Disturbing the Dyson Conjecture (in a GOOD Way)

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Dedicated to Freeman John Dyson and Irving John Good

Abstract

We present a case study in *experimental* yet *rigorous* mathematics by describing an algorithm, fully implemented in both Mathematica and Maple, that *automatically conjectures*, and then *automatically proves*, closed-form expressions extending Dyson's celebrated constant term conjecture.

1 Introduction

Let

$$F_n(\mathbf{x};\mathbf{a};\mathbf{b}) := \prod_{h=1}^{n} x_h^{-b_h} \prod_{1 \leqq i \neq j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j}$$

where $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$, $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$ and

$$c_n(\mathbf{a},\mathbf{b}) = \mathrm{CT}\big(F_n(\mathbf{x};\mathbf{a};\mathbf{b})\big),$$

where $\mathrm{CT}(X)$ denotes the constant term, i.e. the coefficient of $x_1^0 x_2^0 \cdots x_n^0$, in the expression X. The following conjecture is due to Freeman Dyson [Dyson 62, p. 152, Conjecture C]:

Dyson's conjecture. *For positive integers n and nonnegative integers a_i, $1 \leqq i \leqq n$,*

$$\mathrm{CT}\left(F_n(\mathbf{x};\mathbf{a};\mathbf{0})\right) = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!},$$

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where $\mathbf{0} = \langle 0, 0, \ldots, 0\rangle$. Dyson noted that the $n = 1, 2$ cases are trivial and that the $n = 3$ case is equivalent to a hypergeometric summation formula due to A.C. Dixon [Dixon 03]. Dyson proved the $n = 4$ case [Dyson 62, pp. 155–156, Appendix B], and indicated that a similar argument could be used to prove $n = 5$, but that his argument would not work for $n > 5$, and accordingly left $n > 5$ as a conjecture. The conjecture was quickly proved independently by J. Gunson [Gunson 62] and K. Wilson [Wilson 62]. The most compact and elegant proof, however, was supplied by I. J. Good [Good 70]. A combinatorial proof was later given by Zeilberger [Zeilberger 82].

In this paper, we concern ourselves with variations on Dyson's original conjecture where $\mathbf{b}$ can assume (fixed) values other than $\mathbf{0}$. In particular, we have automated, in the accompanying Mathematica package `GoodDyson.m` and analogous Maple package `GoodDyson`, both the act of *conjecturing* an explicit formula for $c_n(\mathbf{a}; \mathbf{b})$ (where $\mathbf{b}$ is a vector of specific integers), and the production of a *proof* of the conjectured form, based on a generalization of Good's ideas. The accompanying Mathematica and Maple packages are available from the authors' web sites `http://www.math.rutgers.edu/~asills` and `http://www.math.rutgers.edu/~zeilberg`.

2 Automating the Conjecturing Process

Given $\mathbf{b} = \langle b_1, \ldots, b_n\rangle$, we guess that the coefficient of $x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ in $F_n(\mathbf{x}; \mathbf{a}; \mathbf{0})$, or equivalently, the constant term of $F_n(\mathbf{x}; \mathbf{a}; \mathbf{b})$, which we denote by $c_n(\mathbf{a}; \mathbf{b})$, can be expressed in the form $d_n(\mathbf{a}; \mathbf{b})$, where

$$d_n(\mathbf{a}; \mathbf{b}) = R_{\mathbf{b}}(\mathbf{a}) \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1! a_2! \cdots a_n!},$$

and $R_{\mathbf{b}}(\mathbf{a})$ is a rational function in the a_i's. Of course, if $\sum_{i=1}^n b_i \neq 0$, then $R_{\mathbf{b}}(\mathbf{a}) = 0$, and in the case of Dyson's original conjecture, we have $R_{\mathbf{0}}(\mathbf{a}) = 1$ for all n.

We programmed a Mathematica function/Maple procedure `GuessRat`, which takes as input a function f, a set of variables $\{a_1, \ldots a_n\}$, and an integer t, and tries to match f to a rational function R in which the sum of the degrees of the numerator and denominator (let us call this the *total degree of* R) is t, using internally generated data. Of course, if no such R of total degree t is found, then the procedure is repeated with a larger t. `GuessRat` could potentially be useful in a wide variety of settings, but for this project we restricted f to the function which extracted the constant term from $F(\mathbf{x}; \mathbf{a}; \mathbf{b}) / \frac{(a_1 + \cdots + a_n)!}{a_1! \cdots a_n!}$ for a specific $\mathbf{b}$ in order to conjecture $R_{\mathbf{b}}(\mathbf{a})$. This is done via the auxiliary function `GuessDysonCoeff`.

Empirical evidence gathered while testing a prototype version of `GuessDysonCoeff` suggested that for each $b_i < 0$, the $R_{\mathbf{b}}(\mathbf{a})$ contained a factor of

$$\frac{1}{(1 + a_i)_{\lfloor b_i/2 \rfloor} (1 + a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n)_{|b_i|}},$$

where $(y)_h$ denotes the rising factorial, which is defined by

$$(y)_h := \begin{cases} y(y+1)(y+2)\cdots(y+h-1), & \text{if } h > 0, \\ 1, & \text{if } h = 0, \\ \frac{1}{(y-1)(y-2)\cdots(y-h)}, & \text{if } h < 0. \end{cases}$$

For example, closed form representation of the constant term of $F_4(\mathbf{x};\mathbf{a};\langle -3, b_2, -1, 4-b_2\rangle)$ contains the factor

$$\frac{a_1(a_1-1)a_3}{(1+a_2+a_3+a_4)(2+a_2+a_3+a_4)(3+a_2+a_3+a_4)(1+a_1+a_2+a_4)}.$$

Accordingly, we modified the f which was sent as input into `GuessRat` via the `GuessDysonCoeff` function, resulting in the output R being of lower total degree, and therefore greatly reducing the time Mathematica/Maple needs to supply a conjecture. For a vector $\mathbf{b}$ whose components sum to zero, let us define the *complexity* of $\mathbf{b}$, $\mathrm{Comp}(\mathbf{b})$, to be the sum of its positive components, or equivalently,

$$\mathrm{Comp}(\mathbf{b}) := \frac{1}{2}\sum_{i=1}^{n} |b_i|.$$

When the complexity of of $\mathbf{b}$ is close to zero, the modified algorithm worked over twenty times faster than the original. For larger complexity, the speedup was even more significant. For instance, in the case $\langle b_1, b_2, b_3\rangle = \langle 4, -2, -2\rangle$, the modified algorithm was over one hundred times faster than the original.

3 The Generalized Good Proof

For some fixed n and $\mathbf{b}$, we want to show that for all $\mathbf{a}$,

$$c_n(\mathbf{a};\mathbf{b}) = d_n(\mathbf{a};\mathbf{b}), \tag{3.1}$$

where an explicit closed form expression $d_n(\mathbf{a};\mathbf{b})$ for $c_n(\mathbf{a};\mathbf{b})$ has been conjectured using Mathematica or Maple. Since the $n = 1$ case is trivial, and the $n = 2$ case

$$c_2(\langle a_1, a_2\rangle; \langle b_1, b_2\rangle) = \begin{cases} (-1)^{b_1}\dfrac{(a_1+a_2)!}{(a_1+b_1)!(a_2-b_1)!}, & \text{if } b_1 = -b_2 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

follows from the binomial theorem, we restrict our attention to $n > 2$.

As in Good's proof [Good 70], for $a_1, a_2, \ldots, a_n \geqq 1$, $F_n(\mathbf{x};\mathbf{a};\mathbf{b})$ satisfies the recursion

$$F_n(\mathbf{x};\mathbf{a};\mathbf{b}) = \sum_{i=1}^{n} F_n(\mathbf{x};\langle a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_n\rangle;\mathbf{b}). \tag{3.3}$$

Applying the constant term operator to both sides of (3.3), we obtain

$$c_n(\mathbf{a};\mathbf{b}) = \sum_{i=1}^{n} c_n(\langle a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_n\rangle;\mathbf{b}). \tag{3.4}$$

Next, we note that for any fixed k, $1 \leqq k \leqq n$,

$$F_n(\mathbf{x};\langle a_1,\ldots,a_{k-1},0,a_{k+1},\ldots,a_n\rangle;\mathbf{b}) = F_{n-1}(\hat{\mathbf{x}}^{(k)},\hat{\mathbf{a}}^{(k)},\mathbf{0}) \left[x_k^{-b_k} \prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i-x_k)^{a_i}}{x_i^{a_i+b_i}} \right], \tag{3.5}$$

where $\hat{\mathbf{x}}^{(k)} = \langle x_1,\ldots,x_{k-1},x_{k+1},\ldots x_n\rangle$ and $\hat{\mathbf{a}}^{(k)} = \langle a_1,\ldots,a_{k-1},a_{k+1},\ldots a_n\rangle$. Notice that on the right hand side of (3.5), we have managed to segregate the factors involving x_k (those in brackets) from those which do not involve x_k. We can therefore let Mathematica or Maple find the explicit Taylor series expansion of $\prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i-x_k)^{a_i}}{x_i^{a_i+b_i}}$ about $x_k = 0$. And so, by extracting the coefficient of x_k^0 on both sides of (3.5), we have, for $1 \leqq k \leqq n$,

$$\begin{aligned} &\text{coeff of } x_k^0 \text{ in } \left[F_n(\mathbf{x};\langle a_1,\ldots,a_{k-1},0,a_{k+1},\ldots,a_n\rangle;\mathbf{b})\right] \\ &= \text{coeff of } x_k^0 \text{ in } \left[\left(x_k^{-b_k} \prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i-x_k)^{a_i}}{x_i^{a_i+b_i}} \right) F_{n-1}(\hat{\mathbf{x}}^{(k)},\hat{\mathbf{a}}^{(k)},\mathbf{0}) \right]. \end{aligned} \tag{3.6}$$

Finally, we apply the constant term operator to both sides of (3.6) to obtain

$$\begin{aligned} &c_n(\langle a_1,\ldots,a_{k-1},0,a_{k+1},\ldots,a_n\rangle;\mathbf{b}) \\ &= P_k \; c_{n-1}(\hat{\mathbf{a}}^{(k)},\mathbf{0}), \end{aligned} \tag{3.7}$$

where P_k is the coefficient of $x_k^{b_k}$ in the Taylor expansion of $\prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i-x_k)^{a_i}}{x_i^{a_i+b_i}}$ about $x_k = 0$. Notice that P_k is a Laurent polynomial in $x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n$, whose coefficients depend on $a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n$. (In the case where $\mathbf{b} = \mathbf{0}$, we have $P_k = 1$, making Good's proof very tidy indeed!) Finally, we note the initial condition

$$c_n(\mathbf{0};\mathbf{b}) = \begin{cases} 1, & \text{if } \mathbf{b} = \mathbf{0} \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

The equation (3.4), the set of n boundary conditions (3.7), together with (3.8) fully determine $c_n(\mathbf{a};\mathbf{b})$. Thus, to prove (3.1), it suffices to show that our conjectured formula $d_n(\mathbf{a};\mathbf{b})$ also satisfies (3.4), (3.7), and (3.8). The equations (3.7) will, in general, depend on $c_{n-1}(\mathbf{a};\mathbf{b})$ for various values of $\mathbf{b}$, and so the boundary conditions will need to be iterated $n-2$ times until (3.7) is expressed fully in terms of c_2's which is given by (3.2).

4 Example

The coefficient of $\frac{x_1^2}{x_2 x_3}$ in the expansion of the Laurent polynomial

$$\left[\left(1-\frac{x_2}{x_1}\right)\left(1-\frac{x_3}{x_1}\right)\right]^{a_1}\left[\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_3}{x_2}\right)\right]^{a_2}\left[\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\right]^{a_3}$$

is

$$d_3(\langle a_1,a_2,a_3\rangle;\langle 2,-1,-1\rangle)=\frac{a_2a_3(2+2a_1+a_2+a_3)(a_1+a_2+a_3)!}{(1+a_1+a_2)(1+a_1+a_3)(1+a_1)a_1!a_2!a_3!}.$$

Good style proof. It is easily verified that for $a_1, a_2, a_3 \geqq 1$,

$$\begin{aligned} F_3(\mathbf{x};\mathbf{a};\langle 2,-1,-1\rangle) &= F_3(\mathbf{x};\langle a_1-1,a_2,a_3\rangle;\langle 2,-1,-1\rangle) \\ &\quad +F_3(\mathbf{x};\langle a_1,a_2-1,a_3\rangle;\langle 2,-1,-1\rangle) \\ &\quad +F_3(\mathbf{x};\langle a_1,a_2,a_3-1\rangle;\langle 2,-1,-1\rangle) \end{aligned} \tag{4.1}$$

Applying the constant term operator to both sides of (3.3), we immediately obtain

$$\begin{aligned} c_3(\mathbf{a};\langle 2,-1,-1\rangle) &= c_3(\langle a_1-1,a_2,a_3\rangle;\langle 2,-1,-1\rangle) \\ &\quad +c_3(\langle a_1,a_2-1,a_3\rangle;\langle 2,-1,-1\rangle) \\ &\quad +c_3(\langle a_1,a_2,a_3-1\rangle;\langle 2,-1,-1\rangle) \end{aligned} \tag{4.2}$$

The boundary conditions are found by `GoodDyson` to be

$$\begin{aligned} c_3(\langle 0,a_2,a_3\rangle;\langle 2,-1,-1\rangle) &= \frac{a_3(a_3-1)}{2}c_2(\langle a_2,a_3\rangle;\langle -1,1\rangle) \\ &\quad +\frac{a_2(a_2-1)}{2}c_2(\langle a_2,a_3\rangle;\langle 1,-1\rangle) \\ &\quad +a_2a_3c_2(\langle a_2,a_3\rangle;\langle 0,0\rangle) && (4.3) \\ c_3(\langle a_1,0,a_3\rangle;\langle 2,-1,-1\rangle) &= 0 && (4.4) \\ c_3(\langle a_1,a_2,0\rangle;\langle 2,-1,-1\rangle) &= 0 && (4.5) \end{aligned}$$

Finally,

$$c_3(\langle 0,0,0\rangle;\langle 2,-1,-1\rangle)=0 \tag{4.6}$$

Since $c_3(\mathbf{a};\langle 2,-1,-1\rangle)$ is uniquely determined by (4.2)–(4.6), and it is easily verified that, in light of (3.2), our conjectured expression $d_3(\mathbf{a};\langle 2,-1,-1\rangle)$ also satisfies (4.2)–(4.6), the proof is complete. □

Full proofs, analogous to the above, can be generated automatically with our Mathematica function/Maple procedure `WritePaper`.

5 Exploiting Symmetry and Algebraic Relations

Suppose $d_n(\mathbf{a};\mathbf{b})$ is known for some particular $\mathbf{b}$. It is then a simple matter to determine $d_n(\mathbf{a};\mathbf{b}')$ for all vectors $\mathbf{b}'$ whose components are a permutation of the components of $\mathbf{b}$: if $\pi_{\mathbf{b}}$ permutes the indices of $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$, so that $\pi_{\mathbf{b}}\mathbf{b} = \mathbf{b}'$, then $d_n(\mathbf{a};\mathbf{b}') = d_n(\pi_{\mathbf{a}}\mathbf{a};\mathbf{b})$. For example, given

$$d_3(\mathbf{a};\langle -1,0,1\rangle) = -\frac{(a_1+a_2+a_3)!a_1}{a_1!a_2!a_3!(1+a_2+a_3)},$$

we immediately know that

$$d_3(\mathbf{a};\langle 1,0,-1\rangle) = -\frac{(a_1+a_2+a_3)!a_3}{a_1!a_2!a_3!(1+a_2+a_1)},$$
$$d_3(\mathbf{a};\langle 0,-1,1\rangle) = -\frac{(a_1+a_2+a_3)!a_2}{a_1!a_2!a_3!(1+a_1+a_3)},$$

etc.

Also, notice that

$$F_n(\mathbf{x};\mathbf{a}+\langle 0,0,\ldots,0,1\rangle;\mathbf{b}) = F_n(\mathbf{x};\mathbf{a};\mathbf{b})\left[\left(1-\frac{x_1}{x_n}\right)\left(1-\frac{x_2}{x_n}\right)\cdots\left(1-\frac{x_{n-1}}{x_n}\right)\right].$$

By expanding out the Laurent polynomial in brackets, distributing it over the $F_n(\mathbf{x};\mathbf{a};\mathbf{b})$ and applying the constant term operator, one can often express a $d_n(\mathbf{x};\mathbf{a};\mathbf{b})$ with a $\mathbf{b}$ of higher complexity as a linear combination of $d_n(\mathbf{x};\mathbf{a};\mathbf{b})$'s with $\mathbf{b}$'s that have previously been calculated and/or have lower complexity. For example,

$$\begin{aligned}
& F_3(\mathbf{x};\mathbf{a}+\langle 0,0,1\rangle;\mathbf{b}) \\
= \; & F_3(\mathbf{x};\mathbf{a};\mathbf{b})\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right) \\
= \; & F_3(\mathbf{x};\mathbf{a};\mathbf{b})\left(1-\frac{x_1}{x_3}-\frac{x_2}{x_3}+\frac{x_1x_2}{x_3^2}\right) \\
= \; & F_3(\mathbf{x};\mathbf{a};\mathbf{b}) - F_3(\mathbf{x};\mathbf{a};\mathbf{b}+\langle -1,0,1\rangle) - F_3(\mathbf{x};\mathbf{a};\mathbf{b}+\langle 0,-1,1\rangle) + F_3(\mathbf{x};\mathbf{a};\mathbf{b}+\langle -1,-1,2\rangle)
\end{aligned}$$

After applying the constant term operator to both sides and solving for the last term, we find that

$$d_3(\mathbf{a};\mathbf{b}+\langle -1,-1,2\rangle) = d_3(\langle a_1,a_2,1+a_3\rangle;\mathbf{b}) - d_3(\mathbf{a};\mathbf{b}) + d_3(\mathbf{a};\mathbf{b}+\langle -1,0,1\rangle) + d_3(\mathbf{a};\mathbf{b}+\langle 0,-1,1\rangle)$$

By systematically taking advantage of the above observations, the procedure `TurboDyson` can find $d_n(\mathbf{a};\mathbf{b})$ for a given n and all $\mathbf{b}$'s of complexity less than or equal to a given complexity C rather quickly. Furthermore, `TurboDyson` stores its findings in an indexed global variable for future use, e.g. a subsequent call to the `WritePaper` procedure.

6 Conclusion

Experimental mathematics, as it is commonly understood, consists of computer-assisted *gathering of data* (usually numeric, but recently also symbolic) that stimulates and inspires *human-made* conjectures, that, in turn, require *human-made proofs*. The novelty of the present research is that all these phases are done by machine, once the initial effort of designing an algorithm, and implementing it, are done by humans. Of course, at present, the general framework for the conjecture, and the *idea* of proof, are still human-made (in our case by two very illustrious humans: Dyson and Good), but all the rest is automatic. We believe that this methodology should be applicable to many other problems.

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