# A DUPLICATE PAIR IN THE SNAPPEA CENSUS 

BENJAMIN A. BURTON


#### Abstract

We identify a duplicate pair in the well-known Callahan-HildebrandWeeks census of cusped finite-volume hyperbolic 3-manifolds. Specifically, the six-tetrahedron non-orientable manifolds x101 and x103 are homeomorphic.


## 1. Introduction

The study of hyperbolic 3-manifolds was transformed by the advent of the software package SnapPea in the early 1990s [10]. At the heart of this software is the cusped hyperbolic census - a collection intended to represent all non-compact finitevolume hyperbolic 3 -manifolds that can be constructed from at most $n$ tetrahedra, for fixed $n$. Hildebrand and Weeks built the first census in 1989 for $n=5$ [6], Callahan, Hildebrand and Weeks extended it to $n=7$ in 1999 [3], and Thistlethwaite recently grew it to $n=8$ in 2010 [8].

To test whether two cusped hyperbolic 3-manifolds are isometric (and thus homeomorphic), SnapPea computes the canonical Epstein-Penner cell decomposition of each [5, 11]: the manifolds are then homeomorphic if and only if their canonical cell decompositions are combinatorially isomorphic.

There is a problem, however: SnapPea works with floating point arithmetic, and is therefore subject to issues such as round-off error and numerical instability. These issues impact upon the isometry / homeomorphism test as follows:

- If SnapPea claims that two manifolds are homeomorphic then this claim is reliable. SnapPea computes canonical cell decompositions using local Pachner moves (bistellar flips) [11, and so obtaining isomorphic cell decompositions means that SnapPea has essentially found a sequence of Pachner moves that transform one manifold into the other.
- If SnapPea claims that two manifolds are not homeomorphic, this claim may be false. Although SnapPea computes cell decompositions that are homeomorphic to their respective input manifolds, these might not be the sought-after Epstein-Penner decompositions, and so two different cell decompositions need not indicate that the two input manifolds are distinct.
As a result, it is theoretically possible that two manifolds with different names in the cusped hyperbolic census could in fact be homeomorphic. Here we show that indeed this happens: the non-orientable manifolds x101 and x103-both described using six tetrahedra in the 1999 census of Callahan, Hildebrand and Weeks-are in fact the same 3-manifold.

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Figure 1. Building x101

The proof that x101 and x103 are homeomorphic is simple: their triangulations from the census are related by just two Pachner moves (or bistellar flips). What is much more remarkable is that this duplicate pair has gone undetected for so long.

In this brief paper we describe the triangulations of x 101 and x 103 , and show why they are homeomorphic. We also describe the canonical cell decompositions that SnapPea computes (at least one of which is erroneous), in order to to better understand how SnapPea reaches its conclusion that the manifolds are distinct.

All computations were performed using the software packages Regina [1, 2] and SnapPea [10] (the latter refers to the mathematical kernel at the heart of both the original SnapPea application and its modern Python-based successor SnapPy [4]).

## 2. The duplicate pair

As is standard nowadays for hyperbolic 3-manifolds, we work with ideal triangulations. An ideal triangulation of a non-compact hyperbolic 3-manifold $\mathcal{M}$ is essentially a collection of $n$ tetrahedra whose faces are affinely identified in pairs using the quotient topology, and where (i) the link of every vertex is a torus or Klein bottle; and (ii) the manifold $\mathcal{M}$ is recovered by removing the vertices of each tetrahedron. For a richer and more precise discussion on ideal triangulations, we refer the reader to [9].

From here onward we use the labels x101 and x103 to denote explicit triangulations (since the underlying manifolds are the same). These triangulations are shipped in the cusped hyperbolic census with both Regina and SnapPy, and readers can use Regina to study their combinatorial structures in detail. Here we give a high-level outline of how each triangulation is constructed.

Definition 2.1 (Triangulation $\times 101$ ). The six-tetrahedron ideal triangulation $\times 101$ is formed as follows.


Figure 2. Building x103
(1) Build a standard five-tetrahedron cube, as shown in Figure 1(a)
(2) Attach an extra tetrahedron to the upper face of the cube to change the diagonal on the boundary, as illustrated in Figure 1(b).
(3) Identify opposite faces of the cube, as illustrated in Figure 1(c)

- join the upper and lower squares with a diagonal reflection;
- join the left and right squares with a horizontal reflection;
- join the front and back squares with a quarter-turn rotation.

In other words, we identify opposite faces according to the following maps:

$$
A B C D \longleftrightarrow G F E H ; \quad A B F E \longleftrightarrow C D H G ; \quad A D H E \longleftrightarrow C G F B .
$$

Definition 2.2 (Triangulation x103). The six-tetrahedron ideal triangulation x103 is formed as follows. Begin with x101, as described above. Let $e$ denote the diagonal edge between the cube and the extra tetrahedron, as marked in Figure 2(a)

This edge has degree four, meeting four distinct tetrahedra to form an octahedron as illustrated in Figure 2(b), Replace this octahedron with a different octahedron that has the same boundary but a different internal diagonal edge, as shown in Figure 2(c) (this operation is known as a 4-4 move).

The final triangulation of the cube is shown in Figure 2(d)] note that the shaded rectangle divides the cube into a pair of three-tetrahedron triangular prisms.

Because the identifications between opposite faces of the cube remain unchanged, the following is immediate:
Observation 2.3. Triangulations $x 101$ and $x 103$ are homeomorphic as 3 -manifolds.

Moreover, we observe that the two triangulations are "combinatorially close". We can express their relationship in terms of the well-known Pachner moves, or bistellar flips [7: since a single 4-4 move can be expressed as a $2-3$ Pachner move followed by a 3-2 Pachner move, we have:


Figure 3. The "canonical" dell decompositions computed by SnapPea

Observation 2.4. Triangulations $x 101$ and $x 103$ are related by just two Pachner moves.

The common manifold that they represent is non-orientable with first homology $H_{1}=\mathbb{Z} \oplus \mathbb{Z}_{2} \oplus \mathbb{Z}_{2}$. SnapPea approximates the hyperbolic volume as $\simeq 5.07470803$.

We finish with a brief explanation as to why SnapPea claims that the underlying manifolds for x101 and x103 are distinct. This is because SnapPea computes different "canonical" cell decompositions for each:

- For $\mathbf{x 1 0 3 , ~ S n a p P e a ~ c o m p u t e s ~ t h e ~ c a n o n i c a l ~ c e l l ~ d e c o m p o s i t i o n ~ t o ~ b e ~ a ~ s i n g l e ~}$ cube with opposite faces identified, as illustrated in Figure 3(b)
- For x101, SnapPea computes a decomposition with two cells: a cube plus a rectangular pillow, joined together as illustrated in Figure 3(a)
The cube for x103 corresponds precisely to the cube illustrated earlier in Figure 1 (c)(i.e., its opposite faces are identified in the same way). The cube and pillow for $\times 101$ correspond to the original five-tetrahedron cube plus the sixth "flat" tetrahedron from Figure 1(b)

The true Epstein-Penner decomposition is indeed the cube found from x103 (Weeks gives a simple argument to this effect based on the symmetry of the cube and its face identifications from Figure 1(c)). For x101, it seems a reasonable conclusion that the numerical errors that SnapPea experiences when computing the canonical cell decomposition are due to the flat tetrahedron that we attach in Figure 1(b)

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School of Mathematics and Physics, The University of Queensland, Brisbane QLD 4072, Australia

E-mail address: bab@maths.uq.edu.au


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