# FOURIER ANALYSIS ON FINITE GROUPS AND THE LOVÁSZ THETA-NUMBER OF CAYLEY GRAPHS 

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#### Abstract

We apply Fourier analysis on finite groups to obtain simplified formulations for the Lovász $\vartheta$-number of a Cayley graph. We put these formulations to use by checking a few cases of a conjecture of Ellis, Friedgut, and Pilpel made in a recent article proving a version of the Erdős-Ko-Rado theorem for $k$-intersecting families of permutations. We also introduce a $q$-analog of the notion of $k$-intersecting families of permutations, and we verify a few cases of the corresponding Erdős-Ko-Rado assertion by computer.


## 1. Introduction

One approach to some problems in extremal combinatorics involves estimating the independence number of a Cayley graph. A classic example is upper bounding sizes of error-correcting codes in Abelian groups. A recent, exciting example is provided by a version of the Erdős-Ko-Rado theorem for permutations proven by Ellis, Friedgut, and Pilpel [4]: If $k$ is a positive integer, $n$ is sufficiently large depending on $k$, and $\mathcal{A}$ is a largest set of permutations on $n$ letters such that any two agree on at least $k$ letters, then $|\mathcal{A}|=(n-k)$ !. This resolved a conjecture of Frankl and Deza from [6] stated in 1977.

The Lovász $\vartheta$-number, introduced in (9, provides an upper bound on the size of an independent set in a general graph. It can be computed by solving a semidefinite program involving $n \times n$-matrices, where $n$ is the cardinality of the vertex set. We specialize the $\vartheta$-number to Cayley graphs and show how the semidefinite program block-diagonalizes to a simpler one involving smaller matrices associated to the irreducible representations of the group. The resulting semidefinite program can be thought of as a "frequency domain" formulation of the $\vartheta$-number. Furthermore, under a sufficient condition on the graph, our semidefinite program collapses to a linear program which can be formulated using only knowledge of the group characters. This condition applies, in particular, for the two examples given above. In fact, one can interpret the arguments in [4] as constructing feasible solutions to the linear program computing the $\vartheta$-number for a particular Cayley graph on the symmetric group.

In 4], the problem of quantifying the dependence of $n$ on $k$ is left open, but they conjecture that the conclusion of their theorem holds when $n \geq 2 k+1$. By explicit computations we verify their conjecture for some small values of $n$ and $k$, and we identify some values for which the $\vartheta$-number does not give a tight enough bound to verify the conjecture, suggesting that other methods will be required to resolve these cases.

[^0]The outline of the paper is as follows: In Section 2 we fix notation and definitions, and recall some basic facts from finite Fourier analysis. In Section 3 we find several reformulations of the Lovász $\vartheta$-function for Cayley graphs by using the group structure on the vertex set. In Section 4 we apply these results in the context of the Ellis-Friedgut-Pilpel conjecture made in 4, and in Section 5 we introduce a $q$-analog of their result as a conjecture, and perform the analogous computations. In Section [6. we show how the machinery developed in Section 3 could also be applied to vertex-transitive graphs.

## 2. Definitions, notation, and background in Fourier analysis

All graphs will be simple and undirected. For any graph $G=(V, E)$, the independence number is the maximum number of pairwise nonadjacent vertices; this maximum will be denoted $\alpha(G)$.

Suppose $\Gamma$ is a finite group. A subset $X \subseteq \Gamma$ will be called a connection set if the unit element $e$ of $\Gamma$ does not belong to $X$, and if $X$ is inverse-closed; that is $x^{-1} \in X$ whenever $x \in X$. For any connection set $X \subseteq \Gamma$, the Cayley graph $\operatorname{Cay}(\Gamma, X)$ is the graph with vertex set $\Gamma$, where two vertices $x$ and $y$ are adjacent if and only if $y^{-1} x \in X$. The defining conditions of a connection set imply that $\operatorname{Cay}(\Gamma, X)$ is an undirected graph without self-loops. Notice that we do not require $X$ to generate $\Gamma$; therefore $\operatorname{Cay}(\Gamma, X)$ need not be connected.

In the following we recall some basic facts from representation theory of finite groups. For a good reference, see for instance Terras [10]. A (finite-dimensional) unitary representation of $\Gamma$ is a group homomorphism $\pi: \Gamma \rightarrow \mathrm{U}\left(d_{\pi}\right)$ where $\mathrm{U}\left(d_{\pi}\right)$ is the group of unitary $d_{\pi} \times d_{\pi}$ matrices. The number $d_{\pi}$ is called the degree of $\pi$. The character of $\pi$ is defined as $\chi_{\pi}(\gamma)=\operatorname{Tr}(\pi(\gamma))$, where $\operatorname{Tr}$ denotes trace. A subspace $M$ of $\mathbb{C}^{d_{\pi}}$ is $\pi$-invariant if $\pi(\gamma) m \in M$ for all $\gamma \in \Gamma$ and $m \in M$. The unitary representation $\pi$ is said to be irreducible if $\{0\}$ and $\mathbb{C}^{d_{\pi}}$ are the only $\pi$-invariant subspaces of $\mathbb{C}^{d_{\pi}}$. Two unitary representations $\pi$ and $\pi^{\prime}$ are (unitarily) equivalent if there is a unitary matrix $T$ such that $T \pi(\gamma)=\pi^{\prime}(\gamma) T$ for all $\gamma \in \Gamma$.

Given two inequivalent irreducible unitary representations $\pi$ and $\pi^{\prime}$, the Schur orthogonality relations give us the following two facts:
(1) $\sum_{\gamma \in \Gamma} \pi_{i j}(\gamma) \overline{\pi_{l k}^{\prime}(\gamma)}=0$, where $\pi_{i j}(\gamma)$ is the $i j$-entry of the matrix $\pi(\gamma)$, and $\pi_{l k}^{\prime}(\gamma)$ is defined analogously;
(2) $\sum_{\gamma \in \Gamma} \pi_{i j}(\gamma) \overline{\pi_{l k}(\gamma)}=\frac{|\Gamma|}{d_{\pi}} \delta_{i l} \delta_{j k}$, where $\delta$ is the Kronecker delta.

These relations are implied by Schur's lemma, which says that if $\pi$ and $\pi^{\prime}$ are irreducible unitary representations, and if $T$ is a matrix for which $T \pi(\gamma)=\pi^{\prime}(\gamma) T$ for all $\gamma \in \Gamma$, then $T$ is either invertible or zero; if $\pi=\pi^{\prime}$, then $T$ is a scalar multiple of the identity matrix.

We fix a set of mutually inequivalent irreducible unitary representations of $\Gamma$, so that each unitary equivalence class has a representative; call this set $\hat{\Gamma}$. This allows us to define the Fourier transform of a function $f: \Gamma \rightarrow \mathbb{C}$ :

$$
\hat{f}(\pi)=\sum_{\gamma \in \Gamma} f(\gamma) \pi(\gamma)
$$

where $\hat{f}(\pi)$ is a complex $d_{\pi} \times d_{\pi}$ matrix. The Fourier inversion formula says we can recover $f$ from its Fourier transform:

$$
f(\gamma)=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\langle\hat{f}(\pi), \pi(\gamma)\rangle
$$

The inner product used here is the trace inner product, defined as $\langle A, B\rangle=\operatorname{Tr}\left(B^{*} A\right)$ for square complex matrices $A$ and $B$ of the same dimension, where $B^{*}$ denotes the conjugate-transpose of $B$.

The convolution of two functions $f: \Gamma \rightarrow \mathbb{C}$ and $g: \Gamma \rightarrow \mathbb{C}$ is defined by

$$
f * g(\gamma)=\sum_{\beta \in \Gamma} f(\beta) g\left(\beta^{-1} \gamma\right)
$$

and the involution of $f$ is defined as $f^{*}(\gamma)=\overline{f\left(\gamma^{-1}\right)}$. It is a fact that $\widehat{f * g}(\pi)=$ $\hat{f}(\pi) \hat{g}(\pi)$, and that $\widehat{f^{*}}(\pi)=\hat{f}(\pi)^{*}$.

A function $f: \Gamma \rightarrow \mathbb{C}$ is of positive type if

$$
\sum_{\gamma \in \Gamma} g * g^{*}(\gamma) f(\gamma) \geq 0
$$

for all functions $g: \Gamma \rightarrow \mathbb{C}$; that is, the sum is a nonnegative real number. We denote by $\mathcal{P}(\Gamma)$ the set of functions on $\Gamma$ of positive type. Notice that $f \in \mathcal{P}(\Gamma)$ if and only if $\bar{f} \in \mathcal{P}(\Gamma)$, where $\bar{f}$ is the pointwise complex-conjugate of $f$. One fact that will be needed later is that $f\left(\gamma^{-1}\right)=\overline{f(\gamma)}$ for all $\gamma \in \Gamma$ when $f$ is of positive type. For a proof of this fact and more information on functions of positive type, see Folland [5, Chapter 3.3].

For vectors $u, v \in \mathbb{C}^{n}$, we use $\langle u, v\rangle$ to denote the usual inner product of $u$ and $v$. An $n \times n$ matrix $A$ with entries from $\mathbb{C}$ will be called positive semidefinite if $\langle A v, v\rangle$ is a nonnegative real number for all $v \in \mathbb{C}^{n}$. Using the polarization identity, it is possible to prove that every positive semidefinite matrix is Hermitian. For each finite set $V$, the set of positive semidefinite matrices with rows and columns indexed on $V$ will be denoted $\mathcal{S}_{\succeq 0}^{V}$. When $V=\{1, \ldots, n\}$, we will use the notation $\mathcal{S}_{\succeq 0}^{n}$ instead. It is a fact that $A \in \mathcal{S}_{\succeq 0}^{n}$ if and only if $\langle A, B\rangle \geq 0$ for all $B \in \mathcal{S}_{\succeq 0}^{n}$; this fact is known as the self-duality of $\mathcal{S}_{\succeq 0}^{n}$.

The following theorem is an application of self-duality, as well as Parseval's identity, which says that

$$
\sum_{\gamma \in \Gamma} f(\gamma) \overline{g(\gamma)}=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\langle\hat{f}(\pi), \hat{g}(\pi)\rangle
$$

for all functions $f$ and $g$ on $\Gamma$ :
Theorem 1 (Bochner's theorem for finite groups). Suppose $\Gamma$ is a finite group and let $f: \Gamma \rightarrow \mathbb{C}$. Then $f$ is of positive type if and only if $\hat{f}(\pi)$ is positive semidefinite for each $\pi \in \hat{\Gamma}$.

Proof. For any two complex-valued functions $f$ and $g$ on $\Gamma$, we have
(1) $\sum_{\gamma \in \Gamma} g * g^{*}(\gamma) \overline{f(\gamma)}=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle\widehat{g * g^{*}}(\pi), \hat{f}(\pi)\right\rangle=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle\hat{g}(\pi) \hat{g}(\pi)^{*}, \hat{f}(\pi)\right\rangle$.

The matrices $\hat{g}(\pi) \hat{g}(\pi)^{*}$ are always positive semidefinite, so (1) is nonnegative if all the matrices $\hat{f}(\pi)$ are positive semidefinite. This gives one direction.

For the other direction, suppose $f: \Gamma \rightarrow \mathbb{C}$ is of positive type, and fix $\pi \in \hat{\Gamma}$. Now let $A \in \mathcal{S}_{\succ 0}^{d_{\pi}}$ be arbitrary, and let $A=B B^{*}$ be the Cholesky decomposition. Define $g: \Gamma \rightarrow \overline{\mathbb{C}}$ by $g(\gamma)=d_{\pi} /|\Gamma|\langle B, \pi(\gamma)\rangle$. By the Schur orthogonality relations (or uniqueness of Fourier coefficients), we have $\hat{g}(\pi)=B$ and $\hat{g}\left(\pi^{\prime}\right)=0$ when $\pi^{\prime}$ and $\pi$ are inequivalent, whence

$$
\hat{g}(\pi) \hat{g}(\pi)^{*}=B B^{*}=A \quad \text { and } \quad \hat{g}\left(\pi^{\prime}\right) \hat{g}\left(\pi^{\prime}\right)^{*}=0
$$

Now (11), which is nonnegative by hypotheses, is equal to $d_{\pi} /|\Gamma|\langle A, \hat{f}(\pi)\rangle$. Since $\pi$ and $A$ were arbitrary, we conclude that $\langle A, \hat{f}(\pi)\rangle \geq 0$ for every $\pi$ and every $A \in \mathcal{S}_{\succeq 0}^{d_{\pi}}$. Self-duality of $\mathcal{S}_{\succeq 0}^{d_{\pi}}$ now implies $\hat{f}(\pi) \in \mathcal{S}_{\succeq 0}^{d_{\pi}}$ for each $\pi \in \hat{\Gamma}$.

## 3. The $\vartheta$-Number of a Cayley graph

Let $G=(V, E)$ be a finite graph. In 9], the Lovász $\vartheta$-number $\vartheta(G)$ of $G$ is defined and a number of equivalent formulations are given. The formulation of $\vartheta(G)$ which will be most important for us is:

$$
\vartheta(G)=\max \left\{\sum_{u, v \in V} A(u, v): A \in \mathcal{S}_{\succeq 0}^{V} \text { real-valued, } \quad \begin{array}{rl} 
& \operatorname{Tr}(A)=1, A(u, v)=0 \text { for }\{u, v\} \in E\} \tag{A}
\end{array}\right.
$$

When $G$ is the Cayley graph $\operatorname{Cay}(\Gamma, X)$, the optimization over matrices in (A) can be replaced with optimization over functions on $\Gamma$, as we proceed to show.

Theorem 2. Suppose $G=\operatorname{Cay}(\Gamma, X)$. Then

$$
\begin{align*}
\vartheta(G)=\max \left\{\sum_{\gamma \in \Gamma} f(\gamma):\right. & f \in \mathcal{P}(\Gamma) \text { real-valued, }  \tag{B}\\
& f(e)=1, f(x)=0 \text { for } x \in X\}
\end{align*}
$$

Before we prove Theorem 2 we require a lemma:
Lemma 3. Suppose $A: \Gamma \times \Gamma \rightarrow \mathbb{C}$ is a Hermitian matrix satisfying $A(\gamma, e)=$ $A(\gamma \beta, \beta)$ for all $\gamma, \beta \in \Gamma$. Define $f: \Gamma \rightarrow \mathbb{C}$ by $f(\gamma)=A(\gamma, e)$. Then for any function $g: \Gamma \rightarrow \mathbb{C}$ we have

$$
\sum_{\gamma \in \Gamma} g * g^{*}(\gamma) f(\gamma)=\sum_{\gamma, \gamma^{\prime} \in \Gamma} g(\gamma) \overline{g\left(\gamma^{\prime}\right)} A\left(\gamma, \gamma^{\prime}\right)
$$

Proof. This follows from a straightforward computation.
Proof of Theorem 圆, For one direction, let $A$ be a feasible solution for (A). Define $\bar{A}: \Gamma \times \Gamma \rightarrow \mathbb{R}$ entrywise by

$$
\bar{A}\left(\gamma, \gamma^{\prime}\right)=\frac{1}{|\Gamma|} \sum_{\beta \in \Gamma} A\left(\gamma \beta, \gamma^{\prime} \beta\right)
$$

Being the average of matrices similar to $A$ (via permutation matrices), the matrix $\bar{A}$ is positive semidefinite, and one now easily checks that $\bar{A}$ is again a feasible solution for (A) having the same objective value as $A$. Moreover, we have $\bar{A}(\gamma, e)=\bar{A}(\gamma \beta, \beta)$ for all $\gamma, \beta \in \Gamma$.

Now define $f: \Gamma \rightarrow \mathbb{R}$ by $f(\gamma)=|\Gamma| \bar{A}(\gamma, e)$. Then $\bar{A}$ and $f /|\Gamma|$ satisfy the hypotheses of Lemma 3, so

$$
\sum_{\gamma \in \Gamma} g * g^{*}(\gamma) f(\gamma)=|\Gamma| \sum_{\gamma, \gamma^{\prime} \in \Gamma} g(\gamma) \overline{g\left(\gamma^{\prime}\right)} \bar{A}\left(\gamma, \gamma^{\prime}\right)
$$

and since $\bar{A}$ is positive semidefinite, it follows that the function $f$ is of positive type. It is easily checked that the other constraints of (B) are satisfied by $f$, and moreover that the objective values are equal:

$$
\sum_{\gamma \in \Gamma} f(\gamma)=|\Gamma| \sum_{\gamma \in \Gamma} \bar{A}(\gamma, e)=\sum_{\gamma, \gamma^{\prime} \in \Gamma} \bar{A}\left(\gamma, \gamma^{\prime}\right)=\sum_{\gamma, \gamma^{\prime} \in \Gamma} A\left(\gamma, \gamma^{\prime}\right) .
$$

For the other direction, we begin with a feasible solution $f: \Gamma \rightarrow \mathbb{R}$ to (B), and we define $A: \Gamma \times \Gamma \rightarrow \mathbb{R}$ by $A(\beta, \gamma)=\frac{1}{|\Gamma|} f\left(\beta \gamma^{-1}\right)$. Then $A$ is a feasible solution to (A) by Lemma 3, and its objective value is $\sum_{\gamma \in \Gamma} f(\gamma)$.

Using Theorem [1 we can also give a (complex) semidefinite programming formulation of (B) using block matrices.

Theorem 4. Suppose $G=\operatorname{Cay}(\Gamma, X)$. Then
(C) $\vartheta(G)=\max \left\{A_{1}: A_{\pi} \in \mathcal{S}_{\succeq 0}^{d_{\pi}}\right.$ for each $\pi \in \hat{\Gamma}$,

$$
\left.\sum_{\pi \in \hat{\Gamma}} d_{\pi} \operatorname{Tr}\left(A_{\pi}\right)=|\Gamma|, \sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle A_{\pi}, \pi(x)\right\rangle=0 \text { for } x \in X\right\}
$$

where $1 \in \hat{\Gamma}$ denotes the trivial representation.
Proof. If $f: \Gamma \rightarrow \mathbb{R}$ is any feasible solution to (B), set $A_{\pi}=\hat{f}(\pi)$ for each $\pi \in \hat{\Gamma}$. By Theorem 1 the matrices $A_{\pi}$ are positive semidefinite. Moreover, one easily checks using the Fourier inversion formula that the other constraints of (C) are satisfied by $\left\{A_{\pi}: \pi \in \hat{\Gamma}\right\}$, and that the objective values are equal: $A_{1}=\sum_{\gamma \in \Gamma} f(\gamma)$.

For the other direction, let $\left\{A_{\pi}: \pi \in \hat{\Gamma}\right\}$ be a feasible solution for (C) and define $g: \Gamma \rightarrow \mathbb{C}$ by

$$
g(\gamma)=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle A_{\pi}, \pi(\gamma)\right\rangle \quad \text { for all } \quad \gamma \in \Gamma
$$

Then $g$ is of positive type by Theorem Now define $f(\gamma)=\frac{1}{2}\left(g(\gamma)+g\left(\gamma^{-1}\right)\right)$ for all $\gamma \in \Gamma$. Then $f$ is real-valued, and that $f$ satisfies all the other constraints of (B) is easily checked using the fact that $X$ is inverse-closed. Moreover

$$
\sum_{\gamma \in \Gamma} f(\gamma)=\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle A_{\pi}, \pi(\gamma)\right\rangle=A_{1}
$$

by the Schur orthogonality relations.
When $\Gamma$ is an Abelian group, then all its irreducible representation are onedimensional. Therefore, the semidefinite program (C) is just a linear program. More generally, (C) is equivalent to a linear program whenever the connection set of the Cayley graph $\operatorname{Cay}(\Gamma, X)$ is closed under conjugation; that is, $\gamma x \gamma^{-1} \in X$ for all $x \in X$ and $\gamma \in \Gamma$. This is the content of the next theorem.

Theorem 5. Let $G$ be the Cayley graph $\operatorname{Cay}(\Gamma, X)$ and suppose that the connection set $X$ is closed under conjugation. Then
(D) $\quad \vartheta(G)=\max \left\{a_{1}: a_{\pi} \geq 0\right.$ for each $\pi \in \hat{\Gamma}$,

$$
\left.\sum_{\pi \in \hat{\Gamma}} d_{\pi}^{2} a_{\pi}=|\Gamma|, \sum_{\pi \in \hat{\Gamma}} d_{\pi} a_{\pi} \chi_{\pi}(x)=0 \text { for } x \in X\right\}
$$

Proof. We prove the equivalence of (C) and (D). Let $\left\{A_{\pi}: \pi \in \hat{\Gamma}\right\}$ be a feasible solution for (C), and for each $\pi$ let

$$
\bar{A}_{\pi}=\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \pi(\gamma) A_{\pi} \pi(\gamma)^{*}
$$

Then $\left\{\bar{A}_{\pi}: \pi \in \hat{\Gamma}\right\}$ is again a solution to (C): If $x \in X$, then

$$
\begin{aligned}
\sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle\bar{A}_{\pi}, \pi(x)\right\rangle & =\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi} \sum_{\gamma \in \Gamma}\left\langle\pi(\gamma) A_{\pi} \pi(\gamma)^{*}, \pi(x)\right\rangle \\
& =\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi} \sum_{\gamma \in \Gamma}\left\langle\pi(\gamma) A_{\pi}, \pi(x \gamma)\right\rangle
\end{aligned}
$$

Since $X$ is closed under conjugation there is a $y \in X$ so that $x \gamma=\gamma y$ holds. Hence, the sum above equals

$$
\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi} \sum_{\gamma \in \Gamma}\left\langle\pi(\gamma) A_{\pi}, \pi(\gamma y)\right\rangle=\frac{1}{|\Gamma|} \sum_{\pi \in \hat{\Gamma}} d_{\pi} \sum_{\gamma \in \Gamma}\left\langle A_{\pi}, \pi(y)\right\rangle=\sum_{\pi \in \hat{\Gamma}} d_{\pi}\left\langle A_{\pi}, \pi(y)\right\rangle=0
$$

Moreover, since $\pi(\gamma) A_{\pi} \pi(\gamma)^{*}$ is similar to $A_{\pi}$ for each $\gamma \in \Gamma$, the matrix $\bar{A}_{\pi}$ is positive semidefinite for each $\pi \in \hat{\Gamma}$ and $\sum_{\pi \in \hat{\Gamma}} d_{\pi} \operatorname{Tr}\left(\bar{A}_{\pi}\right)=|\Gamma|$.

We have constructed $\bar{A}_{\pi}$ so that $\bar{A}_{\pi} \pi(\gamma)=\pi(\gamma) \bar{A}_{\pi}$ for all $\gamma \in \Gamma$. Schur's lemma then implies that $\bar{A}_{\pi}$ is equal to $a_{\pi} I_{d_{\pi}}$ for some scalar $a_{\pi}$ and since $\bar{A}_{\pi}$ is positive semidefinite this scalar is nonnegative. We have $d_{\pi} a_{\pi}=\operatorname{Tr}\left(\bar{A}_{\pi}\right)$ as well as

$$
\left\langle\bar{A}_{\pi}, \pi(\gamma)\right\rangle=a_{\pi} \chi_{\pi}(\gamma) \quad \text { for all } \quad \gamma \in \Gamma
$$

so $\left\{a_{\pi}: \pi \in \hat{\Gamma}\right\}$ is a feasible solution to (D) having objective value $a_{1}=A_{1}$.
For the other direction, we take a feasible solution $\left\{a_{\pi}: \pi \in \hat{\Gamma}\right\}$ to (D), and for each $\pi \in \hat{\Gamma}$, we set $A_{\pi}=a_{\pi} I_{d_{\pi}}$. This is a feasible solution to (C) with objective value $A_{1}=a_{1}$.

Denote the constraint $\sum_{\pi \in \hat{\Gamma}} d_{\pi} a_{\pi} \chi_{\pi}(x)=0$ by $C_{x}(x \in X)$. For computational purposes, the following simplifications can be applied to (D): First, only one of the constraints $\left\{C_{x}, C_{x^{-1}}\right\}$ is needed. Second, since the characters $\chi_{\pi}$ are constant on conjugacy classes, it suffices to keep only the constraints $C_{x}$, with one $x$ per conjugacy class.

## 4. First application: $k$-intersecting Permutations

In this section we apply Theorem 5 to the problem of $k$-intersecting permutations as discussed in the introduction.

Let $S_{n}$ be the symmetric group on $n$ letters. A family $\mathcal{A} \subseteq S_{n}$ is said to be $k$-intersecting if any two permutations in $\mathcal{A}$ agree on at least $k$ elements. That is, a $k$-intersecting family of $S_{n}$ is an independent set in the graph Cay $\left(S_{n}, X_{n, k}\right)$, where

$$
X_{n, k}=\left\{\sigma \in S_{n}: \sigma \text { has strictly less than } k \text { fixed points }\right\}
$$

The set $X_{n, k}$ is closed under conjugation so Theorem 5 applies. One can interpret the method of Ellis, Friedgut, and Pilpel in [4] as constructing an explicit family of feasible solutions to the linear programs which turns out to be optimal for given $k$ and $n$ sufficiently large.

Conjecture 2 of [4] implies that a largest $k$-intersecting family in $S_{n}$ has size

$$
\max _{0 \leq i \leq(n-k) / 2} \mid\left\{\sigma \in S_{n}: \sigma \text { has at least } k+i \text { fixed points in }\{1, \ldots, k+2 i\}\right\} \mid,
$$

which in particular means that the maximum size is $(n-k)$ ! for $n \geq 2 k+1$. We solved the linear program (D) for small values of $n$ and $k$ with the help of a computer. In Table 1 the $(n, k)$-th entry is marked when the $\vartheta$-number gives the conjectured maximum. To evaluate the characters of the symmetric group we used gap [7] and to solve the linear programs we used lrs [1]. Since both software packages only use rational arithmetic our computations are rigorous.


TABLE 1. Computation of $\vartheta\left(\operatorname{Cay}\left(S_{n}, X_{n, k}\right)\right)$

## 5. SECOND APPLICATION: $k$-INTERSECTING INVERTIBLE MATRICES

Here we consider a $q$-analog of the previous application. Let $\Gamma=\operatorname{GL}\left(n, \mathbb{F}_{q}\right)$ be the group of invertible $n \times n$-matrices over the finite field with $q$ elements, where $q$ is a prime power. We say that two matrices $A$ and $B$ in $\operatorname{GL}\left(n, \mathbb{F}_{q}\right) k$-intersect if there is a $k$-dimensional subspace $H$ of $\mathbb{F}_{q}^{n}$ such that $A x=B x$ for all $x \in H$. Given a natural number $k$, let

$$
X_{q, n, k}=\left\{A \in \mathrm{GL}\left(n, \mathbb{F}_{q}\right): \operatorname{rank}(A-I)>n-k\right\}
$$

and consider the Cayley graph $G_{q, n, k}=\operatorname{Cay}\left(\Gamma, X_{q, n, k}\right)$. Independent sets in this graph correspond to $k$-intersecting families of invertible matrices.

The independence number of $G_{q, n, 1}$ was recently calculated by Guo and Wang in [8] (not by computing $\vartheta\left(G_{q, n, 1}\right)$ ).

For any $q$ and $n$, one clearly obtains a lower bound by choosing a nonzero vector $x \in \mathbb{F}_{q}^{n}$ and considering the set $\mathcal{A}$ of all matrices $A \in \mathrm{GL}\left(n, \mathbb{F}_{q}\right)$ such that $A x=x$. One has $|\mathcal{A}|=\prod_{i=1}^{n-1}\left(q^{n}-q^{i}\right)$ by the orbit-stabilizer theorem, and for small values of $n$ and $q$ we found numerically that $\vartheta\left(G_{q, n, 1}\right)$ equals this lower bound. Since $X_{q, n, k}$ is closed under conjugation, $\vartheta\left(G_{q, n, k}\right)$ can be computed by solving the linear program (D).

Conjecture 1. One has $\vartheta\left(G_{q, n, 1}\right)=\alpha\left(G_{q, n, 1}\right)=\prod_{i=1}^{n-1}\left(q^{n}-q^{i}\right)$ for all values of $n$ and $q$.

For $k>1$, we can construct independent sets in a similar way as above: Choose linearly independent vectors $x_{1}, \ldots, x_{k} \in \mathbb{F}_{q}^{n}$ and let $\mathcal{A}$ be the set of all matrices $A \in \operatorname{GL}\left(n, \mathbb{F}_{q}\right)$ such that $A x_{i}=x_{i}$ for $1 \leq i \leq k$. Then $|\mathcal{A}|=\prod_{i=k}^{n-1}\left(q^{n}-q^{i}\right)$. By computing the $\vartheta$-number for small values of $n$ and $q$ (see Table 2) we have evidence that a version of the Erdős-Ko-Rado theorem might also be true in this setting.
Conjecture 2. We conjecture that for each $q, k \in \mathbb{N}$, there exists $n_{0}=n_{0}(q, k) \in \mathbb{N}$ such that $\vartheta\left(G_{q, n, k}\right)=\alpha\left(G_{q, n, k}\right)=\prod_{i=k}^{n-1}\left(q^{n}-q^{i}\right)$ for all $n \geq n_{0}$.

The computations in Table 2have been performed with magma 3] and lpsolve 2]. As the computation of the characters of $\operatorname{GL}\left(n, \mathbb{F}_{q}\right)$ involve irrational numbers we
cannot solve the linear programs with rational arithmetic only. So these computations cannot be considered as rigorous mathematical proofs. Nevertheless we are certain that we placed checkmarks where the exact computation of $\vartheta\left(G_{q, n, k}\right)$ would give an upper bound which is equal to the corresponding lower bound.


TABLE 2. Computation of $\vartheta\left(\operatorname{Cay}\left(\Gamma, X_{q, n, k}\right)\right)$

## 6. Blowing up vertex transitive graphs

The final theorem in this note shows that for the purposes of estimating the independence number of a graph, the theory presented in the preceding sections can be applied not just to Cayley graphs, but also to vertex-transitive graphs.

Theorem 6. Let $G=(V, E)$ be a graph and let $\Gamma$ be a group of automorphisms of $G$. Suppose $\Gamma$ acts transitively on $V$. Then there exists a connection set $X \subseteq \Gamma$ such that

$$
\alpha(G)=\frac{|V|}{|\Gamma|} \alpha(\operatorname{Cay}(\Gamma, X)) .
$$

Proof. Pick a vertex $x_{0} \in V$ and define

$$
X=\left\{\gamma \in \Gamma:\left\{x_{0}, \gamma \cdot x_{0}\right\} \in E\right\}
$$

Then for $\beta, \gamma \in \Gamma$, one has an edge $\{\beta, \gamma\}$ in the Cayley graph Cay $(\Gamma, X)$ if and only if

$$
\gamma^{-1} \beta \in X \Longleftrightarrow\left\{x_{0}, \gamma^{-1} \beta \cdot x_{0}\right\} \in E \Longleftrightarrow\left\{\gamma \cdot x_{0}, \beta \cdot x_{0}\right\} \in E .
$$

Now notice that by the orbit-stabilizer theorem, one has

$$
|\{\gamma \in \Gamma: \gamma \cdot x=x\}|=\frac{|\Gamma|}{|V|} \quad \text { for all } x \in V
$$

and the theorem follows immediately.
Going from $G$ to the Cayley graph $\operatorname{Cay}(\Gamma, X)$ is accomplished using the following procedure: First choose a vertex $x_{0} \in V$ arbitrarily, and let $H$ be the stabilizer subgroup of $x_{0}$ in $\Gamma$. Each vertex $x \in V$ is then replaced with an empty graph on the left coset of $H$ in $\Gamma$ consisting of all those $\gamma \in \Gamma$ such that $\gamma \cdot x_{0}=x$. In other words, the vertex set $V$ is regarded as a $\Gamma$-homogeneous space, and each vertex is "blown up" to an independent set of size $|\Gamma| /|V|$ by replacing it with its inverse image under the projection map.

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