Matrix valued orthogonal polynomials related to SU($N+1$), their algebras of differential operators and the corresponding curves [a]

Recent Trends in Constructive Approximation Theory
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Outline of the talk

- The theory of matrix valued orthogonal polynomials (MVOP) was introduced by M. G. Krein in 1949.
- Systematically studied in the last 15 years.
- In 1997, Durán raised the problem of characterizing MVOP satisfying *second order differential equations.*

  Scalar case: Bochner (1929): Hermite-Laguerre-Jacobi

  $\Rightarrow$**New (non-trivial) matrix examples** (2003): Durán-Grünbaum-Pacharoni-Tirao.
- New phenomena:
  - MVOP satisfying first order differential operators.
  - Richer behavior of the algebra of differential operators.

Background I

- Given a self adjoint positive definite matrix weight function W we define a skew symmetric bilinear form:

$$(P, Q) = \int_a^b P(t)\, W(t)\, Q^*(t)\, dt.$$

- This leads to a family of MVOP $\{P_n\}_{n\geq 0}$ with deg $P_n = n$, non–singular leading coefficient and $(P_n, P_m) = \Theta, n \neq m$.
- Examples considered here satisfy

$$DP_n \equiv P_n''(t)A_2(t) + P_n'(t)A_1(t) + P_n(t)A_0 = \Lambda_n P_n(t).$$

D is *symmetric* if $(DP, Q) = (P, DQ)$.

Background II

Generating examples:

- Solving *symmetry equations*. [Durán–Grünbaum]:

$$
\begin{aligned}
A_2 W &= W A_2^* \\
2(A_2 W)' &= W A_1^* + A_1 W, \\
(A_2 W)'' - (A_1 W)' + A_0 W &= W A_0^*.
\end{aligned}
$$

  with boundary conditions:

$$\lim_{t\to x} W(t)A_2(t) = \Theta = \lim_{t\to x}(W(t)A_1(t) - A_1^*(t)W(t)), \quad \text{for } \; x = a, b.$$

- *Matrix spherical functions* associated with $P_N(\mathbb{C}) = \mathrm{SU}(N+1)/\mathrm{U}(N)$. [Grünbaum–Pacharoni–Tirao].

Goal of this talk: the algebra

- Given a **fixed** family of MVOP $\{P_n\}_{n\geq 0}$ we study the algebra over $\mathbb{C}$:

$$\mathcal{D} = \{D : DP_n(t) = \Lambda_n(D)P_n(t),\ n = 0, 1, 2, \ldots\}$$

where

$$D = \sum_{j=0}^{r} \partial_t^j F_j(t) \quad \text{and} \quad F_j(t) = \sum_{i=0}^{j} B_i^j t^i.$$

- The map

$$\Lambda_n : \mathcal{D} \longrightarrow \mathcal{M}(N, \mathbb{C}), \quad n = 0, 1, 2, \ldots$$

is a *faithful representation*, i. e., $\Lambda_n(D_1 D_2) = \Lambda_n(D_1)\Lambda_n(D_2)$ with $D_1, D_2 \in \mathcal{D}$ and if $\Lambda_n(D) = \Theta$ with $D \in \mathcal{D}$ for all n, then $D = \Theta$.

Scalar case

If $\mathcal{H}$ is the corresponding second order differential operator of the classical orthogonal polynomials (Hermite, Laguerre or Jacobi):

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t)$$

$$tL_n^\alpha(t)'' + (\alpha + 1 - t)L_n^\alpha(t)' = -nL_n^\alpha(t)$$

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha+\beta+2)t)P_n^{(\alpha,\beta)}(t)' =$$

$$-n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(t)$$

then

$$\mathcal{U} = \sum_{i=0}^{k} c_i \mathcal{H}^i,$$

where $c_i \in \mathbb{C}$ and $\mathcal{U}$ is an even order differential operator.

Then $\mathcal{D} =< \mathcal{H} >$

Matrix case

$\mathcal{D}$ can be *non–commutative*, have *more than one generator* and have *relations* among the generators:

- Castro-Grünbaum, *The algebra of differential operators associated to a given family of matrix valued orthogonal polynomials: five instructive examples*, IMRN, (2006).
- Grünbaum-Pacharoni-Tirao, *Matrix valued orthogonal polynomials of the Jacobi type: The role of group representation theory*, Ann. Inst. Fourier, (2005).

The one step example: the initial differential operator

$$W(t) = t^{\alpha}(1-t)^{\beta}T(t)T^{*}(t), \quad \alpha, \beta > -1, \quad \mathcal{H}_1 = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t),$$

$$\begin{aligned}
A_2(t) &= t(1-t)I \\
A_1(t) &= \begin{pmatrix} \alpha+3 & 0 & 0 \\ -1 & \alpha+2 & 0 \\ 0 & -2 & \alpha+1 \end{pmatrix} - t \begin{pmatrix} (\alpha+\beta+4) & 0 & 0 \\ 0 & (\alpha+\beta+5) & 0 \\ 0 & 0 & (\alpha+\beta+6) \end{pmatrix} \\
A_0(t) &= \begin{pmatrix} 0 & 2(\beta-k+1) & 0 \\ 0 & -(\alpha+\beta-k+2) & \beta-k+2 \\ 0 & 0 & -2(\alpha+\beta-k+3) \end{pmatrix}, \quad 1 \le k \le \beta.
\end{aligned}$$

The one step example: polynomial eigenfunctions

Given $\mathcal{H}_1$, there are two ways to generate a sequence of MVOP:

- In terms of the *matrix hypergeometric function*:

$$ {}_2F_1(C, A, B; t) = \sum_{i \geq 0} (C, A, B)_i \frac{t^i}{i!}, $$

  where $(C, A, B)_{i+1} = (C + iI)^{-1}(A + iI)(B + iI)(C, A, B)_i$, $i \geq 0$ and $(C, A, B)_0 = I$, introduced by Tirao in *The matrix valued hypergeometric equation*, Proc. Nat. Acad. Sci. U.S.A., (2003).

- Solving directly the algebraic matrix equations from the differential equation.

The one step example: new differential operator

- **Another** second order differential operator: $\mathcal{H}_2 = B_2(t)\frac{d^2}{dt^2} + B_1(t)\frac{d}{dt} + B_0(t)$,

$$B_2(t) = \begin{pmatrix} t(1-t) & 0 & 0 \\ -t/2 & t(1-t)/2 & 0 \\ 0 & -t & 0 \end{pmatrix}$$

$$B_1(t) = \begin{pmatrix} \alpha+\beta-k+4 & \beta-k+1 & 0 \\ -(\alpha+\beta-k+4)/2 & (\alpha+4)/2 & (\beta-k+2)/2 \\ 0 & -(\alpha+\beta-k+5) & -(\beta-k+2) \end{pmatrix}$$

$$-t\begin{pmatrix} \alpha+\beta+4 & \beta-k+1 & 0 \\ 0 & (\alpha+\beta+5)/2 & (\beta-k+2)/2 \\ 0 & 0 & \alpha+\beta+6 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & -k(\beta-k+1) & 0 \\ 0 & k(\alpha+\beta-k+2)/2 & -k(\beta-k+2)/2 \\ 0 & 0 & k(\alpha+\beta-k+3) \end{pmatrix}, \quad \alpha,\beta > -1, \quad 1 \le k \le \beta.$$

The one step example: relation and characterization

$$\Big(\mathcal{H}_1-\mathcal{H}_2\Big)(\mathcal{H}_2-k(\alpha+\beta-k+3))(\mathcal{H}_1-2\mathcal{H}_2+k(\alpha+\beta-k+1)+\alpha+\beta+2)=\Theta.$$

- None of these factors is zero.
- **Reducible cubic**.
- Characterize the algebra:

order	0	1	2	3	4	5	6	7	8	9	10
dimension	1	0	2	0	3	0	3	0	3	0	3

- **Conjecture**: Every differential operator $\mathcal{U}$ of order $2k$ can be written:

$$\mathcal{U}=\sum_{i,j=0}^{k} c_{i,j}\mathcal{H}_1^i\mathcal{H}_2^j,$$

where $c_{i,j}\in\mathbb{C}$. Then $\mathcal{D}=<\mathcal{H}_1,\mathcal{H}_2>$ and **commutative**.

The two steps example: initial differential operator

$$W(t) = t^{\alpha}(1-t)^{\beta}T(t)T^{*}(t), \quad \alpha, \beta > -1, \quad \mathcal{H}_1 = A_2(t)\frac{d^2}{dt^2} + A_1(t)\frac{d}{dt} + A_0(t),$$

$$\begin{aligned}
A_2(t) &= t(1-t) \\
A_1(t) &= \begin{pmatrix} \alpha+3 & 0 & 0 & 0 \\ -1 & \alpha+2 & 0 & 0 \\ -1 & 0 & \alpha+2 & 0 \\ 0 & -\frac{k_2-k_1+2}{k_2-k_1+1} & -\frac{k_2-k_1}{k_2-k_1+1} & \alpha+1 \end{pmatrix} \\
&\quad -t\begin{pmatrix} \alpha+\beta+4 & 0 & 0 & 0 \\ 0 & \alpha+\beta+5 & 0 & 0 \\ 0 & 0 & \alpha+\beta+5 & \\ 0 & 0 & 0 & \alpha+\beta+6 \end{pmatrix} \\
A_0(t) &= \begin{pmatrix} 0 & \frac{(k_2-k_1+2)(\beta-k_2+1)}{k_2-k_1+1} & \frac{(k_2-k_1)(\beta-k_1+2)}{k_2-k_1+1} & 0 \\ 0 & -(\alpha+\beta+2)+k_2 & 0 & \beta-k_1+2 \\ 0 & 0 & -(\alpha+\beta+3)+k_1 & \beta-k_2+1 \\ 0 & 0 & 0 & -2(\alpha+\beta+3)+k_1+k_2 \end{pmatrix}.
\end{aligned}$$

$$1 \leq k_1 < k_2 \leq \beta$$

The two steps example: 2 new second order diff. operators

- $\mathcal{H}_2 = B_2(t)\frac{d^2}{dt^2} + B_1(t)\frac{d}{dt} + B_0(t), \quad 1 \le k_1 < k_2 \le \beta,$

$$B_2(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_1-k_2-1}{k_1-k_2}t & 0 & \frac{k_1-k_2-1}{k_1-k_2}t(1-t) & 0 \\ 0 & \frac{k_1-k_2-2}{k_1-k_2-1}t & \frac{1}{k_1-k_2-1}t & t(1-t) \end{pmatrix}$$

- $\mathcal{H}_3 = C_2(t)\frac{d^2}{dt^2} + C_1(t)\frac{d}{dt} + C_0(t), \quad 1 \le k_1 < k_2 \le \beta,$

$$C_2(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{k_1-k_2-2}t & \frac{1}{k_1-k_2-2}t(1-t) & 0 & 0 \\ -t & 0 & -t(1-t) & 0 \\ 0 & -t & 0 & -t(1-t) \end{pmatrix}$$

The two steps example: 2 new fourth order diff. operators

order	0	1	2	3	4	5	6	7	8	9	10
dimension	1	0	3	0	6	0	6	0	6	0	6

- At order 4 we get 2 new *non-commutative differential operators* $\mathcal{E}$ and $\mathcal{F}$ given by

$$\mathcal{E} = E_4(t)\frac{d^4}{dt^4} + E_3(t)\frac{d^3}{dt^3} + E_2(t)\frac{d^2}{dt^2} + E_1(t)\frac{d}{dt} + E_0(t),$$

$$\mathcal{F} = F_4(t)\frac{d^4}{dt^4} + F_3(t)\frac{d^3}{dt^3} + F_2(t)\frac{d^2}{dt^2} + F_1(t)\frac{d}{dt} + F_0(t),$$

with $E_4(t)$ and $F_4(t)$ given by:

$$E_4(t) = \frac{(\beta - k_1 + 2)(k_1 - k_2)}{(\beta - k_2 + 1)} t^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{k_1 - k_2 - 2}(1-t) & 0 & \frac{1}{k_1 - k_2 - 2}(1-t)^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{k_1 - k_2 - 1} & 0 & -\frac{1}{k_1 - k_2 - 2}(1-t) & 0 \end{pmatrix}$$

$$F_4(t) = \frac{(\beta - k_2 + 1)(k_1 - k_2 - 2)}{(\beta - k_1 + 2)} t^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{k_1 - k_2}(1-t) & \frac{1}{k_1 - k_2}(1-t)^2 & 0 & 0 \\ \frac{1}{k_2 - k_1 + 1} & \frac{1}{k_2 - k_1 + 1}(1-t) & 0 & 0 \end{pmatrix}.$$

The two steps example: a sample of relations

$$\mathcal{H}_2\big(\mathcal{H}_1 + \mathcal{H}_3 - k_1(\alpha + \beta - k_1 + 3)\big) = \Theta$$

$$\big[(k_1 - k_2)\mathcal{H}_2 + (k_1 - k_2 - 1)\mathcal{H}_3\big]\big[-\mathcal{H}_1 + (k_1 - k_2 - 1)\mathcal{H}_2 + (k_1 - k_2 - 2)\mathcal{H}_3 + (1 + k_2)(\alpha + \beta - k_2 + 2)\big] = \Theta$$

$$\mathcal{H}_1\mathcal{E} + \mathcal{E}\mathcal{H}_3 = \big(k_1(\alpha + \beta - k_1 + 2) + 1 + k_2\big)\mathcal{E}$$

$$\mathcal{E}\mathcal{H}_1 - \mathcal{H}_1\mathcal{E} = (k_1 - k_2 - 1)\mathcal{E}$$

$$\mathcal{F}\mathcal{H}_1 + \mathcal{H}_3\mathcal{F} = \big(k_1(\alpha + \beta - k_1 + 2) + 1 + k_2\big)\mathcal{F}$$

$$\mathcal{H}_1\mathcal{F} - \mathcal{F}\mathcal{H}_1 = (k_1 - k_2 - 1)\mathcal{F}$$

$$\mathcal{H}_2\mathcal{E} = \Theta \quad \text{and} \quad \mathcal{F}\mathcal{H}_2 = \Theta.$$

- **Conjecture**: $\mathcal{D} =< I, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{E}, \mathcal{F} >$ and **non–commutative**.

Closing remarks and future directions

- We hope to find theoretical proofs of these new phenomena.
- The matrix case is **much richer** than the scalar one.
- Possible applications:
  - **Quantum mechanics**: [Durán-Grünbaum] *P A M Dirac meets M G Krein: matrix orthogonal polynomials and Dirac´s equation*, J. Phys. A: Math. Gen. (2006).
  - **Time-and-band limiting**: [Durán-Grünbaum]*A survey on orthogonal matrix polynomials satisfying second order differential equations*, J. Comput. Appl. Math. (2005).