$  of  $O_8$  that is not contained in the Möbius strip mentioned above, and that is not adjacent to the square glueing facet  $Q$ . This Schlegel cap has the  $f$ -vector

$$f(S_7) = (20, 42, 30, 7),$$

while its boundary has the  $f$ -vector

$$f(\partial S_7) = (20, 36, 18).$$

**Step 3.** The same Schlegel cap operation may be performed on the second copy  $O'_8$ . Joining the two copies of the Schlegel cap results in a regular cubical 3-ball  $B_{14}$  with  $f$ -vector

$$f(B_{14}) = (36, 80, 59, 14)$$

whose boundary has the  $f$ -vector

$$f(\partial B_{14}) = (36, 68, 34).$$

The ball  $B_{14}$  again contains the cubical Möbius strip with parallel inner edges of length 9 as an embedded subcomplex (in its 2-skeleton). Compare Figure 17.

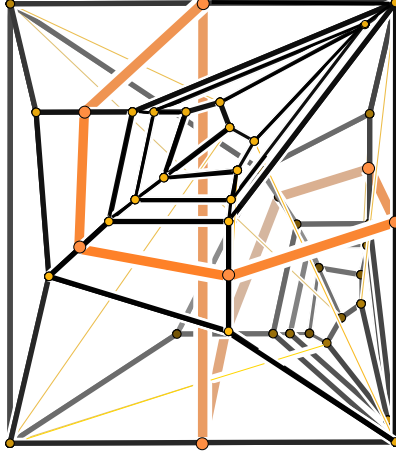


FIGURE 17: The outcome of step 2 of the construction: The cubical convex 3-ball  $\mathcal{B}_{14}$  which contains a Möbius strip with parallel inner edges in the 2-skeleton. (Again, the dual manifold of the Möbius strip is given by the old edges.)

**Step 4.** Now we build the prism over this regular cubical ball, resulting in a regular cubical 4-ball  $\mathcal{B}$  whose  $f$ -vector is

$$f(\mathcal{B}) = (72, 196, 198, 87, 14)$$

and whose support is a cubical 4-polytope  $P_{72} = |\mathcal{B}|$  with two copies of the cubical Möbius strip in its 2-skeleton. Its  $f$ -vector is

$$f(P_{72}) = (72, 196, 186, 62).$$

A further analysis of the dual manifolds shows that there are six dual manifolds in total: one Klein bottle of  $f$ -vector  $(80, 160, 80)$ , and five 2-spheres (four with  $f$ -vector  $(20, 36, 18)$ , one with  $f$ -vector  $(36, 68, 34)$ ). All the spheres are embedded, while the Klein bottle is immersed with five double-intersection curves (embedded 1-spheres), but with no triple points.  $\square$

## 6 Constructing cubifications

A lot of construction techniques for cubifications (see Section 2.2) are available in the CW category. In particular, every cubical CW  $(d-1)$ -sphere  $\mathcal{S}^{d-1}$  with an even number of facets admits a CW cubification, that is a cubical CW  $d$ -ball with boundary  $\mathcal{S}^{d-1}$  (see Thurston [34], Mitchell [24], and Eppstein [9]).

In the polytopal category there is a construction in dimension 3 called the *Hexhoop template* due to Yamakawa & Shimada [35], which is depicted in following figure.



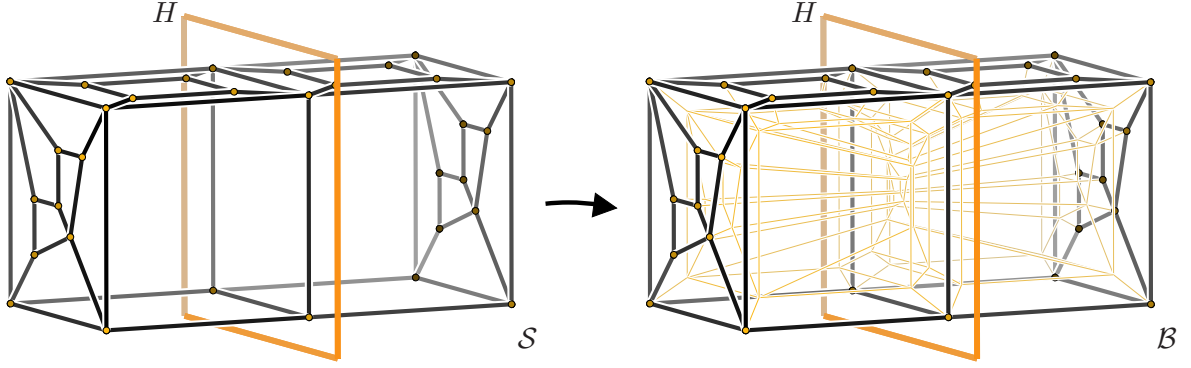


FIGURE 18: The Hexhoop template of Yamakawa & Shimada [35].

Their construction takes as input a 3-polytope  $P$  that is affinely isomorphic to a regular 3-cube, a hyperplane  $H$  and a cubical subdivision  $\mathcal{S}$  of the boundary complex of  $P$  such that  $\mathcal{S}$  is symmetric with respect to  $H$  and  $H$  intersects no facet of  $\mathcal{S}$  in its relative interior. For such a cubical PL 2-sphere  $\mathcal{S}$  they produce a cubification. A 2-dimensional version is shown in Figure 19.

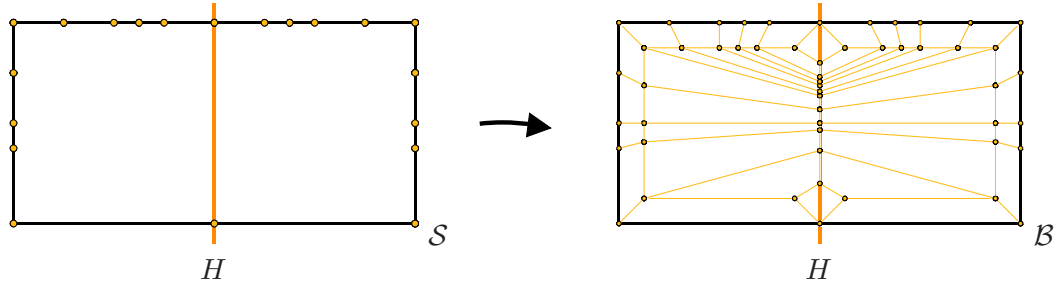


FIGURE 19: A two-dimensional version of the Hexhoop template.

Here we introduce and analyze a *generalized regular Hexhoop template*. This construction can be interpreted as a generalization of the Hexhoop template in several directions: Our approach admits arbitrary geometries, works in any dimension, and yields regular cubifications with “prescribed heights on the boundary” (with a symmetry requirement and with the requirement that the intersection of the symmetry hyperplane and the boundary subdivision is a subcomplex of the boundary subdivision).

Figure 20 displays a 2-dimensional cubification (of a boundary subdivision  $\mathcal{S}$  of a 2-polytope such that  $\mathcal{S}$  is symmetric with respect to a hyperplane  $H$ ) obtained by our construction.

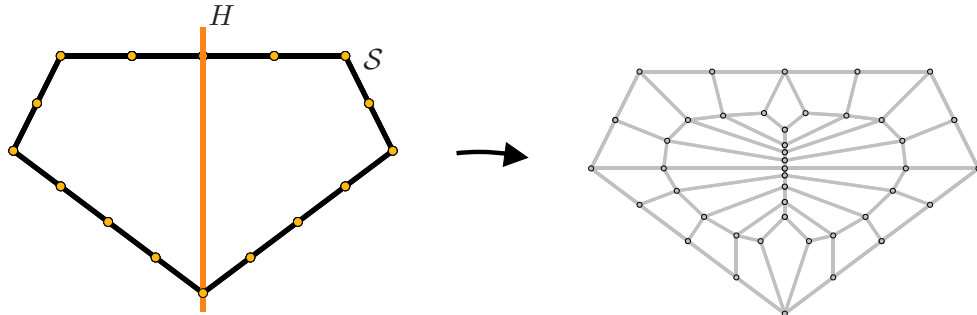


FIGURE 20: A cubification of a boundary subdivision of a pentagon, produced by our *generalized regular Hexhoop* construction.

Not only do we get a cubification, but we may also derive a symmetric lifting function for the cubification that may be quite arbitrarily prescribed on the boundary. The input of our construction is a lifted cubical boundary subdivision  $(\mathcal{S}^{d-1}, \psi)$  of a  $d$ -polytope  $P$ , such that both  $P$  and  $(\mathcal{S}^{d-1}, \psi)$  are symmetric with respect to a hyperplane  $H$ .

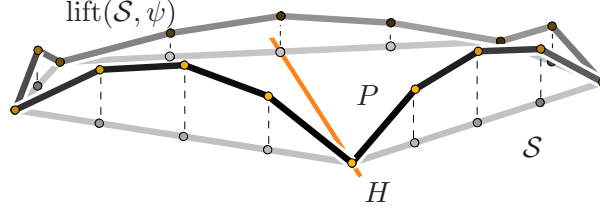


FIGURE 21: An input for the generalized regular Hexhoop construction.

Roughly speaking our approach goes as follows.

- (1) We start with a  $(d + 1)$ -polytope  $T$  that is a *symmetric tent* (defined in Section 6.1) over the given lifted boundary subdivision  $(\mathcal{S}, \psi)$  of the input  $d$ -polytope  $P$ . Such a tent is the convex hull of all ‘lifted vertices’  $(\mathbf{v}, \psi(\mathbf{v})) \in \mathbb{R}^{d+1}$ ,  $\mathbf{v} \in \text{vert}(\mathcal{S})$ , and of two *apex points*  $\mathbf{p}_L, \mathbf{p}_R$ ; the combinatorial structure of  $T$  is as depicted in Figure 22.

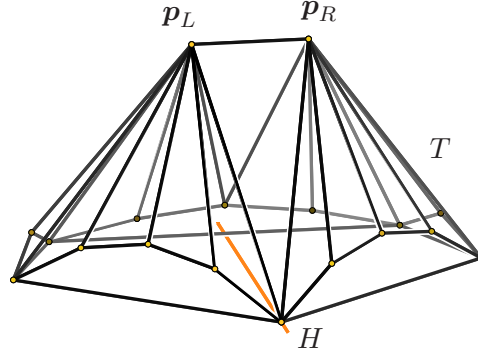


FIGURE 22: A so-called *symmetric tent* over the lifted boundary subdivision  $(\mathcal{S}, \psi)$  of the input  $d$ -polytope  $P$ . Refer Section 6.1 for the precise definition.

- (2) Truncate  $T$  by a hyperplane  $H'$  parallel to  $\text{aff}(P) = \mathbb{R}^d \subset \mathbb{R}^d \times \{0\}$  that separates the lifted points from the apex points, and remove the upper part.
- (3) Add the polytope  $R := \text{cone}(\mathbf{p}_L, Q) \cap \text{cone}(\mathbf{p}_R, Q) \cap H'_+$ , where  $H'_+$  is the halfspace with respect to  $H$  that contains  $\mathbf{p}_L$  and  $\mathbf{p}_R$ . Compare Figure 23.

Throughout this section our figures illustrate the construction for a two-dimensional input polytope which yields two-dimensional complexes in  $\mathbb{R}^3$ . Our running example will be the one of Figure 21. Nevertheless the construction works in higher dimensions. In particular, in Section 7 the generalized regular Hexhoop is used for three-dimensional inputs. A three-dimension cubification is shown at the end of this section, on page 25.

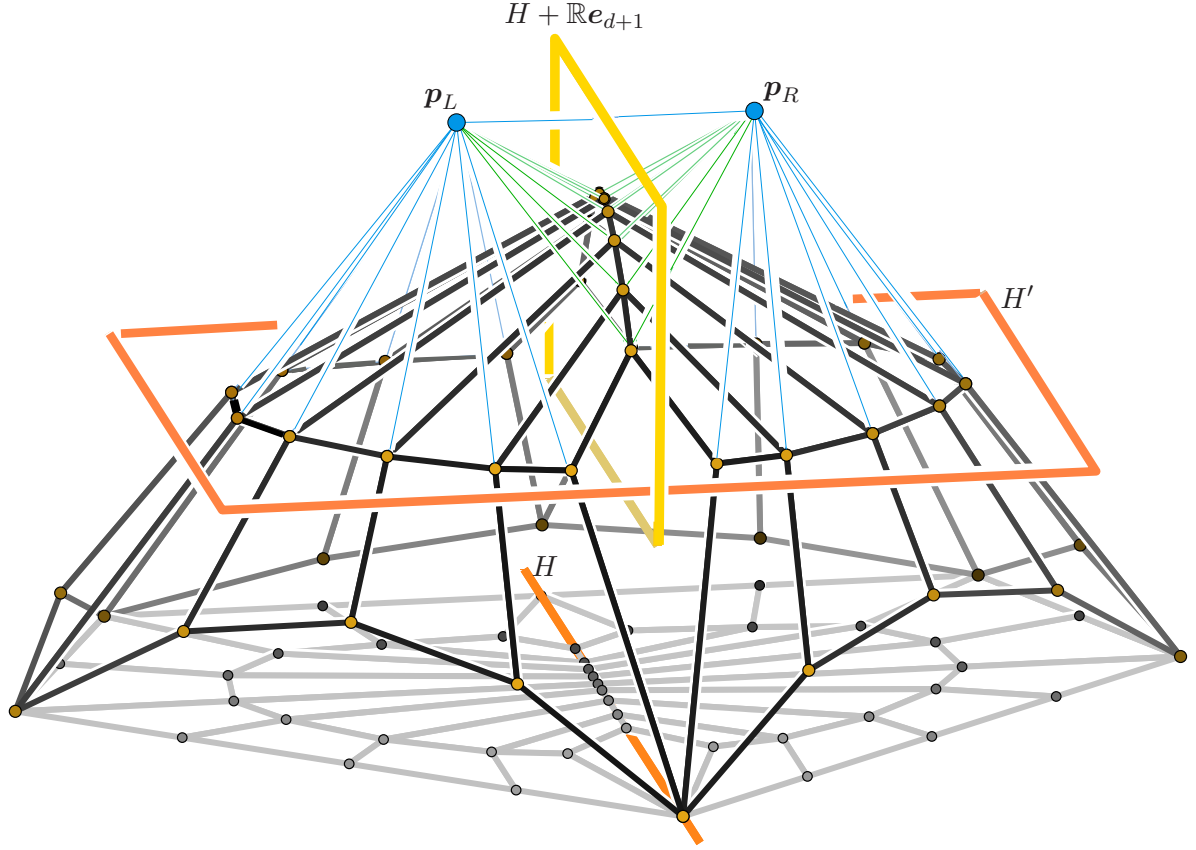


FIGURE 23: Sketch of the generalized regular Hexhoop construction.

### 6.1 Symmetric tent over a lifted boundary subdivision

Let  $P$  be a  $d$ -polytope that is symmetric with respect to a hyperplane  $H$  in  $\mathbb{R}^d$ . Choose a positive halfspace  $H_+$  with respect to  $H$ . Let  $(\mathcal{S}, \psi)$  be a lifted boundary subdivision of  $P$  such that  $\mathcal{S} \cap H$  is a subcomplex of  $\mathcal{S}$ . Define  $\tilde{H} := H + \mathbb{R}e_{d+1}$  which is a symmetry hyperplane for  $P \subset \mathbb{R}^{d+1}$ .

The *symmetric tent* over  $(\mathcal{S}, \psi)$  is the lifted polytopal subdivision  $(\mathcal{T}, \phi)$  of  $P$  given by the upper faces of the polytope

$$T := \text{conv}(P \cup \{\mathbf{p}_L, \mathbf{p}_R\})$$

if  $\mathbf{p}_L, \mathbf{p}_R \in \mathbb{R}^{d+1}$  are two points in  $\mathbb{R}^{d+1}$  that are symmetric with respect to the hyperplane  $\tilde{H}$ , and the upper facets of  $T$  are

- pyramids with apex point  $\mathbf{p}_L$  over facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $\pi(F) \subset H_+$ ,
- pyramids with apex point  $\mathbf{p}_R$  over facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $\pi(F) \subset H_-$ , and
- 2-fold pyramids with apex points  $\mathbf{p}_L, \mathbf{p}_R$  over ridges  $R$  of  $\text{lift}(\mathcal{S}, \psi)$  with  $\pi(R) \subset H$ .

(This requires that  $\mathbf{p}_L \notin \text{aff}(P)$  and  $\pi(\mathbf{p}_L) \in \text{relint}(P \cap H_+)$ .)

**Lemma 6.1.** Assume we are given the following input.

- |                       |  |
|-----------------------|--|
| $P$                   | a convex $d$ -polytope in $\mathbb{R}^d$ ,   |
| $(\mathcal{S}, \psi)$ | a lifted boundary subdivision of $P$ ,   |
| $H$                   | a hyperplane in $\mathbb{R}^d$ such that   |
|                       | <ul style="list-style-type: none"> <li>• <math>P</math> and <math>(\mathcal{S}, \psi)</math> are both symmetric with respect to <math>H</math>, and</li> <li>• <math>\mathcal{S} \cap H</math> is a subcomplex of <math>\mathcal{S}</math>, and</li> </ul> |

- $\mathbf{q}_L, \mathbf{q}_R$  two points in  $P \subset \mathbb{R}^d$  such that
- $\mathbf{q}_L \in \text{relint}(P \cap H_+)$ , and
  - $\mathbf{q}_L, \mathbf{q}_R$  are symmetric with respect to  $\tilde{H}$ .

Then for every sufficiently large height  $h > 0$  the  $T := \text{conv}\{\text{lift}(\mathcal{S}, \psi), \mathbf{p}_L, \mathbf{p}_R\}$  with  $\mathbf{p}_L := (\mathbf{q}_L, h)$  and  $\mathbf{p}_R := (\mathbf{q}_R, h)$  yields a symmetric tent over  $(\mathcal{S}, \psi)$ .

This can be shown for instance by using the Patching Lemma (Lemma 3.2).

## 6.2 Generalized regular Hexhoops

In this section we lay out our generalization of the Hexhoop approach and prove the following existence statement for cubifications.

**Theorem 6.2.** *Assume we are given the following input.*

- $P$  a convex  $d$ -polytope in  $\mathbb{R}^d$ ,  
 $(\mathcal{S}^{d-1}, \psi)$  a lifted cubical boundary subdivision of  $P$ , and  
 $H$  a hyperplane in  $\mathbb{R}^d$  such that
- $P$  and  $(\mathcal{S}^{d-1}, \psi)$  are symmetric with respect to  $H$ , and
  - $\mathcal{S}^{d-1} \cap H$  is a subcomplex of  $\mathcal{S}^{d-1}$ .

Then there exists a lifted cubification  $(\mathcal{B}^d, \phi)$  of  $(\mathcal{S}^{d-1}, \psi)$ .

The proof relies on the following construction.

---

### Construction 5: GENERALIZED REGULAR HEXHOOP

---

**Input:**

- $P$  a convex  $d$ -polytope  $P$  in  $\mathbb{R}^d$ .  
 $(\mathcal{S}^{d-1}, \psi)$  a lifted cubical boundary subdivision of  $P$ .  
 $H$  a hyperplane in  $\mathbb{R}^d$  such that
- $P$  and  $(\mathcal{S}^{d-1}, \psi)$  are symmetric with respect to  $H$ , and
  - $\mathcal{S}^{d-1} \cap H$  is a subcomplex of  $\mathcal{S}^{d-1}$ .

**Output:**

- $(\mathcal{B}^d, \phi)$  a symmetric lifted cubification of  $(\mathcal{S}^{d-1}, \psi)$  given as a cubical  $d$ -ball  $\mathcal{C}'$  in  $\mathbb{R}^{d+1}$ .

- (1) Choose a positive halfspace  $H_+$  with respect to  $H$ , and a point  $\mathbf{q}_L \in \text{relint}(P \cap H_+)$ . Define  $\mathbf{q}_R := \mathbf{p}_L^M$ , where the upper index  $M$  denotes the mirrored copy with respect to  $\tilde{H} = H + \mathbb{R}e_{d+1}$ . By Lemma 6.1 there is a height  $h > 0$  such that

$$T := \text{conv}\{\text{lift}(\mathcal{S}^{d-1}, \psi), \mathbf{p}_L, \mathbf{p}_R\}$$

with  $\mathbf{p}_L := (\mathbf{q}_L, h)$  and  $\mathbf{p}_R := (\mathbf{q}_R, h)$  forms a symmetric tent over  $(\mathcal{S}^{d-1}, \psi)$ .

- (2) Choose a hyperplane  $H'$  parallel to  $\text{aff}(P) \subset \mathbb{R}^d$  that separates  $\{\mathbf{p}_L, \mathbf{p}_R\}$  and  $\text{lift}(\mathcal{S}^{d-1}, \psi)$ . Let  $H'_+$  be the halfspace with respect to  $H'$  that contains  $\mathbf{p}_L$  and  $\mathbf{p}_R$ .

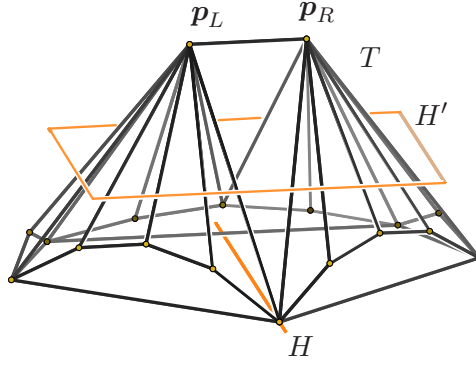


FIGURE 24: Step 2. The hyperplane  $H'$  separates  $\{p_L, p_R\}$  from  $\text{lift}(\mathcal{S}^{d-1}, \psi)$ .

(3) Define the “lower half” of the tent  $T$  as

$$T_- := T \cap H'_-.$$

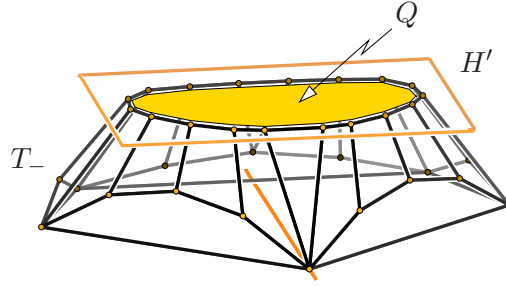


FIGURE 25: Step 3. The “lower half”  $T_-$  of  $T$ .

(4) Define the two  $d$ -polytopes

$$\begin{aligned} Q_L &:= \text{conv} \{v \in \text{vert}(Q) : v \in H_+\}, \\ Q_R &:= \text{conv} \{v \in \text{vert}(Q) : v \in H_-\}, \end{aligned}$$

where  $Q$  denotes the convex  $d$ -polytope  $Q := T \cap H'$ . Furthermore, let  $F_L := H' \cap \text{conv}(p_L, P \cap H)$  denote the unique facet of  $Q_L$  that is not a facet of  $Q$ .

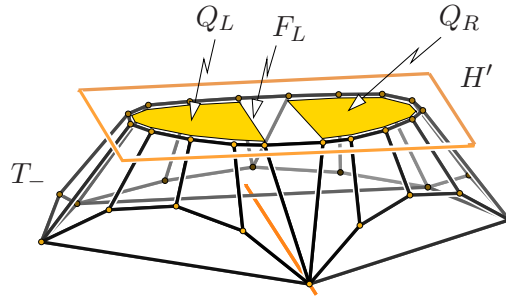


FIGURE 26: Step 4. Define  $Q_L$  and  $Q_R$ .

(5) Construct the polytope  $R := \text{cone}(p_L, Q) \cap \text{cone}(p_R, Q) \cap H'_+$ .

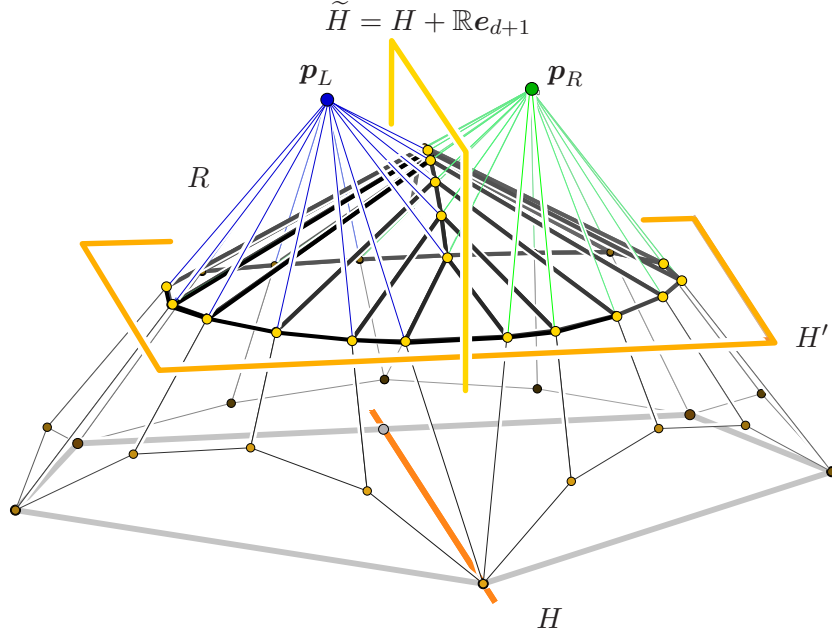


FIGURE 27: Step 5. The polytope  $R := \text{cone}(\mathbf{p}_L, Q) \cap \text{cone}(\mathbf{p}_R, Q) \cap H'_+$ .

The complex  $\mathcal{C}'$  in question is given by the upper facets of the  $(d+1)$ -polytope  $U := \text{conv}(T_- \cup R)$ .

First we analyze the structure of the three polytopes  $Q$ ,  $T_-$  and  $R$ . Afterwards we prove Theorem 6.2.

**Lemma 6.3.** *(Combinatorial structure of  $Q$ )*

The vertex set of  $Q$  consists of

- the points  $\text{conv}(\mathbf{p}_L, \mathbf{v}) \cap H'$  for vertices  $\mathbf{v} \in \text{vert}(\text{lift}(\mathcal{S}, \psi))$  such that  $\pi(\mathbf{v}) \subset H_+$ , and
- the points  $\text{conv}(\mathbf{p}_R, \mathbf{v}) \cap H'$  for vertices  $\mathbf{v} \in \text{vert}(\text{lift}(\mathcal{S}, \psi))$  such that  $\pi(\mathbf{v}) \subset H_-$ .

The set of facets of  $Q$  consists of

- the combinatorial cubes  $\text{conv}(\mathbf{p}_L, F) \cap H'$  for facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}_+$ ,
- the combinatorial cubes  $\text{conv}(\mathbf{p}_R, F) \cap H'$  for facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}_-$ ,
- the combinatorial cubes  $\text{conv}(\mathbf{p}_L, \mathbf{p}_R, F) \cap H'$  for  $(d-2)$ -faces  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}$ .

*Proof.* By the definition of a symmetric tent, the set of upper facets of the symmetric tent  $T$  consists of

- pyramids with apex point  $\mathbf{p}_L$  over facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}_+$ ,
- pyramids with apex point  $\mathbf{p}_R$  over facets  $F$  of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}_-$ , and
- 2-fold pyramids with apex points  $\mathbf{p}_L, \mathbf{p}_R$  over ridges  $R$  of  $\text{lift}(\mathcal{S}, \psi)$  with  $R \subset \tilde{H}$ .

Since  $Q$  is the intersection of  $T$  with  $H$ , the polytope  $Q$  has the vertices and facets listed above. It remains to show that the facets of type (c) are combinatorial cubes. Let  $F$  be a  $(d-2)$ -face of  $\text{lift}(\mathcal{S}, \psi)$  such that  $F \subset \tilde{H}$ . Every point on the facet lies in the convex hull of  $F$  with a unique point on the segment  $[\mathbf{p}_L, \mathbf{p}_R]$ . Thus the facet is combinatorially isomorphic to a prism over  $F$ .  $\square$

Let a  $d$ -dimensional half-cube be the product of a combinatorial  $(d-2)$ -cube and a triangle. A combinatorial half-cube is a polytope combinatorially isomorphic to a half-cube.

**Lemma 6.4.** *(Combinatorial structure of  $T_-$ )*

The vertices of  $T_-$  are the vertices of  $\text{lift}(\mathcal{S}, \psi)$  and the vertices of  $Q$ . Furthermore, the set of upper facets of  $T_-$  consists of

- (a) the combinatorial cubes  $\text{cone}(\mathbf{p}_L, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_+$ ,
- (b) the combinatorial cubes  $\text{cone}(\mathbf{p}_R, F) \cap H'_- \cap (\mathbb{R}^d \times \mathbb{R}_+)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_-$ ,
- (c) the combinatorial half-cubes  $\text{cone}(\mathbf{p}_L, F) \cap \text{cone}(\mathbf{p}_R, F) \cap H'_-$  for facets  $R$  of  $Q$  such that  $R$  intersects  $\tilde{H}$ , and
- (d)  $Q$ .

The set of facet defining hyperplanes of the upper facets of  $T_-$  consists of

- (a)  $\text{aff}(\mathbf{p}_L, F)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_+$ ,
- (b)  $\text{aff}(\mathbf{p}_R, F)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_-$ ,
- (c)  $\text{aff}(\mathbf{p}_L, \mathbf{p}_R, F)$  for facets  $F$  of  $Q$  such that  $F$  intersects  $\tilde{H}$ , and
- (d)  $\text{aff}(Q)$ .

*Proof.* Since  $T_-$  is the intersection of  $T$  with  $H'_-$ , the upper facets of  $T_-$  are given by  $Q$  plus the intersections of the upper facets of  $T$  with  $H'_-$ , and the vertices of  $T_-$  are the vertices of  $T$  and the vertices of  $Q$ .  $\square$

**Lemma 6.5.** *(Combinatorial structure of  $R$ )*

The set of vertices of  $R$  consists of the vertices of  $Q$  and all points in  $V'' := \text{vert}(R) \setminus \text{vert}(Q)$ . Furthermore, the set of (all) facets of  $R$  consists of

- (a) the combinatorial cubes  $\text{conv}(\mathbf{p}_R, F) \cap \tilde{H}_+$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_+$ ,
- (b) the combinatorial cubes  $\text{conv}(\mathbf{p}_L, F) \cap \tilde{H}_+$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_-$ ,
- (c) the combinatorial half-cubes  $\text{conv}(\mathbf{p}_R, F) \cap \text{conv}(\mathbf{p}_L, F)$  for facets  $F$  of  $Q$  such that  $F$  intersects  $\tilde{H}$ , and
- (d)  $Q$ .

The set of facet defining hyperplanes of the facets of  $R$  consists of

- (a)  $\text{aff}(\mathbf{p}_R, F)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_+$ ,
- (b)  $\text{aff}(\mathbf{p}_L, F)$  for facets  $F$  of  $Q$  such that  $F \subset \tilde{H}_-$ ,
- (c)  $\text{aff}(\mathbf{p}_L, \mathbf{p}_R, F)$  for facets  $F$  of  $Q$  such that  $F$  intersects  $\tilde{H}$ , and
- (d)  $\text{aff}(Q)$ .

Now we can prove the correctness of the output of Construction 5.

**Proof of Theorem 6.2.** We show that the complex  $\mathcal{C}'$  given by the upper facets of the polytope  $U$  given by Construction 5 determines a lifted cubification  $(\mathcal{B}^d, \phi)$  of  $(\mathcal{S}^{d-1}, \psi)$ .

First observe that no vertex of  $T_-$  is beyond a facet of  $R$ . Vice versa, no vertex of  $R$  is beyond a facet of  $T_-$ . Hence the boundary of  $U = \text{conv}(T_- \cup R)$  is the union of the two boundaries of the two polytopes, excluding the relative interior of  $Q$ .

Define the vertex sets  $V := \text{vert}(\text{lift}(\mathcal{S}, \psi))$ ,  $V' := \text{vert}(Q)$  and  $V'' := \text{vert}(R) \setminus V'$ . Then we have:

- Each vertex of  $V$  is beneath each facet of  $R$  that is of type (a) or (b).
- Each vertex of  $V''$  is beneath each facet of  $T_-$  that is of type (a) or (b).

Hence these four types of facets are facets of  $U$  that are combinatorial cubes, and the set of vertices of  $U$  is given by the union of  $V, V'$  and  $V''$ . It remains to show that each hyperplane  $\text{aff}(\mathbf{p}_L, \mathbf{p}_R, F)$ , where  $F$  is a facet of  $Q$  that intersects  $\tilde{H}$ , is the affine hull of a cubical facet of  $U$ . To see this, observe that there are two facets  $F_+, F_-$  of  $R, T_-$  respectively, that are both contained in the affine hull of  $F$ . These two facets  $F_+, F_-$  are both half-cubes that intersect in a common  $(d-1)$ -cube, namely  $F$ . Furthermore, all vertices of  $F_+$  and of  $F_-$  that are not contained in  $\text{aff}(F)$  are contained in  $\tilde{H}$ . Hence the union of  $F_+$  and  $F_-$  is a combinatorial cube.

Therefore, every upper facet of  $U$  is a combinatorial cube. Furthermore,  $\pi(R) = \pi(Q)$  and  $\pi(T_-) = |P|$ . This implies that the upper facets of  $U$  determine a lifted cubical subdivision of  $(\mathcal{S}^{d-1}, \psi)$ .  $\square$

**Remark 2 (Dual manifolds).** Up to PL-homeomorphism, we obtain the following dual manifolds:

- $\mathcal{N} \times I$  for each dual manifold  $\mathcal{N}$  (with or without boundary) of  $\mathcal{S}_L$ ,
- two  $(d-1)$ -spheres “around”  $Q, Q^M$ , respectively, where the upper index  $^M$  denotes the mirrored copy.

For the proof, note that the “left half” of  $\mathcal{B}^d$ , which is given by all facets of  $\mathcal{B}^d$  which are contained in  $H^+$ , is combinatorially isomorphic to a prism over  $\mathcal{S}_L$ . (Due to the symmetry of the boundary subdivision  $\mathcal{S}^{d-1}$  the “right half” of  $\mathcal{B}^d$  is combinatorially isomorphic to a prism over  $\mathcal{S}_L$ , too.) See also Figure 28.

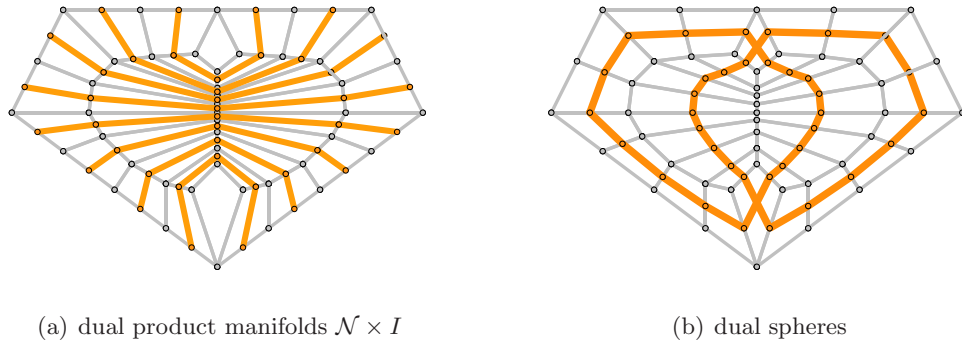


FIGURE 28: The set of dual manifolds of a two-dimensional cubification produced by the generalized regular Hexhoop construction.

If  $\mathcal{N}$  is an embedded dual  $(d-2)$ -sphere of  $\mathcal{S}^{d-1}$  that intersects  $H_+ \setminus H$  and  $H_- \setminus H$ , then there is an embedded dual manifold  $\mathcal{N}'$  of  $\mathcal{B}^d$  which is a  $(d-1)$ -ball with boundary  $\mathcal{N}$ . Compare Figure 29.



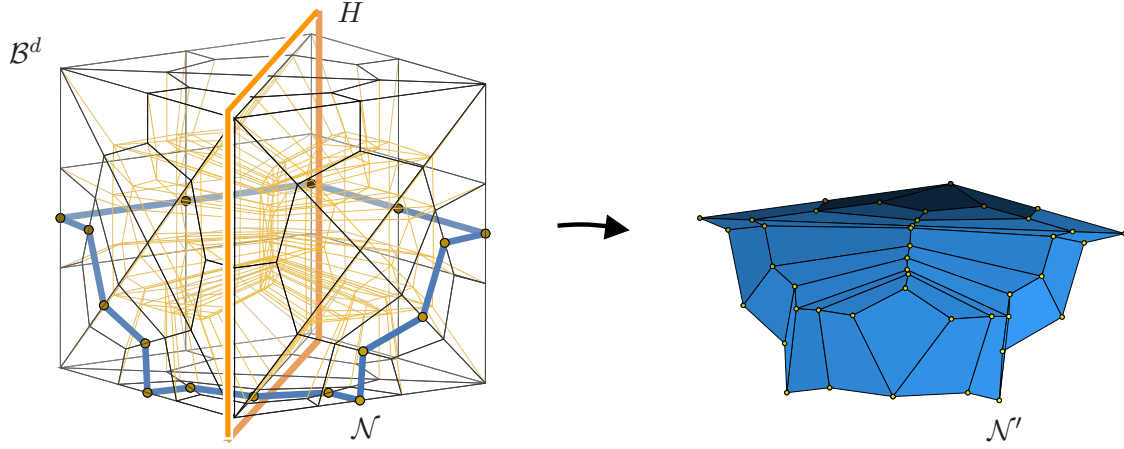


FIGURE 29: A three-dimensional cubification produced by the generalized regular Hexhoop construction. For every embedded dual circle  $\mathcal{N}$  which intersects  $H_+ \setminus H$  and  $H_- \setminus H$ , there is an embedded dual 2-ball  $\mathcal{N}'$  with boundary  $\mathcal{N}$  in the cubification. (This is a cubification for the case “single 5” which is introduced in Section 7.)

## 7 Cubical 4-polytopes with prescribed dual manifold immersions

In this section we combine the construction techniques developed in the previous sections and thus derive our main theorem.

As mentioned in Section 2.5, there seems to be no analog for the last step of the Babson-Chan construction in the polytopal category. For our alternative polytopal constructions we cannot start with arbitrary normal crossing PL-immersions, but we require immersions whose local geometric structure is rather simple. For the version of our constructions presented below, the following assumptions are needed and used (for  $d = 3$ ):

- (1)  $\mathcal{M}^{d-1}$  is a  $(d - 1)$ -dimensional cubical PL-manifold, and  $j : \mathcal{M}^{d-1} \looparrowright \mathbb{R}^d$  is a *grid immersion*, that is, a cubical normal crossing codimension one immersion into  $\mathbb{R}^d$  equipped with the standard unit cube structure (cf. [8]). In particular, for each  $k$ -face  $F$  of  $\mathcal{M}^{d-1}$  the image  $j(F)$  is a  $k$ -face of the skeleton of the standard unit cube structure on  $\mathbb{R}^d$ .
- (2) Moreover, we require that the immersion is *locally symmetric*, that is, that at every vertex  $\mathbf{w}$  of  $j(\mathcal{M}^{d-1})$  there is hyperplane  $H$  through  $\mathbf{w}$  such that for each vertex  $\mathbf{v}$  with  $j(\mathbf{v}) = \mathbf{w}$  the image of the vertex star of  $\mathbf{v}$  is symmetric with respect to  $H$ . Thus, we require that  $H$  is a symmetry hyperplane separately for each of the (up to  $d$ ) local sheets that intersect at  $\mathbf{w}$ .

Such a hyperplane  $H$  is necessarily of the form  $x_i = k$ ,  $x_i + x_j = k$  or  $x_i - x_j = k$ . In the first case we say  $H$  is a *coordinate hyperplane*, and in other cases it is *diagonal*.

### 7.1 From PL immersions to grid immersions

In view of triangulation and approximation methods available in PL and differential topology theory, the above assumptions are not so restrictive.

**Proposition 7.1.** *Every locally flat normal crossing immersion of a compact  $(d - 1)$ -manifold into  $\mathbb{R}^d$  is PL-equivalent to a codimension one grid immersion of a cubification of the manifold into the standard cube subdivision of  $\mathbb{R}^d$ .*

*Proof.* We may replace any PL-immersion of  $\mathcal{M}^{d-1}$  by a simplicial immersion into a suitably fine triangulation of  $\mathbb{R}^d$ . Furthermore, the vertices of  $j(\mathcal{M}^{d-1})$  may be perturbed into general position.

Now we overlay the polyhedron  $j(\mathcal{M}^{d-1})$  with a cube structure of  $\mathbb{R}^d$  of edge length  $\varepsilon$  for suitably small  $\varepsilon > 0$ , such that the vertices of  $j(\mathcal{M}^{d-1})$  are contained in the interiors of distinct  $d$ -cubes.

Then working by induction on the skeleton, within each face of the cube structure, the restriction of  $j(\mathcal{M}^{d-1})$  to a  $k$ -face — which by local flatness consists of one or several  $(k-1)$ -cells — is replaced by a standard cubical lattice version that is supposed to run through the interior of the respective cell, staying away distance  $\varepsilon'$  from the boundary of the cell; here we take different values for  $\varepsilon'$  in the situation where the immersion is not embedded at the vertex in question, that is, comes from several disjoint neighborhoods in  $\mathcal{M}^{d-1}$ .

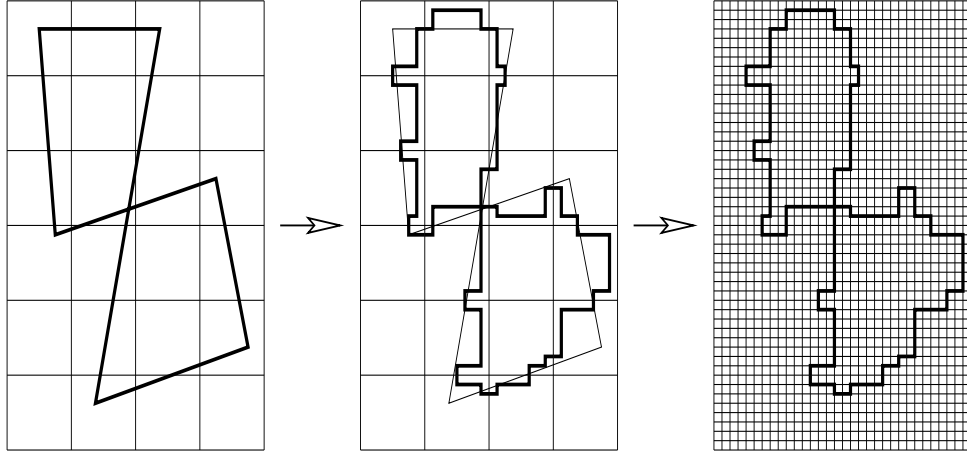
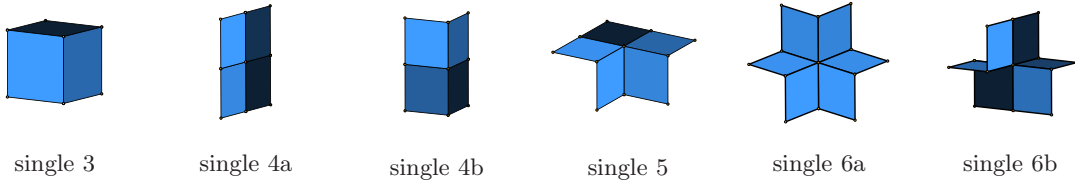


FIGURE 30: Illustration of the proof of Proposition 7.1.

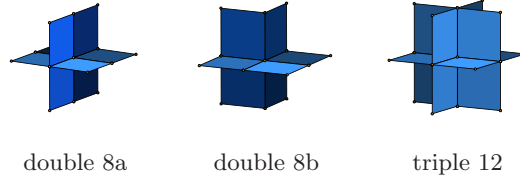
The resulting modified immersion into  $\mathbb{R}^d$  will be cellular with respect to a standard cube subdivision of edge length  $\frac{1}{N}\varepsilon$  for a suitable large  $N$ . Figure 30 illustrates the procedure for  $d = 2$ .  $\square$

## 7.2 Vertex stars of grid immersions of surfaces

There are nine types of vertex stars of grid immersions of surfaces, namely the following five vertex stars of a regular vertex,



plus two vertex stars with double intersection and the vertex star of a triple intersection point:



All these vertex stars satisfy the local symmetry condition, with a single exception, namely the last depicted vertex star (single 6b) of a regular vertex with 6 adjacent quadrangles.

**Proposition 7.2.** *Any grid immersion of a compact cubical 2-manifold into  $\mathbb{R}^3$  is equivalent to a locally symmetric immersion of the same type.*

*Proof.* There is only one type of vertex star that does not have the required symmetry, namely “case 6b”. As indicated in Figure 31, a local modification of the surface solves the problem (with a suitable refinement of the standard cube subdivision).  $\square$

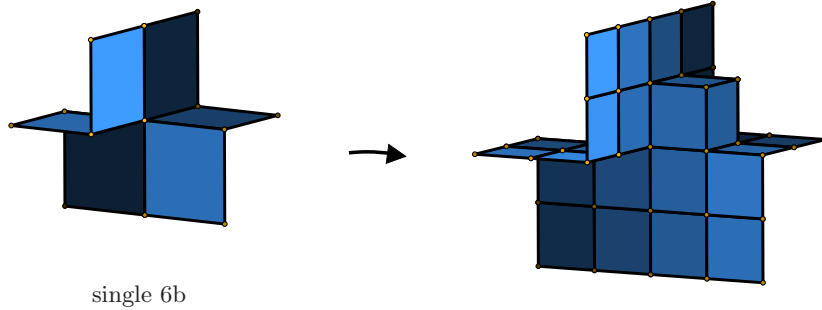


FIGURE 31: Local modification to “repair” the case “single 6a.”

### 7.3 Main theorem (2-manifolds into cubical 4-polytopes)

**Theorem 7.3.** *Let  $j: \mathcal{M} \looparrowright \mathbb{R}^3$  be a locally flat normal crossing immersion of a compact 2-manifold (without boundary)  $\mathcal{M}$  into  $\mathbb{R}^3$ .*

*Then there is a cubical 4-polytope  $P$  with a dual manifold  $\mathcal{M}'$  and associated immersion  $y: \mathcal{M}' \looparrowright |\partial P|$  such that the following conditions are satisfied:*

- (i)  *$\mathcal{M}'$  is a cubical subdivision of  $\mathcal{M}$ , and the immersions  $j$  (considered as a map to  $\mathbb{R}^3 \cup \{\infty\} \cong S^3$ ) and  $y$  are PL-equivalent.*
- (ii) *The number of facets of  $P$  is congruent modulo two to the number  $t(j)$  of triple points of the immersion  $j$ .*
- (iii) *If the given surface  $\mathcal{M}$  is non-orientable and of odd genus, then the cubical 4-polytope  $P$  has an odd number of facets.*

The main ingredient of our proof for this theorem is the following construction of a cubical 3-ball with a prescribed dual manifold immersion.

## 7.4 Cubical 3-balls with a prescribed dual manifold immersions

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### Construction 6: REGULAR CUBICAL 3-BALL WITH A PRESCRIBED DUAL MANIFOLD

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- Input:** A 2-dimensional closed (that is, compact and without boundary) cubical PL-surface  $\mathcal{M}$ , and a locally symmetric codimension one grid immersion  $j: \mathcal{M} \looparrowright |\mathcal{P}_3(\ell_1, \ell_2, \ell_3)| \subset \mathbb{R}^3$ .  
(Without loss of generality one can assume  $j(\mathcal{M}) \subset |\mathcal{P}_3(\ell_1, \ell_2, \ell_3)|$ .)
- Output:** A regular convex 3-ball  $\mathcal{B}$  with a dual manifold  $\mathcal{M}'$  and associated immersion  $y: \mathcal{M}' \looparrowright |\mathcal{B}|$  such that the following conditions are satisfied:
- (i)  $\mathcal{M}'$  is a cubical subdivision of  $\mathcal{M}$ , and the immersions  $j$  and  $y$  are PL-equivalent.
  - (ii) The number of facets of  $\mathcal{B}$  is congruent modulo two to the number  $t(j)$  of triple points of the immersion  $j$ .

The Patching Lemma (Lemma 3.2) will be used to prove that the ball  $\mathcal{B}$  is regular. Therefore, we first construct a raw complex  $\mathcal{A}$ .

#### (0) Raw complex.

As the raw complex  $\mathcal{A}$  we take a copy of the pile of cubes  $\mathcal{P}_3(\ell_1 + 1, \ell_2 + 1, \ell_3 + 1)$  with all vertex coordinates shifted by  $-\frac{1}{2}\mathbb{1}$ . (Hence  $x_i \in \{-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots, \ell_i + \frac{1}{2}\}$  for each vertex  $\mathbf{x} \in \text{vert}(\mathcal{A})$ .)

Due to the local symmetry of the immersion, and the choice of the vertex coordinates of  $\mathcal{A}$ , the following holds:

- ▷ Each vertex of  $j(\mathcal{M})$  is the barycenter of a 3-cube  $C$  of  $\mathcal{A}$ .
- ▷ For each 3-cube  $C$  of  $\mathcal{A}$  the restriction  $(C, j(\mathcal{M}) \cap C)$  is locally symmetric.

The lifted cubical subdivision  $\mathcal{B}$  of  $\mathcal{A}$  is constructed by induction over the skeleton of  $\mathcal{A}$ :

For  $k = 1$  to  $3$  we produce a lifted cubical subdivision  $\mathcal{C}^k$  of the  $k$ -skeleton  $\mathcal{F}_k(\mathcal{A})$ , that is, for any  $k$ -face  $F$  the restriction  $\mathcal{C}^k \cap F$  is a lifted cubical subdivision of  $F$ . Each of these  $\mathcal{C}^k \cap F$  is called a *template*. The lifted cubical subdivision  $\mathcal{B}$  of  $\mathcal{A}$  arises as  $\mathcal{B} := \mathcal{C}^3$ . Let us define the following invariants (for  $k \in \{1, 2, 3\}$ ).

- (I<sub>k</sub>1) *Consistency requirement.*  
For every  $k$ -face  $Q \in \mathcal{F}_k(\mathcal{A})$  and every facet  $F$  of  $Q$ , the induced subdivision  $\mathcal{C}^k \cap F$  equals  $\mathcal{C}^{k-1} \cap F$ .
- (I<sub>k</sub>2) *PL equivalence requirement.*  
For every  $k$ -face  $Q \in \mathcal{F}_k(\mathcal{A})$  and every dual manifold  $\mathcal{N}$  of  $Q$  (with boundary) the cubical subdivision  $\mathcal{C}^k \cap Q$  has a dual manifold that is PL-equivalent to  $j(\mathcal{N}) \cap Q$ .
- (I<sub>k</sub>3) *Symmetry requirement.*  
Every symmetry of  $(Q, j(\mathcal{M}) \cap Q)$  for a  $k$ -face  $Q \in \mathcal{F}_k(\mathcal{A})$  that is a symmetry of each sheet of  $j(\mathcal{M}) \cap Q$  separately is a symmetry of  $(Q, \mathcal{C}^k \cap Q)$ .
- (I<sub>k</sub>4) *Subcomplex requirement.*  
For every diagonal symmetry hyperplane  $H_Q$  of a facet  $Q$  of  $\mathcal{A}$  and every facet  $F$  of  $Q$  the (lifted) induced subdivision  $\mathcal{C}^k \cap (F \cap H)$  is a (lifted) subcomplex of  $\mathcal{C}^k$ .

All of these invariants are maintained while iteratively constructing  $\mathcal{C}^1$  and  $\mathcal{C}^2$ . The resulting lifted cubical subdivision  $\mathcal{C}^3$  of  $\mathcal{A}$  satisfies the conditions (I<sub>3</sub>1) and (I<sub>3</sub>2) but not the other ones.

### (1) Subdivision of edges.

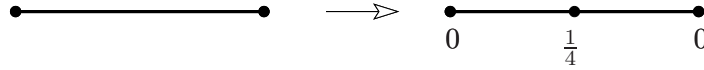
We construct a lifted cubical subdivision  $\mathcal{C}^1$  of the 1-skeleton of  $\mathcal{A}$  by providing for every edge  $e$  of  $\mathcal{A}$  a lifted subdivision  $\mathcal{C}_e^1$ .

Let  $e$  be an edge of  $\mathcal{A}$ .

- If  $e$  is not intersected by the immersed manifold, then we subdivide the edge by an affine copy  $\mathcal{C}_e^1$  of the lifted subdivision  $\mathcal{U}_2 := (\mathcal{U}'_2, h)$ , where  $\mathcal{U}'_2$  is a 1-dimensional complex affinely isomorphic to  $\mathcal{P}_1(2)$ . The vertices of  $\text{lift}(\mathcal{U}'_2, h)$  are

$$\mathbf{v}_1 = (-\frac{1}{2}, 0), \quad \mathbf{v}_2 = (0, \frac{1}{4}), \quad \mathbf{v}_3 = (\frac{1}{2}, 0),$$

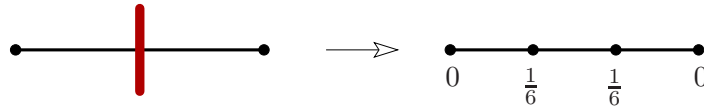
and the two edges are  $(\mathbf{v}_1, \mathbf{v}_2)$  and  $(\mathbf{v}_2, \mathbf{v}_3)$ . The following figure depicts  $(e, j(\mathcal{M}) \cap e)$  and  $\mathcal{U}'_2$  where each vertex of the subdivision is labeled by its height.



- If  $e$  is intersected by the immersed manifold, then we subdivide the edge by an affine copy  $\mathcal{C}_e^1$  of the lifted subdivision  $\mathcal{U}_3 := (\mathcal{U}'_3, h)$ , where  $\mathcal{U}'_3$  is a 1-dimensional complex affinely isomorphic to  $\mathcal{P}_1(3)$ . The vertices of  $\text{lift}(\mathcal{U}'_3, h)$  are

$$\mathbf{v}_1 = (-\frac{1}{2}, 0), \quad \mathbf{v}_2 = (-\frac{1}{6}, \frac{1}{6}), \quad \mathbf{v}_3 = (\frac{1}{6}, \frac{1}{6}), \quad \mathbf{v}_4 = (\frac{1}{2}, 0),$$

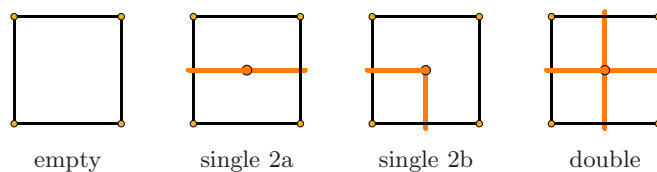
and the edges are  $(\mathbf{v}_i, \mathbf{v}_{i+1})$ , for  $i = 1, 2, 3$ .



Observe that (I<sub>1</sub>1)–(I<sub>1</sub>4) are satisfied.

### (2) Subdivision of 2-faces.

We construct the lifted cubical subdivision  $\mathcal{C}^2$  of the 2-skeleton of  $\mathcal{A}$  by providing for every 2-face (quadrangle)  $Q$  of  $\mathcal{A}$  a lifted cubification  $\mathcal{C}_Q^2$ . Let  $Q$  be a quadrangle of  $\mathcal{A}$ , and  $\mathbf{w}$  the unique vertex of  $j(\mathcal{M})$  that is contained in  $Q$ . There are four possible types of vertex stars of grid immersions of curves:

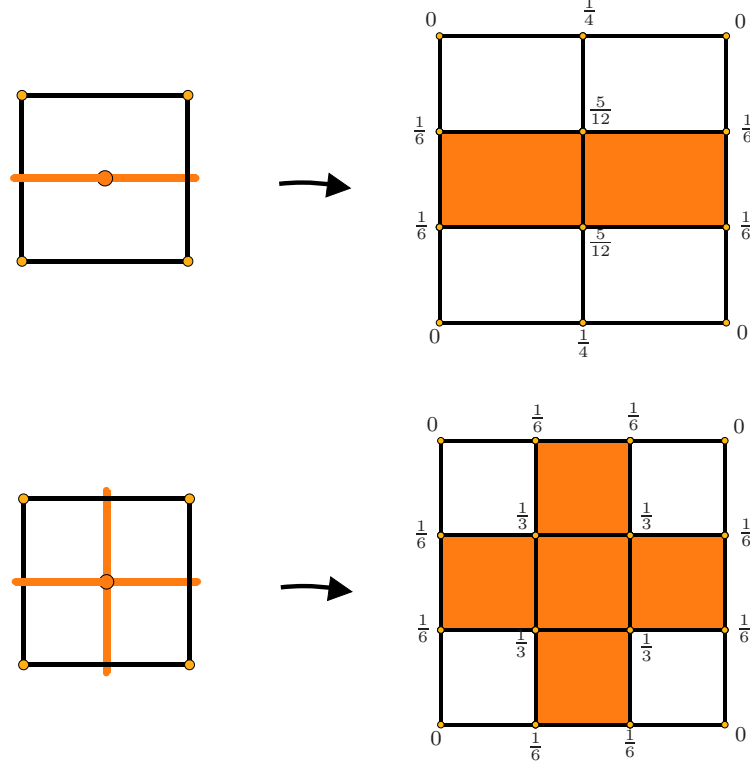


- (a) In the cases “single 2a” and “double” there is a coordinate hyperplane  $H$  such that  $(Q, j(\mathcal{M}) \cap Q)$  is symmetric with respect to  $H$ , and a vertex  $\mathbf{v}$  of  $\mathcal{M}$  such that

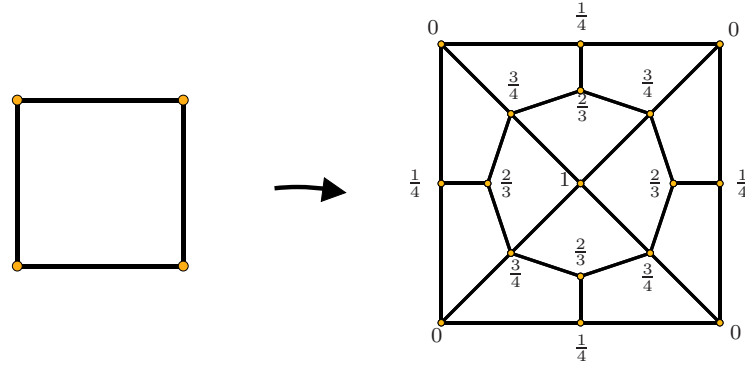
$j(v) = w$  and the image of the vertex star is contained in  $H$ . Let  $F$  be a facet of  $Q$  that does not intersect  $H$ .

Then  $\mathcal{C}_Q^2$  is taken to be an affine copy of the product  $(\mathcal{C}^1 \cap F) \times \mathcal{U}_3$ .

The following figure depicts  $(Q, j(\mathcal{M}) \cap Q)$  and the resulting cubification  $\mathcal{C}_Q^2$  where each vertex of the subdivision is labeled by its height.

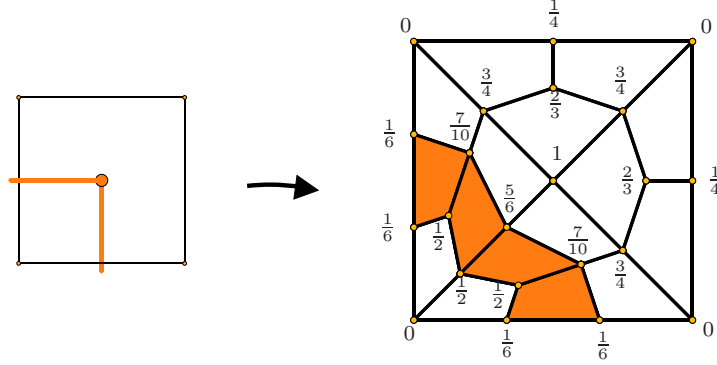


- (b) Assume the immersion does not intersect  $Q$ . In this case  $\mathcal{C}_Q^2$  is taken to be an affine copy of the lifted cubical 2-complex  $\mathcal{V}$ : This complex is given by the upper faces of the 3-polytope whose vertices are  $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0)$ ,  $(\pm\frac{1}{4}, \pm\frac{1}{4}, \frac{3}{4})$ ,  $(0, 0, 1)$ ,  $(0, \pm\frac{1}{2}, \frac{1}{4})$ ,  $(\pm\frac{1}{2}, 0, \frac{1}{4})$ ,  $(0, \pm\frac{1}{3}, \frac{2}{3})$ , and  $(\pm\frac{1}{3}, 0, \frac{2}{3})$ . ( $\mathcal{V}$  arises as the cubical barycentric subdivision of the stellar subdivision of  $[-\frac{1}{2}, \frac{1}{2}]^2$ .)



- (c) In the case “single 2b”, we define  $\mathcal{C}_Q^2$  as an affine copy of the lifted cubical 2-complex  $\mathcal{V}'$ : This complex is given by the upper faces of the 3-polytope whose vertices are  $(\pm\frac{1}{2}, -\frac{1}{2}, 0)$ ,  $(\frac{1}{2}, \frac{1}{2}, 0)$ ,  $(0, 0, 1)$ ,  $(\frac{1}{2}, 0, \frac{1}{4})$ ,  $(\frac{1}{3}, 0, \frac{2}{3})$ ,  $(\frac{1}{4}, \pm\frac{1}{4}, \frac{3}{4})$ ,  $(\pm\frac{1}{6}, -\frac{1}{2}, \frac{1}{6})$ ,  $(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{2})$ ,  $(-\frac{3}{8}, -\frac{1}{8}, \frac{1}{2})$ ,  $(-\frac{1}{6}, -\frac{1}{6}, \frac{5}{6})$ ,  $(\frac{1}{10}, -\frac{3}{10}, \frac{7}{10})$  and their reflections on the hyperplane  $x_1 - x_2 = 0$ .

The complex  $\mathcal{V}'$  is given by  $\mathcal{V}$  truncated by the two hyperplanes  $(1, 5, -3)^T \mathbf{x} = \frac{1}{2}$  and  $(0, \frac{8}{3}, -1)^T \mathbf{x} = \frac{3}{2}$ , and their reflections on the hyperplane  $x_1 - x_2 = 0$ .



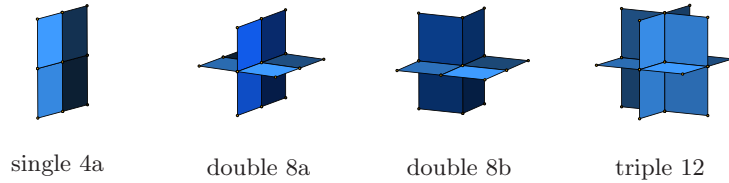
Observe that the conditions (I<sub>21</sub>)–(I<sub>24</sub>) are satisfied.

**(3) Subdivision of 3-cubes.** We construct a lifted cubical subdivision  $\mathcal{C}^3$  of the 3-skeleton of  $\mathcal{A}$  by providing for every facet  $Q$  of  $\mathcal{A}$  a lifted subdivision  $\mathcal{C}_Q^3$ .

Let  $Q$  be a facet of  $\mathcal{A}$  and  $\mathbf{w}$  the unique vertex of  $j(\mathcal{M})$  that is mapped to the barycenter of  $Q$ . Let  $\mathcal{S} := \mathcal{C}^2 \cap Q$  be the induced lifted cubical boundary subdivision of  $Q$ .

We construct a lifted cubification  $\mathcal{C}_Q^3$  of  $\mathcal{S}$  either as a generalized regular Hexhoop or as a product of  $\mathcal{U}_3$  with a lifted cubical subdivision of a facet of  $Q$ .

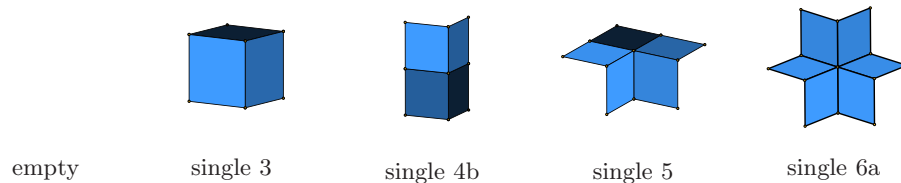
(a) For the following four types of vertex stars it is possible to take a product with  $\mathcal{U}_3$ :



In all these cases there is a symmetry hyperplane  $H$  of  $Q$  of the form  $x_i = k$  such that  $H \cap Q$  is a sheet of  $j(\mathcal{M}) \cap Q$ .

Hence all facets of  $Q$  that intersect  $H$  are subdivided by  $\mathcal{U}_3 \times \mathcal{U}_3$  or  $\mathcal{U}_3 \times \mathcal{U}_2$ . Let  $F$  be one of the two facets of  $Q$  that do not intersect  $H$ . Then the product  $(\mathcal{C}^2 \cap F) \times \mathcal{U}_3$  yields the lifted subdivision  $\mathcal{C}_Q^3$  of  $Q$ . Clearly  $\mathcal{C}_Q^3$  satisfies (I<sub>32</sub>) restricted to  $Q$ .

(b) In the remaining five cases we take a generalized regular Hexhoop with a diagonal hyperplane of symmetry of  $Q$  to produce  $\mathcal{C}_Q^3$ . These cases include the following five cases:



In each of these cases,  $(Q, Q \cap j(\mathcal{M}))$  has a diagonal hyperplane  $H$  of symmetry. This hyperplane  $H$  intersects the relative interior of two facets of  $Q$ . Since (I<sub>2</sub>4) holds, no facet of  $\mathcal{S} = \mathcal{C}^2 \cap Q$  intersects  $H$  in its relative interior. By (I<sub>2</sub>3) the lifted boundary subdivision  $\mathcal{S}$  is symmetric with respect to  $H$ . Hence all preconditions of the generalized regular Hexhoop are satisfied. The resulting cubification  $\mathcal{C}_Q^3$  satisfies (I<sub>3</sub>2) restricted to  $Q$  (see Remark 2).

## 7.5 Correctness

**Proposition 7.4.** *Let  $\mathcal{M}$  be a 2-dimensional closed (that is, compact and without boundary) cubical PL-surface, and  $j: \mathcal{M} \looparrowright \mathbb{R}^3$  a locally symmetric codimension one grid immersion.*

*Then the cubical 3-ball  $\mathcal{B}$  given by Construction 6 has the following properties:*

- (i)  $\mathcal{B}$  is regular, with a lifting function  $\psi$ .
- (ii) There is a dual manifold  $\mathcal{M}'$  of  $\mathcal{B}$  and associated immersion  $y: \mathcal{M}' \looparrowright |\mathcal{B}|$  such that  $\mathcal{M}'$  is a cubical subdivision of  $\mathcal{M}$ , and the immersions  $j$  and  $y$  are PL-equivalent.
- (iii) The number of facets of  $\mathcal{B}$  is congruent modulo two to the number  $t(j)$  of triple points of the immersion  $j$ .
- (iv) There is a lifted cubification  $\mathcal{C}$  of  $(\partial\mathcal{B}, \psi|_{\partial\mathcal{B}})$  with an even number of facets.

*Proof.* (i) *Regularity.* By construction the lifting functions  $\psi_F$ ,  $F \in \text{fac}(\mathcal{A})$ , satisfy the consistency precondition of the patching lemma. Since every pile of cube is regular (compare Section 3.5), the raw complex  $\mathcal{A}$  is regular and the Patching Lemma implies that  $\mathcal{B}$  is regular, too.

(ii) *PL-equivalence of manifolds.* Property (I<sub>3</sub>2) implies that for every facet  $Q \in \text{fac}(\mathcal{A})$  the lifted cubical subdivision  $\mathcal{C}^3 \cap Q$  has a dual manifold  $\mathcal{M}_Q$  that is PL-equivalent to  $j(\mathcal{M}) \cap Q$ . The consistency requirements (I<sub>1</sub>1) – (I<sub>3</sub>1) imply that for each pair  $Q, Q'$  of facets of  $\mathcal{A}$  the dual manifolds  $\mathcal{M}_Q, \mathcal{M}_{Q'}$  coincide, that is,  $j(\mathcal{M}_Q) \cap (Q \cap Q') = j(\mathcal{M}_{Q'}) \cap (Q \cap Q')$ .

Hence the union of all  $\mathcal{M}_Q$ ,  $Q \in \text{fac}(\mathcal{A})$  gives a dual manifold  $\mathcal{M}'$  with associated immersion  $y: \mathcal{M}' \looparrowright |\mathcal{B}|$  such that immersions  $j$  and  $y$  are PL-equivalent.

(iii) *Parity of the number of facets.* For each 3-cube  $Q$  of  $\mathcal{A}$ , its cubification  $\mathcal{C}_Q^3$  is either a product  $\mathcal{C}_F^2 \times \mathcal{U}_3$  (where  $\mathcal{C}_F^2$  is a cubification of a facet  $F$  of  $Q$ ), or the outcome of a generalized regular Hexhoop construction. In the latter case the number of facets of  $\mathcal{C}_Q^3$  is even. In the first case the number of facets depends on the number of 2-faces of  $\mathcal{C}_F^2$ . The number of quadrangles of  $\mathcal{C}_F^2$  is odd if and only if  $j(\mathcal{M}) \cap F$  has a double intersection point. Hence,  $f_3(\mathcal{C}_Q^3)$  is odd if and only if the immersion  $j$  has a triple point in  $Q$ .

(iv) *Alternative cubification.* Applying Construction 6 to  $\mathcal{P}_3(\ell_1, \ell_2, \ell_3)$  without an immersed manifold yields a regular cubification  $\mathcal{C}$  of  $\partial\mathcal{B}$  with the same lifting function as  $\mathcal{B}$  on the boundary. (It is possible to produce a smaller alternative cubification by the cubical barycentric subdivision or by a generalized Hexhoop. See the discussion in Section 8.) Since the immersion  $\emptyset \looparrowright \mathbb{R}^3$  has no triple points the number of facets of  $\mathcal{C}$  is even.  $\square$



## 7.6 Proof of the main theorem

Let  $j: \mathcal{M} \looparrowright \mathbb{R}^3$  be a locally flat normal crossing immersion of a compact  $(d-1)$ -manifold  $\mathcal{M}$  into  $\mathbb{R}^d$ .

By Proposition 7.1 and Proposition 7.2 there is a cubical subdivision  $\mathcal{M}'$  of  $\mathcal{M}$  with a locally symmetric, codimension one grid immersion  $j': \mathcal{M}' \looparrowright \mathbb{R}^3$  such that  $j$  and  $j'$  are PL-equivalent. We construct the convex cubical 3-ball  $\mathcal{B}$  with prescribed dual manifold immersion  $j'$  as described above. By Proposition 7.4(i) the ball  $\mathcal{B}$  is regular, and by Proposition 7.4(iv) there is a cubification  $\mathcal{C}$  of  $\partial\mathcal{B}$  with an even number of facets and the same lifting function on the boundary.

Perform a lifted prism over  $\mathcal{B}$  and  $\mathcal{C}$  (Construction 3 of Section 4.2). This yields a cubical 4-polytope  $P$  with

$$f_3(P) = f_3(\mathcal{B}) + f_3(\mathcal{C}) + f_2(\partial\mathcal{B}).$$

For every dual manifold of  $\mathcal{B}$  there is a PL-equivalent dual manifold of  $P$ . Hence by Proposition 7.4 there is a dual manifold  $\mathcal{M}''$  of  $P$  and associated immersion  $y: \mathcal{M}'' \looparrowright |\partial P|$  such that  $y$  and  $j'$  are PL-equivalent. Condition (i) is satisfied since  $j'$  and  $j$  are PL-equivalent.

To see (ii), observe that for every cubical 3-ball the number of facets of the boundary is even. Hence  $f_2(\partial\mathcal{B})$  is even. Since the number of facets of  $\mathcal{C}$  is even, we obtain

$$f_3(P) \equiv_2 f_3(\mathcal{B}) \equiv_2 t(j).$$

Now consider (iii). By a famous theorem of Banchoff [4] the number of triple points of a normal crossing codimension one immersion of a surface has the same parity as the Euler characteristic. Hence, if  $\mathcal{M}$  is a non-orientable surface of odd genus the number of triple points of  $j$  is odd, which implies that the cubical 4-polytope  $P$  has an odd number of facets.  $\square$

## 7.7 Symmetric templates

The three-dimension templates constructed above, which we call the *standard templates*, do not satisfy the conditions (I<sub>3</sub>3) and (I<sub>3</sub>4). In particular, the condition (I<sub>3</sub>3) is violated by the templates corresponding to the cases “empty”, “single 3”, and “single 6a” (and the condition is satisfied by all others). The cubification for the case “single 5” is illustrated in Figure 29. This cubifications satisfies (I<sub>3</sub>3) since there is only one diagonal symmetry hyperplane.

It is possible to produce an alternative template for the “empty” case by means of the cubical barycentric subdivision. The resulting cubification satisfies both conditions (I<sub>3</sub>3) and (I<sub>3</sub>4), and furthermore, it has less faces — 96 facets, 149 vertices — than the original template.

An alternative cubification for the case “single 3” of full symmetry can be constructed from  $\mathcal{C}''$  by truncating the lifted polytope corresponding to the lifted cubical ball  $\mathcal{C}''$  by some additional hyperplanes.

It is open whether there is a cubification of full symmetry for the case “single 6a.”

## 8 An odd cubical 4-polytope with a dual Boy's surface

**Theorem 8.1.** *Cubical 4-polytopes with odd numbers of facets exist. In particular, there is a cubical 4-polytope of  $f$ -vector*

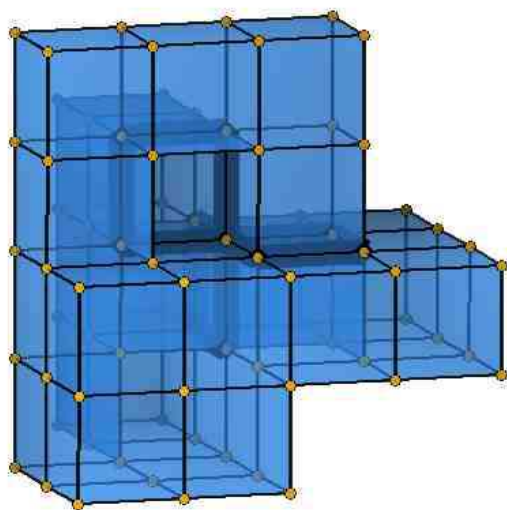
$$f = (19\,520, 56\,186, 54\,999, 18\,333)$$

*that has one dual manifold immersion that is a Boy's surface.*

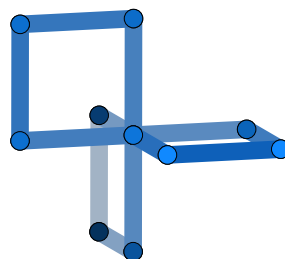
We prove this result by describing in detail the construction of a cubical 4-polytope with an odd number of facets.

### A grid immersion of Boy's surface

The construction starts with a grid immersion (cf. [26]) of Boy's surface, that is, an immersion of the real projective plane with exactly one triple point and three double-intersection curves in a pattern of three loops  $[6, 18, 2]$ . This immersion  $j : \mathcal{M} \looparrowright \mathbb{R}^3$  looks as follows.



(a) grid immersion



(b)

FIGURE 32: A grid immersion the Boy's surface, an immersion of the real projective plane with one triple point and three double-intersection curves in a pattern of three loops ("three-bladed propellor"). In this case, each double-intersection loop is of length four.

The 2-manifold  $\mathcal{M}$  has the  $f$ -vector  $f(\mathcal{M}) = (85, 168, 84)$ , whereas the image of the grid immersion has the  $f$ -vector  $f(j(\mathcal{M})) = (74, 156, 84)$ . The vertex coordinates can be chosen such that the image  $j(\mathcal{M})$  is contained in a pile of cubes  $P_3(4, 4, 4)$ .

### A cubical 3-ball with a dual Boy's surface

We apply Construction 6 to the grid immersion  $j : \mathcal{M} \looparrowright \mathbb{R}^3$  to obtain a cubical 3-ball with a dual Boy's surface, and with an odd number of facets.

Since the image  $j(\mathcal{M})$  is contained in a pile of cubes  $P_3(4, 4, 4)$ , the raw complex  $\mathcal{A}$  given by Construction 6 is affinely isomorphic to  $P_3(5, 5, 5)$ . Hence we have  $5^3 - 74 = 51$  vertices of  $\mathcal{A}$  that are not vertices of  $j(\mathcal{M})$ . We try to give an impression of the subdivision  $\mathcal{C}^2$  of the 2-skeleton of  $\mathcal{A}$  in Figure 33. The  $f$ -vector of  $\mathcal{C}^2$  is  $f = (4\,662, 9\,876, 5\,340)$ .

The subdivision of the boundary of  $\mathcal{A}$  consists of  $150 = 6 \cdot 5 \cdot 5$  copies of the two-dimensional “empty pattern” template. Hence the subdivision of the boundary of  $\mathcal{A}$  (given by  $\mathcal{C}^2 \cap |\partial\mathcal{A}|$ ) has the  $f$ -vector  $f = (1\,802, 3\,600, 1\,800)$ .

The construction of the refinement  $\mathcal{B}$  of  $\mathcal{A}$  depends of the chosen set of templates for dimension 3. First consider the “symmetric set” of templates described in Section 7.7. In this case the  $f$ -vector of  $\mathcal{B}$  is  $f = (15\,915, 45\,080, 43\,299, 14\,133)$ .

In Figure 34 we illustrate the dual Boy’s surface of the cubical 3-ball  $\mathcal{B}$ . It has the  $f$ -vector  $f = (1\,998, 3\,994, 1\,997)$ , and its set of multiple-intersection points has one triple point and three intersection loops of length 16. The ball  $\mathcal{B}$  has 612 dual manifolds is total (339 of them without boundary).

Using the “standard set” of templates yields a cubical ball with 18 281 facets.

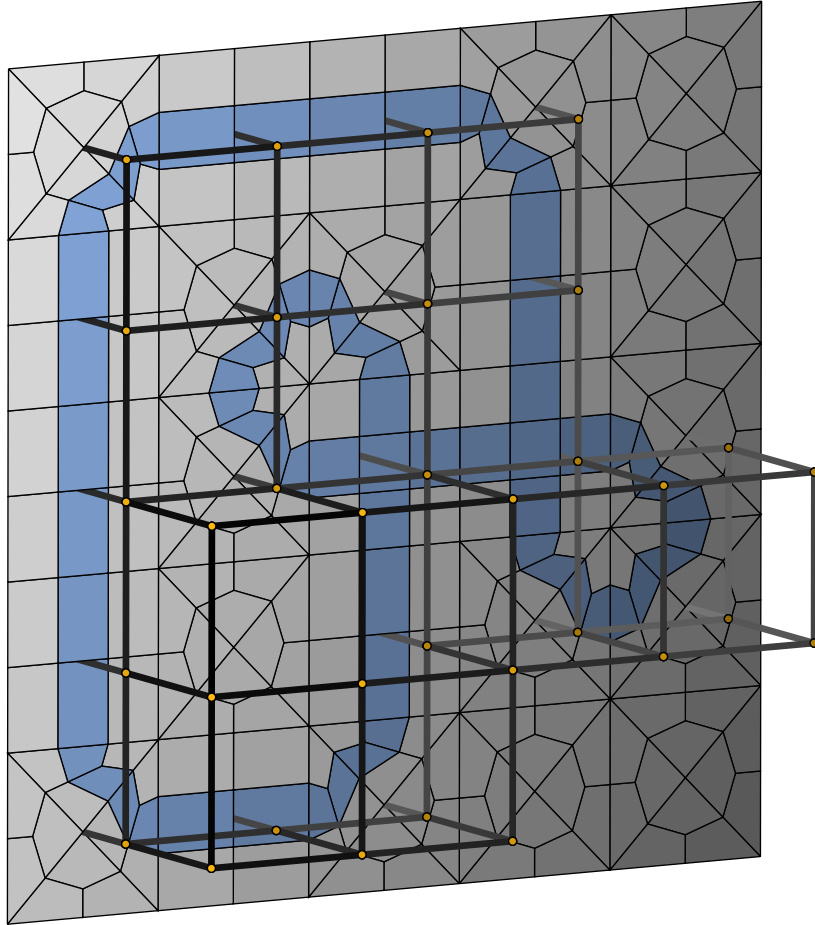


FIGURE 33: A sketch of the cubification of the 2-skeleton of  $\mathcal{A}$ .

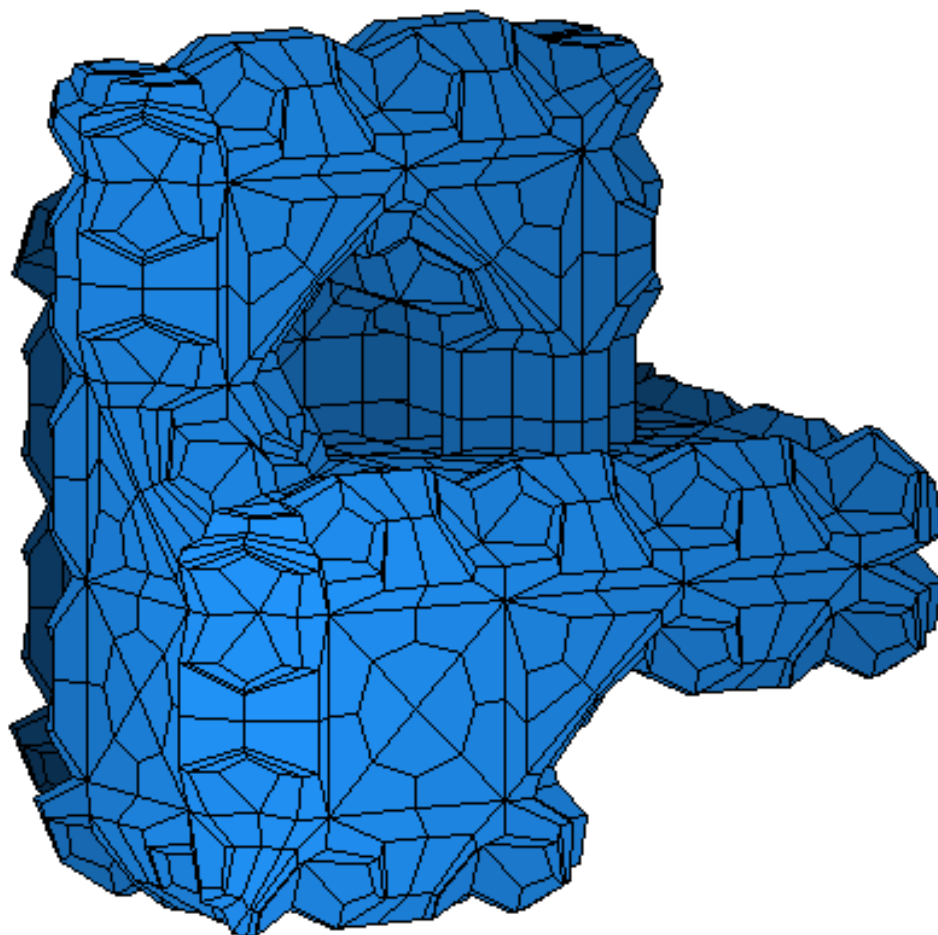


FIGURE 34: The dual Boy's surface of  $f$ -vector  $f = (1\,998, 3\,994, 1\,997)$  of the cubical 3-ball  $\mathcal{B}$ .

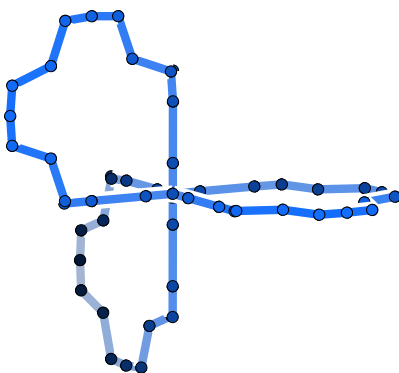


FIGURE 35: The multiple-intersection curve of the dual Boy's surface of the cubical 3-ball  $\mathcal{B}$ .

## A cubical 4-polytope with a dual Boy's surface

An cubification  $\mathcal{B}'$  of  $\partial B$  with an even number of facets is given by subdividing each facet of the raw ball  $\mathcal{A}$  with a cubification for the empty pattern. Using the symmetric cubification for the empty pattern yields a regular cubical 3-ball  $\mathcal{B}_2$  with

$$f(\mathcal{B}_2) = (14\,181, 39\,080, 36\,900, 12\,000).$$

In particular the number of facets is even. By Proposition 7.4 both cubical 3-balls  $\mathcal{B}$  and  $\mathcal{B}'$  have height functions that coincide on the common boundary  $\partial\mathcal{B} = \partial\mathcal{B}'$ .

Hence we can construct the lifted prism construction over  $\mathcal{B}$  and  $\mathcal{B}'$ . The resulting cubical 4-polytope  $P$  has  $f_0(P) = 15\,915 + 14\,181 = 30\,096$  vertices and  $f_3(P) = 14\,133 + 12\,000 + 1\,800 = 27\,933$  facets.

## A smaller cubical 4-polytope with a dual Boy's surface

It is possible to produce a significantly smaller alternative cubification  $\mathcal{B}''$  of  $\partial\mathcal{B}$  with an even number of facets. In the following we sketch the construction of a regular cubification  $\mathcal{B}''$  of  $\partial\mathcal{B}$  with  $f(\mathcal{B}'') = (3\,605, 9\,304, 8\,100, 2\,400)$ . Again, cubical barycentric subdivisions are used.

The resulting cubical 4-polytope  $P'$  has  $f_0 = 19\,520$  vertices and  $f_3 = 18\,333$  facets. A further analysis of the dual manifolds of  $P'$  shows that there are 614 dual manifolds in total: One dual Boy's surface of  $f$ -vector  $f = (1\,998, 3\,994, 1\,997)$ , one immersed surface of genus 20 (immersed with 104 triple points) with  $f$ -vector  $(11\,902, 23\,880, 11\,940)$ , and 612 embedded 2-spheres with various distinct  $f$ -vectors.

## Verification of the instances

All the instances of the cubical 4-polytopes described above were produced electronically. This was done using the `polymake` system by GAWRILOW & JOSWIG [13, 14], a system for the construction and analysis of convex polytopes. Furthermore a number of our own tools for handling cubical complexes are involved. They cover creation, verification, and visualization of cubical complexes (for  $d \in \{2, 3\}$ ).

The instances are available from

<http://www.math.tu-berlin.de/~schwartz/c4p>

Whereas the construction of the instances involves new tools that were writted specifically for this purpose, the verification procedure uses only standard `polymake` tools. All tools used in the verification procedure are parts of `polymake` system which have been used (and thereby verified) by various users over the past years (using a rich variety of classes of polytopes).

The topology of the dual manifolds of our instances was examined using all the following tools:

- A homology calculation code based written by Frank Heckenbach [16].
- The homology calculation code of `topaz`, the *topological application zoo*, which is part of the `polymake` project mentioned above; it covers the construction and analysis of simplicial complexes.
- Our own tool for the calculation of the Euler characteristics.

## 9 Consequences

### 9.1 Lattice of $f$ -vectors of cubical 4-polytopes

By a result of Babson & Chan [3] the  $\mathbb{Z}$ -affine span of  $f$ -vectors of cubical 3-spheres is given by the condition that  $f_0, f_1, f_2 + f_3$  are all even. With the existence of cubical 4-polytopes with an odd number of facets this extends to cubical 4-polytopes.

**Corollary 9.1.** *The  $\mathbb{Z}$ -affine span of  $f$ -vectors of cubical 4-polytopes characterized by*

$$f_0 \equiv_2 f_1 \equiv_2 f_2 + f_3 \equiv_2 0.$$

### 9.2 Cubical 4-polytopes with orientable dual manifolds of prescribed genus

**Corollary 9.2.** *For each  $g \geq 0$ , there is a cubical 4-polytope that has a cubation of the orientable connected 2-manifold  $M_g$  of genus  $g$  as an embedded dual manifold.*

*Proof.* Let  $g \geq 0$  be an integer. There is *grid torus* of  $f$ -vector  $(32, 64, 32)$  and a *grid handle* of  $f$ -vector  $(24, 44, 20)$ . both depicted in Figure 36.

Take the grid torus and  $g - 1$  copies of the grid handle and glue them together. This yields a grid embedding of the oriented surface of genus  $g$  of  $f$ -vector  $(16, 28, 14) + g(16, 36, 18)$ .

Applying the construction in the proof of Theorem 7.3 to this grid embedding yields a cubical 4-polytope with an embedded dual manifold of genus  $g$ .  $\square$

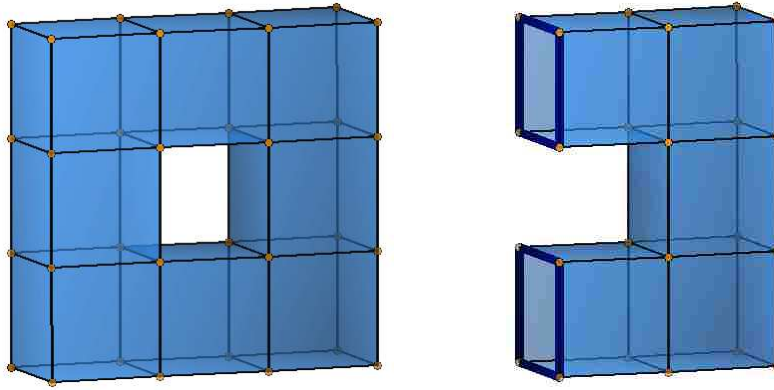


FIGURE 36: An grid embedding of a 2-torus of  $f$ -vector  $f = (32, 64, 32)$  and a grid handle of  $f$ -vector  $f = (24, 44, 20)$ .

Alternatively, cubical 4-polytopes with an orientable dual manifold of prescribed genus can be produced by means of connected sums of copies of the “neighborly cubical” 4-polytope  $C_4^5$  with the graph of a 5-cube (compare Section 2.3).

**Lemma 9.3.** *For each  $g > 0$ , there is a cubical 4-polytope  $P_g$  with the following properties.*

- (i) *The polytope  $P_g$  has exactly one embedded orientable dual 2-manifold  $\mathcal{M}$  of genus  $g$  with  $f$ -vector  $f(\mathcal{M}) = (12g + 4, 28g + 4, 14g + 2)$ .*
- (ii) *There is a facet  $F$  of  $P$  which is not intersected by the image of the dual manifold  $\mathcal{M}$ , and which is affinely regular, that is, there is an affine transformation between  $F$  and the standard cube  $[-1, +1]^3$ .*

- (iii) All other dual manifolds of  $P_g$  are embedded 2-spheres.
- (iv)  $f(P_g) = (24g + 8, 116g + 12, 138g + 6, 46g + 2)$ .

*Proof.* Recall that the neighborly cubical 4-polytope  $C_4^5 := \text{conv}((Q \times 2Q) \cup (2Q \times Q))$ , where  $Q = [-1, +1] \times [-1, +1]$ , has an embedded dual torus  $\mathcal{M}$  (compare Sect. 2.3). Furthermore,  $C_4^5$  has a facet  $F := 4Q \times [0, 5] \times \{0\}$  is affinely regular, and it is not intersected by the embedded dual manifold  $\mathcal{M}$ . Select an arbitrary facet  $G$  of  $C_4^5$  which is intersected by  $\mathcal{M}$ . Due to the symmetry of  $C_4^5$  there is an “opposite” facet  $G'$  of  $C_4^5$  such that  $G'$  is intersected by  $\mathcal{M}$ ,  $G$  and  $G'$  are disjoint, and  $G$  and  $G'$  are congruent. Let  $\phi$  be the congruence which maps  $(G, j(\mathcal{M}) \cap G)$  onto  $(G', j(\mathcal{M}) \cap G')$ , where  $j$  is the canonical PL immersion (in this case an embedding) of  $\mathcal{M}$  into the barycentric subdivision of the boundary of  $C_4^5$ .

Then take  $g$  copies of  $C_4^5$  and glue them together such that each glue operation melts a copy of  $G$  and a copy of  $G'$  (using  $\phi$ ). The resulting polytope  $P_g$  has an embedded dual manifold  $\mathcal{M}_g$  which is obtained by glueing together  $g$  copies of  $\mathcal{M}$ . Furthermore,  $P_g$  has facet  $F'$  which is projectively isomorphic to the facet  $F$  of  $C_4^5$ .  $\square$

This family of cubical 4-polytopes of a given genus has much smaller  $f$ -vector entries than the cubical 4-polytopes as given by Corollary 9.2.

### 9.3 Cubical 4-polytopes with non-orientable dual manifolds of prescribed genus

**Corollary 9.4.** *For each even  $g > 0$ , there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold  $M'_g$  of genus  $g$  as a dual manifold (immersed without triple points and with one double-intersection curve of length 16).*

*Proof.* Let  $g > 0$  be an even integer. Take the grid immersion of the Klein bottle of  $f$ -vector  $f = (52, 108, 56)$  which is depicted in Figure 37. Glue the grid immersion together with  $g - 1$  copies of the grid torus. The resulting grid immersion is non-orientable and of genus  $g$ . The immersion has no triple points and one double-intersection curve of length four. All four double-intersection points are of the case “double 8a.” The double-intersection curve of our cubification made for this case consists of four edges. Hence the cubical 4-polytope given by the construction in the proof of Theorem 7.3 has an immersed non-orientable dual manifold of genus  $g$ , which is immersed without triple points and with one double-intersection curve of length 16.  $\square$

Smaller cubical 4-polytopes with non-orientable cubical 4-polytopes can be produced by means of connected sums of the cubical 4-polytope  $P_{62}$  of Section 5 with a dual Klein bottle, and several copies of the neighborly cubical 4-polytope  $C_4^5$ . (Some “connector cubes” of Lemma 3.4 have to be used.) The resulting cubical 4-polytope has rather small  $f$ -vector entries, but the set of multiple-intersection points consists of five double-intersection curves.

Applying the same proof as above to the grid immersion of Boy’s surface of the previous section yields the following result.

**Corollary 9.5.** *For each odd  $g > 0$ , there is a cubical 4-polytope that has a cubation of the non-orientable connected 2-manifold  $M'_g$  of genus  $g$  as a dual manifold (immersed with one triple point and three double-intersection curves of length 14).*



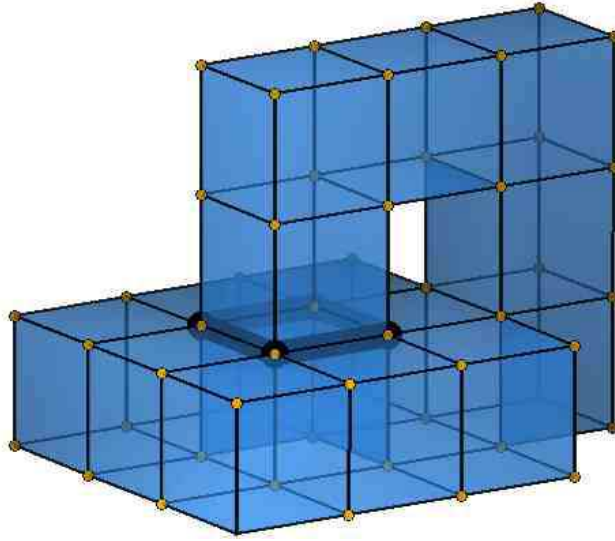


FIGURE 37: A grid immersion with  $f$ -vector  $f = (52, 108, 56)$  of the Klein bottle, an immersion of the non-orientable 2-surface of genus 2 with one double-intersection curve (shown in bold) and without triple points.

## 9.4 Higher-dimensional cubical polytopes with non-orientable dual manifolds

**Corollary 9.6.** *For each  $d \geq 4$  there are cubical  $d$ -polytopes with non-orientable dual manifolds.*

*Proof.* By construction, the 4-dimensional instance  $P_{62}$  of Section 5 comes with a subdivision into a regular cubical 4-ball. Since one of its dual manifolds is not orientable, its 2-skeleton is not edge orientable, i. e. it contains a cubical Möbius strip with parallel inner edges. So if we now iterate the lifted prism construction of Section 4.1, then the resulting cubical  $d$ -polytopes ( $d \geq 4$ ) will contain the 2-skeleton of  $P_{62}$ . By Proposition 2.1 they must also have non-orientable dual manifolds.  $\square$

## 10 Applications to hexa meshing

In the context of computer aided design (CAD) the surface of a workpiece (for instance a part of a car, ship or plane) is often modeled by a *surface mesh*. In order to analyze physical and technical properties of the mesh (and of the workpiece), finite element methods (FEM) are widely used.

Such a surface mesh is either a *topological mesh*, that is in our terminology a 2-dimensional regular CW complex, or a *geometric mesh*, that is, a (pure) 2-dimensional complex with polytopal cells. Common cell types of a surface mesh are triangles (2-simplices) and quadrangles. (A geometric quadrilateral mesh is a 2-dimensional cubical complex, and a topological one is a cubical 2-dimensional regular CW complex.)

In recent years there has been growing interest in volume meshing. Tetrahedral volume meshes (simplicial 3-complexes) are well-understood, whereas there are interesting and challenging open questions both in theory and practice of hexahedral volume meshes, *hexa meshes* for short. (A geometric hexa mesh is a 3-dimensional cubical complex, and a topological one is a cubical 3-dimensional regular CW complex.)

One challenging open question in this context is whether each cubical quadrilateral geometric



surface mesh with an even number of quadrangles admits a geometric hexa mesh. In our terminology this question phrases is whether each cubical PL 2-sphere with an even number of facets admits a cubification. Thurston [34] and Mitchell [24] proved independently that every topological quad mesh with an even number of quadrangles admits a topological hexa mesh. Furthermore, Eppstein showed in [9] that a linear number of topological cubes suffices, and Bern, Eppstein & Erickson proved the existence of a (pseudo-)shellable topological hexa mesh [5].

## 10.1 Parity change

Another interesting question deals with the parity of the number of facets of a mesh. For quad meshes there are several known parity changing operations, that is, operations that change the numbers of facets without changing the boundaries. In [5] Bern, Eppstein & Erickson raised the following questions:

**Question (Bern & Eppstein 2001).**

- (i) Are there geometric quad meshes with geometric hexa meshes of both parities?
- (ii) Is there *parity changing operation* for geometric hexa meshes, that is, an operation that changes the parity of the number of cubes of a cubical 3-ball (without changing the boundary)?

From the existence of a cubical 4-polytope with odd number of facets we obtain positive answers to these questions.

**Corollary 10.1.**

- (i) *Every combinatorial 3-cube has a cubification with an even number of facets. Furthermore, this cubification is regular and even Schlegel.*
- (ii) *Every combinatorial 3-cube is a facet of a cubical 4-polytope with an odd number of facets.*
- (iii) *There is a parity changing operation for geometric hexa meshes.*

*Proof.* To see (ii) let  $F$  be a combinatorial 3-cube, and  $P$  a cubical 4-polytope with an odd number of facets. By Lemma 3.4 there is a combinatorial 4-cube  $C$  that has both  $F$  and a projectively regular 3-cube  $G$  as facets. Let  $F'$  be an arbitrary facet of  $P$ . Then there is a combinatorial 4-cube  $C'$  that has both  $F'$  and a projectively regular 3-cube  $G'$  as facets. Then the connected sum of  $P$  and  $C$  based on the facet  $F'$  yields a cubical 4-polytope  $P'$  with an odd number of facets, and with a projectively regular 3-cube  $G''$  as facet. The connected sum of  $P'$  and  $C$  glueing the facets  $G$  and  $G''$  yields a cubical 4-polytope with an odd number of facets, and with a projective copy of  $F$  as a facet.

The statements (i) and (iii) follow from (ii) via Schlegel diagrams. □

## 10.2 Flip graph connectivity

In analogy to the concept of flips for simplicial (pseudo-)manifolds one can define *cubical flips* for quad or hexa meshes; compare [5]. In the meshing terminology the flip graph is defined as follows. For any domain with boundary mesh, and a type of mesh to use for that domain, define the *flip graph* to be a graph with (infinitely many) vertices corresponding to possible meshes of the domain, and an edge connecting two vertices whenever the corresponding two meshes can be transformed into each other by a single flip.

In this framework, the question concerning a parity changing operation can be phrased as asking for a description of the connected components of the flip graph. As an immediate consequence of the corollary above we obtain the following result.

**Corollary 10.2.** *The cubical flip graph for geometric hexa meshes of a has at least two connected components.*

## 11 Conclusions

In this paper we are primarily concerned with the realization of 2-manifold immersions in terms of cubical 4-polytopes, but the higher-dimensional cases are interesting as well.

For example, one would like to know whether there are cubical 5-polytopes with an odd number of facets. For this we have to realize a normal crossing immersion of 3-manifold into  $S^4$  by a cubical 5-polytope with an odd number of quadruple points. Such immersions exist by an abstract result of Freedman [12] (see also Akhmetev [1]), but more concretely by John Sullivan’s observation (personal communication) that there are regular sphere eversion of the 2-sphere with exactly one quadruple point [32, 27, 11, 23, 33], and from any such one obtains a normal-crossing immersion  $S^3 \looparrowright S^4$  with a single quadruple point.

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All illustrations of polytopes and balls are produced using the `polymake` system of Gawrilow & Joswig [14, 13] and `javaview` of Polthier et. al. [28]. (A small number of figures were drawn with `xfig`.)

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