MODULAR FORMS ON NONCONGRUENCE SUBGROUPS AND ATKIN-SWINNERTON-DYER RELATIONS

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Abstract. We give new examples of noncongruence subgroups $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ whose space of weight 3 cusp forms $S_3(\Gamma)$ admits a basis satisfying the Atkin-Swinnerton-Dyer congruence relations with respect to a weight 3 newform for a certain congruence subgroup.

1. Introduction

A finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$ is *noncongruence* if it does not contain $\Gamma(N)$ for any $N \geq 1$. The study of modular forms on such subgroups was initiated by Atkin and Swinnerton-Dyer who discovered experimentally the congruences now bearing their names [ASwD71]. Subsequently, Scholl proved congruences satisfied by the coefficients of modular forms on noncongruence subgroups [Sch85i, Sch85ii, Sch87, Sch88, Sch93]. A refined conjecture has recently been put forward by Atkin, Li, Long and Yang [LLY03],[ALL05], [LL]. See [LLY05] for a general survey of this.

In this paper we give new examples of noncongruence subgroups having a basis of cuspidal modular forms satisfying the Atkin-Swinnerton-Dyer (ASwD) congruences. We only give experimental evidence of our results, obtained using Magma [BCP97], Mathematica, and PARI [Pari04]. In a later publication, we will give a detailed treatment of one of our examples.

1.1. **Notation.** We assume familiarity with the action of $\mathrm{SL}_2(\mathbf{R})$ on the upper half complex plane $\mathbf{H}$, with congruence subgroups such as $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma^0(N)$, $\Gamma^1(N)$, and with $M_k(\Gamma)$ and $S_k(\Gamma)$ the finite-dimensional vector spaces of modular forms and cusp forms for Γ, and $S_k(\Gamma_0(N), \chi)$ the space of cusp forms with character $\chi : (\mathbf{Z}/N)^* \to \mathbf{C}^*$.

It is well known (see [Shi71] for details) that $S_k(\Gamma_0(N), \chi)$ has a basis of Hecke eigenforms, which have q-expansions

$$f(z) = \sum_{n \geq 1} a_n(f) q^n, \quad \text{where} \quad q = \exp(2\pi i z),$$

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with a_n satisfying the relations

$$a_{np} - a_p a_n + \chi(p) p^{k-1} a_{n/p} = 0, \quad a_n = a_n(f) \tag{1}$$

for all positive integers n and primes $p \not| N$, taking $a_{n/p} = 0$ if $p \not| n$.

1.2. **Atkin–Swinnerton-Dyer congruences.** If Γ is a *noncongruence* subgroup, then $S_k(\Gamma)$ has no basis of forms satisfying (1). Instead, it is conjectured that certain congruences hold, as in the following definition.

Definition 1.2.1 ([LLY03])**.** *Suppose that the noncongruence subgroup Γ has cusp width μ at infinity, and that $h \in S_k(\Gamma)$ has an M-integral $q^{1/\mu}$-expansion $h = \sum a_n(h) q^{n/\mu}$ for some $M \in \mathbf{Z}$. (cf* [Sch85ii, Proposition 5.2]*). Let $f = \sum c_n(f) q^n$ be a normalized newform of weight k, level N, character χ. The forms h and f are said to satisfy the Atkin-Swinnerton-Dyer congruence relation if, for all primes p not dividing MN and for all $n \geq 1$,*

$$(a_{np}(h) - c_p(f) a_n(h) + \chi(p) p^{k-1} a_{n/p}(h))/(np)^{k-1} \tag{2}$$

is integral at all places dividing p.

Definition 1.2.2. *We say that $S_k(\Gamma)$ has an ASwD basis if there is a basis $h_1, ..., h_n$ of $S_k(\Gamma)$ and normalized newforms $f_1, ..., f_n$ such that each pair (h_i, f_i) satisfies the ASwD congruence relation in Definition 1.2.1.*

Note that, in the above definition, the choices of $h_1, ..., h_n$ and of $f_1, ..., f_n$ may depend on the prime number p. There are examples known where the same h_i and f_j work for every prime p (actually all but a finite number of exceptional primes). On the other hand, there are examples known where the choice of the ASwD basis depends on the value of p modulo some modulus N (see examples 2 and 3 in the tables below).

2. Statement of results

2.1. **Tables.** For the noncongruence subgroups Γ considered, there are two main issues addressed:

(1) Modularity of the l-adic Scholl's representation attached to the cusp forms of weight 3, $S_3(\Gamma)$.
(2) Giving a basis of $S_3(\Gamma)$ that satisfies ASwD congruences.

In our cases the dimension of $S_3(\Gamma)$ is 2 so the l-adic representation is 4 dimensional. We find that this 4-dimensional representation breaks up into two 2-dimensional pieces, each of which is isomorphic to the 2-dimensional representations that Deligne constructed for Hecke eigenforms f on congruence subgroups. Thus, each $S_3(\Gamma)$ should be associated to a pair f_1, f_2 of Hecke eigenforms on *congruence* subgroups. In the examples, these are one and the same form, or conjugate forms or base extensions of one form to a quadratic extension of $\mathbf{Q}$.

In Tables 1, 2, 3, 4, we define modular forms h_1, h_2, f, where h_1 and h_2 span $S_3(\Gamma)$ for the noncongruence subgroup Γ given in Definition 3.2.1, and

f is a weight 3 Hecke eigenform for some congruence subgroup. For each group we give a basis (h_1, h_2) of $S_3(\Gamma)$, in some cases depending on the prime p, and a newform f with (h_i, f) satisfying the ASwD congruence relation. Most forms are given in terms of the Dedekind eta function,

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{where } \ q = e^{2\pi i z}. \tag{3}$$

Our experiments support the following:

Theorem 2.1.1. *Let ρ be the l-adic representation constructed by Scholl for $S_3(\Gamma)$ for an appropriate choice of $\mathbf{Q}$-model of the curve X_Γ. For the L-function of the corresponding representations we have*

$$L(s,\rho) = L(s,f)L(s,f) \quad \text{for 1a, 1b,}$$
$$L(s,\rho) = L(s,f)L(s,\overline{f}) \quad \text{for 3a, 3b, 4a, 4b.}$$

In an earlier version of this paper a complete proof for cases 1a and 1b was given. We do not reproduce it here as it is very similar to other published examples. The L-function for examples 2a, 2b exhibits new and interesting features and will be discussed in a future work.

2.2. **The examples.** All the noncongruence subgroups Γ discussed in this paper are of index three inside a congruence subgroup G which itself is one of the index 12 genus 0 subgroups considered by Beauville. Each of these gives rise to a family if elliptic curves $E_G \to X_G = (G\backslash\mathbf{H})^* \cong \mathbf{P}^1(\mathbf{C})$ with ramification over the four cusps of G. For each of these, we select two of the cusps of G to construct a subgroup Γ such that the corresponding covering

$$X_\Gamma \cong \mathbf{P}^1(\mathbf{C}) \longrightarrow X_G \cong \mathbf{P}^1(\mathbf{C})$$

branches only over the two chosen cusps. We describe these coverings in the form $r^3 = m(t)$, where r (resp. t) is a generator of the function field of X_Γ (resp. X_G), i.e., a Hauptmodul, which exists since these curves have genus 0. See table 10. We have also considered arithmetic twists of a given covering gotten by varying some of the constants in the expression of $m(t)$. This leads to different models of Scholl's l-adic representation attached to $S_3(\Gamma)$, i.e., representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ that become isomorphic as representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ for a finite extension $K/\mathbf{Q}$. It is an important point that, in contrast to the case of classical modular curves for congruence subgroups, there are no *canonical* models defined over a number field. Scholl's construction of his l-adic representations depends on a choice of a model. Moreover, this choice is subject to a number of hypotheses: generally that there should be a model defined over $\mathbf{Q}$, and a cusp which is $\mathbf{Q}$-rational. This cusp is used for the expansions of modular forms whose coefficients satisfy ASwD congruences.

The l-adic representations that Scholl constructs that are associated to $S_k(\Gamma)$ for noncongruence subgroups Γ have very different properties from the corresponding representations constructed by Deligne for congruence Γ.

1a. Basis of $S_3(\Gamma_{24.6.1^6})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(4z)^{20}}{\eta(2z)^6}} = q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(4z)^{16}\eta(2z)^6}{\eta(z)^4}} = q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 + \cdots$$

Associated newform in $S_3(\Gamma_0(48), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)\left(\frac{-4}{p}\right)$:

$$f(z) = \frac{\eta(4z)^9\eta(12z)^9}{\eta(2z)^3\eta(6z)^3\eta(8z)^3\eta(24z)^3} = q + 3q^3 - 2q^7 + 9q^9 - 22q^{13} + \ldots$$

The ASwD basis is h_1, h_2.

1b. Basis of $S_3(\Gamma_{8^3.2^3.3^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{20}\eta(8\tau)^4}{\eta(4\tau)^6}} = q^{1/3} - \frac{20}{3}q^{4/3} + \frac{128}{9}q^{7/3} - \frac{400}{81}q^{10/3} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{16}\eta(4\tau)^6}{\eta(8\tau)^4}} = q^{2/3} - \frac{16}{3}q^{14/3} + \frac{38}{9}q^{26/3} + \frac{1696}{81}q^{38/3} + \cdots$$

The associated newform is a twist $f \otimes \chi$ of the f in case 1a.
The ASwD basis is h_1, h_2.

Table 1. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

The main point is that in the congruence case, the Hecke algebra acts and commutes with the Galois action so that the $2d$-dimensional representation ($d = \dim S_k(\Gamma)$) splits into 2-dimensional λ-adic representations. This is no longer the case in general for noncongruence subgroups. It is the case in our examples that the 4-dimensional representations attached to $S_3(\Gamma)$ factor into 2-dimensional pieces. Geometrically this is due to the presence of extra symmetries given by involutions and/or isogenies of our elliptic surfaces.

2.3. **Outline.** In section 3 we define the congruence and noncongruence subgroups we will be working with. Section 4 gives the method we use to construct the noncongruence forms h_1, h_2. Section 5 explains how we computed the traces of Frobenius elements in the l-adic Scholl's representation attached to our group Γ. The main point is to count the number of rational points over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$ of the elliptic modular surface E_Γ. In section 6. we discuss involutions and isogenies of these elliptic surfaces. Finally in section 7 we provide the experimental evidence for the ASwD congruences.

2a. Basis of $S_3(\Gamma_{8^3.6.3.1^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^{10}\eta(8z)^8}{\eta(4z)^4}} = q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{10}\eta(8z)^4}{\eta(2z)^4}} = q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \ldots$$

Newfor m in $S_3(\Gamma_0(432),\chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-4}{p}\right)$:

$f(z) = f_1(12z) + 6\sqrt{2}f_5(12z) + \sqrt{-3}f_7(12z) + 6\sqrt{-6}f_{11}(12z)$,

where

$f_1(z) = \frac{\eta(2z)^3\eta(3z)}{\eta(6z)\eta(z)}E_6(z) \quad f_5(z) = \frac{\eta(z)\eta(2z)^3\eta(3z)^3}{\eta(6z)}$

$f_7(z) = \frac{\eta(6z)^3\eta(z)}{\eta(2z)\eta(3z)}E_6(z) \quad f_{11}(z) = \frac{\eta(3z)\eta(z)^3\eta(6z)^3}{\eta(2z)}$

and $E_6(z) = 1 + 12\sum_{n\geq 1}(\sigma(3n) - 3\sigma(n))q^n$, where $\sigma(n) = \sum_{d|n} d$.

Atkin Swinnerton-Dyer basis:

if $p \equiv 1 \mod 3$ basis is h_1, h_2

if $p \equiv 2 \mod 3$ basis is $h_1 \pm \alpha h_2$, $\alpha^3 = 4$.

2b. Basis of $S_3(\Gamma_{24.3.2^3.1^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{22}\eta(8\tau)^8}{\eta(\tau)^4\eta(4\tau)^8}} = q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{20}\eta(4\tau)^2\eta(8\tau)^4}{\eta(\tau)^8}} = q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 + \cdots$$

The associated new form and the ASwD basis
are given in exactly the same way as in case 2a.

A variant denoted $S_3(\Gamma_{24.3.2^3.1^3B})$ is discussed in section 7.4.3

TABLE 2. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

3a. Basis of $S_3(\Gamma_{18.6.3^3.1^3})$

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^7\eta(6z)^{11}}{\eta(3z)^4}} = q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(3z)^4\eta(6z)^7\eta(2z)^{11}}{\eta(z)^4}} = q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \dots$$

Newform in $S_3(\Gamma_0(243), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)$.

$f(z) = q + 3iq^2 - 5q^4 + 6iq^5 + 11q^7 - 3iq^8 - 18q^{10} + \cdots$

Atkin Swinnerton-Dyer basis:

if $p \equiv 1 \mod 3$ basis is h_1, h_2

if $p \equiv 2 \mod 3$ basis is $h_1 \pm i\sqrt[3]{3}h_2$

3b. Basis of $S_3(\Gamma_{9.6^3.3.2^3})$; $r = q^{1/3}$.

$$h_1(z) = \sqrt[3]{\frac{\eta(\tau)^7\eta(2\tau)^4\eta(3\tau)^{11}}{\eta(6\tau)^4}} = r - \frac{7}{3}r^4 - \frac{19}{9}r^7 + \frac{193}{81}r^{10} + \frac{2306}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(\tau)^{11}\eta(3\tau)^7\eta(6\tau)^4}{\eta(2\tau)^4}} = r^2 - \frac{11}{3}r^5 + \frac{23}{9}r^8 - \frac{13}{81}r^{11} + \cdots$$

The associated new form and the ASwD basis are given in exactly the same way as in case 3a.

Table 3. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

4a. Basis of $S_3(\Gamma_{9.6^4.1^3})$

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^{13}\eta(6z)^{14}}{\eta(2z)^2\eta(3z)^7}} = q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^{14}\eta(6z)^{13}}{\eta(2z)^7\eta(3z)^2}} = q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \ldots$$

Associated newform in $S_3(\Gamma_0(486), \chi)$, where $\chi(\mathrm{Frob}_p) = \left(\frac{-3}{p}\right)$.

$f(z) = q - \sqrt{-2}q^2 - 2q^4 + 3\sqrt{-2}q^5 - 7q^7 + 2\sqrt{-2}q^8 + 6q^{10} - 3\sqrt{-2}q^{11} + 5q^{13}$

Atkin Swinnerton-Dyer basis:

if $p \equiv 1 \mod 3$	basis is	h_1, h_2
if $p \equiv 2 \mod 3$	basis is	$h_1 \pm \sqrt{-2}\sqrt[3]{3}h_2$

4b. Basis of $S_3(\Gamma_{18.3^4.2^3})$; $r = q^{1/3}$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{13}\eta(3\tau)^{14}}{\eta(6\tau)^7\eta(\tau)^2}} = r + \frac{2}{3}r^4 - \frac{28}{9}r^7 - \frac{482}{81}r^{10} - \frac{736}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{14}\eta(3\tau)^{13}}{\eta(6\tau)^2\eta(\tau)^7}} = r^2 + \frac{7}{3}r^5 + \frac{14}{9}r^8 - \frac{148}{81}r^{11} - \frac{1708}{243}r^{14} + \cdots$$

The associated newform is the same as in case 4a.

Atkin Swinnerton-Dyer basis:

if $p \equiv 1 \mod 3$	basis is	h_1, h_2
if $p \equiv 2 \mod 3$	basis is	$h_1 \pm \sqrt{-2}\sqrt[3]{3}h_2$

Table 4. Modular forms for noncongruence subgroups, and associated forms for congruence subgroups.

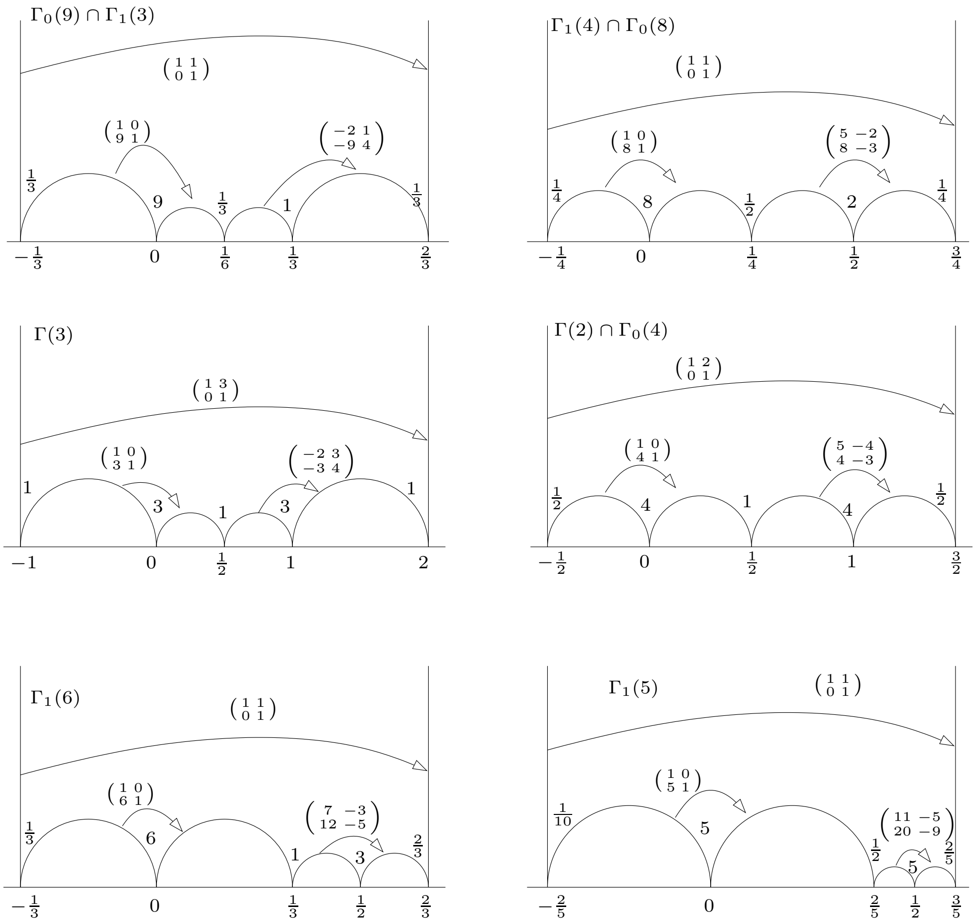

FIGURE 1. Fundamental domains for torsion free index 24 congruence subgroups in $\mathrm{SL}_2(\mathbf{Z})$.

3. Description of the noncongruence subgroups

3.1. **Beauville's families.** We start with certain index 12 genus 0 torsion free congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$, listed in Table 5 [Seb01]. Figure 3.1 shows corresponding fundamental domains and generating matrices.

Table 5 gives equations for the associated families of elliptic curves [Beau82]. Table 6 gives the $a_1, \ldots, a_5$ of the Weierstrass form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. The hauptmodul $t(\tau)$ listed in the table is such that $j(E_{t(\tau)}) = j(\tau)$.

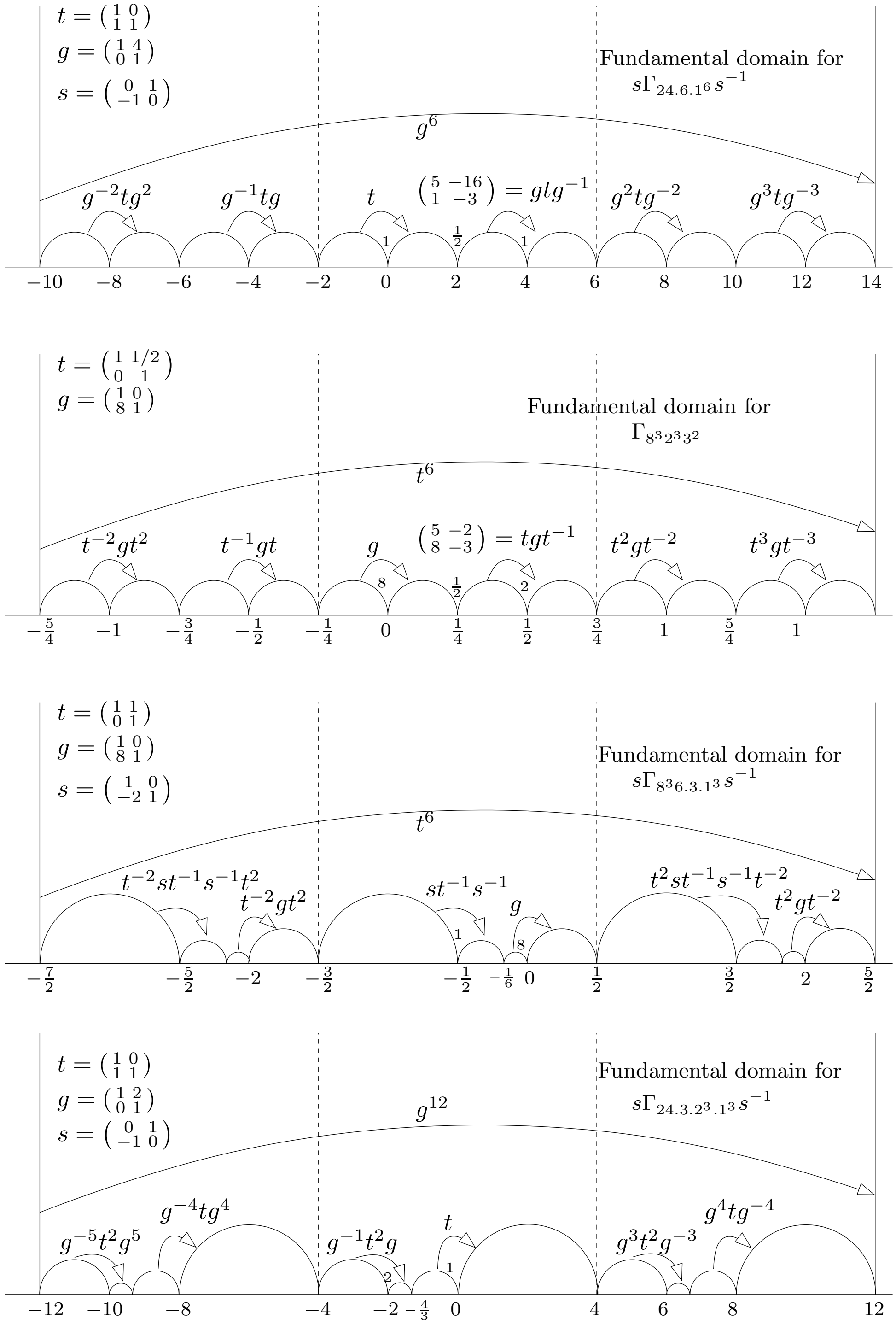

FIGURE 2. Fundamental domains for conjugates of some index 3 subgroups of $\Gamma_0(8) \cap \Gamma_1(4)$.

group	elliptic family	$j-$ invariant
$\Gamma(3)$	$(x^3+y^3+z^3)=txyz$	$\frac{t^3(t^3+216)^3}{(t^3-27)^3}$
$\Gamma(2)\cap\Gamma_1(4)$	$x(x^2+z^2+2zy)=tz(x^2-y^2)$	$\frac{(t^4-t^2+1)^3}{t^4(t-1)^2(t+1)^2}$
$\Gamma^1(5)$	$x(x-z)(y-z)t=y(y-x)z$	$-\frac{(t^4+12t^3+14t^2-12t+1)^3}{t^5(t^2+11t-1)}$
$\Gamma_1(6)$	$(xy+yx+zx)(x+y+z)=txyz$	$\frac{(3t-1)^3(3t^3-3t^2+9t-1)^3}{(t-1)^3t^6(9t-1)}$
$\Gamma_0(8)\cap\Gamma_1(4)$	$(x+y)(xy+z^2)t=4xyz$	$-16\frac{(t^4-16t^2+16)^3}{t^8(t+1)(t-1)}$
$\Gamma_0(9)\cap\Gamma_1(3)$	$(x^2y+y^2z+z^2x)=txyz$	$\frac{t^3(t^3-24)^3}{t^3-27}$

Table 5. Data for Beauville's elliptic surfaces.

level	Coefficients of Weierstrass form					t as a
	a_1	a_2	a_3	a_4	a_6	Hauptmodul
3	0	t^2	0	$-72t$	$-8(4t^2+27)$	$\frac{\eta\left(\frac{1}{3}\tau\right)^3}{\eta(3\tau)^3}+3$
4	0	$4+4t^2$	0	$16t^2$	0	$\frac{1}{2}\frac{\eta(\tau)^{12}}{\eta(2\tau)^8\eta\left(\frac{1}{2}\tau\right)^4}$
5	$t+1$	t	t	0	0	$q^{\frac{1}{5}}\prod_{\substack{n=0\\ e=1,-1}}^{\infty}\left(\frac{\left(1-q^{n+e\frac{1}{5}}\right)}{\left(1-q^{n+e\frac{2}{5}}\right)}\right)^5$
6	$t+1$	$t-t^2$	$t-t^2$	0	0	$\frac{1}{9}\frac{\eta(6\tau)^4\eta(\tau)^8}{\eta(3\tau)^8\eta(2\tau)^4}$
8	4	t^2	$4t^2$	0	0	$\frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}$
9	0	t^2	0	$8t$	16	$27\frac{\eta(9\tau)^3}{\eta(\tau)^3}+3$

Table 6. Weierstrass equations for Beauville's elliptic families.

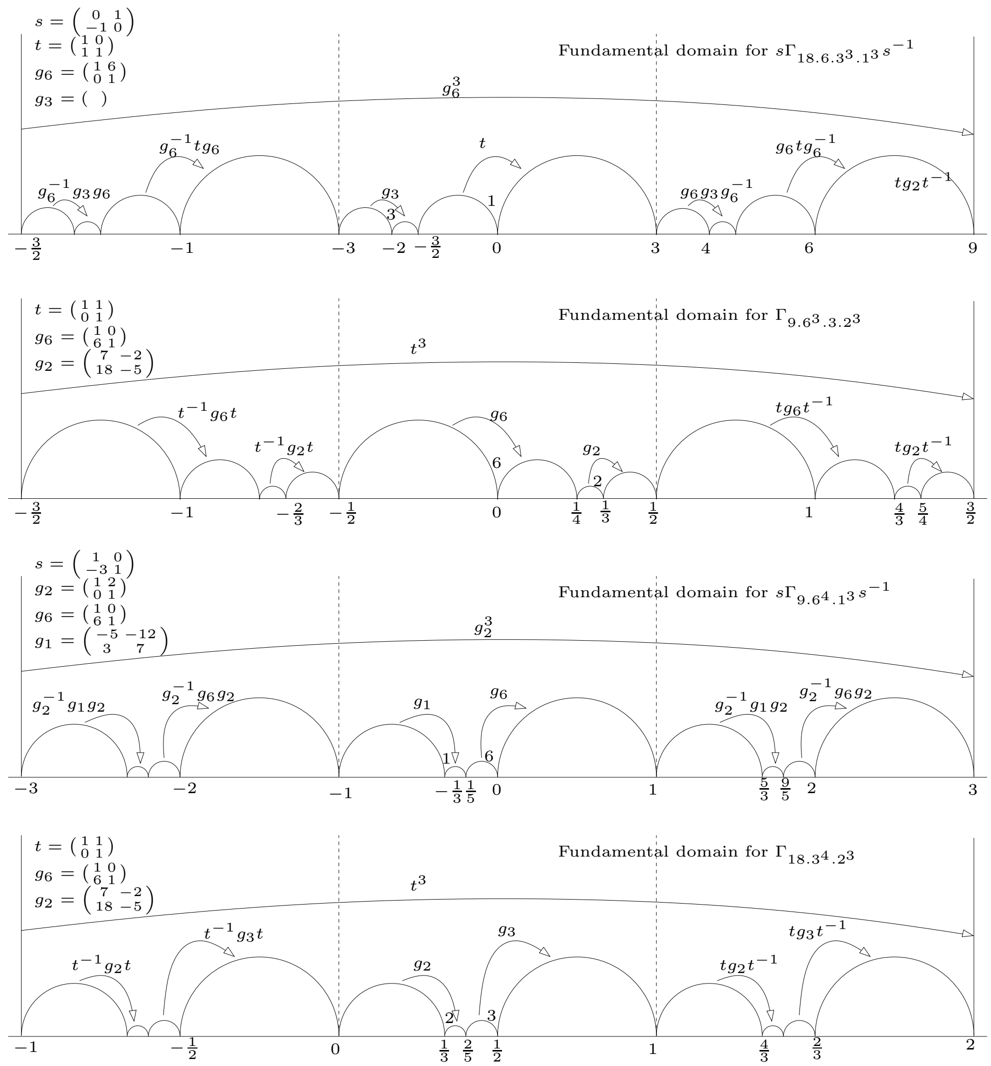

FIGURE 3. Fundamental domains for conjugates of some index 3 subgroups of $\Gamma_1(6)$.

cusps and subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$				
cusp τ	∞	0	$\frac{1}{2}$	$\frac{1}{4}$
width	1	8	2	1
subgroup	ramified cusps indicated by ✓			
$\Gamma_{24.6.1^6}$		✓	✓	
$\Gamma_{8^3.2^3.3^2}$	✓			✓
$\Gamma_{8^3.6.3.1^3}$			✓	✓
$\Gamma_{24.3.2^3.1^3}$		✓		✓

cusps and subgroups of $\Gamma_1(6)$				
cusp τ	∞	0	$\frac{1}{2}$	$\frac{1}{3}$
width	1	6	3	2
subgroup	ramified cusps indicated by ✓			
$\Gamma_{18.6.3^3.1^3}$		✓		✓
$\Gamma_{9.6^3.3.2^3}$	✓		✓	
$\Gamma_{9.6^4.1^3}$			✓	✓
$\Gamma_{18.3^4.2^3}$	✓	✓		

TABLE 7. Ramification points of triple covers of $X(\Gamma_0(8)\cap\Gamma_1(4))$ and $X(\Gamma_1(6))$, with corresponding subgroups.

3.2. **The noncongruence subgroups.** We will work with certain index 3 normal subgroups of $\Gamma_1(6)$ and $\Gamma_0(8)\cap\Gamma_1(4)$. The case $\Gamma_1(5)$ has been studied in [LLY03]. The fundamental domain of Γ is a union of three copies of a fundamental domain for G, corresponding to the three cosets of Γ in G. From the fundamental domains, shown in Figures 2 and 3, we obtain generators and cusp widths [Kul91], allowing us to make the following definition.

Definition 3.2.1. *We let* $\Gamma_{24.6.1^6}$, $\Gamma_{8^36.3.1^3}$, $\Gamma_{24.3.2^3.1^3}$, $\Gamma_{8^32^33^2}$ *be index* 3 *genus* 0 *subgroups of* $\Gamma_0(8)\cap\Gamma_1(4)$, *and* $\Gamma_{18.6.3^3.1^3}, \Gamma_{9.6^4.1^3}, \Gamma_{9.6^3.3.2^3}$, $\Gamma_{18.3^4.2^3}$ *index* 3 *genus* 0 *subgroups of* $\Gamma_1(6)$, *defined by their generators as follows:*

Γ	*generators*
$\Gamma_{24.6.1^6}$	$\begin{pmatrix} 1 & 0 \\ 24 & 1 \end{pmatrix}, \begin{pmatrix} 9 & -1 \\ 64 & -7 \end{pmatrix}, \begin{pmatrix} 5 & -1 \\ 16 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ 16 & 5 \end{pmatrix}, \begin{pmatrix} -7 & -1 \\ 64 & 9 \end{pmatrix}, \begin{pmatrix} -11 & -1 \\ 144 & 13 \end{pmatrix}.$
$\Gamma_{8^32^33^2}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -7 & -8 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ 8 & -3 \end{pmatrix}, \begin{pmatrix} 9 & -8 \\ 8 & -7 \end{pmatrix}, \begin{pmatrix} 13 & -18 \\ 8 & -11 \end{pmatrix}.$
$\Gamma_{8^36.3.1^3}$	$\begin{pmatrix} -11 & 6 \\ -24 & 13 \end{pmatrix}, \begin{pmatrix} 41 & -25 \\ 64 & -39 \end{pmatrix}, \begin{pmatrix} 49 & -32 \\ 72 & -47 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 25 & -9 \\ 64 & -23 \end{pmatrix}, \begin{pmatrix} 81 & -32 \\ 200 & -79 \end{pmatrix}.$
$\Gamma_{24.3.2^3.1^3}$	$\begin{pmatrix} 1 & 0 \\ 24 & 1 \end{pmatrix}, \begin{pmatrix} 21 & -2 \\ 200 & -19 \end{pmatrix}, \begin{pmatrix} 9 & -1 \\ 64 & -7 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ 8 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -11 & -2 \\ 72 & 13 \end{pmatrix}, \begin{pmatrix} -7 & -1 \\ 64 & 9 \end{pmatrix}.$
$\Gamma_{18.6.3^3.1^3}$	$\begin{pmatrix} 1 & 0 \\ 18 & 1 \end{pmatrix}, \begin{pmatrix} 25 & -3 \\ 192 & -23 \end{pmatrix}, \begin{pmatrix} 7 & -1 \\ 36 & -5 \end{pmatrix}, \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix}, \begin{pmatrix} -5 & -1 \\ 36 & 7 \end{pmatrix}.$
$\Gamma_{9.6^3.3.2^3}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -5 & -6 \\ 6 & 7 \end{pmatrix}, \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 7 & -2 \\ 18 & -5 \end{pmatrix}, \begin{pmatrix} 7 & -6 \\ 6 & -5 \end{pmatrix}, \begin{pmatrix} 25 & -32 \\ 18 & -23 \end{pmatrix}.$
$\Gamma_{9.6^4.1^3}$	$\begin{pmatrix} -17 & 6 \\ -54 & 19 \end{pmatrix}, \begin{pmatrix} 127 & -49 \\ 324 & -125 \end{pmatrix}, \begin{pmatrix} 61 & -24 \\ 150 & -59 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 91 & -25 \\ 324 & -89 \end{pmatrix}, \begin{pmatrix} 85 & -24 \\ 294 & -83 \end{pmatrix}.$
$\Gamma_{18.3^4.2^3}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix}, \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix}, \begin{pmatrix} 7 & -2 \\ 18 & -5 \end{pmatrix}, \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix}, \begin{pmatrix} 25 & -32 \\ 18 & -23 \end{pmatrix}, \begin{pmatrix} 19 & -27 \\ 12 & -17 \end{pmatrix}.$

By comparing cusp widths, in Tables 11 and 12, with possible cusp widths of congruence subgroups in Table 8, we obtain the following result.

Theorem 3.2.2. *The groups in Definition 3.2.1 are noncongruence subgroups.*

$6-6-6-6-3-3-3-3$
$9-9-9-3-3-1-1-1$
$9-9-3-3-3-3-3-3$
$10-10-5-5-2-2-1-1$
$18-9-2-2-2-1-1-1$
$27-3-1-1-1-1-1-1$

TABLE 8. Possible cusp widths of index 36 genus zero torsion free subgroups of $\mathrm{PSL}_2(\mathbf{Z})$, taken from [Seb01, §7, Table 2].

Values of t_8				
cusp c	∞	0	$\frac{1}{2}$	$\frac{1}{4}$
$t_8(c)$	1	0	∞	-1

Values of t_6				
cusp c	∞	0	$\frac{1}{2}$	$\frac{1}{3}$
$t_8(c)$	$\frac{1}{9}$	0	1	∞

TABLE 9. Values of Hauptmoduln at cusps.

3.3. **Hauptmoduln and covering maps.** Throughout this paper we fix our choice of identification of $X(\Gamma_0(8)\cap\Gamma_1(4))$ and $X(\Gamma_1(6))$ with the projective line $\mathbf{P}^1$, with parameter t_8 and t_6 respectively. As functions of z in the upperhalf complex plane, $t_8(z)$ and $t_6(z)$ are given terms of the Dedekind eta function, as listed in the last column of Table 6:

$$t_8(z)=\frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}},\quad\text{and}\quad t_6(z)=\frac{1}{9}\frac{\eta(6\tau)^4\eta(\tau)^8}{\eta(3\tau)^8\eta(2\tau)^4}.$$

The values of these functions at the cusps are as in Table 9.

Since the ramification points of the covering maps $\Gamma\setminus\mathbf{H}\to G\setminus\mathbf{H}$ are at cusps as in Table 7, the covering maps are given in each case by a map

$$r\mapsto r^3=m(t),$$

where the maps m corresponding to each of our subgroups are as in Table 10.

subgroup	$m(t)$	$m^{-1}(r^3)$
$\Gamma_{24.6.1^6}$	t	r^3
$\Gamma_{8^3.2^3.3^2}$	$\frac{1+t}{1-t}$	$\frac{r^3-1}{r^3+1}$
$\Gamma_{8^3.6.3.1^3}$	$\frac{t+1}{4}$	$4r^3-1$
$\Gamma_{24.3.2^3.1^3}$	$\frac{2(1+t)}{t}$	$\frac{2}{r^3-2}$

subgroup	$m(t)$	$m^{-1}(r^3)$
$\Gamma_{18.6.3^3.1^3}$	$t/9$	$9r^3$
$\Gamma_{9.6^3.3.2^3}$	$\frac{1-9t}{3-3t}$	$\frac{1-3r^3}{9-3r^3}$
$\Gamma_{9.6^4.1^3}$	$\frac{8}{3-3t}$	$1-\frac{8}{3r^3}$
$\Gamma_{18.3^4.2^3}$	$\frac{1-9t}{24t}$	$\frac{1}{24r^3+9}$

TABLE 10. Covering maps corresponding to subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$ and $\Gamma_1(6)$.

4. Constructing elements of $S_3(\Gamma)$

4.1. **Dimension.** For odd k, Shimura [Shi71, Theorem 2.25] gives the following formula for $\dim S_k(\Gamma)$ for a genus g subgroup $\Gamma \notin -I$ of $\mathrm{SL}_2(\mathbf{Z})$:

$$\dim S_k(\Gamma) = (k-1)(g-1) + \frac{1}{2}(k-2)u + \frac{1}{2}(k-1)u' + \sum_{i=1}^{r} k\frac{e_i - 1}{2e_i}.$$

The e_i are orders of elliptic points, u is the number of regular cusps, and u' the number of irregular cusps. Using this formula, we find that

$$\dim S_3(\Gamma) = 2,$$

for Γ equal to any of the groups in Definition 3.2.1.

4.2. **Method of constructing elements of $S_3(\Gamma)$.** Suppose that Γ has index 3 in G, one of the groups in Table 5, and that the corresponding covering is ramified at cusps c_1 and c_2. Let t be a Hauptmodul for G, e.g., as in [CN79]. By a transformation, take t with $t(c_1) = 0$ and $t(c_2) = \infty$. Then $\sqrt[3]{t}$ is a Hauptmodul for Γ. Let $f \in M_3(G)$. Then $\sqrt[3]{t}f \in A_3(\Gamma)$. If f is zero where t has poles, then $\sqrt[3]{t}f$ and $\sqrt[3]{t^2}f$ are in $S_3(\Gamma)$. We give modular forms in terms of the Dedekind eta function, using the data given by Martin [M96]. Explicit details of the forms and their poles and zeros are given in Tables 11 and 12, and the q-expansions are given in Tables 13 and 14.

forms for $\Gamma_0(8)\cap\Gamma_1(4)$	weight	$\frac{1}{2}(2)$	$0(8)$	$\infty(1)$	$\frac{1}{4}(1)$
	cusps (and widths) / order of vanishing				
$t=\frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}=1-8q+32q^2+\cdots$	0	-1	1	0	0
$\frac{t+1}{2}=\frac{\eta(z)^4\eta(4z)^{14}}{\eta(8z)^4\eta(2z)^{14}}=1-4q+16q^2+\cdots$	0	-1	0	0	1
$\frac{t+1}{2t}=\frac{\eta(4z)^{10}}{\eta(8z)^4\eta(2z)^2\eta(z)^4}=1+4q+16q^2+\cdots$	0	0	-1	0	1
$\frac{4(t+1)}{(1-t)}=\frac{\eta(4z)^{12}}{\eta(8z)^8\eta(2z)^4}$	0	0	0	-1	1
$E_a=\frac{\eta(4z)^4\eta(2z)^6}{\eta(z)^4}$	3	1	0	1	1
$E_b=\left(\frac{2t}{t+1}\right)E_a=\frac{\eta(2z)^8\eta(8z)^4}{\eta(4z)^6}$	3	1	1	1	0

forms for $\Gamma_{24.6.1^6}$	cusps	$1/2$	0	$-\frac{1}{8}$	∞	$\frac{1}{8}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$
	width	6	24	1	1	1	1	1	1
	weight	order of vanishing of form at cusps							
$\sqrt[3]{t}$	0	-1	1	0	0	0	0	0	0
E_a	3	3	0	1	1	1	1	1	1
$t^{1/3}E_a$	3	2	1	1	1	1	1	1	1
$t^{2/3}E_a$	3	1	2	1	1	1	1	1	1

forms for $\Gamma_{8^3 6.3.1^3}$	cusps	$1/2$	$\frac{2}{5}$	0	$\frac{2}{3}$	$\frac{3}{8}$	∞	$\frac{5}{8}$	$1/4$
	width	6	8	8	8	1	1	1	3
	weight	order of vanishing of form at cusps							
$r_1=\sqrt[3]{\frac{t+1}{2}}$	0	-1	0	0	0	0	0	0	1
E_b	3	3	1	1	1	1	1	1	0
r_1E_b	3	2	1	1	1	1	1	1	1
$r_1^2E_b$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{24.3.2^3.1^3}$	cusps	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{10}$	0	$-\frac{1}{8}$	∞	$\frac{1}{8}$	$1/4$
	width	2	2	2	24	1	1	1	3
	weight	order of vanishing of form at cusps							
$r_2=\sqrt[3]{\frac{(t+1)}{2t}}$	0	0	0	0	-1	0	0	0	1
E_b	3	1	1	1	3	1	1	1	0
r_2E_b	3	1	1	1	2	1	1	1	1
$r_2^2E_b$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{8^3 2^3 3^2}$	cusps	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	-1	0	1	∞	$\frac{1}{4}$
	width	2	2	2	8	8	8	3	3
	weight	order of vanishing of form at cusps							
$r_3=\sqrt[3]{\frac{4(t+1)}{(t-1)}}$	0	0	0	0	0	0	0	-1	1
E_b	3	1	1	1	1	1	1	3	0
r_3E_b	3	1	1	1	1	1	1	2	1
$r_3^2E_b$	3	1	1	1	1	1	1	1	2

TABLE 11. Orders of vanishingat cusps for forms for $\Gamma_0(8)\cap\Gamma_1(4)$ and for subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$.

cusps (and widths)		$\infty(1)$	$0(6)$	$\frac{1}{2}(3)$	$\frac{1}{3}(2)$
forms for $\Gamma_1(6)$	weight	order of vanishing			
$a = \frac{\eta(z)\eta(6z)^6}{\eta(2z)^2\eta(3z)^3} = q - q^2 + q^3 + q^4 + \cdots$	1	1	0	0	0
$b = \frac{\eta(2z)\eta(3z)^6}{\eta(z)^2\eta(6z)^3} = 1 + 2q + 4q^2 + 2q^3 + \cdots$	1	0	0	0	1
$c = \frac{\eta(3z)\eta(2z)^6}{\eta(6z)^2\eta(z)^3} = 1 + 3q + 3q^2 + 3q^3 + \cdots$	1	0	0	1	0
$d = \frac{\eta(6z)\eta(z)^6}{\eta(3z)^2\eta(2z)^3} = 1 - 6q + 12q^2 - 6q^3 \cdots$	1	0	1	0	0
$r_0 = b/d = 1 + 8q + 40q^2 + 152q^3 + \cdots$	0	0	-1	0	1
$r_1 = b/c = 8\frac{r_0}{(9r_0-1)} = 1 - q + 4q^2 + \cdots$	0	0	0	-1	1
$r_2 = a/c = \frac{(r_0-1)}{(9r_0-1)} = q - 4q^2 + 10q^3 \cdots$	0	1	0	-1	0
$r_3 = a/d = \frac{1}{8}(r_0 - 1) = q + 5q^2 + 19q^3 \cdots$	0	1	-1	0	0
$acd = q - 4q^2 + q^3 + 16q^4 + \cdots$	3	1	1	1	0
$bcd = 1 - q - 5q^2 - q^3 + 11q^4 + \cdots$	3	0	1	1	1

forms for $\Gamma_{18.6.3^3.1^3}$	cusps	$\frac{1}{6}$	∞	$-\frac{1}{6}$	0	$\frac{1}{8}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{3}$
	width	1	1	1	18	3	3	3	6
	weight	order of vanishing of form at cusps							
$\sqrt[3]{b/d}$	0	0	0	0	-1	0	0	0	1
acd	3	1	1	1	3	1	1	1	0
$(\sqrt[3]{b/d})acd$	3	1	1	1	2	1	1	1	1
$(\sqrt[3]{b/d})^2acd$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{9.6^4.1^3}$	cusps	$\frac{5}{18}$	∞	$\frac{7}{18}$	$\frac{2}{5}$	0	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{1}{3}$
	width	1	1	1	6	6	6	9	6
	weight	order of vanishing of form at cusps							
$\sqrt[3]{b/c}$	0	0	0	0	0	0	0	-1	1
acd	3	1	1	1	1	1	1	3	0
$(\sqrt[3]{b/c})acd$	3	1	1	1	1	1	1	2	1
$(\sqrt[3]{b/c})^2acd$	3	1	1	1	1	1	1	1	2

forms for $\Gamma_{9.6^3.3.2^3}$	cusps	∞	-1	0	1	$\frac{1}{2}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
	width	3	6	6	6	9	2	2	2
	weight	order of vanishing of form at cusps							
$\sqrt[3]{a/c}$	0	1	0	0	0	-1	0	0	0
bcd	3	0	1	1	1	3	1	1	1
$(\sqrt[3]{a/c})bcd$	3	1	1	1	1	2	1	1	1
$(\sqrt[3]{a/c})^2bcd$	3	2	1	1	1	1	1	1	1

forms for $\Gamma_{18.3^4.2^3}$	cusps	∞	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$
	width	3	18	3	3	3	2	2	2
	weight	order of vanishing of form at cusps							
$\sqrt[3]{a/d}$	0	1	-1	0	0	0	0	0	0
bcd	3	0	3	1	1	1	1	1	1
$(\sqrt[3]{a/d})bcd$	3	1	2	1	1	1	1	1	1
$(\sqrt[3]{a/d})^2bcd$	3	2	1	1	1	1	1	1	1

TABLE 12. Orders of vanishing at cusps for forms for subgroups of $\Gamma_1(6)$.

$\Gamma_{24.6.1^6}$	
$\sqrt[3]{\eta(\tau)^{-4}\eta(2\tau)^6\eta(4\tau)^{16}}$	$= q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 - \frac{850}{243}q^5 - \frac{3488}{729}q^6 - \frac{5968}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(\tau)^{4}\eta(2\tau)^{-6}\eta(4\tau)^{20}}$	$= q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 - \frac{850}{243}q^5 + \frac{3488}{729}q^6 - \frac{5968}{6561}q^7 + \cdots$
$\Gamma_{8^36.3.1^3}$	
$\sqrt[3]{\eta(\tau)^{4}\eta(2\tau)^{10}\eta(4\tau)^{-4}\eta(8\tau)^8}$	$= q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 - \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(\tau)^{8}\eta(2\tau)^{-4}\eta(4\tau)^{10}\eta(8\tau)^4}$	$= q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots$
$\Gamma_{24.3.2^3.1^3}$	
$\sqrt[3]{\eta(\tau)^{-4}\eta(2\tau)^{22}\eta(4\tau)^{-8}\eta(8\tau)^8}$	$= q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots$
$\sqrt[3]{\eta(\tau)^{-8}\eta(2\tau)^{20}\eta(4\tau)^{2}\eta(8\tau)^4}$	$= q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 - \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots$
$\Gamma_{8^32^33^2}$	
$\sqrt[3]{\eta(2\tau)^{20}\eta(4\tau)^{-6}\eta(8\tau)^4}$	$= q^{2/3} - \frac{20}{3}q^{8/3} + \frac{128}{9}q^{14/3} - \frac{400}{81}q^{20/3} + \cdots$
$\sqrt[3]{\eta(2\tau)^{16}\eta(4\tau)^{6}\eta(8\tau)^{-4}}$	$= q^{1/3} - \frac{16}{3}q^{7/3} + \frac{38}{9}q^{13/3} + \frac{1696}{81}q^{19/3} + \cdots$

TABLE 13. q-expansions of basis of forms for $S_3(\Gamma)$ for four subgroups of $\Gamma_0(8)\cap\Gamma_1(4)$

$\Gamma_{18.6.3^3.1^3}$		
$ab^{1/3}cd^{2/3}$	$= \sqrt[3]{\eta(\tau)^4\eta(2\tau)^7\eta(3\tau)^{-4}\eta(6\tau)^{11}}$	$= q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots$
$ab^{2/3}cd^{1/3}$	$= \sqrt[3]{\eta(\tau)^{-4}\eta(2\tau)^{11}\eta(3\tau)^{4}\eta(6\tau)^{7}}$	$= q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \cdots$
$\Gamma_{9.6^4.1^3}$		
$ab^{1/3}c^{2/3}d$	$= \sqrt[3]{\eta(\tau)^{13}\eta(2\tau)^{-2}\eta(3\tau)^{-7}\eta(6\tau)^{14}}$	$= q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots$
$ab^{2/3}c^{1/3}d$	$= \sqrt[3]{\eta(\tau)^{14}\eta(2\tau)^{-7}\eta(3\tau)^{-2}\eta(6\tau)^{13}}$	$= q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \cdots$
$\Gamma_{9.6^3.3.2^3}$		
$a^{1/3}bc^{2/3}d$	$= \sqrt[3]{\eta(\tau)^{7}\eta(2\tau)^{4}\eta(3\tau)^{11}\eta(6\tau)^{-4}}$	$= q^{\frac{1}{3}} - \frac{7}{3}q^{\frac{4}{3}} - \frac{19}{9}q^{\frac{7}{3}} + \frac{193}{81}q^{\frac{10}{3}} + \frac{2306}{243}q^{\frac{13}{3}} + \cdots$
$a^{2/3}bc^{1/3}d$	$= \sqrt[3]{\eta(\tau)^{11}\eta(2\tau)^{-4}\eta(3\tau)^{7}\eta(6\tau)^{4}}$	$= q^{\frac{2}{3}} - \frac{11}{3}q^{\frac{5}{3}} + \frac{23}{9}q^{\frac{8}{3}} - \frac{13}{81}q^{\frac{11}{3}} + \frac{2495}{243}q^{\frac{14}{3}} + \cdots$
$\Gamma_{18.3^4.2^3}$		
$a^{1/3}bcd^{2/3}$	$= \sqrt[3]{\eta(\tau)^{-2}\eta(2\tau)^{13}\eta(3\tau)^{14}\eta(6\tau)^{-7}}$	$= q^{\frac{1}{3}} + \frac{2}{3}q^{\frac{4}{3}} - \frac{28}{9}q^{\frac{7}{3}} - \frac{482}{81}q^{\frac{10}{3}} - \frac{736}{243}q^{\frac{13}{3}} + \cdots$
$a^{2/3}bcd^{1/3}$	$= \sqrt[3]{\eta(\tau)^{-7}\eta(2\tau)^{14}\eta(3\tau)^{13}\eta(6\tau)^{-2}}$	$= q^{\frac{2}{3}} + \frac{7}{3}q^{\frac{5}{3}} + \frac{14}{9}q^{\frac{8}{3}} - \frac{148}{81}q^{\frac{11}{3}} - \frac{1708}{243}q^{\frac{14}{3}} + \cdots$

TABLE 14. Basis of weight three cusp forms for some index 3 subgroups of $\Gamma_1(6)$. a, b, c, d are eta products as in Table 12.

5. Traces and Point Counting

As described by Scholl, corresponding to each of these families, we have a representation on parabolic cohomology:

$$\rho = \rho_l : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to H^1(X(\Gamma), j_* R^1 f_* \mathbf{Q}_l). \tag{4}$$

Here

$$E^\circ(\Gamma) \xrightarrow{f} Y(\Gamma) \xhookrightarrow{j} X(\Gamma),$$

with

$$Y(\Gamma) = \Gamma\backslash\mathbf{H}, \quad X(\Gamma) = (\Gamma\backslash\mathbf{H})^*$$

$E^\circ(\Gamma)$ be a family of elliptic curves over $Y(\Gamma)$. We let $\mathcal{F} = j_* R^1 f_* \mathbf{Q}_l$, an l-adic sheaf for the étale topology on $X(\Gamma)$. We computed the traces of the Frobenius elements of this representation via point counting, as in [LLY03] and [ALL05].

5.1. **Equations for elliptic surfaces associated with the noncongruence subgroups.** As in section Section 3.1, associated to $\Gamma_0(8)\cap\Gamma_1(4)$ and $\Gamma_1(6)$, we have families of elliptic curves $E_8(t)$ and $E_6(t)$ as given in Table 6:

$$E_8(t): \qquad y^2 + 4xy + 4t^2 y = x^3 + t^2 x^2 \tag{5}$$

$$E_6(t): \qquad y^2 + (t+1)xy + (t-t^2)y = x^3 + (t-t^2)x^2. \tag{6}$$

Thus we have elliptic surfaces E_8 and E_6, with fibrations

$$f_8 : E_8 \to X(\Gamma_0(8)\cap\Gamma_1(4))$$

and

$$f_6 : E_6 \to X(\Gamma_1(6)),$$

with fibres given by $f_8^{-1}(t) = E_8(t)$ and $f_6^{-1}(t) = E_6(t)$.

By composing the covering maps given in Table 10 with the fibrations f_8 or f_6, associated with our noncongruence subgroups we have the families of elliptic curves given in Table 15. Our notation is explained by example: The elliptic surface $E(\Gamma_{8^3.2^3.3^3})$ corresponding to $\Gamma_{8^3.2^3.3^2}$ has a fibration

$$f : E(\Gamma_{8^3.2^3.3^3}) \to X(\Gamma_{8^3.2^3.3^3}),$$

with fiber $f^{-1}(r)$ having an equation

$$y^2 + 4xy + 4\left(\frac{r^3-1}{r^3+1}\right)^2 y = x^3 + 4\left(\frac{r^3-1}{r^3+1}\right)^2 x^2,$$

i.e., the t in (5) is replaced by $m^{-1}(r^3) = \frac{r^3-1}{r^3+1}$, where $m(t) = \frac{1+t}{1-t}$. This family of elliptic curve is denoted by $E_8\left(\frac{r^3-1}{r^3+1}\right)$. The other families are constructed and denoted in a similar way.

We computed the traces of Frobenius by summing local terms using:

Theorem 5.1.1.

$$\mathrm{Tr}(\mathrm{Frob}_q\,|H^1(X(\Gamma),\mathcal{F})) = -\sum_{x\in X(\mathbf{F}_q)} \mathrm{Tr}(\mathrm{Frob}_q\,|\mathcal{F}_x).$$

group	family of curves
$\Gamma_{24.6.1^6}$	$E_8(r^3)$
$\Gamma_{8^3 2^3 3^2}$	$E_8\left(\frac{r^3-1}{r^3+1}\right)$
$\Gamma_{8^3 6.3.1^3}$	$E_8(4r^3-1)$
$\Gamma_{24.3.2^3.1^3}$	$E_8\left(\frac{2}{r^3-2}\right)$

group	family of curves
$\Gamma_{18.6.3^3.1^3}$	$E_6(9r^3)$
$\Gamma_{9.6^3.3.2^3}$	$E_6\left(\frac{1-3r^3}{9-3r^3}\right)$
$\Gamma_{9.6^4.1^3}$	$E_6\left(1-\frac{8}{9r^3}\right)$
$\Gamma_{18.3^4.2^3}$	$E_6\left(\frac{1}{9(8r^3+1)}\right)$

TABLE 15. Families of elliptic curves $E_n(m^{-1}(r^3))$ corresponding to certain noncongruence subgroups.

Proof. This follows from Grothendieck-Lefschetz trace formula because the other terms $H^i(X(\Gamma), \mathcal{F}))$, $i \neq 1$ are zero. $\square$

The following is also well known:

Theorem 5.1.2. $\mathrm{Tr}(\mathrm{Frob}_q\,|\mathcal{F}_x)$ *may be computed according to the following:*

(1) *If the fiber E_x is smooth, then*

$$\mathrm{Tr}(\mathrm{Frob}_q|\mathcal{F}_x) = \mathrm{Tr}(\mathrm{Frob}_q|H^1(E_x, \mathbf{Q}_l)) = q + 1 - \#E_x(\mathbf{F}_q).$$

(2) *If the fiber E_x is singular, then Tate's algorithm tells us that*

$$\mathrm{Tr}(\mathrm{Frob}_q|\mathcal{F}_x) = \begin{cases} 1 \textit{ if the fiber is split multiplicative.} \\ -1 \textit{ if the fiber is nonsplit multiplicative.} \\ 0 \textit{ if the fiber is additive.} \end{cases}$$

(3) *If E is a singular curve over a field with characteristic not 2 or 3, given by an equation*

$$E : y^2 = x^3 + ax + b,$$

then the reduction type of E is determined as follows:

$$\left.\begin{array}{l} \textit{additive} \\ \textit{split multiplicative} \\ \textit{nonsplit multiplicative} \end{array}\right\} \textit{if } -2ab \textit{ is} \left\{\begin{array}{l} 0 \textit{ in } k \\ \textit{a nonzero square in } k \\ \textit{not a square in } k \end{array}\right.$$

In order to apply part (3) of the above result, we need to transform $E_8(t)$ and $E_6(t)$ in to the simplified Weierstrass form $y^2 = x^3 + ax + b$. We obtain the following curves, isomorphic to the originals, over any field of characteristic not 2 or 3.

$$(7)\quad \widetilde{E}_8 : \qquad y^2 = x^3 - 27(t^4 - 16t^2 + 16)x + 54(t^2 - 2)(t^4 + 32t^2 - 32)$$

$$(8)\quad \widetilde{E}_6 : \qquad y^3 = x^3 - 2^4 3^3(3t - 1)(3t^3 - 3t^2 + 9t - 1)x \\ -2^7 3^3(3t^2 + 6t - 1)(9t^4 - 36t^3 + 30t^2 - 12t + 1)$$

Thus one may compute values of the trace by using the above result, for example with MAGMA. The results for a range of values of p and various covers of E_8 and E_6 are given in Table 16.

Group	Equation	p	5	7	11	13	17	19	23	73
$\Gamma_{24.6.1^6}$	$E_8(r^3)$	Tr_p	0	4	0	-44	0	52	0	-92
		Tr_{p^2}	100	-188	484	292	1156	-92	2116	-17084
$\Gamma_{8^3 2^3 3^2}$	$E_8\left(\frac{r^3-1}{r^3+1}\right)$	Tr_p	0	-4	0	-44	0	-52	0	-92
		Tr_{p^2}	100	-188	484	292	1156	-92	2116	-17084
$\Gamma_{8^3 6.3.1^3}$	$E_8(r^3-1)$	Tr_p	0	-3	0	13	0	33	0	-71
		Tr_{p^2}	-44	-95	52	169	1012	-359	-1772	5617
	$E_8(2r^3-1)$	Tr_p	0	3	0	13	0	-33	0	-71
		Tr_{p^2}	-44	-95	52	169	1012	-359	-1772	5617
	$E_8(4r^3-1)$	Tr_p	0	0	0	-26	0	0	0	142
		Tr_{p^2}	-44	190	52	-338	1012	718	-1772	-11234
$\Gamma_{24.3.2^3.1^3}$	$E_8\left(\frac{2}{r^3-2}\right)$	Tr_p	0	0	0	-26	0	0	0	142
		Tr_{p^2}	-44	190	52	-338	1012	718	-1772	-11234
$\Gamma_{18.6.3^3.1^3}$	$E_6(3r^3)$	Tr_p	0	-11	0	-5	0	19	0	76
		Tr_{p^2}	28	-23	196	313	508	361	316	-18428
$\Gamma_{18.6.3^3.1^3}$	$E_6(9r^3)$	Tr_p	0	22	0	10	0	-38	0	76
		Tr_{p^2}	28	46	196	-626	508	-722	316	-18428
$\Gamma_{9.6^3.3.2^3}$	$E_6\left(\frac{1-3r^3}{9-3r^3}\right)$	Tr_p	0	22	0	10	0	-38	0	76
		Tr_{p^2}	28	46	196	-626	508	-722	316	-18428
$\Gamma_{9.6^4.1^3}$	$E_6\left(1-\frac{24}{r^3}\right)$	Tr_p	0	7	0	-5	0	-17	0	-248
		Tr_{p^2}	64	49	448	313	-140	433	1972	9436
	$E_6\left(1-\frac{8}{3r^3}\right)$	Tr_p	0	-14	0	10	0	34	0	-248
		Tr_{p^2}	64	-98	448	-626	-140	-866	1972	9436
$\Gamma_{18.3^4.2^3}$	$E_6\left(\frac{1}{24r^3+9}\right)$	Tr_p	0	-14	0	10	0	34	0	-248
		Tr_{p^2}	64	-98	448	-626	-140	-866	1972	9436

TABLE 16. Table of Tr $\rho^*(\mathrm{Frob}_p)$.

6. Involutions and Isogenies

6.1. **Involutions.** The four dimensional representations on $H^1(X(\Gamma), \mathcal{F}_\Gamma)$ in fact split into two 2-dimensional Galois representations. We can achieve this splitting by using an involution on $\Gamma \setminus \mathbf{H}$ which extends to either an automorphism or isogeny on the elliptic surface.

For each family given in Table 15 by an equation $E_n(r)$, corresponding to a covering $r^3 = m(t)$, we have involutions i and ι of t and r, given in Table 17, such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "r\mapsto\iota(r)"] \arrow[d, "r\mapsto r^3=m(t)"'] & \mathbb{P}^1 \arrow[d, "r\mapsto r^3=m(t)"] \\ \mathbb{P}^1 \arrow[r, "t\mapsto i(t)"'] & \mathbb{P}^1 \end{tikzcd}$$

Furthermore, if c_1, c_2 are the ramified cusps of the map $r \mapsto r^3 = m(t)$, and c_3, c_4 are the unramified cusps, then i fixes the sets $\{c_1, c_2\}$ and $\{c_3, c_4\}$. This means that the involution i lifts to an involution ι of r, as indicated in Table 17. To check these are the correct maps, one just needs to verify that $(\iota(\sqrt[3]{m(t)}))^3 = m(i(t))$, which is simple algebra.

6.2. **Isogenies.** The involutions i of modular curves given in Table 17 lift to maps

$$\begin{aligned} \tilde{i} : E_n &\rightarrow E_n \\ \tilde{i} : (t, x, y) \in E_n(t) &\mapsto (i(t),\ i_x(t, x, y),\ i_y(t, x, y)), \end{aligned} \tag{9}$$

where $n = 8$ or 6, which restrict to isogenies between the fibres of the corresponding family of elliptic curves (given by (5) and (6)). From the isogenies of the families $E_6(t)$, $E_8(t)$, one can obtain the isogenies on the families $E_6(m^{-1}(r^3))$, $E_8(m^{-1}(r^3))$, lifting ι to $\tilde{\iota}$. These isogenies will give rise to involutions on the level of cohomology.

To show that two curves $E(t)$ and $E(i(t))$ are isogenous by an isogeny of degree d, it suffices to show that $\Phi_d(j(E(t)), j(E(i(t)))) = 0$, where Φ_d is the dth modular polynomial. The isogeny can be explicitly determined by Velu's methodfrom a subgroup of order d on $E(t)$. Although the algorithms involved are well known and not difficult theoretically, in practice they should be carried out with the help of a computer program, such as Magma [BCP97], because of the large number of of terms in the polynomials involved. For example, Φ_8 is a polynomial in two variables of degree 20 with 141 terms; Φ_n can be found in a Magma database using the command `ClassicalModularPolynomial(n)` for $1 \leq n \leq 17$.

Although it's not important to know the isogeny exactly, we do need to know the field over which the map is defined. This information was computed with the assistance of Magma, and is given in Table 18. The polynomials given in this table are such that their roots are the x-coordinates of points in the kernel of the isogeny.

Involutions i of $X(\Gamma_0(8)\cap\Gamma_1(4))$, and ι of $X(\Gamma)$, for $\Gamma\subset\Gamma_0(8)\cap\Gamma_1(4)$					
subgroup Γ	values of τ and t where cover ramifies		$r^3 = m(t)$	involutions of t and r	
	τ	$t(\tau)$		$i : t \mapsto$	$\iota : r \mapsto$
$\Gamma_{24.6.1^6}$	$1/2, 0$	$\infty, 0$	t	$-t$	$-r$
$\Gamma_{8^2.2^3.3^2}$	$\infty, 1/4$	$1, -1$	$\frac{t+1}{1-t}$	$1/t$	$-r$
$\Gamma_{8^3 6.3.1^3}$	$1/2, 1/4$	$\infty, -1$	$\frac{t+1}{4}$	$\frac{1-t}{1+t}$	$\frac{1}{2r}$
$\Gamma_{24.3.2^3.1^3}$	$0, 1/4$	$0, -1$	$\frac{2(1+t)}{t}$	$\frac{t+1}{t-1}$	$\frac{2}{r}$

Involutions i of $X(\Gamma_1(6))$, and ι of $X(\Gamma)$ for $\Gamma\subset\Gamma_1(6)$					
subgroup Γ	values of τ and t where cover ramifies		$r^3 = m(t)$	involutions of t and r	
	τ	$t(\tau)$		$i : t \mapsto$	$\iota : r \mapsto$
$\Gamma_{18.6.3^3.1^3}$	$1/3, 0$	$\infty, 0$	$t/9$	$\frac{1}{9t}$	$\frac{1}{9r}$
$\Gamma_{9.6^3.3.2^3}$	$\infty, 1/2$	$\frac{1}{9}, 1$	$\frac{1-9t}{3(1-t)}$	$\frac{1}{9t}$	$\frac{1}{r}$
$\Gamma_{9.6^4.1^3}$	$1/2, 1/3$	$1, \infty$	$\frac{8}{3(1-t)}$	$\frac{1-9t}{9-9t}$	$\frac{2}{r}$
$\Gamma_{18.3^4.2^3}$	$\infty, 0$	$\frac{1}{9}, 0$	$\frac{1-9t}{24t}$	$\frac{1-9t}{9-9t}$	$\frac{1}{2r}$

TABLE 17. Involutions of modular curves $\Gamma\backslash\mathbf{H}$. For $\Gamma_0(8)\cap\Gamma_1(4)$, $t(\tau) = \frac{\eta(z)^8\eta(4z)^4}{\eta(2z)^{12}}$, and for $\Gamma_1(6)$, $t(\tau) = \frac{1}{9}\frac{\eta(6\tau)^4\eta(\tau)^8}{\eta(3\tau)^8\eta(2\tau)^4}$, as in Tables 6, 11, and 12.

6.3. **Isogenous relationships between families.** In the previous section we showed how involutions give rise to isogenies on the fibres, which will resulting in involutions on the cohomology of each family. There are also isogenous maps between families, which explain our groupings into pairs of cases, which was originally based on the relationships between traces seen in Table 16. Combining the relations between curves we already have, we

subgroup	$i(t)$	d	polynomial defining kernel of isogeny	$\tilde{\imath}$'s field of definition
Level 8 cases				
$\Gamma_{24.6.1^6}$	$-t$	1	$-$	$\mathbf{Q}$
$\Gamma_{8^2.2^3.3^2}$	$1/t$	4	$(x+t^2)x$	$\mathbf{Q}$
$\Gamma_{8^36.3.1^3}$	$\frac{1-t}{1+t}$	8	$(x^2-4tx-4t^3)(x+t^2)x$	$\mathbf{Q}[\sqrt{-1}]$
$\Gamma_{24.3.2^3.1^3}$	$\frac{t+1}{t-1}$	8	$(x^2+4tx+4t^3)(x+t^2)x$	$\mathbf{Q}[\sqrt{-1}]$
Level 6 cases				
$\Gamma_{18.6.3^3.1^3}, \Gamma_{9.6^4.1^3}$	$\frac{1}{9t}$	3	$x-t^2+t$	$\mathbf{Q}[\sqrt{-3}]$
$\Gamma_{9.6^3.1^3}, \Gamma_{18.3^4.2^3}$	$\frac{1-9t}{9-9t}$	6	$(x-t^2+t)x(x+t)$	$\mathbf{Q}[\sqrt{-3}]$

TABLE 18. Data concerning involutions i and ι of Table 17, lifted to maps $\tilde{\imath}$ of families of curves, defining isogenies of degree d on fibres. In particular, $\Phi_d(j(E_n(i(t))), j(E_n(t))) = 0$ where n is the level, and Φ_d is the dth modular polynomial.

find that

$$\begin{aligned}
\Phi_8\left(j\left(E_6\left(\frac{t-1}{t+1}\right)\right), j\left(E_8\left(\phi_1(t)\right)\right)\right) &= 0\\
\Phi_8\left(j(E_8(4t-1)), j\left(E_8\left(\frac{2}{\phi_2(t)-2}\right)\right)\right) &= 0\\
\Phi_6\left(j\left(E_6\left(\frac{1-3t}{9-3t}\right)\right), j\left(E_6\left(9\phi_3(t)\right)\right)\right) &= 0\\
\Phi_3\left(j\left(E_6\left(1-\frac{8}{3t}\right)\right), j\left(E_6\left(\frac{1}{9-24\phi_4(t)}\right)\right)\right) &= 0,
\end{aligned}$$

where $\phi_1(t) = \phi(2) = 1/t, \phi_3(t) = t/3, \phi_4(t) = -1/t$. This may also be checked directly with MAGMA. Thus the maps ϕ_i between the bases lift to isogenies on the fibres between families. Replacing t by r^3 in these equations does not change the relationships, so this also holds for the covers, and these maps induce isomorphisms on the level of cohomology. Refer to Table 16 for which cover corresponds to which group.

7. Experimental data for the ASwD congruences

The strategy for finding an ASwD basis is the following: For our noncongruence subgroup Γ, we have found a basis h_1, h_2 for $S_3(\Gamma)$. We have also found a Hecke eigenform $f \in S_3(\Gamma_0, \chi)$ for some congruence subgroup Γ_0. Let a_n and b_n respectively be the expansion coefficients of h_1 and h_2. Let A_n be the expansion coefficients of f. We consider two possible situations.

7.1. **Case 1.**

In the simplest case, h_1, h_2 is already an ASwD basis. This case occurs in section 7.3. So for good primes p and integers n with $p \nmid n$

$$a_{pn} \equiv A_p a_n \bmod p^2 \quad \text{and} \quad b_{pn} \equiv A_p b_n \bmod p^2, \tag{10}$$

which implies, for p fixed and n varying with $a_n \neq 0$ and $b_n \neq 0$,

$$a_{pn}/a_n \equiv \text{constant} \bmod p^2 \quad \text{and} \quad b_{pn}/b_n \equiv \text{constant} \bmod p^2. \tag{11}$$

So, our test for whether h_1, h_2 is an ASwD basis is to check whether a_{pn}/a_n and b_{pn}/b_n take constant values for fixed p and varying n, with np less than some fixed bound. If this holds, then we also consider this to be evidence that h_1, h_2 is an ASwD basis. We can make this conclusion regardless of whether f is known.

In the case $n = 1$, since $a_1 = b_1 = 1$, (11) implies that

$$a_p \equiv A_p \bmod p^2 \quad \text{and} \quad b_p \equiv A_p \bmod p^2. \tag{12}$$

In order to determine the associated congruence modular form, we test whether (12) holds for small primes for the candidate form f. This is what happens in subsection 7.3.1.

In some cases, to get congruences, f needs to be replaced by $f \otimes \chi$ for some character χ. Then A_p will be replaced by $A_p\chi(p)$ in (12), so this phenomena can be recognized by checking whether A_p/a_p and A_p/b_p are roots of unity. This happens in subsection 7.3.2. However, we have not worked out what the character χ is.

7.2. **Case 2.**

In most of our examples examples, it turns out that the ASwD basis depends on the congruence class of the prime p modulo some small integer. It turns out that for some primes, (11) holds for the values tested, in which case h_1, h_2 is assumed to be the ASwD basis, but for other primes, this does not hold.

If (11) does not hold for some prime p, then we will assume that for this prime, an ASwD basis consists of linear combinations of the form $h_1 + \alpha h_2$, where α is an algebraic number of small degree, such that for integers n with $p \nmid n$. the expansion coefficients satisfy

$$a_{pn} + \alpha b_{pn} \equiv A_p(a_n + \alpha b_n) \bmod p^2. \tag{13}$$

A priori, α depends on p, though we will see that in the examples we are considering, evidence suggests that it only depends on the congruence class of p modulo a small integer.

For (13) to hold, it is sufficient, but not necessary, that

$$(14) \qquad a_{pn} \equiv A_p \alpha b_n \mod p^2, \text{ and } \alpha b_{pn} \equiv A_p a_n \mod p^2,$$

which, assuming all the terms are non-zero, implies that $a_{pn}/b_n = A_p\alpha_p$ and $b_{pn}/a_n = A_p/\alpha_p$, So if (11) does not hold as n varies, we test whether

$$(15) \quad \frac{a_{np}}{b_n} \equiv \text{constant} \mod p^2 \quad \text{and} \quad \frac{b_{np}}{a_n} \equiv \text{constant} \mod p^2.$$

If this holds, the values of α and A_p mod p^2, up to sign, are determined by

$$(16) \quad \alpha^2 \equiv \frac{a_{np}}{b_n} \Big/ \frac{b_{np}}{a_n} \mod p^2, \quad \text{and} \quad A_p^2 \equiv \frac{a_{np}}{b_n}\frac{b_{np}}{a_n} \mod p^2.$$

For p for which (15) holds, there are two solutions to (16) for α, and the ASwD basis has the form $h_1+\alpha h_2, h_1-\alpha h_2$. We expect that α only depends on p modulo some small integer. Since α is expected to be an algebraic integer, but not an integer, it may be difficult to guess the value of α, from $\alpha \mod p^2$. So we also look at powers of $\alpha \mod p^2$, and if for some small power these are constant as p varies, then we deduce a value of α. Once α is determined, $A_p \mod p^2$ is determined, if this agrees with the coefficients of our congruence modular form, then we take this as evidence that $h_1 + \alpha h_2, h_1 - \alpha h_2$ is an ASwD basis with f the associated new form. As for case 1, we will also test whether the A_p must be multiplied some root of unity, presumably the value $\chi(p)$ for some character χ, though again, we have not determined the character in question.

7.3. **Examples associated with newform in $S_3(\Gamma_0(48), \chi)$.** For $\Gamma_{24.6.1^6}$ and $\Gamma_{8^3.2^3.3^3}$, evidence suggests that the associated congruence form is as follows, with the first few A_p as in Table 19.

$$(17)\ f(z) = \frac{\eta(4z)^9\eta(12z)^9}{\eta(2z)^3\eta(6z)^3\eta(8z)^3\eta(24z)^3} = q + 3q^3 - 2q^7 + 9q^9 - 22q^{13} - 26q^{19} - 6q^{21} + 25q^{25} + \ldots$$

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
a_p	0	−2	0	−22	0	−26	0	0	46	26	0	22	0	0	0	74	−122

TABLE 19. First few coefficients A_p for newform for $S_3(\Gamma_0(48), \chi)$.

7.3.1. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{24.6.1^6}$. We have shown previously that $S_3(\Gamma_{24.6.1^6})$ has a basis

$$(18)\ h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(4z)^{20}}{\eta(2z)^6}} = q - \frac{4}{3}q^2 + \frac{8}{9}q^3 - \frac{176}{81}q^4 - \frac{850}{243}q^5 \cdots$$

$$(19)\ h_2(z) = \sqrt[3]{\frac{\eta(4z)^{16}\eta(2z)^6}{\eta(z)^4}} = q + \frac{4}{3}q^2 + \frac{8}{9}q^3 + \frac{176}{81}q^4 - \frac{850}{243}q^5 \cdots$$

The first few prime coefficients of these forms are:

p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$\frac{8}{9}$	$-\frac{850}{243}$	$-\frac{5968}{6561}$	$-\frac{35104520}{4782969}$	$\frac{952141694}{129140163}$	$-\frac{206256733102}{31381059609}$	$\frac{60201506159720}{2541865828329}$
b_p	$\frac{4}{3}$	$\frac{8}{9}$	$-\frac{850}{243}$	$-\frac{5968}{6561}$	$-\frac{35104520}{4782969}$	$\frac{952141694}{129140163}$	$-\frac{206256733102}{31381059609}$	$\frac{60201506159720}{2541865828329}$

p	5	7	11	13	17	19	23	29	31	37	41	43	47
$a_{np}/a_n \mod p^2$	0	47	0	147	0	335	0	0	46	26	0	22	0
$b_{np}/b_n \mod p^2$	0	47	0	147	0	335	0	0	46	26	0	22	0

TABLE 20. values of $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ for primes $p \geq 5$ and integers n, with $pn \leq 500$. These agree mod p^2 with values in Table 19.

Since the ratios a_{np}/a_n and b_{np}/b_n, given in Table 20 appear to be constant, and the numbers in Tables 19 and 20 agree modulo p^2, we conclude that the ASwD basis of $S_3(\Gamma_{24.6.1^6})$ is h_1, h_2, as given by (18) and (19) for all primes, with f in (17) being the associated congruence form.

7.3.2. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{8^3.2^3.3^3}$. Basis of $S_3(\Gamma_{8^3.2^3.3^3})$, written in terms of $r = q^{1/3}$ and $s = q^{2/3}$.

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{20}\eta(8\tau)^4}{\eta(4\tau)^6}} = \sum_{n\geq 1} a_n s^n = s - \frac{20}{3}s^4 + \frac{128}{9}s^7 - \frac{400}{81}s^{10} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{16}\eta(4\tau)^6}{\eta(8\tau)^4}} = \sum_{n\geq 1} b_n r^n = r - \frac{16}{3}r^7 + \frac{38}{9}r^{13} + \frac{1696}{81}r^{19} + \cdots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$\frac{128}{9}$	0	$-\frac{3454}{243}$	0	$-\frac{38656}{6561}$
b_p	0	0	0	$-\frac{16}{3}$	0	$\frac{38}{9}$	0	$\frac{1696}{81}$

Our computations show that the ratios $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ remain constant for fixed p, for values of pn up to 500. We can write these ratios in terms of ω, a sixth root of 1 mod p^2, as in Table 21. In this table we also tabulate ω, and the order of ω as an element of $(\mathbf{Z}/p^2\mathbf{Z})^\times$.

Since the values of a_{np}/a_n and b_{np}/b_n are constant over the ranges computed, we conjecture that h_1, h_2 is an ASwD basis for all primes. Comparing these values with the coefficients of f, we conjecture that the associated congruence form is $f \otimes \chi$ where χ is a certain Hecke character.

p	$\frac{a_{np}}{a_n}$ mod p^2		$\frac{b_{np}}{b_n}$ mod p^2		ω	$o(\omega)$
7	36	$= -2\omega$	11	$= -2\omega^{-1}$	31	6
11	0		0			
13	168	$= -22\omega$	23	$= -22\omega^{-1}$	146	3
17	0		0			
19	11	$= -26\omega$	324	$= -26\omega^{-1}$	69	6
23	0		0			
29	0		0			
31	915	$= 46\omega$	915	$= 46\omega^{-1}$	-1	2
37	47	$= 26\omega$	1296	$= 26\omega^{-1}$	581	3
41	0		0			
43	1827	$= 22\omega$	1827	$= 22\omega^{-1}$	-1	2
47	0		0			

TABLE 21. values of $\frac{a_{np}}{a_n}$ and $\frac{b_{np}}{b_n}$ for $\Gamma_{8^3.2^3.3^3}$, for primes $p \geq 5$ and integers n, with $pn \leq 500$, in terms of a 6th root of unity, ω, with order $o(\omega)$. Compare with values in Table 19.

7.4. **Examples associated with newform in $S_3(\Gamma_0(432), \chi)$.** For $\Gamma_{8^3.6.3.1^3}$ and $\Gamma_{24.3.2^3.1^3}$ evidence suggests that the associated congruence form is

$$f(z) = q + 6\sqrt{2}q^5 + \sqrt{-3}q^7 + 6\sqrt{-6}q^{11} + 13q^{13} - 6\sqrt{2}q^{17} + 11\sqrt{-3}q^{19} - 18\sqrt{-6}q^{23} + 47q^{25} - 24\sqrt{2}q^{29} + \cdots \tag{20}$$

The first few A_p are given in Table 22, where they are divided by either 1, $\sqrt{2}$, $\sqrt{3}$, or $\sqrt{-6}$, for easy readability

p	5	7	11	13	17	19	23	29	31	37	41	43	47
A_p				13						35			
$A_p/\sqrt{2}$	6				-6			-24			0		
$A_p/\sqrt{-3}$		1				11			24			-24	
$A_p/\sqrt{-6}$			6				-18						6

TABLE 22. Coefficients of f in (20) and (21).

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The form f can be given in terms of eta products and an Eisenstein series as follows:

$$f(z) = f_1(12z) + 6\sqrt{2}f_5(12z) + \sqrt{-3}f_7(12z) + 6\sqrt{-6}f_{11}(12z), \tag{21}$$

where

$$f_1(z) = \frac{\eta(2z)^3\eta(3z)}{\eta(6z)\eta(z)}E_6(z) \tag{22}$$

$$f_5(z) = \frac{\eta(z)\eta(2z)^3\eta(3z)^3}{\eta(6z)} \tag{23}$$

$$f_7(z) = \frac{\eta(6z)^3\eta(z)}{\eta(2z)\eta(3z)}E_6(z) \tag{24}$$

$$f_{11}(z) = \frac{\eta(3z)\eta(z)^3\eta(6z)^3}{\eta(2z)} \tag{25}$$

$$\text{where } E_6(z) = 1 + 12\sum_{n\geq 1}(\sigma(3n) - 3\sigma(n))q^n, \tag{26}$$

and $\sigma(n) = \sum_{d|n} d$.

7.4.1. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{8^3.6.3.1^3}$. We have seen that a basis of $S_3(\Gamma_{8^3.6.3.1^3})$ can be given by:

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^{10}\eta(8z)^8}{\eta(4z)^4}} = \sum_{n\geq 1} a_n q^n = q - \frac{4}{3}q^2 - \frac{40}{9}q^3 + \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{10}\eta(8z)^4}{\eta(2z)^4}} = \sum_{n\geq 1} b_n q^n = q - \frac{8}{3}q^2 + \frac{8}{9}q^3 + \frac{32}{81}q^4 - \frac{82}{243}q^5 + \dots$$

The first few prime coefficients of h_1 and h_2 are as follows:

p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$-\frac{40}{9}$	$\frac{1454}{243}$	$-\frac{13168}{6561}$	$\frac{38671144}{4782969}$	$-\frac{2230795138}{129140163}$	$-\frac{418720079278}{31381059609}$	$\frac{30660416258552}{2541865828329}$
b_p	$-\frac{8}{3}$	$\frac{8}{9}$	$-\frac{82}{243}$	$-\frac{24400}{6561}$	$\frac{16345336}{4782969}$	$\frac{1236747902}{129140163}$	$\frac{842483994194}{31381059609}$	$-\frac{34758650729368}{2541865828329}$

For $p \equiv 1 \mod 3$, our data suggests that a_{pn}/a_p and b_{pn}/b_n remain constant as n varies, with values as in Table 23. This means we are in case 1, described in subsection 7.1. Experimentally, we noted that for these p we always have $\left(\frac{a_{pn}}{a_p}/\frac{b_{pn}}{b_n}\right)^6 \equiv 1 \mod p^2$ (excluding the case $p = 13$, when $a_{pn} \equiv b_{pn} \equiv 0 \mod 13$). We also checked that $\frac{a_{pn}}{a_p} \times \frac{b_{pn}}{b_n} \equiv A_p^2 \mod p^2$ where the A_p are as in Table 22. The first observation indicates that these two forms correspond to congruence forms which are twists of each other by an order 6 character, and the second observation indicates that the congruence form is the f given by (20). Using these two observations, we write the ratios a_{np}/a_n and b_{np}/b_n in the factored forms in Table 23. The values of ω, a sixth root of 1, and the values used for $\sqrt{3} \mod p^2$ are also tabulated.

Based on these experiments, we conjecture that the Atkin Swinnerton-Dyer basis of $S_3(\Gamma_{8^3.6.3.1^3})$ when $p \equiv 1 \mod 3$ is h_1, h_2, and the associated congruence forms are $f \otimes \chi$ and $f \otimes \chi^{-1}$ for a certain Hecke character.

p	$\frac{a_{np}}{a_n}$ mod p^2		$\frac{b_{np}}{b_n}$ mod p^2		$\sqrt{-3}$	ω
7	17	$=\omega^{-4}\sqrt{-3}$	29	$=\omega^{-2}\sqrt{-3}$	37	$\sqrt[4]{-18}$
13	52	$=\omega^{-2}13$	130	$=\omega^{2}13$		$\sqrt{23}$
19	48	$=\omega^{-2}11\sqrt{-3}$	346	$=\omega^{-4}11\sqrt{-3}$	137	$\sqrt{69}$
31	915	$=\omega^{6}24\sqrt{-3}$	46	$=24\sqrt{-3}$	82	$\sqrt[6]{-1}$
37	165	$=\omega^{-4}35$	1169	$=\omega^{4}35$		$\sqrt[4]{581}$
43	11	$=-\omega^{6}24\sqrt{-3}$	1838	$=-24\sqrt{-3}$	1002	$\sqrt[6]{-1}$

TABLE 23. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \mod 3$, for h_1 and h_2 for $\Gamma_{8^3.6.3.1^3}$, in terms of A_p in Table 22.

p	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2	$(\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n})^6 \equiv \alpha^3$	$\frac{a_{np}}{b_n}\frac{b_{np}}{a_n} \equiv A_p^2$
5	3	1	4	$-2\cdot 6^2$
11	84	32	4	$-6\cdot 6^2$
17	278	243	4	$-2\cdot 6^2$
23	335	130	4	$-6\cdot 18^2$
29	272	441	4	$-2\cdot 24^2$
41	0	0		
47	302	760	4	$-6\cdot 6^2$

TABLE 24. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \mod 3$, for h_1 and h_2 for $\Gamma_{8^3.6.3.1^3}$, with α as in (16), and A_p (experimentally) as in Table 22.

From the data in Table 24, following the explanation of Section 7.2, the Atkin Swinnerton-Dyer basis of $S_3(\Gamma_{8^3.6.3.1^3})$ when $p \equiv 1 \mod 3$ should be h_1, h_2, and when $p \equiv 2 \mod 3$, it should consist of forms of the form $h_1 + \alpha h_2$ with $\alpha^3 = 4$.

7.4.2. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{24.3.2^3.1^3}$. Basis of $S_3(\Gamma_{24.3.2^3.1^3})$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{22}\eta(8\tau)^8}{\eta(\tau)^4\eta(4\tau)^8}} = q + \frac{4}{3}q^2 - \frac{40}{9}q^3 - \frac{400}{81}q^4 + \frac{1454}{243}q^5 + \frac{1888}{729}q^6 - \frac{13168}{6561}q^7 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{20}\eta(4\tau)^2\eta(8\tau)^4}{\eta(\tau)^8}} = q + \frac{8}{3}q^2 + \frac{8}{9}q^3 - \frac{32}{81}q^4 - \frac{82}{243}q^5 - \frac{5440}{729}q^6 - \frac{24400}{6561}q^7 + \cdots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	$\frac{4}{3}$	$-\frac{40}{9}$	$\frac{1454}{243}$	$-\frac{13168}{6561}$	$\frac{38671144}{4782969}$	$-\frac{2230795138}{129140163}$	$-\frac{418720079278}{31381059609}$	$\frac{30660416258552}{2541865828329}$
b_p	$\frac{8}{3}$	$\frac{8}{9}$	$-\frac{82}{243}$	$-\frac{24400}{6561}$	$\frac{16345336}{4782969}$	$\frac{1236747902}{129140163}$	$\frac{842483994194}{31381059609}$	$-\frac{34758650729368}{2541865828329}$

Note that up to sign these are identical to the coefficients of the forms given for the $\Gamma_{8^3.6.3.1^3}$ case, and so the ASwD basis is expected to be the same as

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$
7	17	29
13	52	130
19	48	346
31	915	46
37	165	1169
43	11	1838

TABLE 25. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \mod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3})$. These values are the same as those in Table 23.

p	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$ mod p^2
5	3	1
11	84	32
17	278	243
23	335	130
29	272	441
41	0	0
47	302	760

TABLE 26. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \mod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3})$. These values are the same as those in Table 24.

in the $\Gamma_{8^3.6.3.1^3}$ case, namely h_1, h_2 when $p \equiv 1 \mod 3$ and $h_1 + \alpha h_2$ with $\alpha^3 = 4$ when $p \equiv 2 \mod 3$.

7.4.3. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{24.3.2^3.1^3B}$. This is a conjugate of the $S_3(\Gamma_{24.3.2^3.1^3})$ example by the involution

$$W_8 = \begin{pmatrix} 0 & -1 \\ 8 & 0 \end{pmatrix}.$$

Basis of $S_3(\Gamma_{24.3.2^3.1^3B})$ in terms of $r = q^{1/3}$.

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^8\eta(4z)^{22}}{\eta(8z)^4\eta(8z)^8}} = \sum_{n\geq 1} a_n r^n = r^2 - \frac{8}{3}r^5 + \frac{20}{9}r^8 - \frac{256}{81}r^{11} - \frac{64}{243}r^{14} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^4\eta(2z)^2\eta(4z)^{20}}{\eta(8z)^8}} = \sum_{n\geq 1} b_n r^n = r - \frac{4}{3}r^4 - \frac{16}{9}r^7 + \frac{112}{81}r^{10} + \ldots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19	23	29	31
a_p	1	0	$-\frac{8}{3}$	0	$-\frac{256}{81}$	0	$\frac{7984}{729}$	0	$\frac{172544}{19683}$	$-\frac{18907736}{1594323}$	0
b_p	0	0	0	$-\frac{16}{9}$	0	$-\frac{1534}{243}$	0	$\frac{78560}{6561}$	0	0	$-\frac{126424784}{4782969}$

p	a_{np}/a_n	b_{np}/b_n	a_{np}/b_n	b_{np}/a_n	ω	i	$\sqrt[3]{2}$
7	$32 = -\sqrt{-3}\cdot\omega^2$	$20 = \sqrt{-3}\cdot\omega$			18	$\sqrt{-3} = 12$	
13	$52 = -13\cdot\omega$	$130 = -13\cdot\omega^2$			22	$\sqrt{-3} = 45$	
19	$313 = 11\sqrt{-3}\cdot\omega$	$15 = -11\sqrt{-3}\cdot\omega^2$			68	$\sqrt{-3} = 137$	
31	$46 = 24\sqrt{-3}$	$915 = -24\sqrt{-3}$			439	$\sqrt{-3} = 82$	
37	$165 = 35\cdot\omega^2$	$1169 = 35\cdot\omega$			581		
43	$1838 = 24\sqrt{-3}$	$11 = -24\sqrt{-3}$			423	$\sqrt{-3} = 847$	

TABLE 27. Values of a_{np}/a_n and b_{np}/b_n for $p \equiv 1 \mod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3B})$.

p	a_{np}/a_n	b_{np}/b_n	a_{np}/b_n	b_{np}/a_n	ω	i	$\sqrt[3]{2}$
5			$14 = 6\sqrt{-2}\cdot\frac{\sqrt{2}}{2\sqrt[3]{2}}$	$2 = 6\sqrt{-2}\cdot\frac{2\sqrt[3]{2}}{\sqrt{2}}$		$i = 7$	3
7	$32 = -\sqrt{-3}\cdot\omega^2$	$20 = \sqrt{-3}\cdot\omega$			18	$\sqrt{-3} = 12$	
11			$79 = 6\sqrt{-6}\cdot\frac{\sqrt{-2}}{2\sqrt[3]{2}}$	$57 = 6\sqrt{-6}\cdot\frac{2\sqrt[3]{2}}{\sqrt{-2}}$		$\sqrt{3} = 27$	73
13	$52 = -13\cdot\omega$	$130 = -13\cdot\omega^2$			22	$\sqrt{-3} = 45$	
17			$139 = 6\sqrt{-2}\cdot\frac{\sqrt{2}}{2\sqrt[3]{2}}$	$197 = 6\sqrt{-2}\cdot\frac{2\sqrt[3]{2}}{\sqrt{2}}$		$i = 38$	195
19	$313 = 11\sqrt{-3}\cdot\omega$	$15 = -11\sqrt{-3}\cdot\omega^2$			68	$\sqrt{-3} = 137$	
23			$97 = -18\sqrt{-6}\cdot\frac{\sqrt{-2}}{2\sqrt[3]{2}}$	$269 = -18\sqrt{-6}\cdot\frac{2\sqrt[3]{2}}{\sqrt{-2}}$		$\sqrt{3} = 223$	384
29			$136 = -24\sqrt{-2}\cdot\frac{\sqrt{2}}{2\sqrt[3]{2}}$	$41 = -24\sqrt{-2}\cdot\frac{2\sqrt[3]{2}}{\sqrt{2}}$		$i = 800$	403
31	$46 = 24\sqrt{-3}$	$915 = -24\sqrt{-3}$			439	$\sqrt{-3} = 82$	
37	$165 = 35\cdot\omega^2$	$1169 = 35\cdot\omega$			581		
41			0	0			
43	$1838 = 24\sqrt{-3}$	$11 = -24\sqrt{-3}$			423	$\sqrt{-3} = 847$	
47			$2058 = 6\sqrt{-6}\cdot\frac{\sqrt{-2}}{2\sqrt[3]{2}}$	$689 \quad 6\sqrt{-6}\cdot\frac{2\sqrt[3]{2}}{\sqrt{-2}}$		$\sqrt{3} = 270$	1854

TABLE 28. Values of a_{np}/b_n and b_{np}/a_n for $p \equiv 2 \mod 3$, for h_1 and h_2 for $S_3(\Gamma_{24.3.2^3.1^3B})$.

Ratios when terms are non-zero:

Atkin Swinnerton-Dyer basis:

$$\text{if } p \equiv 1 \mod 3 \quad \text{basis is} \quad h_1, h_2$$
$$\text{if } p \equiv 5 \mod 12 \quad \text{basis is} \quad h_1 \pm \frac{\sqrt{2}}{2\sqrt[3]{2}} h_2$$
$$\text{if } p \equiv 11 \mod 12 \quad \text{basis is} \quad h_1 \pm \frac{\sqrt{-2}}{2\sqrt[3]{2}} h_2$$

7.5. **Examples associated with newform in** $S_3(\Gamma_0(243), \chi)$**.**

$$f(z) = q + 3iq^2 - 5q^4 + 6iq^5 + 11q^7 - 3iq^8 - 18q^{10} + 12iq^{11} + \cdots$$

where i is a root of $x^2+1=0$. Note, the corresponding Galois representation is a twist of the representation corresponding to $E_6(3r^3)$.

The first few prime coefficients $\tilde{A}_p$ of this form are as follows:

p	5	7	11	13	17	19	23	29	31	37
$\tilde{A}_p$	$6i$	11	$12i$	5	$-18i$	-19	$-30i$	$48i$	-13	17

7.5.1. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{18.6.3^3.1^3}$. Basis of $S_3(\Gamma_{18.6.3^3.1^3})$

$$\begin{aligned} h_1(z) &= \sqrt[3]{\frac{\eta(z)^4\eta(2z)^7\eta(6z)^{11}}{\eta(3z)^4}} = \sum_{n\geq 1} a_n q^n = q - \frac{4}{3}q^2 - \frac{31}{9}q^3 + \frac{400}{81}q^4 + \frac{104}{243}q^5 + \cdots \\ h_2(z) &= \sqrt[3]{\frac{\eta(3z)^4\eta(6z)^7\eta(2z)^{11}}{\eta(z)^4}} = \sum_{n\geq 1} b_n q^n = q + \frac{4}{3}q^2 - \frac{7}{9}q^3 - \frac{112}{81}q^4 - \frac{616}{243}q^5 + \dots \end{aligned}$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	$-\frac{4}{3}$	$-\frac{31}{9}$	$\frac{104}{243}$	$\frac{44018}{6561}$	$-\frac{38654696}{4782969}$	$-\frac{1857609346}{129140163}$	$\frac{362933655200}{31381059609}$	$-\frac{33243449873158}{2541865828329}$
b_p	$\frac{4}{3}$	$-\frac{7}{9}$	$-\frac{616}{243}$	$-\frac{15886}{6561}$	$\frac{43656424}{4782969}$	$-\frac{343807618}{129140163}$	$-\frac{100695940768}{31381059609}$	$\frac{19258418018042}{2541865828329}$

with a_n and b_n the coefficients of the non-congruence forms given above. The following ratios, all computed mod p^2, appear to be constant as n varies, for the given ps. The table shows the constants; if no entry is shown, this means the ratio is not constant in this case.

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$	mod p^2
5			3	13	
7	36	2			
11			13	82	
13	54	110			
17			279	148	
19	228	152			
23			130	400	
29			296	515	
31	915	59			
37	1058	294			

Case I: $p \equiv 1 \mod 3$. These ratios are a special case of the Atkin-Swinnerton-Dyer type relation, e.g., $a_{7n}/a_n \equiv 36 \mod 7^2$ can be written as

$$a_{7n} - 36a_n + 7^2 a_{n/p} \equiv 0 \mod 7^2.$$

So, for $p \equiv 1 \mod 3$, it looks like h_1 and h_2 form an Atkin Swinnerton-Dyer basis.

Note that for p in the above table with $p \equiv 1 \mod p$, except for the case $p = 19$, we have $(\frac{a_{np}}{a_n}/\frac{b_{np}}{b_n})^3 \equiv 1 \mod p^2$.

It's not surprising that this relation holds, since the ratios ought to be the values of A_p given above, which we can see should always be ω or ω^2 in these cases, including for $p = 19$.

The reason the congruence does not hold for $p = 19$ is that in this case we have $\omega, \omega^2 \equiv 68, 292 \mod 19^2$, and $\alpha_1 = -19\omega, \alpha_2 = -19\omega^2 \equiv 152, 228 \mod 19^2$, so we only have that $\alpha_1/19 \equiv \omega \mod 19$, $\alpha_2/19 \equiv \omega^2 \mod 19$, i.e., the ratio satisfies $(\frac{a_{19n}}{a_n}/\frac{b_{19n}}{b_n})^3 \equiv 1 \mod 19$, which we can check is true. Case II: $p \equiv 2 \mod 3$. Observation: when $p \equiv 2 \mod 3$ we always have $(\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n})^3 \equiv -9 \mod p^2$.

Suppose that the Atkin Swinnerton-Dyer basis is $h_1 + \alpha h_2$, then (writing $\alpha_p = \alpha \mod p^2$) we would have

$$a_{pn} + \alpha_p b_{pn} \equiv A_p(a_n + \alpha_p b_n) \mod p^2,$$

and suppose we in fact have

$$a_{pn} \equiv A_p \alpha_p b_n \mod p^2, \text{ and } \alpha_p b_{pn} \equiv A_p a_n \mod p^2,$$

then this implies that $a_{pn}/b_n = A_p\alpha_p$ and $b_{pn}/a_n = A_p/\alpha_p$, so $\alpha_p^2 \equiv \frac{a_{np}}{b_n}/\frac{b_{np}}{a_n}$, so from the above observation we expect $\alpha^6 \equiv -9 \mod p^2$, i.e., $\alpha \equiv \sqrt[3]{3}i \mod p^2$, so it seems that for $p \equiv 2 \mod 3$ we should have Atkin Swinnerton-Dyer basis consisting of forms of the form $h_1 + \alpha h_2$, where $\alpha^6 = -9$.

The value of A_p is given by $A_p \equiv \pm\sqrt{\frac{a_{np}}{b_n}\frac{b_{np}}{a_n}} \mod p^2$, whereas the values for $p \equiv 1 \mod 3$ are those already in the table above. From the values in the above table, we compute the following table of A_ps, with no particular order given to the two possible values. In this table, we write e.g., $A_p \equiv 6i \mod 25$ to mean that $A_p^2 \equiv -36 \mod 25$, etc, and ω means $\omega^2 + \omega + 1 \equiv 0 \mod p^2$.

p	5	7	11	13	17	19	23	29	31	37
A_p	$6i$	11ω	$12i$	5ω	$18i$	-19ω	$30i$	$48i$	-13ω	17ω
mod p^2	$-6i$	$11\omega^2$	$-12i$	$5\omega^2$	$-18i$	$-19\omega^2$	$-30i$	$-48i$	$-13\omega^2$	$17\omega^2$

7.5.2. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{9.6^3.3.2^3}$. Basis of $S_3(\Gamma_{9.6^3.3.2^3})$ in terms of $r = q^{1/3}$.

$$h_1(z) = \sqrt[3]{\frac{\eta(\tau)^7\eta(2\tau)^4\eta(3\tau)^{11}}{\eta(6\tau)^4}} = \sum_{n\geq 1} a_n r^n = r - \frac{7}{3}r^4 - \frac{19}{9}r^7 + \frac{193}{81}r^{10} + \frac{2306}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(\tau)^{11}\eta(3\tau)^7\eta(6\tau)^4}{\eta(2\tau)^4}} = \sum_{n\geq 1} b_n r^n = r^2 - \frac{11}{3}r^5 + \frac{23}{9}r^8 - \frac{13}{81}r^{11} + \cdots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$-\frac{19}{9}$	0	$\frac{2306}{243}$	0	$-\frac{151696}{6561}$
b_p	1	0	$-\frac{11}{3}$	0	$-\frac{13}{81}$	0	$-\frac{7130}{729}$	0

First few prime coefficients mod p^2. Notice that these are either zero or the same as in the $\Gamma_{18.6.3^3.1^3}$ case.

Ratios of coefficients, (when all terms are non-zero), all numbers given mod p^2. When $p \equiv 2 \mod 3$, there is a unique cube root mod p^2 of any integer, so the given value of $\sqrt[3]{3}$ is unique. i means the square root of -1.

p	a_{np}/a_n	b_{np}/b_n	a_{np}/b_n	b_{np}/a_n	ω	ω^2	$\sqrt[3]{3}$
5			$3 \quad = 6i \cdot i\sqrt[3]{3}$	$13 \quad = 6i/i\sqrt[3]{3}$			12
7	$36 \quad = 11 \cdot \omega^2$	$2 \quad = 11 \cdot \omega$			18	30	
11			$13 \quad = 12i \cdot i\sqrt[3]{3}$	$82 \quad = 12i/i\sqrt[3]{3}$			9
13	$54 \quad = 5 \cdot \omega^2$	$110 \quad = 5 \cdot \omega$			22	146	
17			$279 \quad = -18i \cdot i\sqrt[3]{3}$	$148 \quad = -18i/i\sqrt[3]{3}$			160
19	$228 \quad = -19 \cdot \omega^2$	$152 \quad = -19 \cdot \omega$			68	292	
23			$130 \quad = -30i \cdot i\sqrt[3]{3}$	$400 \quad = -30i/i\sqrt[3]{3}$			357
29			$296 \quad = 48i \cdot \sqrt[3]{3}$	$515 \quad = 48i/i\sqrt[3]{3}$			134
31	$915 \quad = -13 \cdot \omega^2$	$59 \quad = -13 \cdot \omega$			439	521	
37	$1058 \quad = 17 \cdot \omega^2$	$294 \quad = 17 \cdot \omega$			581	787	
41			$1384 \quad = -30i \cdot i\sqrt[3]{3}$	$869 \quad = -30i/i\sqrt[3]{3}$			1503
43	$1173 \quad = 29 \cdot \omega^2$	$647 \quad = 29 \cdot \omega$			1425	423	
47			$155 \quad = -24i \cdot i\sqrt[3]{3}$	$1906 \quad = -24i/i\sqrt[3]{3}$			1203

The above table indicates that when $p \equiv 1 \mod 3$, we have

$$a_{np} - A_p\omega^2 a_n \equiv 0 \mod p^2 \qquad \text{and} \qquad b_{np} - A_p\omega b_n \equiv 0 \mod p^2$$

for certain A_p, indicating h_1, h_2 is an ASWD-basis in this case.

Note that this relation only hold when terms are non zero. E.g., $b_1 = 0$, so we can't have $b_p + A_p b_1 \equiv 0 \mod p$ for any p with $b_p \neq 0$.

For $p \equiv 2 \mod 3$, the above table indicates that we have

$$\left(a_{np} + i\sqrt[3]{3}b_{np}\right) + iA_p\left(a_n + i\sqrt[3]{3}b_n\right) \equiv 0 \mod p^2$$
$$\left(a_{np} - i\sqrt[3]{3}b_{np}\right) - iA_p\left(a_n - i\sqrt[3]{3}b_n\right) \equiv 0 \mod p^2,$$

so $h_1 + i\sqrt[3]{3}h_2$, $h_1 - i\sqrt[3]{3}h_2$ should be the ASWD-basis in this case. (shouldn't make any difference which cube root of three is taken)

7.6. **Examples associated with newform in** $S_3(\Gamma_0(48), \chi)$..

$$\begin{aligned} f(z) \quad = \quad & q - \sqrt{-2}q^2 - 2q^4 + 3\sqrt{-2}q^5 - 7q^7 + 2\sqrt{-2}q^8 + 6q^{10} - 3\sqrt{-2}q^{11} + 5q^{13} \\ & +7\sqrt{-2}q^{14} + 4q^{16} - 18\sqrt{-2}q^{17} + 17q^{19} - 6\sqrt{-2}q^{20} - 6q^{22} - 6\sqrt{-2}q^{23} \\ & +7q^{25} - 5\sqrt{-2}q^{26} + 14q^{28} - 39\sqrt{-2}q^{29} + 59q^{31} - 4\sqrt{-2}q^{32} - 36q^{34} + \cdots \end{aligned}$$

First few coefficients a_p, First few prime coefficients, divided by either 1 or $3\sqrt{-2}$, for easy readability

p	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
a_p		-7		5		17			59	-19		47				-4	-46
$\frac{a_p}{3\sqrt{-2}}$	1		-1		-6		-2	-13			13		-19	9	-5		

7.6.1. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{9.6^4.1^3}$. Basis of $S_3(\Gamma_{9.6^4.1^3})$

$$h_1(z) = \sqrt[3]{\frac{\eta(z)^{13}\eta(6z)^{14}}{\eta(2z)^2\eta(3z)^7}} = \sum_{n\geq 1} a_n q^n = q - \frac{13}{3}q^2 + \frac{32}{9}q^3 + \frac{670}{81}q^4 - \frac{3577}{243}q^5 + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(z)^{14}\eta(6z)^{13}}{\eta(2z)^7\eta(3z)^2}} = \sum_{n\geq 1} b_n q^n = q - \frac{14}{3}q^2 + \frac{56}{9}q^3 - \frac{58}{81}q^4 + \frac{266}{243}q^5 + \ldots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	$-\frac{13}{3}$	$\frac{32}{9}$	$-\frac{3577}{243}$	$\frac{38780}{6561}$	$\frac{97488844}{4782969}$	$-\frac{198000616}{129140163}$	$\frac{1030071452831}{31381059609}$	$-\frac{91038813695632}{2541865828329}$
b_p	$-\frac{14}{3}$	$\frac{56}{9}$	$\frac{266}{243}$	$-\frac{1036}{6561}$	$\frac{24235144}{4782969}$	$-\frac{2216727472}{129140163}$	$-\frac{894269035558}{31381059609}$	$\frac{97467805305080}{2541865828329}$

p	$\frac{a_{np}}{a_n}$	$\frac{b_{np}}{b_n}$	$\frac{a_{np}}{b_n}$	$\frac{b_{np}}{a_n}$	mod p^2	$(\frac{a_{np}}{a_n}/\frac{b_{np}}{b_n})^3$	$\frac{a_{np}}{a_n}\frac{b_{np}}{b_n}$	$(\frac{a_{np}}{b_n}/\frac{b_{np}}{a_n})^6$	$\frac{a_{np}}{b_n}\frac{b_{np}}{a_n}$
5			11	12				4	-18
7	35	21				1	0		
11			94	41				75	-18
13	54	110				1	5^2		
17			10	282				69	$-18\cdot 6^2$
19	271	73				1	17^2		
23			503	369				522	$-18\cdot 2^2$
29			661	101				724	$-18\cdot 13^2$
31	948	915				1	59^2		
37	106	1282				1	19^2		
41			1463	1587				1656	$-18\cdot 13^2$
43	1391	411				1	47^2		
47			2117	887				519	$-18\cdot 19^2$

7.6.2. *Atkin Swinnerton-Dyer congruences for* $\Gamma_{18.3^4.2^3}$. Basis of $S_3(\Gamma_{18.3^4.2^3})$, in terms of $r = q^{1/3}$:

$$h_1(z) = \sqrt[3]{\frac{\eta(2\tau)^{13}\eta(3\tau)^{14}}{\eta(6\tau)^7\eta(\tau)^2}} = \sum_{n\geq 1} a_n r^n = r + \frac{2}{3}r^4 - \frac{28}{9}r^7 - \frac{482}{81}r^{10} - \frac{736}{243}r^{13} + \cdots$$

$$h_2(z) = \sqrt[3]{\frac{\eta(2\tau)^{14}\eta(3\tau)^{13}}{\eta(6\tau)^2\eta(\tau)^7}} = \sum_{n\geq 1} b_n q^n = r^2 + \frac{7}{3}r^5 + \frac{14}{9}r^8 - \frac{148}{81}r^{11} - \frac{1708}{243}r^{14} + \cdots$$

First few prime coefficients:

p	2	3	5	7	11	13	17	19
a_p	0	0	0	$-\frac{28}{9}$	0	$-\frac{736}{243}$	0	$\frac{120680}{6561}$
b_p	1	0	$\frac{7}{3}$	0	$-\frac{148}{81}$	0	$-\frac{4529}{729}$	0

p	a_{np}/a_n	b_{np}/b_n	a_{np}/b_n	b_{np}/a_n	ω	ω^2	$\sqrt[3]{3}$
5			$3 \quad = -1 \cdot 6\sqrt[3]{3}$	$19 \quad = 1 \cdot 3/\sqrt[3]{3}$			12
7	$35 \quad = -7 \cdot \omega^2$	$21 \quad = -7 \cdot \omega$			18	30	
11			$54 \quad = 1 \cdot 6\sqrt[3]{3}$	$40 \quad = -1 \cdot 3/\sqrt[3]{3}$			9
13	$54 \quad = 5 \cdot \omega^2$	$110 \quad = 5 \cdot \omega$			22	146	
17			$269 \quad = 6 \cdot 6\sqrt[3]{3}$	$148 \quad = -6 \cdot 3/\sqrt[3]{3}$			160
19	$271 \quad = 17 \cdot \omega^2$	$73 \quad = 17 \cdot \omega$			68	292	
23			$52 \quad = 2 \cdot 6\sqrt[3]{3}$	$80 \quad = -2 \cdot 3/\sqrt[3]{3}$			357
29			$360 \quad = 13 \cdot 6\sqrt[3]{3}$	$370 \quad = -13 \cdot 3/\sqrt[3]{3}$			134
31	$948 \quad = 59 \cdot \omega^2$	$915 \quad = 59 \cdot \omega$			439	521	
37	$106 \quad = -19 \cdot \omega^2$	$1282 \quad = -19 \cdot \omega$			581	787	
41			$436 \quad = -13 \cdot 6\sqrt[3]{3}$	$47 \quad = 13 \cdot 3/\sqrt[3]{3}$			1503
43	$1391 \quad = 47 \cdot \omega^2$	$411 \quad = 47 \cdot \omega$			1425	423	
47			$184 \quad = 19 \cdot 6\sqrt[3]{3}$	$661 \quad = -19 \cdot 3/\sqrt[3]{3}$			1203

When $p \equiv 1 \mod 3$, we see the ASWD-basis should be h_1, h_2.

For $p \equiv 2 \mod 3$, the congruences (which only hold when all terms are non-zero)

$$a_{np}/b_p \equiv -\alpha_p \cdot 6\sqrt[3]{3} \qquad \text{and} \qquad b_{np}/a_p \equiv \alpha_p \cdot 3/\sqrt[3]{3}$$

should be rewritten in terms of u, where $u^2 = -2$, writing $-6 = 3u \cdot u$, so we have

$$a_{np}/b_p \equiv \alpha_p 3u \cdot u\sqrt[3]{3} \qquad \text{and} \qquad b_{np}/a_p \equiv \alpha_p 3u/u\sqrt[3]{3}.$$

These imply that $a_{np} \equiv \alpha_p 3u \cdot u\sqrt[3]{3} b_p \qquad \text{and} \qquad u\sqrt[3]{3} b_{np} \equiv \alpha_p 3u a_p$. so

$$a_{np} + u\sqrt[3]{3} b_{np} \equiv \alpha_p 3u(\cdot u\sqrt[3]{3} b_p + a_p),$$

which holds for u replaced with $-u$, so the ASWD-basis should be $h_1 \pm \sqrt{-2}\sqrt[2]{3} h_2$.

References

[ALL05] Atkin, A. O. L.; Li, Wen-Ching Winnie; Long, Ling *On Atkin and Swinnerton-Dyer congruence relations. II.* Math. Ann. 340 (2008), no. 2, 335–358.

[ASwD71] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Modular forms on non-congruence subgroups*, Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1971, pp. 1–25.

[Beau82] A. Beauville, *Les familles stable de courbes elliptiques sur* $\mathbf{P}^1$ *admettant quatre fibres singulières*, C.R. Acad. Sci. **294** (1982), 657-660.

[CN79] J. H. Conway and S. P. Norton, *Monstrous moonshine.* Bull. London Math. Soc. **11** (1979), no. 3, 308–339.

[Kul91] R. S. Kulkarni, *An arithmetic-geometric method in the study of the subgroups of the modular group.* Amer. J. Math. **113** (1991), no. 6, 1053–1133

[LLY03] Li, Wen-Ching Winnie; Long, Ling; Yang, Zifeng *On Atkin-Swinnerton-Dyer congruence relations.* J. Number Theory 113 (2005), no. 1, 117–148.

[LLY05] Li, Wen-Ching Winnie; Long, Ling; Yang, Zifeng *Modular forms for noncongruence subgroups.* Q. J. Pure Appl. Math. 1 (2005), no. 1, 205–221.

[LL] Long, Ling. *On Atkin-Swinnerton-Dyer congruence relations(3)*, preprint, arXiv:math/0701310

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I: The user language*, 1997, http://www.maths.usyd.edu.au:8000/u/magma/, pp. 235–265.

[M96] Y. Martin, *Multiplicative η-quotients*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4825–4856; MR1376550 (97d:11070)

[Pari04] PARI/GP, version 2.1.5, Bordeaux, 2004, http://pari.math.u-bordeaux.fr/.

[Sch85i] A. J. Scholl, *A trace formula for F-crystals.* Invent. Math., **79** (1985), 31-48.

[Sch85ii] —–, *Modular forms and deRham cohomology; Atkin-Swinnerton-Dyer congruences.* Invent. Math., **79** (1985), 49-77.

[Sch87] —–, *Modular forms on noncongruence subgroups.* Séminaire de Théorie des Nombres, Paris 1985-86, 199-206, Progr. Math. 71, Birkhäuser, Boston, MA, 1987.

[Sch88] —–, *The l-adic representations attached to a certain non-congruence subgroup*, J. Reine Angew. Math. **392** (1988), 1–15.

[Sch93] —–, *The l-adic representations attached to non-congruence subgroups II*, preprint, 1993.

[Sch97] —–, *On the Hecke algebra of a noncongruence subgroup*, Bull. London Math.Soc. **29** (1997), 395-399.

[Seb01] A. Sebbar, *Classification of torsion-free genus zero congruence groups*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2517–2527 (electronic).

[Shi71] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1.

[Stein] W. Stein *The Modular Forms database* http://modular.fas.harvard.edu/Tables

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