# ON THE INTEGRAL CARATHEODORY PROPERTY 

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#### Abstract

In this note we document the existence of a finitely generated rational cone that is not covered by its unimodular Hilbert subcones, but satisfies the integral Carathéodory property. We explain the algorithms that decide these properties and describe our experimental approach that led to the discovery of the examples.


## 1. Introduction

Let $C \subset \mathbb{R}^{d}$ be a finitely generated rational cone, i. e. the set of all linear combinations $a_{1} x_{1}+\cdots+a_{n} x_{n}$ of rational vectors $x_{1}, \ldots, x_{d}$ with coefficients from $\mathbb{R}_{+}$. We can of course assume that $x_{i} \in \mathbb{Z}^{d}, i=1, \ldots, n$. In this note a cone is always supposed to be rational and finitely generated. Moreover, we will assume that $C$ is pointed: if $x,-x \in C$, then $x=0$. Finally, it is tacitly understood that $C$ has full dimension $d$.

The monoid $M(C)=C \cap \mathbb{Z}^{d}$ is finitely generated by Gordan's lemma (for example, see [2, Section 2.A]). Since $C$ is pointed, $M$ is a positive monoid so that 0 is the only invertible element in $M(C)$.

It is not hard to see that $M(C)$ has a unique minimal system of generators that we call its Hilbert basis, denoted by $\operatorname{Hilb}(M(C))$ or simply $\operatorname{Hilb}(C)$. It consists of those elements $z \neq 0$ of $M(C)$ that have no decomposition $z=x+y$ in $M(C)$ with $y, z \neq 0$.

We want to discuss combinatorial conditions on $\operatorname{Hilb}(C)$ expressing that $C$ or $M(C)$ is covered by certain "simple" subcones or submonoids, respectively. To this end we define a $u$-subcone of $C$ to be a subcone generated by vectors $x_{1}, \ldots, x_{d} \in \operatorname{Hilb}(C)$ that form a basis of the group $\mathbb{Z}^{d}$. In particular, $x_{1}, \ldots, x_{d}$ are linearly independent, and if just this weaker condition is satisfied, then the cone $S$ generated by $x_{1}, \ldots, x_{d}$ is called an $f$-subcone. In this case we let $\Gamma(S)$ denote the subgroup of $\mathbb{Z}^{d}$ generated by $x_{1}, \ldots, x_{d}$ and $\Sigma(S)$ the submonoid of $\mathbb{Z}^{d}$ generated by $x_{1}, \ldots, x_{d}$. Note that $S$ is a $u$-subcone if and only if $\Gamma(S)=\mathbb{Z}^{d}$, or, equivalently, $\Sigma(S)=S \cap \mathbb{Z}^{d}$.

One says that $C$ satisfies (UHC) if $C$ is the union of its $u$-subcones. The letter $U$ stands for unimodular, H reminds us of the condition that the generators of the $u$-subcones belong to $\operatorname{Hilb}(C)$, and C simply stands for cover.

A weaker condition than (UHC) is the integral Carathéodory property (ICP). One says that $C$ has (ICP) if every element of $M(C)$ can be written as a linear combination of at most $d$ elements $x_{i} \in \operatorname{Hilb}(C)$ with integral nonnegative coefficients $a_{i}$. The terminology is motivated by Carathéodory's theorem: let $y_{1}, \ldots, y_{m}$ be a minimal system of generators of the cone $C$; then every element $y \in C$ is a linear combination $y=a_{1} y_{i_{1}}+\cdots+a_{d} y_{i_{d}}$ with nonnegative real coefficients.

Both (UHC) and (ICP) can be formulated more generally for positive affine monoids $M \subset \mathbb{Z}^{d}$. However, it is easy to see that every monoid $M$ satisfying (UHC) is given in
the form $M=\mathbb{R}_{+} M \cap \mathbb{Z}^{d}$. By a theorem of Bruns and Gubeladze [1, Theorem 6.1] the same holds true if $M$ satisfies (ICP), provided the group $\operatorname{gp}(M)$ generated by $M$ equals $\mathbb{Z}^{d}$. In loc. cit. it is also shown that (ICP) is equivalent to the formally stronger condition that $M$ is the union of its submonoids $\Sigma(S)$. (This condition is called (FHC) in [1].) The equivalence is crucial for our note, and therefore we reproduce the statement and its proof in Theorem 2 .

While we view (UHC) and (ICP) as structural properties of (normal) affine monoids, these properties have first been discussed in the context of integer programming: see Cook, Fonlupt and Schrijver [6] and Sebő [8].

It was asked by Sebő [8] whether every cone $C$ has (ICP) or (UHC), and he proved that (UHC) holds if $d \leq 3$. He actually proved a stronger statement: $C$ has a triangulation by $u$-subcones. A counterexample to (UHC) in dimension 6, called $C_{10}$ in the following, was found by Bruns and Gubeladze [1], and then verified to violate (ICP), too, in cooperation with Henk, Martin, and Weismantel [3]. Despite the existence of the counterexample, one can fairly say, at least heuristically, that almost all cones satisfy (UHC).

It remained an open problem whether (UHC) is strictly stronger than (ICP). In this note we want to document the existence of cones that satisfy (ICP) but fail (UHC), explain the algorithms that decide (UHC) and (ICP), and describe the experimental approach that led to the discovery of the examples.

All our experiments seem to indicate that $C_{10}$ is the core counterexample to (ICP) and (UHC). In fact, all counterexamples to these properties that we have been found contain it. It would be very desirable indeed to clarify the situation in dimensions 4 and 5.

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## 2. Deciding UHC

Let us say that $x \in C$ is $u$-covered if it is contained in a $u$-subcone. A subset of $C$ is $u$-covered if each of its elements is $u$-covered. Using this simple terminology, we can describe an algorithm deciding (UHC); see Table 1 .

In the algorithm we use a function named split. It decomposes $D$ along a support hyperplane $H$ of a $u$-subcone $U$ such that $D \cap H^{>} \neq \emptyset$ as well as $D \cap H^{<} \neq \emptyset$. Such a hyperplane does indeed exist if $\operatorname{int}(D) \cap \operatorname{int}(U) \neq \emptyset$, but $D \not \subset U$. The cones produced are $D_{1}=D \cap H^{+}$and $D_{2}=D \cap H^{-}$. (The open halfspaces determined by $H$ are denoted by $H^{>}$and $H^{<}$, and $H^{+}$and $H^{-}$are the corresponding closed halfspaces.)

It is easy to see that the algorithm terminates: there are only finitely many hyperplanes by which we split subcones. Therefore only finitely many subcones can be created.

In order to check the correctness of the algorithm, observe that at each step in the for loop in Unicover none of the $u$-subcones $U_{1}, \ldots, U_{i-1}$ of $C$ intersects the interior of $D$. In fact, the index $i$ is only increased if $\operatorname{int}(D) \cap \operatorname{int}\left(U_{i}\right)=\emptyset$. (In the recursive call the index $i$ that has been reached at the parent level starts the for loop at the child level.)

Thus, when the loop terminates with $i=N+1$, none of the $u$-subcones of $C$ intersects the interior of $D$. So $D$, and consequently $C$, indeed contains a vector that is not $u$ covered. Conversely, if $C$ contains such a vector $x$, then on each level of the recursion tree

```
Unicover \((D, n)\)
    for \(i \leftarrow n\) to \(N\)
    do
        if \(D \subset U_{i}\)
        then return
        if \(\operatorname{int}(D) \cap \operatorname{int}\left(U_{i}\right) \neq \emptyset\)
        then \(\left(D_{1}, D_{2}\right) \leftarrow \operatorname{split}\left(D, U_{i}\right)\)
            Unicover \(\left(D_{1}, i\right)\)
            Unicover \(\left(D_{2}, i\right)\)
            return
output ( \(D\) not \(u\)-covered )
return
```

main()
1 Create the list $U_{1}, \ldots, U_{N}$ of $u$-subcones of $C$
$2 \operatorname{unicover}(C, 1)$

Table 1. An algorithm deciding UHC


Figure 1. The function Split
one finds a subcone $D$ containing $x$, and for such $D$ the condition $D \subset U_{i}$ can never be satisfied. Therefore there exists an end node of the recursion tree at which the loop is left with $i=N+1$.

The pseudocode in Table 1 is a somewhat simplistic sketch of the actual implementation since it is necessary to cope with substantial memory requirements. For example, the list $U_{1}, \ldots, U_{N}$ is not produced a priori, but extended whenever necessary and always kept as small as possible. Moreover, all allocated memory is recycled carefully within the program.

Instead of starting the covering algorithm with the full cone $C$, the actual implementation uses the output of a preprocessor that computes several triangulations $\Delta_{1}, \ldots, \Delta_{t}$ of $C$, The input to unicover (and also to caradec below) is the list of intersections $D_{1} \cap \cdots \cap D_{t}$ where $D_{i}$ is a nonunimodular simplicial cone in $\Delta_{i}$.

## 3. DECIDING ICP

Let us first fix some terminology that parallels that for (UHC). An element $x \in C \cap \mathbb{Z}^{d}$ is $f$-covered if it belongs to one of the monoids $\Sigma(S)$ where $S$ is an $f$-subcone, and a subset of $C$ is $f$-covered if each of its elements is $f$-covered.

The lemma contains the basic criterion by which we can check that $C$ is $f$-covered. In the theorem following it, we will then see that this property is equivalent to (ICP).

## Lemma 1.

(a) Let $G_{1}, \ldots, G_{n}$ be subgroups of $\mathbb{Z}^{d}$, and let $N$ be a residue class of $\mathbb{Z}^{d}$ modulo $G_{1} \cap \cdots \cap G_{n}$. Then $N \subset G_{1} \cup \cdots \cup G_{n}$ if and only if $N \cap\left(G_{1} \cup \cdots \cup G_{n}\right) \neq \emptyset$.
(b) Let $S_{1}, \ldots, S_{n}$ be $f$-subcones of $C$, each containing the d-dimensional subcone $D$ of C. If every residue class of $\mathbb{Z}^{d}$ modulo $\Gamma\left(S_{1}\right) \cap \cdots \cap \Gamma\left(S_{n}\right)$ meets $\Gamma\left(S_{1}\right) \cup \cdots \cup$ $\Gamma\left(S_{n}\right)$, then $D$ is $f$-covered.
(c) Let $D$ be a d-dimensional subcone of $C$ with the following property: for every $f$-subcone $S$ either $D \subset S$ or $\operatorname{int}(D) \cap \operatorname{int}(S)=\emptyset$. Furthermore let $G_{D}$ be the intersection of the groups $\Gamma(S), S \supset D$, and $H_{D}$ their union. Then $D$ is $f$-covered if and only if every residue class of $\mathbb{Z}^{d}$ modulo $G_{D}$ meets $H_{D}$.
Proof. (a) Suppose that $N \cap\left(G_{1} \cup \cdots \cup G_{n}\right) \neq \emptyset$, and let $x$ be an element in the intersection, $x \in G_{i}$. The subgroup $G^{\prime}=G_{1} \cap \cdots \cap G_{n}$ is contained in $G_{i}$, and so $N=x+G^{\prime} \subset G_{i}$. The converse implication is trivial.
(b) Let $x \in D \cap \mathbb{Z}^{d}$. It follows from (a) that $x \in \Gamma\left(S_{i}\right)$ for some $i$. But $x \in S_{i}$, too. Therefore $x \in \Gamma(S) \cap S_{i}=\Sigma\left(S_{i}\right)$. (At this point we use that the generators of $S_{i}$ are linearly independent.)
(c) It only remains to show the necessity of the condition. For it we only need to observe that every residue class $N$ of $\mathbb{Z}^{d}$ modulo $G_{D}$ meets $\operatorname{int}(D)$. By hypothesis on $D$, an element $x \in N \cap \operatorname{int}(D)$ is $f$-covered if and only if $x \in \Sigma(S)$ for some $f$-subcone $S$ containing $D$.

We include the next theorem and its proof for the convenience of the reader. It is a simplified version of [1, Theorem 6.1] whose proof contains the crucial ideas for the algorithm deciding (ICP).
Theorem 2. Let $M \subset \mathbb{Z}^{d}$ be a positive affine monoid such that $\operatorname{gp}(M)=\mathbb{Z}^{d}$. If $M$ satisfies (ICP), then $M=\mathbb{R}_{+} M \cap \mathbb{Z}^{d}$, and every element of $M$ is $f$-covered.
Proof. We dissect $C=\mathbb{R}_{+} M$ along all the support hyperplanes of the cones spanned by linearly independent vectors $x_{1}, \ldots, x_{d}$ of $\operatorname{Hilb}(M)$ into elementary subcones. Set $\bar{M}=$ $\mathbb{R}_{+} M \cap \mathbb{Z}^{d}$ and choose $x \in \bar{M}$. Suppose that $x$ has no representation as a linear combination $x=a_{1} y_{1}+\cdots+a_{d} y_{d}, a_{1}, \ldots, a_{d} \in \mathbb{Z}_{+}$and $y_{1}, \ldots, y_{d} \in \operatorname{Hilb}(M)$ linearly independent. The element $x$ belongs to one of the elementary subcones $D$, and as in the proof of the lemma it follows that there exists a finite index subgroup $G$ of $\mathbb{Z}^{d}$ such that no element $z$ of $(x+G) \cap \operatorname{int}(D)$ has a representation $a_{1} y_{1}+\cdots+a_{d} y_{d}$ with $a_{1}, \ldots, a_{d} \in \mathbb{Z}_{+}$and $y_{1}, \ldots, y_{d} \in \operatorname{Hilb}(M)$ linearly independent.

The crucial point is that $\bar{M} \backslash M$ is contained in the union of finitely many hyperplanes (see [2, Section 2.B]), and the same applies to all elements of $M$ that are linear combinations of linearly dependent elements of $\operatorname{Hilb}(M)$. But $(x+G) \cap \operatorname{int}(D)$ is not contained
in the union of finitely many hyperplanes, and so must contain elements of $M$. This is impossible if $M$ satisfies (ICP).

```
\(\operatorname{CARADEC}(D, G, \mathscr{R}, n)\)
    for \(i \leftarrow n\) to \(N\)
    do
        if \(D \subset S_{i}\)
            then \(\mathscr{R}^{\prime} \leftarrow \emptyset\)
                    for \(\left(x \in \mathscr{R}, y \in G /\left(G \cap \Gamma\left(S_{i}\right)\right)\right)\)
                    do
                    if \(x+y \notin \Gamma\left(S_{i}\right)\)
                        then \(\mathscr{R}^{\prime}=\mathscr{R}^{\prime} \cup\{x+y\}\)
            \(G \leftarrow G \cap \Gamma\left(S_{i}\right), \quad \mathscr{R} \leftarrow \mathscr{R}^{\prime}\)
            if \(\mathscr{R}=\emptyset\)
                    then return
        if \(D \not \subset S_{i}\) and \(\operatorname{int}(D) \cap \operatorname{int}\left(S_{i}\right) \neq \emptyset\)
            then \(\left(D_{1}, D_{2}\right) \leftarrow \operatorname{split}\left(D, S_{i}\right)\)
                \(\operatorname{Caradec}\left(D_{1}, G, \mathscr{R}, i\right)\)
                \(\operatorname{caradec}\left(D_{2}, G, \mathscr{R}, i\right)\)
            return
    output( \(D\) not \(f\)-covered )
    return
main()
    1 Create the list \(S_{1}, \ldots, S_{N}\) of \(f\)-subcones of \(C\)
    \(2 \operatorname{caradec}\left(C, \mathbb{Z}^{d},\{0\}, 1\right)\)
```

TAble 2. An algorithm deciding ICP

For the algorithm deciding (ICP) we have to enrich our data structure by those components that have shown up in the proof of the lemma. Subcones are replaced by triples ( $D, G, \mathscr{R}$ ) where $D$ is a subcone of $C, G$ is a finite index subgroup of $\mathbb{Z}^{d}$, and $\mathscr{R}$ is a list of residue classes in $\mathbb{Z}^{d} / G$. In $\mathscr{R}$ each residue class is represented by a single vector that belongs to it, and in the algorithm (see Table 2) the loop

$$
\text { for }\left(x \in \mathscr{R}, y \in G /\left(G \cap \Gamma\left(S_{i}\right)\right)\right)
$$

runs over all elements of $\mathscr{R} \times G /\left(G \cap \Gamma\left(S_{i}\right)\right)$.
Again it is clear that the algorithm terminates after finitely many steps: the number of hyperplanes that we can use to split subcones of $C$ is still finite (though larger than for (UHC)).

The crucial point for caradec is that at each step in the loop the $f$-cones $S_{1}, \ldots, S_{N}$ satisfy the following conditions:
(1) for each $j \leq i-1$ either $S_{j} \supset D$ or $\operatorname{int}\left(S_{j}\right) \cap \operatorname{int}(D)=\emptyset$;
(2) $G$ is the intersection of all groups $\Gamma\left(S_{j}\right), j \leq i-1$, for which $D \subset S_{j}$;
(3) $\mathscr{R}$ is the list of those residue classes in $\mathbb{Z}^{d} / G$ that are not contained in the union of the groups $\Gamma\left(S_{j}\right), j \leq i-1, D \subset S_{j}$.
We have only to check that these conditions remain satisfied when $S_{i}$ is tested against $D$. To this end let $S_{k_{1}}, \ldots, S_{k_{m}}$ be those among $S_{1}, \ldots, S_{i-1}$ that contain $D$.

If $\operatorname{int}\left(S_{i}\right) \cap \operatorname{int}(D)=\emptyset$, then $D \not \subset S_{i}$, and this case is done.
If $\operatorname{int}\left(S_{i}\right) \cap \operatorname{int}(D) \neq \emptyset$, but $D \not \subset S_{i}$, then $i$ is not increased (!) and all three conditions are inherited by both $D_{1}$ and $D_{2}$ : among the $S_{j}, j \leq i-1$, exactly $S_{k_{1}}, \ldots, S_{k_{m}}$ contain $D_{1}$ or $D_{2}$, simply because $S_{j} \supset D_{1}$ or $S_{j} \supset D_{1} \operatorname{implies} \operatorname{int}\left(S_{j}\right) \cap \operatorname{int}(D) \neq \emptyset$, and so $S_{j} \supset D$.

But if $S_{i} \supset D$, the bookkeeping is also correct. Evidently $G$ is replaced by the correct group $G \cap \Gamma\left(S_{i}\right)$. Next observe that all residue classes of $G \cap \Gamma\left(S_{i}\right)$ that are contained in residue classes modulo $G$ not appearing in $\mathscr{R}$ remain in $\Gamma\left(S_{k_{1}}\right) \cup \cdots \cup \Gamma\left(S_{k_{m}}\right)$. On the other hand, those that refine elements of $\mathscr{R}$ must belong to $\Gamma\left(S_{i}\right)$ to be in $\Gamma\left(S_{k_{1}}\right) \cup \cdots \cup \Gamma\left(S_{k_{m}}\right) \cup$ $\Gamma\left(S_{i}\right)$. The correctness of the algorithm follows now immediately from Lemma 1

The biggest hurdle for it are the lists $\mathscr{R}$ of residue classes that usually become extremely long already in dimension 6. Moreover, along each branch of the recursion tree, several of them must be kept in memory. (This problem cannot be eliminated by a nonrecursive implementation.)

The growth of the list $\mathscr{R}$ can be estimated. Set

$$
e=\#\left(G /\left(G \cap \Gamma\left(S_{i}\right)\right)\right) \quad \text { and } \quad e^{\prime}=\#\left(\mathbb{Z}^{d} / \Gamma\left(S_{i}\right)\right)
$$

Each element $x \in \mathscr{R}$ is involved in $e$ vectors $x+y$. At most one of them lies in $\Gamma\left(S_{i}\right)$ since the vectors $y$ belong to pairwise different residue classes modulo $\Gamma\left(S_{i}\right)$. Therefore

$$
\#\left(\mathscr{R}^{\prime}\right) \geq(e-1) \#(\mathscr{R})
$$

If the elements of $\mathscr{R}$ are randomly distributed over the residue classes of $\mathbb{Z}^{d}$ modulo $\Gamma\left(S_{i}\right)$, then the expected share of vectors $x+y \in \Gamma\left(S_{i}\right)$ drops to $1 / e^{\prime}$.

Instead of keeping the lists of residue classes in memory, one could alternatively try to follow the recursion tree along the whole list $S_{1}, \ldots, S_{N}$, compute only $G$ along each branch and test the residue classes one by one only at the end nodes. However, this approach seems unfeasible since it derives no advantage from the case $e=1$, which fortunately happens frequently and often stops the recursion before the end of $S_{1}, \ldots, S_{N}$ is reached.

The list $S_{1}, \ldots, S_{N}$ is actually scanned in growing order of the determinants of the $S_{i}$. This has turned out very effective, at least for those cones that satisfy (ICP). In fact, all cones in Table 5 with (ICP) are covered by $f$-subcones of determinant $\leq 2$.

In addition to Caradec we use a Monte Carlo approach for disproving (ICP). It reads the output of UNICOVER, computes a large number of vectors in the non- $u$-covered subcones of $C$ and tests whether they are $f$-covered.

Remark 3. CARADEC provides us with a precise measure for the failure of (ICP), namely the ratios $\#(\mathscr{R}) / \#(G)$ at the end nodes of the recursion tree. For the cone $C_{10}$ (Table 3) there is precisely one end node with $\mathscr{R} \neq \emptyset$, and the ratio is $32 / 15552=1 / 486$. The number of non- $f$-covered vectors in the Monte Carlo test confirms the ratio rather precisely.

For the cone $C_{15}^{\prime \prime}$ (Table 5) there is again a single group with $\mathscr{R} \neq \emptyset$ and ratio

$$
2,4468,480 / 2,286,144,000,000 \approx 1.07 / 10^{6}
$$

So the Monte Carlo test cannot be expected to be conclusive with $<10^{6}$ test vectors.

## 4. The Search

Let us recapitulate an important notion from [1]. An element $x$ of $\operatorname{Hilb}(C)$ is called destructive if $H^{\prime}=\operatorname{Hilb}(C) \backslash\{x\}$ is not the Hilbert basis of $\mathbb{R}_{+} H^{\prime}$. We say that $C$ is tight if every element of $\operatorname{Hilb}(C)$ is destructive. The crucial role of tight cones for (UHC) and (ICP) is illuminated by the following lemma [1, Corollary 2.3].

Lemma 4. Let C be a cone that is a counterexample to (UHC) or (ICP). Suppose C is minimal first with respect to dimension and second with respect to $\# \operatorname{Hilb}(C)$. Then $C$ is tight.
Remark 5. Updating the information in [1] we mention that tight cones exist in all dimensions $d \geq 3$. The first 3 -dimensional tight cone was found by P. Dueck. The smallest such cone found by the author has a Hilbert basis of 19 elements. The elements of the Hilbert basis in the extreme rays form a regular hexagon (with respect to the action of $\mathrm{GL}_{3}(\mathbb{Z})$ ) so that the cone has the dihedral group $D_{6}$ as its automorphism group. The regularity is in indication that it may be the smallest possible tight cone. (Here and in the following the automorphism group of a cone $C$ is always understood to be the automorphism group of the monoid $C \cap \mathbb{Z}^{d}$.)

Our search for counterexamples has been based on the crucial Lemma 4. We produce a set of random vectors, consider them as the generating set of a cone $C$, and then use a program named shrink to remove nondestructive elements of $\operatorname{Hilb}(C)$ until a tight cone is reached. (Shrink is based on the same algorithm as normaliz; see [4, 5].) Almost always, $C$ shrinks to the 0 -cone, but sometimes a nontrivial tight cone emerges. Then UNICOVER, and possibly CARADEC, are invoked.

When we started the search in spring 1998, we used cones over randomly generated lattice parallelepipeds. In May 1998 the search stopped with the counterexample $C_{10}$. Its Hilbert basis is shown in Table 3. The cone $C_{10}$ has 27 support hyperplanes.

$$
\begin{array}{lr}
z_{1}=(0,1,0,0,0,0) & z_{6}=(1,0,2,1,1,2) \\
z_{2}=(0,0,1,0,0,0) & z_{7}=(1,2,0,2,1,1) \\
z_{3}=(0,0,0,1,0,0) & z_{8}=(1,1,2,0,2,1) \\
z_{4}=(0,0,0,0,1,0) & z_{9}=(1,1,1,2,0,2) \\
z_{5}=(0,0,0,0,0,1) & z_{10}=(1,2,1,1,2,0)
\end{array}
$$

Table 3. $\operatorname{Hilb}\left(C_{10}\right)$

The reader should note that for the questions considered in this note we can always replace a given cone $C$ by $\phi(C)$ where $\phi$ is an arbitrary transformation in $\mathrm{GL}_{d}(\mathbb{Z})$. In this sense, $C$ stands for a class of cones that are isomorphic under an integral isomorphism of $\mathbb{R}^{d}$. We express this fact by speaking of different embeddings of a cone $C$.

While unicover showed that $C_{10}$ fails (UHC), it was then verified in cooperation with Henk, Martin, and Weismantel that $C_{10}$ also fails (ICP). (caradec was not written before

September 2006.) See Bruns and Gubeladze [1] and Bruns et al. [3] for more information on $C_{10}$.

The automorphism group of $C_{10}$ is remarkably large: it is the Frobenius group $F_{20}$ of order 20, which acts transitively on $z_{1}, \ldots, z_{10}$. ( $F_{20}$ is the semidirect product of $\mathbb{Z}_{5}$ with its automorphism group $\mathbb{Z}_{5}^{*} \cong \mathbb{Z}_{4}$.) From the embedding above one can see that at least the dihedral group $D_{5} \subset F_{20}$ is acting on $C_{10}$. All the remaining 10 automorphisms have order 4 and swap $\left\{z_{1}, \ldots, z_{5}\right\}$ with $\left\{z_{6}, \ldots, z_{10}\right\}$. Moreover, $z_{1}, \ldots, z_{10}$ all lie in the hyperplane given by $-5 \zeta_{1}+\zeta_{2}+\cdots+\zeta_{6}=1$. The convex hulls of $\left\{z_{1}, \ldots, z_{5}\right\}$ and $\left\{z_{6} \ldots, z_{10}\right\}$ are both simplices of dimension 4.

Remark 6. It was communicated to us by F. Santos that the lattice polytope spanned by $\operatorname{Hilb}\left(C_{10}\right)$ is a projection of the Ohsugi-Hibi polytope [7]. The projection leads to the following description of $C_{10} \cap \mathbb{Z}^{6}$. Consider the complete graph $K_{5}$ and decompose it into 2 cycles of length 5 as shown in Figure 2. Now choose the incidence vectors


Figure 2. Cycle decomposition of $K_{5}$
$(1,1,0,0,0)$ etc. of the edges in the first cycle and prefix them with 0 . Then prefix the incidence vectors $(1,0,1,0,0)$ etc. of the second cycle with 1 . The resulting 10 vectors in $\mathbb{Z}^{6}$ generate a monoid $M$ isomorphic with $C_{10} \cap \mathbb{Z}^{6}$. While this description is even more aesthetic than the one in Table 3, it has the disadvantage that $\mathrm{gp}(M)$ is of index 2 in $\mathbb{Z}^{6}$.

In the summer of 1998 a second counterexample $C_{12}$ to (UHC) and (ICP) emerged. It has a Hilbert basis of 12 elements. We continued the search for two more years. The frustrating outcome was that $C_{10}$ appeared over and over again, but no new counterexample showed up (and even $C_{12}$ did not return until November 22, 2006).

The project was taken up again at the end of 2004 when our department had installed a dual processor Opteron system with very fast integer arithmetic. Nevertheless, the outcome of the search remained as disappointing as it had been before.

Finally, in August 2006 we did what should have been done long before, namely compare $C_{12}$ with $C_{10}$ : it turned out that $\operatorname{Hilb}\left(C_{12}\right)$ (in an embedding that had to be found!) extends $\operatorname{Hilb}\left(C_{10}\right)$ by two vectors. Relative to $C_{10}$, this finding explained why $C_{12}$ fails (UHC) and (ICP), too: the extra $u$-subcones and $f$-subcones are not sufficient to cover all integral vectors in $C_{10}$. (It also shows that one cannot speed up shrinking by removing two vectors at a time.)

However, this not very surprising a posteriori insight made it suddenly clear that there might be many interesting objects in the vicinity of $C_{10}$. Especially, when we approach
$C_{10}$ along a shrink path, why should the stronger property (UHC) not be lost before (ICP)? After a modification of Shrink we also applied Unicover to the, say, 6 last non-tight approximations to $C_{10}$, and within hours many new non-(UHC) cones emerged. Several of them defeated all Monte Carlo attacks on (ICP). It became clear that caradec had to be implemented, and it indeed recognized many non-(UHC), but (ICP) cones.

Ironically, within a few weeks after we had given up our narrow-minded insistence on checking only tight cones, two new non-(UHC) such cones surfaced, both of them satisfying (ICP). They appear as $C_{12}^{\prime}$ and $C_{15}$ in Table 4. Since all these cones contain

$$
\begin{array}{rlrl}
C_{12}: & z_{11}^{\prime}=(2,2,1,4,1,3) & C_{15}: & w_{1}=(2,1,0,5,1,5) \\
& z_{12}^{\prime}=(2,3,1,4,1,2) & w_{2}=(1,0,-1,4,0,4) \\
& & w_{3}=(0,0,-1,1,0,1) \\
C_{12}^{\prime}: & z_{11}^{\prime \prime}=(0,-1,2,-1,-1,2) & & w_{4}=(2,1,2,3,2,4) \\
& z_{12}^{\prime \prime}=(1,0,3,0,0,3) & w_{5}=(1,1,0,3,1,2) .
\end{array}
$$

Table 4. Additional vectors in $\operatorname{Hilb}\left(C_{12}\right), \operatorname{Hilb}\left(C_{12}^{\prime}\right), \operatorname{Hilb}\left(C_{15}\right)$
$C_{10}$, we list only the extra vectors that complement the Hilbert basis of $C_{10}$ (using the embedding given in Table 3). The numbers of support hyperplanes are 39 for $C_{12}, 40$ for $C_{12}^{\prime}$, and 36 for $C_{15}$.

The most interesting after $C_{10}$ undoubtedly is $C_{12}^{\prime}$, not only because it satisfies (ICP). Its automorphism group - certainly invisible from the embedding given - is again the Frobenius group $F_{20}$. It is clear that $F_{20}$ cannot act transitively on $\operatorname{Hilb}\left(C_{12}^{\prime}\right)$, which rather decomposes into an orbit of 10 elements and one of 2 . However, this is by no means an extension of the action of $F_{20}$ on $C_{10}$ since the orbit of two elements is $\left\{z_{1}, z_{5}\right\}$ ! Only a subgroup isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ restricts to $C_{10}$, showing that there are 5 conjugate embeddings of $C_{10}$ into $C_{12}^{\prime}$, and each of them contains $z_{1}, z_{5}$.

The convex hulls of $\left\{z_{1}, \ldots, z_{5}, z_{11}^{\prime \prime}\right\}$ and $\left\{z_{6} \ldots, z_{10}, z_{12}^{\prime \prime}\right\}$ are both bipyramids over a tetrahedron. The bipyramids are situated in the parallel hyperplanes with the equations $\zeta_{1}=0$ and $\zeta_{1}=1$. Both $C_{12}$ and $C_{12}^{\prime}$ have their Hilbert bases in the hyperplane spanned by $\operatorname{Hilb}\left(C_{10}\right)$ but this is not true for $C_{15}$.

Table 5 lists all the 11 tight non-(UHC) cones of dimension 6 that have been found by December 17, 2006, including those mentioned already. In the last column we indicate whether the Hilbert basis is contained in a hyperplane.

The smallest non-(UHC), but (ICP) cone we have found has a Hilbert basis of 11 elements, and the largest has a Hilbert basis of 24 elements (most likely (ICP)). Like all the others they are extensions of $C_{10}$ which seems to be the core obstruction to (ICP) and (UHC).

While we have the implications (UHC) $\Longrightarrow$ (ICP) $\Longrightarrow M=\mathbb{R}_{+} M \cap \mathbb{Z}^{d}$ (under the condition that $\operatorname{gp}(M)=\mathbb{Z}^{d}$ ), it is now clear that the converse implications do not hold. However, it remains an open problem whether all cones in dimensions 4 and 5 have (ICP) or even (UHC).

|  | \#Hilb | \#Supp | ICP | Aut | flat |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $C_{10}$ | 10 | 27 | no | $F_{20}$ | yes |
| $C_{12}$ | 12 | 39 | no | $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ | yes |
| $C_{12}^{\prime}$ | 12 | 40 | yes | $F_{20}$ | yes |
| $C_{14}$ | 14 | 34 | yes | $\{\mathrm{id}\}$ | yes |
| $C_{14}^{\prime}$ | 14 | 39 | no | $\{\mathrm{id}\}$ | yes |
| $C_{14}^{\prime \prime}$ | 14 | 42 | yes | $\mathbb{Z}_{2}$ | no |
| $C_{15}$ | 15 | 36 | yes | $\{\mathrm{id}\}$ | no |
| $C_{15^{\prime}}$ | 15 | 36 | yes | $\{\mathrm{id}\}$ | yes |
| $C_{15}^{\prime \prime}$ | 15 | 44 | no | $\{\mathrm{id}\}$ | yes |
| $C_{16}$ | 16 | 49 | no | $\mathbb{Z}_{2}$ | yes |
| $C_{16}^{\prime}$ | 16 | 36 | no | $\mathbb{Z}_{2}$ | yes |

Table 5. Tight non-(UHC) cones

## 5. Computational issues

All programs have been written in C. The tight cones in Section 4 were found by the Opteron (O) system mentioned above. It runs Linux, and the executables have been produced by the gcc compiler. In the following we will also mention computations on two other systems, the author's Intel Core 2660 (C2) with Windows XP and the DJGPP port of gcc, and the University of Osnabrück's Itanium (I) system with Linux and the Intel compiler icc. The machines ( O ) and ( C 2 ) are close to each other in speed; (I) is somewhat slower (for integer arithmetic), but has very large memory ( 32 GB ).

Some of the equipment used in the 1998 computations is still accessible. This allowed us to measure the gain in speed by improved hardware: the factor is $\geq 40$. Moreover, a better implementation of SHRINK yields an acceleration by a factor $\geq 3$. In other words, 4 months of the 1998 search take now a single day.

The number of cones shrunk by Shrink per second depends very much on the parameters used for their creation. The performance for 6 -dimensional cones generated by random $0-1$-vectors, whose number varies between 6 and 26, is about 1000 per second on (O) or (C2). Cones over 5-dimensional parallelotopes of Euclidean volume $\leq 30$ are shrunk at a rate of 0.6 per second. The output of tight cones is nevertheless comparable.

The cones in Section 4 are light food for unicover. For example, the running time for $C_{14}^{\prime \prime}$ on (C2) is 1.7 seconds. About 2.5 million vectors are created, but the list of vectors in memory simultaneously is bounded by 15,000 .

While all the other programs use 32 bit arithmetic, caradec is set to 64 bit. It has no problem with all the cones mentioned, as long as they have (ICP), simply because in all cases they are $f$-covered by cones of determinant $\leq 2$. The running time for $C_{14}^{\prime \prime}$ on (C2) is 6.9 seconds.

Despite of its sometimes enormous appetite for memory, CARADEC has also been successful for all the cones of Table 5 that lack (ICP), with the exception of $C_{16}$ and $C_{16}^{\prime}$. It failed for these cones though it was allowed 200 million vectors in memory.

The (ICP) property of $C_{16}$ and $C_{16}^{\prime}$ was falsified by the Monte Carlo method with 1 million test vectors for each of the non-(UHC) subcones produced by unicover (17.4 seconds on (C2) for $C_{16}$ ).

The longest successful run of caradec with a negative result was $C_{15}^{\prime \prime}: 1,990$ seconds on (I), 2.1 billion vectors, 110 million simultaneously. The Monte Carlo method does the job with 1 million vectors for the single non-(UHC) subcone in 2.7 seconds on (C2). See Section 3 for further data $C_{15}^{\prime \prime}$.

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