# Statistics of Different Reduction Types of Fermat Curves

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#### Abstract

We present some theoretic bounds and algorithms concerning the statistics of different reduction types in the family of Fermat curves  $Y^p = X^s(1-X)$ , where p is prime and  $s = 1, \ldots, p-2$ .

### 1 Introduction

For a prime p, we define  $S_p = \{1, \ldots, p-2\}$ , and consider the family of curves

 $\mathcal{F}_s: Y^p = X^s(1-X), \qquad s \in \mathcal{S}_p,$ 

over the algebraic closure of  $\mathbb{Q}$ .

It has been shown by McCallum [16] (see also [15, 17, 18]) that there is a direct link between the properties of the reduction of  $\mathcal{F}_s$  modulo p (in particular to tame, wild split and wild non-split reductions) and the *Fermat quotients*. We recall that for a prime p and an integer u with gcd(u, p) = 1 the Fermat quotient  $q_p(u)$  is defined by the conditions:

$$q_p(u) \equiv \frac{u^{p-1} - 1}{p} \pmod{p}$$
 and  $0 \le q_p(u) < p$ .

We define the sequence of Legendre symbols:

$$\vartheta_{p,s} = \left(\frac{2s(s+1)q_p\left(s^s/(s+1)^{s+1}\right)}{p}\right), \qquad s \in \mathcal{S}_p.$$

By a result of McCallum [16] (see also [15, 17, 18]), the curve  $F_s$  is tame, wild split or wild non-split depending on whether  $\vartheta_{p,s} = 0$ , 1 or -1, respectively.

For  $\vartheta = 0, \pm 1$ , we define  $N_{\vartheta}(p)$  as the number of integers  $s \in S_p$  with  $\vartheta_{p,s} = \vartheta$ . A natural conjecture is that  $2s(s+1)q_p(s^s/(s+1)^{s+1})$  behaves uniformly randomly modulo p, and that the values for the various s are independent. Therefore one expects  $\vartheta_{p,s} = \pm 1$  both occur about half the time, and  $\vartheta_{p,s} = 0$  occurs for about 1/p values of s on average. In other words, one expects that  $N_0(p)$  behaves like a Poisson random variable with mean 1, and that  $N_{\pm 1}(p)$  behave like normal random variables with mean p/2 and variance p/2, as  $p \to \infty$ .

We show that these heuristics are not quite correct: there are *relations* among the  $\vartheta_{p,s}$ , so that the independence hypothesis must be modified. After adjusting our heuristics to take into account these relations, we find excellent numerical agreement with a table of values of  $N_{0,\pm 1}(p)$  for  $p < 10^7$ .

We also prove that  $N_0(p) = O(p^{2/3})$ . This confirms that  $\vartheta_{p,s} = 0$  occurs relatively rarely. Unfortunately this bound is much weaker than the bounds suggested by our numerical experiments.

Clearly the question about the distribution of the values of  $N_{\pm 1}(p)$  is essentially a question about bounding a certain sum of Legendre symbols with Fermat quotients. Recently there has been some progress in the area of estimating multiplicative character sums with Fermat quotients  $q_p(u)$ , see [3, 24, 25, 26]. However, the results and methods of these papers do not seem to apply to sums corresponding to  $\vartheta_{p,s}$ .

#### 2 Preparations

Our bound of  $N_0(p)$  is based on the following bound of Heath-Brown [11, Lemma 2] (see also [19]). Let

$$f(u) = \sum_{j=1}^{p-1} \frac{u^j}{j}.$$

**Lemma 1.** For any integer r the congruence

$$f(u) \equiv r \pmod{p}, \qquad 2 \le u \le p-1,$$

has  $O(p^{2/3})$  solutions.

We also recall some notions and the results from the uniform distribution theory, see [6]. As usual, we define the *discrepancy*  $\Delta(\Gamma)$  of a sequence  $\Gamma = (\gamma_m)_{m=1}^M$  of M (not necessarily distinct) points in the unit interval [0, 1] by

$$\Delta(\Gamma) = \sup_{0 \le \gamma \le 1} \left| \frac{I_{\Gamma}(\gamma)}{M} - \gamma \right|,$$

where  $I_{\Gamma}(\gamma)$  is the number of points  $\gamma_m$  of  $\Gamma$  with  $\gamma_m \leq \gamma$ .

We trivially have:

**Lemma 2.** Let  $\Phi = (\varphi_m)_{m=1}^M$  and  $\Psi = (\psi_m)_{m=1}^M$  be two sequences of M points in the unit interval [0, 1] with  $|\varphi_m - \psi_m| \leq \delta$ ,  $m = 1, \ldots, M$ . Then for their discrepancies we have  $\Delta(\Phi) = \Delta(\Psi) + O(\delta)$ .

The following result about the distribution of modular inverses is wellknown, see [27, Section 3] for a survey of several similar estimates. It has appeared in many works, and follows instantly from the Weil bound for Kloosterman sums, see [12, Theorem 11.11].

**Lemma 3.** For any prime  $\ell$  and integers  $0 < K < K+R < \ell$  the discrepancy of the sequence  $\overline{r}/\ell$ ,  $r \in [K, K+R]$  is  $O(R^{-1}\ell^{1/2}(\log \ell)^2)$ , where  $\overline{r}$  is defined by the congruence

$$\overline{r}r \equiv 1 \pmod{\ell}, \qquad 1 \leq \overline{r} \leq \ell - 1.$$

Let, as usual,  $\pi(X)$  denote the number of primes  $\ell \leq X$ . Then we have the following lower bound, due to Baker, Harman and Pintz [1] for primes in short intervals, see also [12, Section 10.5] for several related results. **Lemma 4.** There in an absolute constant c > 0 such that for  $Y \ge X^{0.525}$  we have  $\pi(X + Y) - \pi(X) \ge cY/\log Y$ .

Finally we need some algorithmic results. We measure the complexity of our algorithms in the so-called RAM model of computation, see [10] for a background.

Since we are mostly interested in theoretic estimates, we always assume that fast arithmetic of long integers is used and in particular any arithmetic operation on *n*-bit integers can be performed in time  $n^{1+o(1)}$  as  $n \to \infty$ , see [10, Theorem 8.23].

Besides we recall that for any prime p and integers a, b and k, all at most n-bits long, we can

- compute the residue  $a^k \pmod{p}$  in time  $n^{2+o(1)}$ , see [10, Section 4.3].
- compute the Legendre symbol (a/p) in time  $n^{1+o(1)}$ , see [2];
- find an integer solution (u, v) of the linear equation au bv = 1 in time  $n^{1+o(1)}$ , see [20].

We use these estimates throughout Sections 4 and 5.

## **3** Bounding $N_0(p)$

The upper bound for  $N_0(p)$  is immediate from Lemma 1.

Theorem 5. We have

$$N_0(p) = O(p^{2/3}).$$

*Proof.* It is easy to verify that

$$q_p(uv) \equiv q_p(u) + q_p(v) \pmod{p} \tag{1}$$

for any integers u and v with gcd(uv, p) = 1, see, for example, [7, Equation (3)].

Therefore

$$q_p \left( s^s / (s+1)^{s+1} \right) \equiv sq_p(s) - (s+1)q_p(s+1) \equiv \frac{s^p - (s+1)^p + 1}{p} \pmod{p}.$$
(2)

It is shown by Heath-Brown [11, Section 1] that for  $1 \le s \le p-2$ 

$$\frac{s^p - (s+1)^p + 1}{p} \equiv f(s+1) \pmod{p}$$

where f(u) is as in Section 2. Applying Lemma 1 we conclude the proof.  $\Box$ 

Therefore the curve  $Y^p = X^s(1-X)$  is tame for at most  $O(p^{2/3})$  positive integers  $s \le p-1$ .

It has been shown in the proof of [18, Lemma 6.1] that for  $p \equiv 1 \pmod{3}$ we have  $\vartheta_{p,s} = 0$  for both roots s of the congruence  $s^2 + s + 1 \equiv 0 \pmod{p}$ . So  $N_0(p) \ge 2$  for  $p \equiv 1 \pmod{3}$ . We are not aware of any other lower bounds.

### 4 Computing $N_{\vartheta}(p)$

We start with an observation that the known algorithmic results presented in in Section 2 imply that the values of  $N_{0,\pm 1}(p)$  can be computed directly from the definition in time and space

$$T = p(\log p)^{2+o(1)} \quad \text{and} \quad S = O(\log p),$$

respectively. Indeed, this follows instantly from the congruence

$$q_p\left(s^s/(s+1)^{s+1}\right) \equiv sq_p(s) - (s+1)q_p(s+1) \pmod{p}$$

that is based on (1) and which we have used in the proof of Theorem 5.

If memory is not of concern, we can simply compute the table of the values of  $q_p(s)$ ,  $s = 1, \ldots, p-2$ , in O(1) arithmetic operations modulo p per value, see [21, Theorem 7]. After this, using fast arithmetic for the Legendre symbol, we can compute  $N_{0,\pm 1}(p)$ , in time and space

$$T = p(\log p)^{1+o(1)} \quad \text{and} \quad S = O(p \log p),$$

respectively.

Furthermore, using [21, Algorithm 8] one can have some trade-off between the space complexity and running time of computing  $N_{\vartheta}(p)$ . Indeed, for any parameter  $Z \ge 2$ , we can evaluate in time  $pZ^{-1}(\log p)^{1+o(1)}$  a certain table (of size  $O(pZ^{-1}\log p))$  such that after this for each  $s = 1, \ldots, p-2$  we can compute  $\vartheta_{p,s}$  in time  $(\log p)^{1+o(1)} \log Z$ , see [21, Theorem 9]. Thus for any  $Z \ge 2$ , we can compute  $N_{0,\pm 1}(p)$  in time and space

$$T = p(\log p)^{1+o(1)} \log Z \quad \text{and} \quad S = O\left(pZ^{-1} \log p\right),$$

respectively. Clearly Z = p corresponds to the above trivial algorithm. However, taking  $Z = \exp(\sqrt{\log p})$  we see that we can compute  $N_{0,\pm 1}(p)$  in time  $p(\log p)^{3/2+o(1)}$  and space  $p \exp(-(1+o(1))\sqrt{\log p})$ .

## 5 Approximating $N_{\vartheta}(p)$

We now design Quasi-Monte Carlo type algorithms that evaluate  $\vartheta_{p,s}$  on a sequence of s that is asymptotically uniformly distributed in the interval [1, p-2] and have much more modest space requirements that the algorithms of Section 4.

Let

$$U = \left\lceil p^{1/2} \right\rceil$$
 and  $\Delta = \left\lceil p^{3/8} \log p \right\rceil$ .

We now precompute and store the table Q of values  $q_p(w)$ ,  $1 \le w \le U$ , which can be done in in time and space

$$T = p^{1/2} (\log p)^{2+o(1)}$$
 and  $S = O(p^{1/2} \log p),$  (3)

respectively. This is the cost of preprocessing.

Let  $\mathcal{L}$  be the set of primes  $\ell \in [U - \Delta, U]$  and let  $\mathcal{R}$  be the set of integers  $r \in [U - 3\Delta, U - 2\Delta]$ . We now recall the complexity bounds of Section 2 and proceed as follows:

Algorithm 6 (Using Linear Equations).

**Step 1:** Select at random a prime  $\ell \in \mathcal{L}$  and an integer  $r \in \mathcal{R}$ .

- **Step 2:** Find positive integers u < r and  $v < \ell$  with  $rv \ell u = 1$  in time  $(\log p)^{1+o(1)}$ .
- Step 3: Set  $s = \ell u$ .

**Step 4:** Using the precomputed table Q and applying (1), we compute

$$q_p\left(s^s/(s+1)^{s+1}\right) \equiv sq_p(\ell)q_p(u) - (s+1)q_p(r)q_p(v) \pmod{p},$$

and then  $\vartheta_{p,s}$ , in time  $(\log p)^{1+o(1)}$ .

Thus after the preprocessing with the cost given by (3) we compute  $\vartheta_{p,s}$ for every  $s \in \mathcal{S}$  in essentially linear time, where  $\mathcal{S}$  is the set of s generated at Step 2 of the above algorithm. That is,

$$\mathcal{S} = \{ s = \ell u : rv - \ell u = 1, \ell \in \mathcal{L}, r \in \mathcal{R}, 0 < u < r \}.$$

We now show that indeed the above algorithm samples s in a reasonably uniform fashion.

**Theorem 7.** The discrepancy of the sequence  $(s/p)_{s\in\mathcal{S}}$  is  $O\left(p^{-1/8}\log p\right)$ .

*Proof.* First of all we note that Lemma 4 applies to the set  $\mathcal{L}$  so it is not empty and contains at least  $c_0\Delta/\log p$  primes, for some absolute constant  $c_0 > 0$ . Furthermore, distinct pairs  $(\ell, r) \in \mathcal{L} \times \mathcal{R}$  lead to distinct products  $s = \ell u$ .

Clearly for  $\ell \in \mathcal{L}$  and  $r \in \mathcal{R}$  we have

$$\ell r = U^2 + O(U\Delta) = p + O(U\Delta).$$

Therefore,

$$\frac{s}{p} = \frac{\ell u}{p} = \frac{rv}{p} - \frac{1}{p} = \frac{rv}{\ell r + O(U\Delta)} - \frac{1}{p} = \frac{v}{\ell} + O\left(\frac{U\Delta}{p}\right)$$

Using Lemmas 2 and 3, we see that the discrepancy of the sequence of fractions s/p with  $s \in S$  corresponding to a given value of  $\ell$  is

$$O\left(\Delta^{-1}U^{1/2}(\log U)^2 + U\Delta p^{-1}\right) = O\left(p^{-1/8}\log p\right).$$

Obviously the discrepancy of the entire sequence  $(s/p)_{s\in\mathcal{S}}$  also satisfies the same bound.

Unfortunately the proof of Theorem 7 takes no advantage of averaging over  $\ell$ . So, this certainly can be a way to obtain a further improvement. In particular, it is possible that the switching over argument of Fouvry [8] maybe of help here.

The above approach is based on constructing suitable values of s from solutions of linear equations with the coefficients  $\ell$  and r running independently through some prescribed sets. There are various possible modifications that may lead to algorithmic advantages, and there are a variety of results [4, 5, 9, 22, 23] that can be used to prove the analogues of Theorem 7 about the uniformity of distribution of the corresponding values of s.

Clearly Algorithm 6 makes sense only if one intends to compute  $\vartheta_{p,s}$  for at least  $p^{1/2} \log p$  values of s.

### 6 Relations between $\vartheta_{p,s}$

Define permutations  $F, G: \mathcal{S}_p \to \mathcal{S}_p$  by

$$F(s) \equiv -1 - s \pmod{p}, \qquad G(s) \equiv 1/s \pmod{p}.$$

One checks that  $F^2 = G^2 = (FG)^3 = \mathrm{id}_{S_p}$ , so the group H generated by F and G contains at most six distinct permutations, namely 1, F, G, FG, GF, and FGF (= GFG). Furthermore, F, G and FGF each have a single fixed point (respectively -1/2, 1, and -2 modulo p), and the fixed points of FG and GF are the roots of  $x^2 + x + 1 = 0 \pmod{p}$  (if they exist).

From these facts one can easily determine the number of orbits of  $S_p$ under H, as follows. Assume that  $p \ge 11$ , so that -1/2, 1, -2, and the roots of  $x^2 + x + 1 = 0 \pmod{p}$ , are all distinct modulo p. First suppose that  $p = 1 \pmod{3}$ . Let  $s_0$  and  $s_1$  be the roots of  $x^2 + x + 1 = 0 \pmod{p}$ . Note that F and G interchange  $s_0$  and  $s_1$ . Thus there are precisely (p + 5)/6 orbits, namely  $\{-1/2, 1, -2\}, \{s_0, s_1\}, \text{ and } (p-7)/6$  orbits of order 6. Now suppose that  $p = 2 \pmod{3}$ . Then  $x^2 + x + 1 = 0 \pmod{p}$  has no roots, and we obtain (p+1)/6 orbits, namely  $\{-1/2, 1, -2\}$  and (p-5)/6 orbits of order 6.

In all cases, we see that there are p/6 + O(1) orbits. Next we show that  $\vartheta_{p,s}$  is *constant* on each orbit.

**Theorem 8.** We have  $\vartheta_{p,F(s)} = \vartheta_{p,s}$  and  $\vartheta_{p,G(s)} = \vartheta_{p,s}$ .

*Proof.* We will use the fact that for any  $u, v \in \mathbb{Z}$ , with  $u \neq 0 \pmod{p}$ ,

$$q_p(u+vp) \equiv \frac{(u+vp)^{p-1}-1}{p}$$
$$\equiv \frac{u^{p-1}+(p-1)u^{p-2}vp-1}{p}$$
$$\equiv q_p(u) - \frac{v}{u} \pmod{p},$$
(4)

see also [7, Equation (2)]. For the first relation,

$$\begin{split} \vartheta_{p,F(s)} &= \left(\frac{2(-s-1)(-s)\left((-s-1)q_p(p-1-s)-(-s)q_p(p-s)\right)}{p}\right) \\ &= \left(\frac{2s(s+1)\left(sq_p(s-p)-(s+1)q_p(s+1-p)\right)}{p}\right). \end{split}$$

The result then follows from

$$q_p(s-p) \equiv q_p(s) - \frac{1}{s} \pmod{p}$$

and

$$q_p(s+1-p) \equiv q_p(s+1) - \frac{1}{s+1} \pmod{p}.$$

For the second relation,

$$\vartheta_{p,G(s)} = \left(\frac{2s^{-1}(s^{-1}+1)\left(\frac{1}{s}q_p(G(s)) - (s^{-1}+1)q_p(G(s)+1)\right)}{p}\right)$$
$$= \left(\frac{2s(s+1)\left(q_p(G(s)) - (s+1)q_p(G(s)+1)\right)}{p}\right)$$

since  $(s^4/p) = 1$ . Now let sG(s) = 1 + kp for some  $k \in \mathbb{Z}$ . Then

$$q_p(G(s)) = q_p(sG(s)) - q_p(s) = q_p(1) - k - q_p(s) = -k - q_p(s) \pmod{p}$$

and

$$q_p(G(s)+1) \equiv q_p(sG(s)+s) - q_p(s) = q_p(s+1) - \frac{k}{s+1} - q_p(s) \pmod{p}.$$

Therefore

$$q_p(G(s)) - (s+1)q_p(G(s)+1) \equiv sq_p(s) - (s+1)q_p(s+1) \pmod{p},$$

and  $\vartheta_{p,G(s)} = \vartheta_{p,s}$  as desired.

A natural question is whether there is some geometric explanation for the above relations. For example, are there maps between  $\mathcal{F}_s$ ,  $\mathcal{F}_{F(s)}$  and  $\mathcal{F}_{G(s)}$  that force them to have the same reduction type?

### 7 Numerical Results

We have computed  $N_{0,\pm 1}(p)$  for all  $p < 10^7$  using a C implementation of a fairly naive algorithm.

Table 1 gives a statistical summary of the distribution of  $N_1(p)$ . The data for  $N_{-1}(p)$  is similar and is not shown. The table has been constructed in

k	$E(X^k)$	$E(N^k)$
1	-0.00085	0
2	0.99979	1
3	0.00051	0
4	3.00059	3
5	0.00403	0
6	14.92162	15
7	-0.07897	0
8	102.90932	105

Table 1: Moments of normalised  $N_1(p)$ , for  $3 \le p < 10^7$ 

the following way. Following the results of the previous section,  $N_1(p)$  should behave like a sum of p/6 independent random variables that take the values 6 and 0 with equal probability. Each such variable has mean 3 and variance 9, so under this assumption we expect  $N_1(p)$  to have mean 3(p/6) = p/2 and variance 9(p/6) = 3p/2. We treat each prime p as an 'observation' of  $N_1(p)$ , and define a normalised random variable

$$X = \frac{N_1(p) - p/2}{\sqrt{3p/2}}.$$

Table 1 shows the moments of X, compared to the moments of the standard normal distribution. The closeness of the fit strongly supports the assumptions of our model.

Table 2 summarises the behaviour of  $N_0(p)$ . If we assume that  $\vartheta_{p,s}$  takes the value 0 with probability 1/p on each of the p/6 + O(1) orbits, then we expect  $N_0(p)/6$  to behave like a Poisson random variable with mean 1/6. In Table 2, the column  $T_2(k)$  counts the number of primes  $p = 2 \pmod{3}$  such that  $N_0(p) = 6k$ ; it closely matches the last column, which shows the value predicted by the Poisson model. For  $p = 1 \pmod{3}$  we must modify this slightly, because we know that automatically  $\vartheta_{p,s} = 0$  when s is one of the roots of  $s^2 + s + 1 = 0 \pmod{p}$  (see Section 3). This effectively increases  $N_0(p)$  by two. In the table, we correspondingly define  $T_1(k)$  to be the number of primes  $p = 1 \pmod{3}$  such that  $N_0(p) = 6k+2$ . Again this closely matches the Poisson model.

Table 2: Frequency table for  $N_0(p)$ , for  $5 \le p < 10^7$ ,  $p \ne 1093, 3511$ 

k	$T_1(k)$	$T_2(k)$	Poisson prediction
0	281486	281127	281277
1	46619	47088	46879
2	3860	3923	3906
3	217	231	217.03
4	10	14	9.043

Finally, one computes that

$$\vartheta_{p,1} = \left(\frac{-2q_p(2)}{p}\right)$$

By definition, the latter is zero modulo p if and only if p is a Wieferich prime. There are only two known Wieferich primes, namely 1093 and 3511. For these two primes,  $\vartheta_{p,s}$  is zero on the orbit  $\{1, -1/2, -2\}$ . In fact  $N_0(3511) = 5$  and  $N_0(1093) = 17$ .

### 8 Comments

We remark that in the range of our calculations of  $N_{0,\pm 1}(p)$ , none of the asymptotically faster algorithms of Sections 4 and 5 were used. We however believe that these algorithms are not only of theoretic interest and can become more practically useful for large values of p.

We note that the definition of  $\vartheta_{p,s}$  makes sense for any integer s with gcd(s(s+1), p) = 1 and then it obviously becomes a periodic function of s with period  $p^2$ . Furthermore, a more careful analysis shows that it is periodic with period p. Indeed, using (4), we see that

$$(u+p)q_p(u+p) \equiv uq_p(u+p) \equiv uq_p(u) - 1.$$

Now, recalling (2), we derive

$$q_p \left( (s+p)^{s+p}/(s+p+1)^{s+p+1} \right) \\ \equiv (s+p)q_p(s+p) - (s+p+1)q_p(s+p+1) \\ \equiv sq_p(s) - (s+1)q_p(s+1) \equiv q_p \left( s^s/(s+1)^{s+1} \right) \pmod{p}.$$

This explains why it is enough to study the values of  $\vartheta_{p,s}$  only for  $s \in S_p$ . This can also be used in numerical tests as one can use any values of s for which the relevant values of  $q_p$  are easy to are easy to compute.

For instance, assume that we want to compute  $N_{0,\pm 1}(p)$  for many primes  $p \leq X$ . Then we can try to find a large set S of  $s \in [1, X]$  (together with their factorisations) such that s and s + 1 are Y-smooth (that is, whose prime factors are all less than Y). Then, for each p, we compute and store the values of  $q_p(\ell)$  for all primes  $\ell \leq Y$ . After this  $\vartheta_{p,s}$ ,  $s \in S$ , can be computed very rapidly. Certainly for this approach to work, we need to know for what Y there many such integers s and how they are distributed in [1, X]. However, we remark that the questions of counting and generating smooth pairs (s, s + 1) is apparently very difficult, see [13, Section 6].

We can expand further the set of potentially "friendly" test points by considering triples (u, v, w) of Y-smooth positive integers  $u, v, w \leq X$ . After this we compute  $t \equiv u/w \pmod{p^2}$  and then compute  $s \in S_p$  with  $s \equiv t \pmod{p}$ . We now have  $\vartheta_{p,s} = \vartheta_{p,t}$ , while t is a "friendly" value. Unfortunately counting and generating such triples (u, v, w) is still a difficult problem, see [13, 14].

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