NUMERICAL EVIDENCE TOWARD A 2-ADIC EQUIVARIANT "MAIN CONJECTURE"

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1. The conjecture

Let K be a totally real finite Galois extension of $\mathbb Q$ with Galois group G dihedral of order 8, and suppose that $\sqrt{2}$ is not in K. Fix a finite set S of primes of $\mathbb Q$ including $2, \infty$ and all primes that ramify in K. Let C be the cyclic subgroup of G of order 4 and F the fixed field of C acting on K. Fix a 2-adic unit $u \equiv 5 \mod 8\mathbb{Z}_2$.

Write $L_F(s,\chi)$ for the 2-adic L-functions, normalized as in [W], of the 2-adic characters χ of C or, equivalently by class field theory, of the corresponding 2-adic primitive ray class characters. We always work with their S-truncated forms

$$L_{F,S}(s,\chi) = L_F(s,\chi) \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})} \langle N(\mathfrak{p}) \rangle^{1-s} \right)$$

where \mathfrak{p} runs through all primes of F above $S\setminus\{2,\infty\}$, and $\langle \rangle : \mathbb{Z}_2^{\times} \to 1+4\mathbb{Z}_2$ is the unique function with $\langle x\rangle x^{-1} \in \{-1,1\}$ for all x. Now our interest is in the 2-adic function

$$f_1(s) = \frac{\rho_{F,S}\log(u)}{8(u^{1-s} - 1)} + \frac{1}{8} \left(L_{F,S}(s, 1) + L_{F,S}(s, \beta^2) - 2L_{F,S}(s, \beta) \right)$$

where β is a faithful irreducible 2-adic character of C and

$$\rho_{F,S} = \lim_{s \to 1} (s-1) L_{F,S}(s,1).$$

It follows from known results that $\frac{1}{2}\rho_{F,S} \in \mathbb{Z}_2$ and that $f_1(s)$ is an *Iwasawa* analytic function of $s \in \mathbb{Z}_2$, in the sense of [R]. This means that there is a unique power series $F_1(T) \in \mathbb{Z}_2[[T]]$ so that

$$F_1(u^n - 1) = f_1(1 - n)$$
 for $n = 1, 2, 3, \dots$

The conjecture we want to test is

Conjecture 1.

$$\frac{1}{2}\rho_{F,S} \in 4\mathbb{Z}_2$$
 and $F_1(T) \in 4\mathbb{Z}_2[[T]].$

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Testing the conjecture amounts to calculating $\frac{1}{2}\rho_{F,S}$ and (many of) the power series coefficients of

$$F_1(T) = \sum_{j=1}^{\infty} x_j T^{j-1}$$

modulo $4\mathbb{Z}_2$. Were the conjecture false we would expect to find a counterexample in this way.

The idea of the calculation is, roughly, to express the coefficients of the power series $F_1(T)$ as integrals over suitable 2-adic continuous functions with respect to the measures used to construct the 2-adic L-functions.

The conjecture has been tested for 60 fields K determined by the size of their discriminant and the splitting of 2 in the field F. For this purpose, it is convenient to replace the datum K by F together with the ray class characters of F which determine K (cf §5). A description of the results is in §6: they are affirmative.

Where does $f_1(s)$ come from? It is an example which arises by attempting to refine the Main Conjecture of Iwasawa theory. This connection will be discussed next in order to prove that $F_1(T)$ is in $\mathbb{Z}_2[[T]]$.

2. The motivation

The Main Conjecture of classical Iwasawa theory was proved by Wiles [W] for odd prime numbers ℓ . More recently [RW2], an equivariant "main conjecture" has been proposed, which would both generalize and refine the classical one for the same ℓ . When a certain μ -invariant vanishes, as is expected for odd ℓ (by a conjecture of Iwasawa), this equivariant "main conjecture", up to its uniqueness assertion, depends only on properties of ℓ -adic L-functions, by Theorem A of [RW3].

The point is that it is possible to numerically test this Theorem A property of ℓ -adic L-functions, at least in simple special cases when it may be expressed in terms of congruences and the special values of these L-functions can be computed. The conjecture of §1 is perhaps the simplest non-abelian example when this happens, but with the price of taking $\ell=2$. Although there are some uncertainties about the formulation of the "main conjecture" for $\ell=2$, partly because [W] no longer applies, it seems clearer what the 2-adic analogue of the Theorem A properties of L-functions should be, in view of their "extra" 2-power divisibilities [DR].

More precisely, let $L_{k,S} \in \operatorname{Hom}^*(R_\ell(G_\infty), \mathcal{Q}^c(\Gamma_k)^\times)$ be the "power series" valued function of ℓ -adic characters χ of $G_\infty = \operatorname{Gal}(K_\infty/k)$ defined in §4 of [RW2]. This is made from the values of ℓ -adic L-functions by viewing them as a quotient of Iwasawa analytic functions, by the proof of Proposition 11 in [RW2]. When $\ell \neq 2$, the vanishing of the μ -invariant mentioned above means precisely that the coefficients of these power series have no nontrivial common divisor; and the Theorem A property of L-functions is that then $L_{k,S}$ is in $\operatorname{Det}(K_1(\Lambda(G_\infty)_{\bullet}))$ (see next section for precise definitions).

When $\ell = 2$, we can still form $L_{k,S}$, but now its values at characters χ of degree 1 have numerators divisible by $2^{[k:\mathbb{Q}]}$, because of (4.8), (4.9) of [R]. Define

$$\widetilde{L}_{k,S}(\chi) = 2^{-[k:\mathbb{Q}]\chi(1)} L_{k,S}(\chi)$$

for all 2-adic characters χ of G_{∞} , so that the deflation and restriction property of Proposition 12 of [RW2] are maintained. Then the analogous coprimality condition on coefficients of numerator, denominator of the values $\widetilde{L}_{k,S}(\chi)$ will be referred to as vanishing of the $\widetilde{\mu}$ -invariant of K_{∞}/k : the Theorem A property we want to test is therefore

Conjecture 2.

$$\widetilde{L}_{k,S}$$
 is in $\operatorname{Det}\left(K_1(\Lambda(G_\infty)_{\bullet})\right)$.

Remark 2.1. a) When Conjecture 2 holds, then $\widetilde{L}_{k,S}(\chi)$ is in $\Lambda^c(\Gamma_k)^{\times}_{\bullet}$ for all $\chi \in R_2(G_{\infty})$, implying the vanishing of the $\widetilde{\mu}$ -invariant of K_{∞}/k .

b) For $\ell \neq 2$, some cases of the equivariant "main conjecture" have recently been proved ([RW]).

3. Interpreting Conjecture 2 as a congruence

We now specialize to the situation of §1, so use the notation of its first paragraph, in order to exhibit a congruence equivalent to Conjecture 2 (see Figure 1).

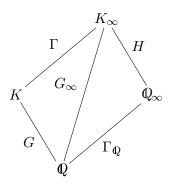


FIGURE 1.

Let \mathbb{Q}_{∞} be the cyclotomic \mathbb{Z}_2 -extension of \mathbb{Q} , i.e. the maximal totally real subfield of the field obtained from \mathbb{Q} by adjoining all 2-power roots of unity, and set $\Gamma_{\mathbb{Q}} = \operatorname{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q}) \simeq \mathbb{Z}_2$. Let $K_{\infty} = K\mathbb{Q}_{\infty}$, noting that $K \cap \mathbb{Q}_{\infty} = \mathbb{Q}$ follows from $\sqrt{2} \notin K$, and set $G_{\infty} = \operatorname{Gal}(K_{\infty}/\mathbb{Q})$. Defining $\Gamma = \ker(G_{\infty} \to G)$, $H = \ker(G_{\infty} \to \Gamma_{\mathbb{Q}})$, we now have $H \hookrightarrow G_{\infty} \twoheadrightarrow \Gamma_{\mathbb{Q}}$ in the notation of [RW2].

Since $G_{\infty} = \Gamma \times H$ with $\Gamma \simeq \Gamma_{\mathbb{Q}}$ and $H \simeq G$ dihedral of order 8 we can understand the structure of

$$\Lambda(G_{\infty})_{\bullet} = \Lambda(\Gamma)_{\bullet} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[H] = \Lambda(\Gamma)_{\bullet}[H]$$

where \bullet means "invert all elements of $\Lambda(\Gamma) \setminus 2\Lambda(\Gamma)$."

Namely, choose σ, τ in G so that $C = \langle \tau \rangle$ with $\sigma^2 = 1$, $\sigma \tau \sigma^{-1} = \tau^{-1}$ and extend them to K_{∞} , with trivial action on \mathbb{Q}_{∞} , to get s, t respectively. Then the abelianization of H is $H^{ab} = H/\langle t^2 \rangle$ and we get a pullback diagram

$$\Lambda(G_{\infty})_{\bullet} = \Lambda(\Gamma)_{\bullet}[H] \longrightarrow (\Lambda(\Gamma)_{\bullet}(\zeta_{4})) * \langle s \rangle$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Gamma(G_{\infty}^{ab})_{\bullet} = \Lambda(\Gamma)_{\bullet}[H^{ab}] \longrightarrow \Lambda(\Gamma)_{\bullet}[H^{ab}]/2\Lambda(\Gamma)_{\bullet}[H^{ab}]$$

where the upper right term is the crossed product order with $\Lambda(\Gamma)_{\bullet}$ -basis $1, \zeta_4, \widetilde{s}, \zeta_4 \widetilde{s}$ with $\zeta_4^2 = -1$, $\widetilde{s}^2 = 1$, $\widetilde{s}\zeta_4 = \zeta_4^{-1}\widetilde{s} = -\zeta_4 \widetilde{s}$ and the top map takes t, s to ζ_4, \widetilde{s} respectively, the right map takes ζ_4, \widetilde{s} to t^{ab}, s^{ab} . This diagram originates in the pullback diagram for the cyclic group $\langle t \rangle$ of order 4, then going to the dihedral group ring $\mathbb{Z}_2[H]$ by incorporating the action of s, and finally applying $\Lambda(\Gamma)_{\bullet} \otimes_{\mathbb{Z}_2} -$.

We now turn to getting the first version of our congruence in terms of the pullback diagram above. This is possible since $R^{\times} \to K_1(R)$ is surjective for all the rings considered there. We also simplify notation a little by setting $\mathfrak{A} = (\Lambda(\Gamma)_{\bullet}(\zeta_4)) * \langle s \rangle$ and writing $\widetilde{L}_{k,S}$ as $\widetilde{L}_{K_{\infty}/k}$, because we will now have to vary the fields and S is fixed anyway. The dihedral group G has 4 degree 1 irreducible characters $1, \eta, \nu, \eta \nu$ with $\eta(\tau) = 1$, $\nu(\sigma) = 1$ and a unique degree 2 irreducible α , which we view as characters of G_{∞} by inflation.

Proposition 3.1. Let K^{ab}_{∞} be the fixed field of $\langle t^2 \rangle$, hence $\operatorname{Gal}(K^{ab}_{\infty}/\mathbb{Q}) = G^{ab}_{\infty}$. Then

- a) $\widetilde{L}_{K_{\infty}^{ab}/\mathbb{Q}} = \operatorname{Det}(\widetilde{\Theta}^{ab}) \text{ for some } \widetilde{\Theta}^{ab} \in \Lambda(G_{\infty}^{ab})^{\times}_{\bullet}$
- b) $\widetilde{L}_{K_{\infty}/\mathbb{Q}} \in \operatorname{Det}(K_1(\Lambda(G_{\infty})_{\bullet}))$ if, and only if, any $y \in \mathfrak{A}$ mapping to $\widetilde{\Theta}^{ab} \mod 2$ in $\Lambda(G_{\infty}^{ab})_{\bullet}/2\Lambda(G_{\infty}^{ab})_{\bullet}$ has

$$nr(y) \equiv \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha) \mod 4\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$$

where nr is the reduced norm of (the total ring of fractions of) \mathfrak{A} to its centre $\Lambda(\Gamma)_{\bullet}$ and we identify $\Lambda(\Gamma)_{\bullet}$ with $\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$ via $\Gamma \stackrel{\simeq}{\to} \Gamma_{\mathbb{Q}}$.

Proof. a) The vanishing of $\widetilde{\mu}$ for K_{∞}/\mathbb{Q} , in the sense of §2, is known by [FW], i.e. $\widetilde{L}_{K_{\infty}^{ab}/\mathbb{Q}}(\chi)$ is a unit in $\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$ for all 2-adic characters χ of G_{∞}^{ab} . By the proof of Theorem 9 in [RW3] we have $L_{K_{\infty}^{ab}/\mathbb{Q}} = \mathrm{Det}(\lambda)$ with $\lambda \in \Lambda(G_{\infty}^{ab})_{\bullet}$ the pseudomeasure of Serre. The point is then that $\lambda = 2\widetilde{\Theta}^{ab}$ with $\widetilde{\Theta}^{ab} \in \Lambda(G_{\infty}^{ab})_{\bullet}$, which follows from Theorem 3.1b) of [R], because of Theorem 4.1 (loc.cit.) and the relation between λ and μ_c discussed just after it. Then $\widetilde{L}_{K_{\infty}^{ab}/\mathbb{Q}} = \mathrm{Det}(\widetilde{\Theta}^{ab})$ and now the proof of the Corollary to Theorem 9 in [RW3] shows that $\widetilde{\Theta}^{ab}$ is a unit of $\Lambda(G_{\infty}^{ab})$.

b) Claim: $nr(1+2\mathfrak{A})=1+4\Lambda(\Gamma)_{\bullet}$.

Proof of the claim. If $x = a1 + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ with $a, b, c, d \in \Lambda(\Gamma)_{\bullet}$, one computes $nr(x) = (a^2 + b^2) - (c^2 + d^2)$ from which $nr(1 + 2\mathfrak{A}) \subseteq 1 + 4\Lambda(\Gamma)_{\bullet}$; equality follows from $nr((1 + 2a) + 2a\widetilde{s}) = (1 + 2a)^2 - (2a)^2 = 1 + 4a$ for $a \in \Lambda(\Gamma)_{\bullet}$.

Suppose first that the congruence for $\widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha)$ holds. Start with $\widetilde{\Theta}^{ab}$ from a) in the lower left corner of the pullback square and map it to $\widetilde{\Theta}^{ab}$ mod 2 in the lower right corner. Choosing any $y_0 \in \mathfrak{A}$ mapping to $\widetilde{\Theta}^{ab}$ mod 2, we note that $y_0 \in \mathfrak{A}^{\times}$ because the maps in the pullback diagram are ring homomorphisms and the kernel $2\mathfrak{A}$ of the right one is contained in the radical of \mathfrak{A} . Thus $nr(y_0) \in \Lambda(\Gamma)^{\times}_{\bullet}$ has $nr(y_0)^{-1}\widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha) \in 1 + 4\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$ by the congruence, hence, by the Claim, $nr(y_0)^{-1}\widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha) = nr(z)$, $z \in 1 + 2\mathfrak{A}$. So $y_1 = y_0 z$ is another lift of $\widetilde{\Theta}^{ab} \mod 2$ and $nr(y_1) = \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha)$. By the pullback diagram we get $Y \in \Lambda(G_{\infty})^{\times}_{\bullet}$ which maps to $\widetilde{\Theta}^{ab}$ and y_1 , where $nr(y_1) = \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha)$.

It follows that $\operatorname{Det} Y = \widetilde{L}_{K_{\infty}/\mathbb{Q}}$. To see this we check that their values agree at every irreducible character χ of G_{∞} ; it even suffices to check it on the characters $1, \eta, \nu, \eta\nu, \alpha$ of G by Theorem 8 and Proposition 11 of [RW2], because every irreducible character of G_{∞} is obtained from these by multiplying by a character of type W. It works for the characters $1, \eta, \nu, \eta\nu$ of G_{∞}^{ab} by Proposition 12, 1b) (loc.cit.) since $\operatorname{defl}(Y) = \widetilde{\Theta}^{ab}$ and $\operatorname{Det} \widetilde{\Theta}^{ab} = \widetilde{L}_{K_{\infty}^{ab}/\mathbb{Q}}$ by a). Finally, $(\operatorname{Det} Y)(\alpha) = j_{\alpha}(nr(Y)) = nr(y_1) = \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha)$ by the commutative triangle before Theorem 8 (loc.cit.), the definition of j_{α} , and $G_{\infty} = \Gamma \times H$.

The converse depends on related ingredients. More precisely, $\widetilde{L}_{K_{\infty}/\mathbb{Q}} \in \text{Det } K_1\big((\Lambda G_{\infty})_{\bullet}\big)$ implies $\widetilde{L}_{K_{\infty}/\mathbb{Q}} = \text{Det } Y$ with $Y \in (\Lambda G_{\infty})_{\bullet}^{\times}$ by surjectivity of $(\Lambda G_{\infty})_{\bullet}^{\times} \to K_1\big((\Lambda G_{\infty})_{\bullet}\big)$. Since $(\Lambda G_{\infty}^{ab})_{\bullet}^{\times} \to K_1\big((\Lambda G_{\infty}^{ab})_{\bullet}\big)$ is an isomorphism, we get defl $Y = \widetilde{\Theta}^{ab}$ in $\Lambda(G_{\infty}^{ab})^{\times}$. Letting $y_1 \in \mathfrak{A}^{\times}$ be the image of Y in the pullback diagram, it follows that $nr(y_1) = \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha)$ and that y_1 maps to $\widetilde{\Theta}^{ab} \mod 2$ in $\Lambda(G_{\infty}^{ab})_{\bullet}/2\Lambda(G_{\infty}^{ab})_{\bullet}$. Given any y as in b), then $y_1^{-1}y$ maps to 1 hence is in $1+2\mathfrak{A}$ and our congruence follows from the Claim on applying nr.

4. Rewriting the congruence in testable form

Set
$$F_0 = \frac{\widetilde{L}_{K_{\infty}/F,S}(1) + \widetilde{L}_{K_{\infty}/F,S}(\beta^2)}{2} - \widetilde{L}_{K_{\infty}/F,S}(\beta)$$
.

Proposition 4.1. a) F_0 is in $\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$

b)
$$\widetilde{L}_{K_{\infty}/\mathbb{Q}} \in \text{Det } K_1(\Lambda(G_{\infty})_{\bullet}) \text{ if, and only if, } F_0 \in 4\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$$

Proof. Note that $\operatorname{ind}_C^G 1_C = 1_G + \eta$, $\operatorname{ind}_C^G \beta^2 = \nu + \eta \nu$, $\operatorname{ind}_C^G \beta = \alpha$. When we inflate β to a character of $\operatorname{Gal}(K_\infty/F)$ then $\operatorname{ind}_{\operatorname{Gal}(K_\infty/F)}^{G_\infty}\beta = \alpha$ with α inflated to G_∞ , etc.

By Proposition 3.1 of the previous section we can write $\widetilde{L}_{K^{ab}_{\infty}/\mathbb{Q}} = \operatorname{Det}(\widetilde{\Theta}^{ab})$ with

$$\widetilde{\Theta}^{ab} = a + bt^{ab} + cs^{ab} + ds^{ab}t^{ab}$$

for some a, b, c, d in $\Lambda(\Gamma)_{\bullet}$. It follows that

$$\begin{split} \widetilde{L}_{K_{\infty}/\mathbb{Q}}(1) &= a+b+c+d \\ \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\eta) &= a+b-c-d \\ \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\nu) &= a-b+c-d \\ \widetilde{L}_{K_{\infty}/\mathbb{Q}}(\eta\nu) &= a-b-c+d. \end{split}$$

Form $y = a + b\zeta_4 + c\widetilde{s} + d\zeta_4\widetilde{s}$ in $(\Lambda(\Gamma)_{\bullet}(\zeta_4)) * \langle s \rangle$. By the computation in the Claim in the proof of Proposition 3.1, we have

$$\begin{split} nr(y) &= (a+c)(a-c) + (b+d)(b-d) \\ &= \frac{\widetilde{L}_{\mathbb{Q}}(1) + \widetilde{L}_{\mathbb{Q}}(\nu)}{2} \, \frac{\widetilde{L}_{\mathbb{Q}}(\eta) + \widetilde{L}_{\mathbb{Q}}(\eta\nu)}{2} + \frac{\widetilde{L}_{\mathbb{Q}}(1) - \widetilde{L}_{\mathbb{Q}}(\nu)}{2} \, \frac{\widetilde{L}_{\mathbb{Q}}(\eta) - \widetilde{L}_{\mathbb{Q}}(\eta\nu)}{2} \\ &= \frac{1}{4} \left(\widetilde{L}_{\mathbb{Q}}(1+\eta) + \widetilde{L}_{\mathbb{Q}}(1+\eta\nu) + \widetilde{L}_{\mathbb{Q}}(\nu+\eta) + \widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu) \right) \\ &+ \frac{1}{4} \left(\widetilde{L}_{\mathbb{Q}}(1+\eta) - \widetilde{L}_{\mathbb{Q}}(1+\eta\nu) - \widetilde{L}_{\mathbb{Q}}(\nu+\eta) + \widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu) \right) \\ &= \frac{\widetilde{L}_{\mathbb{Q}}(1+\eta) + \widetilde{L}_{\mathbb{Q}}(\nu+\eta\nu)}{2} = \frac{\widetilde{L}_{F}(1) + \widetilde{L}_{F}(\beta^{2})}{2} \, , \end{split}$$

because

$$\widetilde{L}_{K_{\infty}/\mathbb{Q}}(\operatorname{ind}_{\mathrm{Gal}(K_{\infty}/F)}^{G_{\infty}}\chi)=\widetilde{L}_{K_{\infty}/F}(\chi)$$

for all characters χ of $\operatorname{Gal}(K_{\infty}/F)$. Thus also $\widetilde{L}_{K_{\infty}/\mathbb{Q}}(\alpha) = \widetilde{L}_{K_{\infty}/F}(\beta)$, so we now have shown that

$$F_0 = nr(y) - L_{K_{\infty}}/\mathbf{0}(\alpha)$$

proving a), since $\widetilde{L}_{K_{\infty}/F}(\beta) \in (\Lambda\Gamma_F)_{\bullet}$ by §2, as β has degree 1.

Moreover, the image of y under the right arrow of the pullback diagram of §3 equals $\widetilde{\Theta}^{ab} \mod 2$, by construction, hence b) follows directly from Proposition 3.1b).

Remark 4.2. Considering F_0 in $\Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$, instead of its natural home $\Lambda(\Gamma_F)_{\bullet}$, is done to be consistent with the identification in b) of Proposition 3.1, via the natural isomorphisms $\Gamma \to \Gamma_F \to \Gamma_{\mathbb{Q}}$: this is the sense in which $L_{K_{\infty}/\mathbb{Q}}(\alpha) = L_{K_{\infty}/F}(\beta)$.

The congruence $F_0 \equiv 0 \mod 4\Lambda(\Gamma_{\mathbb Q})_{\bullet}$ can now be put in the more testable form of Conjecture 1. Let $\gamma_{\mathbb Q}$ be the generator of $\Gamma_{\mathbb Q}$ which, when extended to $\mathbb Q(\sqrt{-1})$ as the identity, acts on all 2-power roots of unity in $\mathbb Q_{\infty}(\sqrt{-1})$ by raising them to the $u^{\rm th}$ power, where $u \equiv 5 \mod 8\mathbb Z_2$ as fixed before. Then the Iwasawa isomorphism $\Lambda(\Gamma_{\mathbb Q}) \simeq \mathbb Z_2[[T]]$, under which $\gamma_{\mathbb Q}-1$ corresponds

to T, makes $F_0 \in \Lambda(\Gamma_{\mathbb{Q}})_{\bullet}$ correspond to some $F_0(T) \in \mathbb{Z}_1[[T]]_{\bullet}$ and the congruence of Proposition 4.1b) to

$$F_0(T) \equiv 0 \mod 4\mathbb{Z}_2[[T]]$$

Since β is an abelian character, we know that $\widetilde{L}_{F,S}(\beta^2)$, $\widetilde{L}_{F,S}(\beta)$ correspond to elements of $\mathbb{Z}_2[[T]]$, not just $\mathbb{Z}_2[[T]]_{\bullet}$ (cf. §4 of [RW2]), and $\widetilde{L}_F(1)$ to one of $T^{-1}\mathbb{Z}_2[[T]]$. We thus have

$$F_0(T) = \frac{x_0}{T} + \sum_{j=1}^{\infty} x_j T^{j-1}$$

with $x_j \in \mathbb{Z}_2$ for all $j \geq 0$.

By the interpolation definition of $(\widetilde{L}_{F,S}(\beta^i))(T)$ (cf §4 of [R]), it follows that

$$F_0(u^s-1) = \frac{1}{2} \left(\frac{L_{F,S}(1-s,1)}{4} + \frac{L_{F,S}(1-s,\beta^2)}{4} - 2 \frac{L_{F,S}(1-s,\beta)}{4} \right) = f_0(1-s).$$

This implies

$$x_0 = -\frac{\rho_{F,S}\log(u)}{8} ,$$

because the left side is

$$\lim_{T \to 0} TF_0(T) = \lim_{s \to 1} \frac{u^{1-s} - 1}{s - 1} (s - 1) f_0(s) = -\log(u) \lim_{s \to 1} (s - 1) \frac{L_{F,S}(s, 1)}{8}$$

as required. Note that $u \equiv 5 \mod 8$ implies that $\frac{\log(u)}{4}$ is a 2-adic unit, hence $\frac{1}{2} \rho_{F,S} \in \mathbb{Z}_2$ is in $4\mathbb{Z}_2$ if, and only if, $x_0 \in 4\mathbb{Z}_2$.

Define
$$F_1(T) = F_0(T) - x_0 T^{-1} = \sum_{j=1}^{\infty} x_j T^{j-1} \in \mathbb{Z}_2[[T]]$$
. It follows that

$$F_1(u^s - 1) = -\frac{x_0}{u^s - 1} + F_0(u^s - 1) = \frac{\rho_{F,S} \log(u)}{8(u^s - 1)} + f_0(1 - s)$$

which is $f_1(1-s)$, with f_1 as in §1, thus reconciling the notation $F_1(T)$ here with that there. Thus Conjecture 1 of §1 is equivalent to Conjecture 2 of §2 for the special case K_{∞}/\mathbb{Q} of §1.

5. Testing Conjecture 1

Let χ be a 2-adic character of the Galois group C of K/F and let \mathfrak{f} be the conductor of K/F. By class field theory, we view χ as a map on the group of ideals relatively prime to \mathfrak{f} . Fix a prime ideal \mathfrak{c} not dividing \mathfrak{f} . For \mathfrak{a} , a fractional ideal relatively prime to \mathfrak{c} and \mathfrak{f} , let $\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s)$ denote the associated 2-adic twisted partial zeta function [PCN]. Thus, we have

$$L_{F,S}(s,\chi) = \frac{1}{\chi(\mathfrak{c})\langle N\mathfrak{c}\rangle^{1-s}-1} \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}}\langle N\mathfrak{p}\rangle^{1-s}\right) \sum_{\sigma \in G} \chi(\sigma)^{-1} \mathcal{Z}_{\mathfrak{f}}(\mathfrak{a}_{\sigma}^{-1},\mathfrak{c};s)$$

where \mathfrak{p} runs through the prime ideals of F in S not dividing $2\mathfrak{f}$, \mathfrak{a}_{σ} is a (fixed) integral ideal coprime with $2\mathfrak{f}\mathfrak{c}$ whose Artin symbol is σ

Denote the ring of integers of F by \mathcal{O}_F and let $\gamma \in \mathcal{O}_F$ be such that $\mathcal{O}_F = \mathbb{Z} + \gamma \mathbb{Z}$. In [Rob] (see also [BBJR] for a slightly different presentation), it is shown that the function $\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s)$ is defined by the following integral

$$\mathcal{Z}_{\mathfrak{f}}(\mathfrak{a},\mathfrak{c};s) = \int \frac{\langle N\mathfrak{a} N(x_1 + x_2\gamma)\rangle^{1-s}}{N\mathfrak{a} N(x_1 + x_2\gamma)} d\mu_{\mathfrak{a}}(x_1, x_2)$$

where the integration domain is \mathbb{Z}_2^2 , $\langle \rangle$ is extended to \mathbb{Z}_2 by $\langle x \rangle = 0$ if $x \in 2\mathbb{Z}_2$, and the measure $\mu_{\mathfrak{a}}$ is a measure of norm 1 (depending also on γ , \mathfrak{f} and \mathfrak{c}).

Assume now, as we can do without loss of generality, that the ideal \mathfrak{c} is such that $\langle N\mathfrak{c} \rangle \equiv 5 \pmod{8\mathbb{Z}_2}$ and take $u = \langle N\mathfrak{c} \rangle$. For $s \in \mathbb{Z}_2$, we let $t = t(s) = u^s - 1 \in 4\mathbb{Z}_2$, so that $s = \log(1+t)/\log(u)$. For $x \in \mathbb{Z}_2^{\times}$, one can check readily that

$$\langle x \rangle^s = \left(u^{\mathcal{L}(x)} \right)^s = (1 + u^s - 1)^{\mathcal{L}(x)} = \sum_{n \ge 0} \binom{\mathcal{L}(x)}{n} t^n$$

where $\mathcal{L}(x) = \log \langle x \rangle / \log u \in \mathbb{Z}_2$. For $x \in \mathbb{Z}_2^{\times}$, we set

$$L(x;T) = \sum_{n>0} {\binom{\mathcal{L}(x)}{n}} T^n \in \mathbb{Z}_2[[T]]$$

and L(x;T)=0 if $x\in 2\mathbb{Z}_2$. Now, we define

$$R(\mathfrak{a},\mathfrak{c};T) = \int \frac{L(N\mathfrak{a} N(x_1 + x_2\gamma);T)}{N\mathfrak{a} N(x_1 + x_2\gamma)} d\mu_{\mathfrak{a}}(x_1,x_2) \in \mathbb{Z}_2[[T]]$$

$$B(\chi;T) = \chi(\mathfrak{c})(T+1) - 1 \in \mathbb{Z}_2[\chi][T],$$

$$A(\chi;T) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}} L(N\mathfrak{p};T) \right) \sum_{\sigma \in G} \chi(\sigma)^{-1} R(\mathfrak{a}_{\sigma}^{-1},\mathfrak{c};T) \in \mathbb{Z}_{2}[\chi][[T]]$$

where \mathfrak{p} runs through the prime ideals of F in S not dividing $2\mathfrak{f}$.

Proposition 5.1. We have, for all $s \in \mathbb{Z}_2$

$$L_{F,S}(1-s,\chi) = \frac{A(\chi; u^s - 1)}{B(\chi; u^s - 1)}.$$

We now specialize to our situation. For that, we need to make the additional assumption that $\beta^2(\mathfrak{c}) = -1$, so $\beta(\mathfrak{c})$ is a fourth root of unity in \mathbb{Q}_2^c that we will denote by i. Thus, we have

$$B(1;T) = T$$
, $B(\beta;T) = i(T+1) - 1$,
 $B(\beta^2;T) = -T - 2$, $B(\beta^3;T) = -i(T+1) - 1$.

Let $x \mapsto \bar{x}$ be the \mathbb{Q}_2 -automorphism of $\mathbb{Q}_2(i)$ sending i to -i. Then we have $\overline{L_{F,S}(1-s,\beta)} = L_{F,S}(1-s,\beta^3)$ by the expression of $L_{F,S}(s,\chi)$ given at the

beginning of the section since the twisted partial zeta functions have values of \mathbb{Q}_2 and $\bar{\beta} = \beta^3$. And furthermore,

$$L_{F,S}(s,\beta^3) = L_{\mathbb{Q},S}(s,\operatorname{Ind}_C^G(\beta^3)) = L_{\mathbb{Q},S}(s,\operatorname{Ind}_C^G(\beta)) = L_{F,S}(s,\beta).$$

Therefore, by Prop. 5.1, we deduce that

$$A(\beta; u^{s} - 1) + \bar{A}(\beta; u^{s} - 1) = (B(\beta; T) + B(\beta^{3}; T)) L_{F,S}(1 - s, \beta)$$

= $-2L_{F,S}(1 - s, \beta)$.

Since

$$f_1(s) = \frac{\rho_{F,S} \log u}{8(u^{1-s} - 1)} + \frac{1}{8} \left(L_{F,S}(s, 1) + L_{F,S}(s, \beta^2) - 2L_{F,S}(s, \beta) \right)$$

we find that

$$F_1(T) = \frac{\rho_{F,S} \log u}{8T} + \frac{1}{8} \left(\frac{A(1;T)}{T} - \frac{A(\beta^2;T)}{T+2} + A(\beta;T) + \bar{A}(\beta,T) \right)$$

is such that $F_1(u^n - 1) = f_1(1 - n)$ for n = 1, 2, 3, ...

The conjecture that we wish to check states that

$$\frac{1}{2}\rho_{F,S} \in 4\mathbb{Z}_2$$
 and $F_1(T) \in 4\mathbb{Z}_2[[T]].$

Now define $D(T) = 8T(T+2)F_1(T)$, so that

$$D(T) = (T+2) \left(\rho_{F,S} \log u + A(1;T) \right) - TA(\beta^2;T) + T(T+2) \left(A(\beta;T) + \bar{A}(\beta,T) \right).$$

We can now give a final reformulation of the conjecture which is the one that we actually tested.

Conjecture 3.

$$\rho_{F,S} \in 8\mathbb{Z}_2 \quad and \quad D(T) \in 32\mathbb{Z}_2[[T]]$$

The computation of $\rho_{F,S}$ is done using the following formula [Col]

$$\rho_{F,S} = 2 h_F R_F d_F^{-1/2} \prod_{\mathfrak{p}} (1 - 1/N(\mathfrak{p}))$$

where h_F , R_F , d_F are respectively the class number, 2-adic regulator and discriminant of F and \mathfrak{p} runs through all primes of F above 2. Note that although R_F and $d_F^{-1/2}$ are only defined up to sign, the quantity $R_F d_F^{-1/2}$ is uniquely determined in the following way: Let ι be the embedding of F into \mathbb{R} for which $\sqrt{d_F}$ is positive and let ε be the fundamental unit of F such that $\iota(\varepsilon) > 1$. Then for any embedding g of F into \mathbb{Q}_2^c , we have

$$R_F d_F^{-1/2} = \frac{\log_2 g(\varepsilon)}{g(\sqrt{d})}.$$

Now, for the computation of D(T), the only difficult part is the computations of the $R(\mathfrak{a},\mathfrak{c};T)$. The measures $\mu_{\mathfrak{a}}$ are computed explicitly using

the methods of [Rob] (see also [BBJR]), that is we construct a power series $M_{\mathfrak{a}}(X_1, X_2)$ in $\mathbb{Q}_2[X_1, X_2]$ with integral coefficients, such that

$$\int (1+t_1)^{x_1} (1+t_2)^{x_2} d\mu_{\mathfrak{A}}(x_1, x_2) = M_{\mathfrak{A}}(t_1, t_2) \quad \text{ for all } t_1, t_2 \in 2\mathbb{Z}_2.$$

In particular, if f is a continuous function on \mathbb{Z}_2^2 with values in \mathbb{C}_2 and Mahler expansion

$$f(x_1, x_2) = \sum_{n_1, n_2 \ge 0} f_{n_1, n_2} \binom{x_1}{n_1} \binom{x_2}{n_2}$$

then we have

$$\int f(x_1, x_2) \, d\mu_{\mathfrak{A}}(x_1, x_2) = \sum_{n_1, n_2 \ge 0} f_{n_1, n_2} m_{n_1, n_2}$$

where
$$M_{\mathfrak{A}}(X_1, X_2) = \sum_{n_1, n_2 \ge 0} m_{n_1, n_2} X_1^{n_1} X_2^{n_2}$$
.

We compute this way the first few coefficients of the power series $A(\chi;T)$, for $\chi = \beta^j$, j = 0, 1, 2, 3, and then deduce the first coefficients of D(T) to see if they do indeed belong to $32\mathbb{Z}_2[[T]]$. We found that this was indeed always the case; see next section for more details.

To conclude this section, we remark that, in fact, we do not need the above formula to compute $\rho_{F,S}$ since the constant coefficient of A(1;T) is $-\rho_{F,S} \log u$. (This can be seen directly from the expression of x_0 given at the end of Section 4 or using the fact that D(T) has zero constant coefficient since $F_1(T) \in \mathbb{Z}_2[[T]]$.) However, we did compute it using this formula since it then provides a neat way to check that (at least one coefficient of) A(1;T) is correct.

6. The numerical verifications

We have tested the conjecture in 60 examples. The examples are separated in three subcases of 20 examples according to the way 2 decomposes in the quadratic subfield F: ramified, split or inert. In each subcase, the examples are actually the first 20 extensions K/\mathbb{Q} of the suitable form of the smallest discriminant. These are given in the following three tables of Figure 2 where the entries are: the discriminant d_F of F, the conductor \mathfrak{f} of K/F (which is always a rational integer) and the discriminant d_K of K. In each example, we have computed $\rho_{F,S}$ and the first 30 coefficients of D(T) to a precision of at least 2^8 and checked that they satisfy the conjecture.

We now give an example, namely the smallest example for the discriminant of K. We have $F = \mathbb{Q}(\sqrt{145})$ and K is the Hilbert class field of F. The prime 2 is split in F/\mathbb{Q} and the primes above 2 in F are inert in K/F. We compute $\rho_{F,S}$ and find that

$$\rho_{F,S} \equiv 2^7 \pmod{2^8}$$

2 ramified in F				2 inert in F			2 split in F		
d_F	f	d_K	d_F	f	d_K	d_F	f	d_K	
44	3	2732361984	445	1	39213900625	145	1	442050625	
156	2	9475854336	5	21	53603825625	41	5	44152515625	
220	2	37480960000	205	3	143054150625	505	1	65037750625	
12	14	39033114624	221	3	193220905761	689	1	225360027841	
156	4	151613669376	61	5	216341265625	777	1	364488705441	
380	2	333621760000	205	4	452121760000	793	1	395451064801	
152	3	389136420864	221	4	610673479936	17	13	403139914489	
24	11	587761422336	901	1	659020863601	897	1	647395642881	
876	1	588865925376	29	15	895152515625	905	1	670801950625	
220	4	599695360000	1045	1	1192518600625	305	3	700945700625	
444	2	621801639936	5	16	1911029760000	377	3	1636252863921	
12	28	624529833984	109	5	2205596265625	1145	1	1718786550625	
44	12	699484667904	1221	1	2222606887281	145	8	1810639360000	
92	6	835600748544	29	20	2829124000000	305	4	2215334560000	
60	8	849346560000	29	13	3413910296329	1313	1	2972069112961	
44	10	937024000000	205	7	4240407600625	377	4	5171367076096	
12	19	975543388416	149	5	7701318765625	545	3	7146131900625	
12	26	1601419382784	1677	1	7909194404241	17	21	7163272192041	
44	15	1707726240000	21	19	9149529982761	1705	1	8450794350625	
1164	1	1835743170816	341	3	9857006530569	329	3	8 541 047 165 049	

Figure 2.

Using the method of the previous section, we compute the first 30 coefficients of the power series $A(\cdot;T)$ to a 2-adic precision of 2^8 . We get

$$\begin{split} A(1;T) &\equiv 2^2 \left(16T + 57T^3 + 44T^4 + 8T^5 + 40T^6 + 21T^7 + 40T^8 + 30T^9 \right. \\ &+ 16T^{10} + 49T^{11} + 56T^{12} + 29T^{13} + 32T^{14} + 50T^{15} \\ &+ 62T^{16} + 47T^{17} + 48T^{18} + 60T^{19} + 32T^{20} + 16T^{21} \\ &+ 8T^{22} + 21T^{23} + 30T^{24} + 26T^{25} + 2T^{26} + 9T^{27} \\ &+ 56T^{28} + 34T^{29}\right) + O(T^{30}) \pmod{2^8} \end{split}$$

$$\begin{split} A(\beta;T) &\equiv 2^2 \big((28+1124i) + (36+1728i)T + (47+45i)T^2 + (56+153i)T^3 \\ &\quad + (46+154i)T^4 + (56+282i)T^5 + (55+433i)T^6 \\ &\quad + (54+435i)T^7 + (40+386i)T^8 + (48+392i)T^9 \\ &\quad + (63+65i)T^{10} + (48+257i)T^{11} + (63+161i)T^{12} \\ &\quad + (20+477i)T^{13} + (38+182i)T^{14} + (56+66i)T^{15} \\ &\quad + (37+35i)T^{16} + (6+341i)T^{17} + (20+446i)T^{18} \\ &\quad + (40+412i)T^{19} + 368iT^{20} + (56+336i)T^{21} \\ &\quad + (61+291i)T^{22} + (40+427i)T^{23} + (34+38i)T^{24} \\ &\quad + (48+94i)T^{25} + (9+47i)T^{26} + (6+497i)T^{27} \\ &\quad + (40+42i)T^{28} + (44+52i)T^{29}) + O(T^{30}) \pmod{2^8} \end{split}$$

$$\begin{split} A(\beta^2;T) &\equiv 2^2 \big(32 + 32T + 22T^2 + 39T^3 + 36T^4 + 20T^5 + 62T^6 + 27T^7 \\ &\quad + 16T^8 + 62T^9 + 46T^{10} + 23T^{11} + 30T^{12} + 51T^{13} \\ &\quad + 4T^{14} + 2T^{15} + 56T^{16} + 33T^{17} + 44T^{18} + 12T^{19} \\ &\quad + 40T^{20} + 8T^{21} + 54T^{22} + 11T^{23} + 34T^{24} + 42T^{25} \\ &\quad + 43T^{27} + 56T^{28} + 46T^{29} \big) + O(T^{30}) \pmod{2^8} \end{split}$$

Therefore

$$\begin{split} D(T) &\equiv 2^5 \left(6T + 7T^2 + 4T^3 + 5T^4 + 4T^7 + 2T^8 + 4T^9 + 2T^{10} + 4T^{11} \right. \\ &+ T^{12} + 6T^{13} + 7T^{14} + 3T^{16} + 5T^{17} + 2T^{18} + 3T^{19} \\ &+ 7T^{20} + 5T^{21} + 7T^{22} + 4T^{23} + 4T^{24} + T^{25} + 7T^{26} \\ &+ 3T^{27} + 7T^{28} + 6T^{29} \right) + O(T^{30}) \pmod{2^8} \end{split}$$

and the conjecture is satisfied by the first 30 coefficients of the series D associated to the extension.

Note, as a final remark, that we have tested the conjecture in the same way for 30 additional examples where F is real quadratic, K/F is cyclic of order 4 but K is not a dihedral extension of \mathbb{Q} (either K/\mathbb{Q} is not Galois or its Galois group is not the dihedral group of order 8). In all of these examples, we found that the conjecture was not satisfied, that is either $\rho_{F,S}$ did not belong to $8\mathbb{Z}_2$ or one of the first 30 coefficients of the associated power series D did not belong to $32\mathbb{Z}_2$.

References

- [BBJR] A. Besser, P. Buckingham, R. de Jeu and X.-F. Roblot, On the p-adic Beilinson conjecture for number fields, Pure Appl. Math. Q. 5 (2009), 375-434.
- [PCN] P. Cassou-Noguès, Valeurs aux entiers négatifs des fonctions zêta p-adiques, Invent. Math. 51 (1979), 29-59.
- [Col] P. Colmez, Résidu en s=1 des fonctions zêta p-adiques, Invent. Math. **91** (1988), 371–389.
- [DR] P. Deligne and K.A. Ribet, Values of abelian L-functions at negative integers over totally real fields, Invent. Math. 59 (1980), 227-286.
- [FW] B. Ferrero and L. Washington, The Iwasawa invariant μ_p vanishes for abelian fields, Annals of Math. 109 (1979), 377-395.
- [R] K.A. Ribet, Report on p-adic L-functions over totally real fields, Société Mathém. de France, Astérisque 61 (1979), 177-192.
- [Rob] X.-F. Roblot, Computing p-adic L-functions of totally real number fields, in preparation
- [RW2] J. Ritter and A. Weiss, Toward equivariant Iwasawa theory, II, Indag. Mathem. N.S. 15 (2004), 549-572.
- [RW3] J. Ritter and A. Weiss, Toward equivariant Iwasawa theorem, III, Math. Ann. 336 (2006), 27-49.
- [RW] J. Ritter and A. Weiss, Equivariant Iwasawa theory: an example, Documenta Math. 13 (2008), 117-129.

[W] A. Wiles, The Iwasawa conjecture for totally real fields, Annals of Math. 131 (1990), 493-540.

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