# A new multidimensional slow continued fraction algorithm and stepped surface 

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#### Abstract

We give a new algorithm of slow continued fraction expansion related to any real cubic number field as a 2-dimensional version of the Farey map. Using our algorithm, we can find the generators of dual substitutions (so-called tiling substitutions) for any stepped surface for any cubic direction.


## 1 Introduction

The main topics of this paper are
(i) to find a good algorithm of slow continued fraction expansion of dimension 2 by which the expansion of $\bar{\alpha}=\left(\alpha_{1}, \alpha_{2}\right)$ is always expected to be periodic for any $\mathbb{Q}$-basis $\overline{\bar{\alpha}}=\left(1, \alpha_{1}, \alpha_{2}\right)$ of an arbitrarily given real cubic number field $K$ such that the unimodular

[^0]matrix $\mathcal{P e r}$ coming from a period of the expansion has the minimal polynomial of a Pisot number as its characteristic polynomial,
and
(ii) to find a set of generators of dual substitutions for the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ of dimension 2 for any $\mathbb{Q}$-basis $\overline{\bar{\alpha}}$ of $K$ and to give a finite description (or an effective construction) in terms of six dual substitutions coming from the continued fraction expansion obtained by our algorithms, see Section 3, 5 for notation.

Notice that by the symmetry of the lattice $\mathbb{Z}^{s}$, we may assume that $\alpha_{0}, \alpha_{1}, \alpha_{2}>0$ with $\alpha_{0}+\alpha_{1}+\alpha_{2}=1$.

The stepped surface $\mathscr{S}(\overline{\bar{\alpha}}),\left(\overline{\bar{\alpha}}=\left(\alpha_{0}, \alpha_{1}, \ldots, \alpha_{s}\right)\right)$ was introduced as a $s$-dimensional version of the sturmian word, cf. [Ito and Ohtsuki 93, Ito and Ohtsuki 94, Arnoux and Ito 01]. The stepped surface of dimension $s=2$ is of special interest, since it is not only a geometrical object related to an aperiodic tiling, but also it has a connection with number theoretical problems related to simultaneous Diophantine approximations which are best possible up to constant, cf. Ito et al. 03. If $\overline{\bar{\alpha}} \in K^{s+1}$ is a $\mathbb{Q}$-basis of certain real algebraic number field $K$ of degree $s+1$, there is a beautiful connection between the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ and the continued fraction expansion of $\bar{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{s}\right) \in$ $K^{s}$ provided that the continued fraction is periodic related to the Jacobi-Perron algorithm or the Brun algorithm (including the so-called modified Jacobi-Perron algorithm as its 2-dimensional case). There appeared many papers concerning the construction of the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ for $\bar{\alpha}$ having a periodic continued fraction expansion, cf. Ito and Ohtsuki 93, Ito and Ohtsuki 94, Fujita et al. 00, Arnoux and Ito 01, Arnoux et al. 02, Fernique 05, Berthé and Fernique 11.

On the other hand, it has been a difficult problem to construct even a part of the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ for some $\mathbb{Q}$-basis $\overline{\bar{\alpha}}$ of some real algebraic number field $K$, since there have not been a good deterministic algorithm to get a periodic continued fraction of $\bar{\alpha}$ for $s \geq 2$. In a series of papers (Tamura 04, Tamura and Yasutomi 09, Tamura and Yasutomi 10, Tamura and Yasutomi 12, Tamura and Yasutomi 11, Tamura and Yasutomi 12]) we made some good candidates of continued fraction algorithms for $1 \leq s \leq 4$, equipped with a value function $v$ (see Section (2) by which the algorithms become deterministic.

The properties/conditions of the matrix $\mathcal{P e r}$ mentioned in (i), i.e., the unimodularity, the Pisot property and the irreducibility are essential; in fact, under these conditions, the stepped surface becomes finitely descriptive, cf. Theorem 5 in Fernique 05 due to T. Fernique.

Concerning our algorithm of continued fraction expansion of dimension $s=2$, which is expected to have the properties mentioned in (i) above, let us consider the case where
$s=1$ for a while. The algorithm $([0,1], T, \varepsilon)$ and its modified version are considered by many authors (for example see [Ito 89, Ito and Yasutomi 90]), where the transformation $T$ on the interval $[0,1]$ is defined by

$$
T(x):= \begin{cases}\frac{x}{1-x} & \text { if } x \in I_{0}:=\left[0, \frac{1}{2}\right] \\ \frac{2 x-1}{x} & \text { if } x \in I_{1}:=\left(\frac{1}{2}, 1\right]\end{cases}
$$

$\varepsilon:[0,1] \rightarrow\{0,1\}$ with $x \in I_{\varepsilon(x)}$. Let $x=\left[0 ; k_{1}, k_{2}, \ldots\right]$ be the simple continued fraction expansion, and let $F$ be the transformation on $(0,1]$ defined by $F(x)=\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor$. Then, one can see that $T^{k_{1}+k_{2}}=F^{2}(x)$ holds. In this sense, $([0,1], T, \varepsilon)$ can be considered as a kind of slow continued fraction algorithm. $([0,1], T, \varepsilon)$ also has a connection with the Farey partition. The famous Lagrange's theorem says that, if we consider the restriction $T_{K}$ of $T$ on $[0,1] \cap K$ for a real quadratic number field $K$, then every element $\alpha \in[0,1] \cap K$ becomes a periodic point of $T_{K}$, i.e., there exists $m \neq n \in \mathbb{Z}_{>0}$ such that $T_{K}^{m}(\alpha)=$ $T_{K}^{n}(\alpha)$. The expansion obtained by the slow continued fraction algorithm ( $[0,1], T, \varepsilon$ ) can be considered as an infinite word $\varepsilon(x) \varepsilon(T(x)) \varepsilon\left(T^{2}(x)\right) \cdots$ over an alphabet $\{0,1\}$. Consequently, the generators of the dual substitutions on the stepped surface of dimension 1 (the sturmian word of dimension 1 ) consist of $\#\{0,1\}=2$ primitive substitutions. We can extend the algorithm $([0,1], T, \varepsilon)$ to certain multidimensional algorithms.

In Section 2, we define a new deterministic algorithms of slow (additive) continued fraction of dimension 2 equipped with a value function $v=v_{r}(\alpha, \beta, i, j)$. The resulting expansion by our algorithm can be considered as an infinite word over an alphabet Ind given by

$$
\text { Ind }:=\{(i, j) \mid i, j \in\{0,1,2\}, i \neq j\}
$$

consequently the generators of the dual substitutions on the stepped surface of dimension 2 consist of $\# \operatorname{Ind}=6$ primitive substitutions.

Theorem 2.2 says the additivity of our algorithms. Theorems 2.5, 2.6, 2.7 give some admissibility conditions of the expansion obtained by our algorithms. Proposition 2.8 says the convergence of the continued fractions.

Theorem 2.9 gives infinitely many examples of periodic expansions obtained by one of the algorithms.

In Section 3, we translate the expansion obtained by our algorithms into canonical representations of continued fractions of dimension 2, cf. Theorem 3.2. We also give reduction rules by which we can make acceleration of our continued fractions.

In Section 4, we made the periodicity test for one of our algorithms $(r=5 / 2)$, cf. Table 4. We also gave an experiment, by using PC for pure cubic extensions $K=$
$\mathbb{Q}(\sqrt[3]{d}), 2 \leq d \leq 10000, \sqrt[3]{d} \notin \mathbb{Q}$, which supports Conjecture 7.1, that is a cubic version of Lagrange's theorem. We also checked that 18797 continued fraction expansions obtained by our algorithm with $r=5 / 2$ coming from the set $N_{15}$ are periodic, and the matrix $\mathcal{P e r}$ always has the minimal polynomial of a Pisot number as its characteristic polynomial. Such a Pisot property was supported by another independent experiment for around 10000 continued fractions obtained by random generation of totally real cubic number fields.

In Section 5, we give some experiments which describe the generating process of the whole part of some stepped surfaces in terms of dual substitutions (or tiling substitutions) related to some real cubic number field $K$ (including both totally real fields and fields having complex embeddings).

In Section 7, we give two conjectures. Under these two conjectures together with Fernique's result (Theorem 5 in Fernique 05 mentioned above) we shall see that the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ becomes finitely descriptive by using only six dual substitutions for any $\mathbb{Q}$-basis $\overline{\bar{\alpha}}$ of any given real cubic number field, see Conclusion 7.5.

## 2 A new algorithm

In what follows, $K$ denotes arbitrarily chosen fixed real cubic number field unless otherwise mentioned. We put

$$
\Delta_{K}:=\left\{\begin{array}{l|l}
(\alpha, \beta) \in K^{2} & \begin{array}{l}
1, \alpha, \beta \text { are linearly independent over } \mathbb{Q} \\
0<\alpha, \beta \text { and } \alpha+\beta<1
\end{array}
\end{array}\right\}
$$

We need a following lemma.
Lemma 2.1. Let $p, q \in \mathbb{Z}^{+},(p, q)=1$ and $p \not \equiv 0(\bmod 3)$. Then,

$$
\frac{|N(\alpha)|}{\alpha^{p / q}}=\frac{|N(\beta)|}{\beta^{p / q}} \text { implies } \alpha=\beta,
$$

for all $\alpha, \beta>0$ such that $\alpha, \beta \in K \backslash \mathbb{Q}$.
Proof. We suppose that $\frac{|N(\alpha)|}{\alpha^{p / q}}=\frac{|N(\beta)|}{\beta^{p / q}}$ holds for $\alpha, \beta \in K$ with $\alpha, \beta \notin \mathbb{Q}$ and $\alpha, \beta>0$. Let $\zeta=\alpha / \beta$. Then, we have $\zeta^{p}=|N(\zeta)|^{q}$. Therefore, we see that $N(\zeta)^{p}=|N(\zeta)|^{3 q}$. Since $N(\zeta)$ is a rational number and $p$ is not divisible by 3 , we see $|N(\zeta)|=1$. Thus, we have $\zeta^{p}=1$, which implies $\zeta= \pm 1$. Since $\alpha, \beta>0, \zeta=1$.

We put

$$
\text { Ind }:=\{(i, j) \mid i, j \in\{0,1,2\}, i \neq j\} .
$$

We denote domains $\Delta$ and $\triangle(i, j)$ for $(i, j) \in I n d$ by

$$
\begin{aligned}
& \Delta:=\left\{(x, y) \in \mathbb{R}^{2} \mid x, y \geq 0, x+y \leq 1\right\} \\
& \triangle(1,2):=\{(x, y) \in \Delta \mid x \geq y\} \\
& \triangle(2,1):=\{(x, y) \in \Delta \mid x \leq y\} \\
& \triangle(0,1):=\{(x, y) \in \Delta \mid 2 x+y-1 \leq 0\} \\
& \triangle(1,0):=\{(x, y) \in \Delta \mid 2 x+y-1 \geq 0\} \\
& \triangle(0,2):=\{(x, y) \in \Delta \mid x+2 y-1 \leq 0\} \\
& \triangle(2,0):=\{(x, y) \in \Delta \mid x+2 y-1 \geq 0\}
\end{aligned}
$$

(see Figure 11).


Figure 1: The domains $\triangle(i, j),(i, j) \in \operatorname{Ind}$.
For each $(i, j) \in I n d$, let us introduce the maps $T_{(i, j)}: \Delta(i, j) \rightarrow \Delta$ as follows:

$$
\begin{aligned}
T_{(1,2)}(x, y) & :=\left(\frac{x-y}{1-y}, \frac{y}{1-y}\right), \\
T_{(2,1)}(x, y) & :=\left(\frac{x}{1-x}, \frac{y-x}{1-x}\right), \\
T_{(0,1)}(x, y) & :=\left(\frac{x}{1-x}, \frac{y}{1-x}\right), \\
T_{(1,0)}(x, y) & :=\left(\frac{2 x+y-1}{x+y}, \frac{y}{x+y}\right), \\
T_{(0,2)}(x, y) & :=\left(\frac{x}{1-y}, \frac{y}{1-y}\right), \\
T_{(2,0)}(x, y) & :=\left(\frac{x}{x+y}, \frac{x+2 y-1}{x+y}\right) .
\end{aligned}
$$

We define the value $v_{r}(\alpha, \beta, i, j)$ for $r \in \mathbb{R}^{+},(\alpha, \beta) \in \Delta_{K}$ and $i, j \in\{0,1,2\}$ with
$i \neq j$ as follows:

$$
v_{r}(\alpha, \beta, i, j):= \begin{cases}\frac{\left|\alpha^{r} \beta^{r}\right|}{|N(\alpha) N(\beta)|} & \text { if }\{i, j\}=\{1,2\}, \\ \frac{\left|\alpha^{r}(1-\alpha-\beta)^{r}\right|}{|N(\alpha) N(1-\alpha-\beta)|} & \text { if }\{i, j\}=\{0,1\}, \\ \frac{\left|\beta^{r}(1-\alpha-\beta)^{r}\right|}{|N(\beta) N(1-\alpha-\beta)|}, & \text { if }\{i, j\}=\{0,2\} .\end{cases}
$$

In what follows, we suppose that $r=p / q$ with $p, q \in \mathbb{Z}^{+},(p, q)=1$ and $p \not \equiv 0 \bmod 3$ in this paper.

It follows from Lemma 2.1 that the element $\left(i_{0}, j_{0}\right) \in$ Ind is uniquely determined by $v_{r}\left(\alpha, \beta, i_{0}, j_{0}\right)=\max \left\{v_{r}(\alpha, \beta, i, j)\right\}$. We define $\varepsilon(\alpha, \beta)=\varepsilon_{K}(\alpha, \beta)$ for $(\alpha, \beta) \in \Delta_{K}$ by

$$
\varepsilon(\alpha, \beta):= \begin{cases}(1,2) & \text { if }\left\{i_{0}, j_{0}\right\}=\{1,2\} \text { and }(\alpha, \beta) \in \triangle(1,2) \\ (2,1) & \text { if }\left\{i_{0}, j_{0}\right\}=\{1,2\} \text { and }(\alpha, \beta) \in \triangle(2,1) \\ (0,1) & \text { if }\left\{i_{0}, j_{0}\right\}=\{0,1\} \text { and }(\alpha, \beta) \in \triangle(0,1), \\ (1,0) & \text { if }\left\{i_{0}, j_{0}\right\}=\{0,1\} \text { and }(\alpha, \beta) \in \triangle(1,0), \\ (0,2) & \text { if }\left\{i_{0}, j_{0}\right\}=\{0,2\} \text { and }(\alpha, \beta) \in \triangle(0,2) \\ (2,0) & \text { if }\left\{i_{0}, j_{0}\right\}=\{0,2\} \text { and }(\alpha, \beta) \in \triangle(2,0) .\end{cases}
$$

Notice that $\varepsilon(\alpha, \beta)$ is well-defined since $1, \alpha, \beta$ is linearly independent over $\mathbb{Q}$. We define the transformation $T=T_{K}=T_{K, r}$ on $\Delta_{K}$ by

$$
T(\alpha, \beta):=T_{\left(i_{0}, j_{0}\right)}(\alpha, \beta) \text { if } \varepsilon(\alpha, \beta)=\left(i_{0}, j_{0}\right)
$$

Thus, we have seen that an algorithm $\left(\Delta_{K}, T, \varepsilon\right)$ can be defined. We put

$$
\begin{array}{ll}
A_{(1,2)}:=\left(\begin{array}{lll}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right), & A_{(2,1)}:=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{array}\right), \\
A_{(0,1)}:=\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), & A_{(1,0)}:=\left(\begin{array}{ccc}
2 & -1 & -1 \\
1 & 0 & -1 \\
0 & 0 & 1
\end{array}\right), \\
A_{(0,2)}:=\left(\begin{array}{lll}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), & A_{(2,0)}:=\left(\begin{array}{ccc}
2 & -1 & -1 \\
0 & 1 & 0 \\
1 & -1 & 0
\end{array}\right) . \tag{3}
\end{array}
$$

For $n \in \mathbb{Z}_{\geq 0}$ we define $\left(\alpha_{n}, \beta_{n}\right)=T^{n}(\alpha, \beta)$. For $n \in \mathbb{Z}_{>0}$ we define

$$
M_{n}(\alpha, \beta)=\left(\begin{array}{ccc}
p_{n}^{\prime \prime}(\alpha, \beta) & p_{n}^{\prime}(\alpha, \beta) & p_{n}(\alpha, \beta)  \tag{4}\\
q_{n}^{\prime \prime}(\alpha, \beta) & q_{n}^{\prime}(\alpha, \beta) & q_{n}(\alpha, \beta) \\
r_{n}^{\prime \prime}(\alpha, \beta) & r_{n}^{\prime}(\alpha, \beta) & r_{n}(\alpha, \beta)
\end{array}\right):=A_{\varepsilon\left(\alpha_{0}, \beta_{0}\right)} \cdots A_{\varepsilon\left(\alpha_{n-1}, \beta_{n-1}\right)} S,
$$

where

$$
S:=\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

Then, we have following Theorem 2.2.
Theorem 2.2. For $(\alpha, \beta) \in \Delta_{K}$ and $n \in \mathbb{Z}_{\geq 0}$

$$
\begin{align*}
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(1,2) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right)+\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right),
\end{array}\right.  \tag{5}\\
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(2,1) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right)+\left(p_{n}, q_{n}, r_{n}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right),
\end{array}\right.  \tag{6}\\
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(0,1) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right)+\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right),
\end{array}\right.  \tag{7}\\
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(1,0) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right)+\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right),
\end{array}\right.  \tag{8}\\
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(0,2) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right)+\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right),
\end{array}\right.  \tag{9}\\
& \varepsilon\left(\alpha_{n}, \beta_{n}\right)=(2,0) \Rightarrow\left\{\begin{array}{l}
\left(p_{n+1}, q_{n+1}, r_{n+1}\right)=\left(p_{n}, q_{n}, r_{n}\right), \\
\left(p_{n+1}^{\prime}, q_{n+1}^{\prime}, r_{n+1}^{\prime}\right)=\left(p_{n}^{\prime}, q_{n}^{\prime}, r_{n}^{\prime}\right), \\
\left(p_{n+1}^{\prime \prime}, q_{n+1}^{\prime \prime}, r_{n+1}^{\prime \prime}\right)=\left(p_{n}^{\prime \prime}, q_{n}^{\prime \prime}, r_{n}^{\prime \prime}\right)+\left(p_{n}, q_{n}, r_{n}\right) .
\end{array}\right. \tag{10}
\end{align*}
$$

Proof. Let $\varepsilon\left(\alpha_{n}, \beta_{n}\right)=(1,2)$. Then, we get

$$
\begin{aligned}
& \left(\begin{array}{lll}
p_{n+1}^{\prime \prime}(\alpha, \beta) & p_{n+1}^{\prime}(\alpha, \beta) & p_{n+1}(\alpha, \beta) \\
q_{n+1}^{\prime \prime}(\alpha, \beta) & q_{n+1}^{\prime}(\alpha, \beta) & q_{n+1}(\alpha, \beta) \\
r_{n+1}^{\prime \prime}(\alpha, \beta) & r_{n+1}^{\prime}(\alpha, \beta) & r_{n+1}(\alpha, \beta)
\end{array}\right) \\
& =A_{\varepsilon\left(\alpha_{0}, \beta_{0}\right)} \ldots A_{\varepsilon\left(\alpha_{n}, \beta_{n}\right)} S=A_{\varepsilon\left(\alpha_{0}, \beta_{0}\right)} \ldots A_{\varepsilon\left(\alpha_{n-1}, \beta_{n-1}\right)}\left(\begin{array}{lll}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right) \\
& \quad=\left(\begin{array}{lll}
p_{n}^{\prime \prime}(\alpha, \beta) & p_{n}^{\prime}(\alpha, \beta) & p_{n}(\alpha, \beta)+p_{n}^{\prime}(\alpha, \beta) \\
q_{n}^{\prime \prime}(\alpha, \beta) & q_{n}^{\prime}(\alpha, \beta) & q_{n}(\alpha, \beta)+q_{n}^{\prime}(\alpha, \beta) \\
r_{n}^{\prime \prime}(\alpha, \beta) & r_{n}^{\prime}(\alpha, \beta) & r_{n}(\alpha, \beta)+r_{n}^{\prime}(\alpha, \beta)
\end{array}\right) .
\end{aligned}
$$

Thus, we have (5). We have (6)-(9) in the similar manner.

We shall give some theorems concerning matrices $A_{(i, j)}$. We need some definitions. Let $\mathbb{P}_{2}(\mathbb{R})$ be the projective space of dimension 2 over $\mathbb{R}$, i.e.,

$$
\mathbb{P}_{2}(\mathbb{R})=\left(\mathbb{R}^{3} \backslash\{\overline{\overline{0}}\}\right) / \sim,
$$

where $\sim$ is an equivalence relation defined by

$$
\overline{\bar{x}} \sim \overline{\bar{y}} \quad\left(\overline{\bar{x}}, \overline{\bar{y}} \in \mathbb{R}^{3} \backslash\{\overline{\overline{0}}\}\right) \Leftrightarrow 0 \neq \exists c \in \mathbb{R} \text { such that } \overline{\bar{x}}=c \overline{\bar{y}} .
$$

We mean by $f: X-\rightarrow Y$ a "map" from a set $X$ to a set $Y$ with some exceptional elements $x \in X$ for which the value $f(x)$ is not defined. For a matrix $A \in M_{3}(\mathbb{R})$ (which denotes the set of $3 \times 3$ matrices of real components), a map

$$
A^{\text {proj }}: \mathbb{P}_{2}(\mathbb{R})-\rightarrow \mathbb{P}_{2}(\mathbb{R})
$$

can be defined by

$$
A^{\left.\operatorname{proj}^{( } \kappa(\overline{\bar{x}})\right):=\kappa A(\overline{\bar{x}}), ~}
$$

where $\kappa(\overline{\bar{x}}):=\{c \overline{\bar{x}} \mid 0 \neq c \in \mathbb{R}\} \in \mathbb{P}_{2}(\mathbb{R})$. Notice that the map $A^{p r o j}$ is well-defined and

$$
\begin{equation*}
(A B)^{p r o j}=A^{p r o j} B^{p r o j} \tag{11}
\end{equation*}
$$

holds for $A, B \in M_{3}(\mathbb{R})$. We define two maps $\pi: \mathbb{P}_{2}(\mathbb{R})-\rightarrow \mathbb{R}^{2}$ and $\iota: \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ by

$$
\begin{aligned}
\pi(\kappa(\overline{\bar{x}})) & :=\frac{1}{x^{(0)}}\binom{x^{(1)}}{x^{(2)}} \quad \text { for } \overline{\bar{x}}=^{t}\left(x^{(0)}, x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{3}, \\
\iota(\bar{x}) & :={ }^{t}\left(1, x^{(1)}, x^{(2)}\right) \quad \text { for } \bar{x}=\left(x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{2} .
\end{aligned}
$$

Then, the linear fractional map $A^{f r a c}: \mathbb{R}^{2}-\rightarrow \mathbb{R}^{2}\left(A \in M_{3}(\mathbb{R})\right)$ can be defined by

$$
A^{\text {frac }}(\bar{x}):=\pi A^{\operatorname{proj}_{\kappa \iota}(\bar{x}) .}
$$

Notice that this map is also well-defined and the diagram (see Figure 2) commutes.


Figure 2: The commutative diagram with respect to $A^{\text {proj }}$ and $A^{\text {frac. }}$.

Hence, in view of (11), we get the following.

## Lemma 2.3.

$$
(A B)^{f r a c}=A^{f r a c} B^{f r a c} \quad\left(A, B \in M_{3}(\mathbb{R})\right)
$$

We easily see following Theorem 2.4.
Theorem 2.4. For each $(i, j) \in \operatorname{Ind}, A_{(i, j)}^{f r a c} \circ T_{(i, j)}\left(\right.$ resp. $\left.T_{(i, j)} \circ A_{(i, j)}^{f r a c}\right)$ is an identity map on $\triangle(i, j)$ (resp., $\Delta$ ).

For each $(\alpha, \beta) \in \Delta_{K}$ and $n \in \mathbb{Z}_{\geq 0}$ we define $\delta_{n}(\alpha, \beta)$ by the set of all inner points in a triangle with the edge points $\left(q_{n} / p_{n}, r_{n} / p_{n}\right),\left(q_{n}^{\prime} / p_{n}^{\prime}, r_{n}^{\prime} / p_{n}^{\prime}\right)$ and $\left(q_{n}^{\prime \prime} / p_{n}^{\prime \prime}, r_{n}^{\prime \prime} / p_{n}^{\prime \prime}\right)$. From Lemma 2.3 and Theorem [2.4 we have following:

Theorem 2.5. Let $(\alpha, \beta) \in \Delta_{K}$. For each $n \in \mathbb{Z}_{\geq 0},(\alpha, \beta) \in \delta_{n}(\alpha, \beta)$ holds.

The following theorem describes an admissibility of the sequence $\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0,1, \ldots}$ obtained by the algorithm $\left(\Delta_{K}, T, \varepsilon\right)$.

Theorem 2.6. Let $(\alpha, \beta)=\left(\alpha_{0}, \beta_{0}\right)$ and $n \in \mathbb{Z}_{\geq 0}$. Then, both

$$
\begin{aligned}
& \varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right) \neq\left(i^{\prime}, \theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right)\right) \\
& \text { and } \\
& \varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right) \neq\left(\theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right), i^{\prime}\right)
\end{aligned}
$$

hold, where $\varepsilon\left(\alpha_{n}, \beta_{n}\right)=\left(i^{\prime}, j^{\prime}\right)$ and $\theta(i, j):=k \in\{0,1,2\}$ with $k \neq i$ and $k \neq j$, so that there are 12 forbidden words $\varepsilon\left(\alpha_{n}, \beta_{n}\right) \varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right)$ (see Table 1).

Table 1: The forbidden words of length 2.

| $\varepsilon\left(\alpha_{n}, \beta_{n}\right)$ | $\varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right)$ |
| :---: | :---: |
| $(1,2)$ | $(0,1),(1,0)$ |
| $(2,1)$ | $(0,2),(2,0)$ |
| $(0,1)$ | $(0,2),(2,0)$ |
| $(1,0)$ | $(1,2),(2,1)$ |
| $(0,2)$ | $(0,1),(1,0)$ |
| $(2,0)$ | $(1,2),(2,1)$ |

Proof. First, we suppose that $\varepsilon\left(\alpha_{n}, \beta_{n}\right)=(2,1)$. Let $\gamma_{n}=1-\alpha_{n}-\beta_{n}$. From the definition of the value function, we have

$$
\begin{equation*}
\frac{\left|\alpha_{n}^{r} \beta_{n}^{r}\right|}{\left|N\left(\alpha_{n}\right) N\left(\beta_{n}\right)\right|}>\frac{\left|\alpha_{n}^{r} \gamma_{n}^{r}\right|}{\left|N\left(\alpha_{n}\right) N\left(\gamma_{n}\right)\right|}, \quad \frac{\left|\alpha_{n}^{r} \beta_{n}^{r}\right|}{\left|N\left(\alpha_{n}\right) N\left(\beta_{n}\right)\right|}>\frac{\left|\beta_{n}^{r} \gamma_{n}^{r}\right|}{\left|N\left(\beta_{n}\right) N\left(\gamma_{n}\right)\right|} \tag{12}
\end{equation*}
$$

(12) is equivalent to

$$
\begin{equation*}
\frac{\left|\beta_{n}\right|^{r}}{\left|N\left(\beta_{n}\right)\right|}>\frac{\left|\gamma_{n}\right|^{r}}{\left|N\left(\gamma_{n}\right)\right|}, \quad \frac{\left|\alpha_{n}\right|^{r}}{\left|N\left(\alpha_{n}\right)\right|}>\frac{\left|\gamma_{n}\right|^{r}}{\left|N\left(\gamma_{n}\right)\right|^{2}} . \tag{13}
\end{equation*}
$$

Moreover, let $\left(\alpha_{n+1}, \beta_{n+1}\right)$ and $\gamma_{n+1}$ be

$$
\begin{aligned}
\left(\alpha_{n+1}, \beta_{n+1}\right) & :=T\left(\alpha_{n}, \beta_{n}\right)=\left(\frac{\alpha_{n}}{1-\alpha_{n}}, \frac{\beta_{n}-\alpha_{n}}{1-\alpha_{n}}\right) \\
\gamma_{n+1} & :=1-\alpha_{n+1}-\beta_{n+1}=\frac{\gamma_{n}}{1-\alpha_{n}}
\end{aligned}
$$

We suppose that $\varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right)=\left(2, \theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right)\right)$ or $\varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right)=$ $\left(\theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right), 2\right)$, i.e., $\varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right) \in\{(2,0),(0,2)\}$. Then, from the analogous above discussion, we get

$$
\begin{aligned}
& \frac{\left|\beta_{n+1}^{r} \gamma_{n+1}^{r}\right|}{\left|N\left(\beta_{n+1}\right) N\left(\gamma_{n+1}\right)\right|}>\frac{\left|\alpha_{n+1}^{r} \beta_{n+1}^{r}\right|}{\left|N\left(\alpha_{n+1}\right) N\left(\beta_{n+1}\right)\right|}, \\
& \frac{\left|\beta_{n+1}^{r} \gamma_{n+1}^{r}\right|}{\left|N\left(\beta_{n+1}\right) N\left(\gamma_{n+1}\right)\right|}>\frac{\left|\alpha_{n+1}^{r} \gamma_{n+1}^{r}\right|}{\left|N\left(\alpha_{n+1}\right) N\left(\gamma_{n+1}\right)\right|},
\end{aligned}
$$

i.e.,

$$
\begin{equation*}
\frac{\left|\gamma_{n+1}^{r}\right|}{\left|N\left(\gamma_{n+1}\right)\right|}>\frac{\left|\alpha_{n+1}^{r}\right|}{\left|N\left(\alpha_{n+1}\right)\right|}, \quad \frac{\left|\beta_{n+1}^{r}\right|}{\left|N\left(\beta_{n+1}\right)\right|}>\frac{\left|\alpha_{n+1}^{r}\right|}{\left|N\left(\alpha_{n+1}\right)\right|} \tag{14}
\end{equation*}
$$

From $N(\alpha \beta)=N(\alpha) N(\beta)$,(14) is written by

$$
\frac{\left|\frac{\gamma_{n}}{1-\alpha_{n}}\right|^{r}}{\left|N\left(\frac{\gamma_{n}}{1-\alpha_{n}}\right)\right|}>\frac{\left|\frac{\alpha_{n}}{1-\alpha_{n}}\right|^{r}}{\left|N\left(\frac{\alpha_{n}}{1-\alpha_{n}}\right)\right|}, \quad \frac{\left|\frac{\beta_{n}-\alpha_{n}}{1-\alpha_{n}}\right|^{r}}{\left|N\left(\frac{\beta_{n}-\alpha_{n}}{1-\alpha_{n}}\right)\right|}>\frac{\left|\frac{\alpha_{n}}{1-\alpha_{n}}\right|^{r}}{\left|N\left(\frac{\alpha_{n}}{1-\alpha_{n}}\right)\right|}
$$

i.e.,

$$
\begin{equation*}
\frac{\left|\gamma_{n}\right|^{r}}{\left|N\left(\gamma_{n}\right)\right|}>\frac{\left|\alpha_{n}\right|^{r}}{\left|N\left(\alpha_{n}\right)\right|}, \quad \frac{\left|\beta_{n}-\alpha_{n}\right|^{r}}{\left|N\left(\beta_{n}-\alpha_{n}\right)\right|}>\frac{\left|\alpha_{n}\right|^{r}}{\left|N\left(\alpha_{n}\right)\right|} \tag{15}
\end{equation*}
$$

But, (15) contradicts (13). Therefore,

$$
\varepsilon\left(\alpha_{n+1}, \beta_{n+1}\right) \neq\left(2, \theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right)\right),\left(\theta\left(\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right), 2\right)
$$

holds. The other cases can be proved analogously.

Theorem 2.6] says that there are some forbidden words $(i, j)$ in the sequences $\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ obtained by the algorithm $\left(\Delta_{K}, T, \varepsilon\right)$. On the other hand, there exists $(\alpha, \beta) \in \Delta_{K}$ such that the other words of length 2 except for the forbidden words eventually appear in the sequences $\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ for any real cubic field $K$ :

Theorem 2.7. For $(i, j),(k, l) \in \operatorname{Ind}$, if $(k, l) \neq(i, \theta(i, j))$ and $(k, l) \neq(\theta(i, j), i)$, then there exists $(\alpha, \beta) \in \Delta_{K}$ such that $\varepsilon_{K}(\alpha, \beta)=(i, j)$ and $\varepsilon_{K}\left(T_{K}(\alpha, \beta)\right)=(k, l)$.

Proof. Let $\lambda$ be the root of $x^{3}-5 x+1$ with $\lambda>1$. Let $K_{1}=\mathbb{Q}(\lambda)$. We note that $K_{1}$ is a totally real cubic field. By the direct calculation one can check Table 2 given below:

Table 2: (totally real case) The words of length 2 which are not forbidden by Table 1 eventually occur.

| $(\zeta, \eta)$ | $\varepsilon_{K_{1}}(\zeta, \eta)$ | $T_{K_{1}}(\zeta, \eta)$ | $\varepsilon_{K_{1}}\left(T_{K_{1}}(\zeta, \eta)\right)$ |
| :---: | :---: | :---: | :---: |
| $\left(5 / 39+7 \lambda / 39-2 \lambda^{2} / 39\right.$, | $(2,1)$ | $\left(6 / 37+6 \lambda / 37-\lambda^{2} / 37\right.$, | $(2,1)$ |
| $\left.2 / 39-5 \lambda / 39+7 \lambda^{2} / 39\right)$ |  | $\left.-5 / 37-5 \lambda / 37+7 \lambda^{2} / 37\right)$ |  |
| $\left(6 / 37+6 \lambda / 37-\lambda^{2} / 37\right.$, | $(2,1)$ | $(1 / 5+\lambda / 5$, | $(1,0)$ |
| $\left.-5 / 37-5 \lambda / 37+7 \lambda^{2} / 37\right)$ |  | $\left.-2 / 5-\lambda / 5+\lambda^{2} / 5\right)$ |  |
| $\left(-16 / 15+\lambda / 15+4 \lambda^{2} / 15\right.$, | $(2,1)$ | $\left(-9 / 17+\lambda / 17+3 \lambda^{2} / 17\right.$, | $(0,1)$ |
| $\left.4 / 5+\lambda / 5-\lambda^{2} / 5\right)$ |  | $\left.15 / 17+4 \lambda / 17-5 \lambda^{2} / 17\right)$ |  |
| $\left(4 / 5+\lambda / 5-\lambda^{2} / 5\right.$, | $(2,1)$ | $\left(5-\lambda^{2}\right.$, | $(1,2)$ |
| $\left.-4 / 15+4 \lambda / 15+\lambda^{2} / 15\right)$ |  | $\left.-19 / 3+\lambda / 3+4 \lambda^{2} / 3\right)$ |  |

We suppose that $K$ is a totally real cubic field. Let ${ }_{1} \rho_{K},{ }_{2} \rho_{K}$ be distinct embeddings from $K$ to $\mathbb{R}$ over $\mathbb{Q}$ different from the non trivial embedding. We define a mapping $\rho_{K}$ from $\Delta_{K}$ to $\Delta \times \mathbb{R}^{4}$ as follows. For $(\alpha, \beta) \in \Delta_{K}$,

$$
\rho_{K}(\alpha, \beta):=\left(\alpha, \beta,{ }_{1} \rho_{K}(\alpha),{ }_{1} \rho_{K}(\beta),{ }_{2} \rho_{K}(\alpha),{ }_{2} \rho_{K}(\beta)\right) .
$$

Then, it is not difficult to see that $\rho_{K}\left(\Delta_{K}\right)$ is dense in $\Delta \times \mathbb{R}^{4}$ by virtue of algebraic number theory (for example see Chapter 1 in [Neukirch 99]). We put $\left(\alpha^{\prime}, \beta^{\prime}\right)=(5 / 39+$ $\left.7 \lambda / 39-2 \lambda^{2} / 39,2 / 39-5 \lambda / 39+7 \lambda^{2} / 39\right)$ which is in Table 2. Then, by the analogous discussion in Theorem 2.6, one can show

$$
\begin{equation*}
\frac{\left|\beta^{\prime}\right|^{r}}{\left|N\left(\beta^{\prime}\right)\right|}>\frac{\left|\gamma^{\prime}\right|^{r}}{\left|N\left(\gamma^{\prime}\right)\right|}, \quad \frac{\left|\alpha^{\prime}\right|^{r}}{\left|N\left(\alpha^{\prime}\right)\right|}>\frac{\left|\gamma^{\prime}\right|^{r}}{\left|N\left(\gamma^{\prime}\right)\right|} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta^{\prime}>2 \alpha^{\prime}, \quad \frac{\left|\beta^{\prime}-\alpha^{\prime}\right|^{r}}{\left|N\left(\beta^{\prime}-\alpha^{\prime}\right)\right|}>\frac{\left|\gamma^{\prime}\right|^{r}}{\left|N\left(\gamma^{\prime}\right)\right|}, \tag{17}
\end{equation*}
$$

where $\gamma^{\prime}=1-\alpha^{\prime}-\beta^{\prime}$. We see that there exists $\delta>0$ such that $\left|x-\alpha^{\prime}\right|<\delta,\left|y-\beta^{\prime}\right|<\delta$, $\left|x^{\prime}-{ }_{1} \rho_{K_{1}}\left(\alpha^{\prime}\right)\right|<\delta,\left|y^{\prime}-{ }_{1} \rho_{K_{1}}\left(\beta^{\prime}\right)\right|<\delta$, $\left|x^{\prime \prime}-{ }_{2} \rho_{K_{1}}\left(\alpha^{\prime}\right)\right|<\delta$ and $\left|y^{\prime \prime}-{ }_{2} \rho_{K_{1}}\left(\beta^{\prime}\right)\right|<\delta$ implies

$$
\begin{equation*}
\frac{|y|^{r}}{\left|y y^{\prime} y^{\prime \prime}\right|}>\frac{|z|^{r}}{\left|z z^{\prime} z^{\prime \prime}\right|}, \quad \frac{|x|^{r}}{\left|x x^{\prime} x^{\prime \prime}\right|}>\frac{|z|^{r}}{\left|z z^{\prime} z^{\prime \prime}\right|} \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
y>2 x, \quad \frac{|y-x|^{r}}{\left|(y-x)\left(y^{\prime}-x^{\prime}\right)\left(y^{\prime \prime}-x^{\prime \prime}\right)\right|}>\frac{|z|^{r}}{\left|z z^{\prime} z^{\prime \prime}\right|}, \tag{19}
\end{equation*}
$$

where $z=1-x-y, z^{\prime}=1-x^{\prime}-y^{\prime}$ and $z^{\prime \prime}=1-x^{\prime \prime}-y^{\prime \prime}$ for every $\left(x, y, x^{\prime}, y^{\prime}, x^{\prime \prime}, y^{\prime \prime}\right) \in$ $\Delta \times \mathbb{R}^{4}$. Since $\rho_{K}\left(\Delta_{K}\right)$ is dense in $\Delta \times \mathbb{R}^{4}$, there exists an element $(\alpha, \beta) \in \Delta_{K}$ such that

$$
\begin{equation*}
\frac{|\beta|^{r}}{|N(\beta)|}>\frac{|\gamma|^{r}}{|N(\gamma)|}, \quad \frac{|\alpha|^{r}}{|N(\alpha)|}>\frac{|\gamma|^{r}}{|N(\gamma)|} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta>2 \alpha, \quad \frac{|\beta-\alpha|^{r}}{|N(\beta-\alpha)|}>\frac{|\gamma|^{r}}{|N(\gamma)|}, \tag{21}
\end{equation*}
$$

where $\gamma=1-\alpha-\beta$. From (20) and (21), it follows $\varepsilon_{K}(\alpha, \beta)=(2,1)$ and $\varepsilon_{K}\left(T_{K}(\alpha, \beta)\right)=$ $(2,1)$. By the analogous above discussion, we see that for each $(i, j) \in\{(1,2),(1,0),(0,1)\}$ there exists $\left(\alpha^{\prime \prime}, \beta^{\prime \prime}\right) \in \Delta_{K}$ such that $\varepsilon_{K}\left(\alpha^{\prime \prime}, \beta^{\prime \prime}\right)=(2,1)$ and $\varepsilon_{K}\left(T_{K}\left(\alpha^{\prime \prime}, \beta^{\prime \prime}\right)\right)=(i, j)$. By applying permutations of the coordinates of $(\gamma, \alpha, \beta)$ we get Theorem 2.7 for the totally real cubic field $K$. We consider the case where $K$ has complex embeddings. Let $\mu$ be the real root of $x^{3}-5$. Put $K_{2}=\mathbb{Q}(\mu)$. The direct calculation implies Table 3,

Table 3: (not totally real case) The words of length 2 which are not forbidden by Table 1 eventually occur.

| $(\zeta, \eta)$ | $\varepsilon_{K_{2}}(\zeta, \eta)$ | $T_{K_{2}}(\zeta, \eta)$ | $\varepsilon_{K_{2}}\left(T_{K_{2}}(\zeta, \eta)\right)$ |
| :---: | :---: | :---: | :---: |
| $\left(1 / 4+\mu / 4-3 \mu^{2} / 20\right.$, | $(2,1)$ | $\left(-2 / 17+6 \mu / 17-\mu^{2} / 17\right.$, | $(2,1)$ |
| $\left.\mu^{2} / 5\right)$ |  | $\left.8 / 17-7 \mu / 17+4 \mu^{2} / 17\right)$ |  |
| $\left(-2 / 17+6 \mu / 17-\mu^{2} / 17\right.$, | $(2,1)$ | $\left(-3 / 26+7 \mu / 26+\mu^{2} / 26\right.$, | $(1,0)$ |
| $\left.8 / 17-7 \mu / 17+4 \mu^{2} / 17\right)$ |  | $\left.10 / 13-6 \mu / 13+\mu^{2} / 13\right)$ |  |
| $\left(-1 / 6+\mu / 6+\mu^{2} / 30\right.$, | $(2,1)$ | $\left(-1 / 11+3 \mu / 22+\mu^{2} / 22\right.$, | $(0,1)$ |
| $\left.11 / 12-5 \mu / 12+7 \mu^{2} / 60\right)$ |  | $\left.10 / 11-4 \mu / 11+\mu^{2} / 22\right)$ |  |
| $\left(1-\mu^{2} / 5\right.$, | $(2,1)$ | $(-1+\mu$, | $(1,2)$ |
| $\left.1 / 2+\mu / 2-3 \mu^{2} / 10\right)$ |  | $\left.-1 / 2-\mu / 2+\mu^{2} / 2\right)$ |  |

Using Table 3, we can show Theorem 2.7 for the case where $K$ is not a totally real cubic field.

For the periodic continued fraction obtained by this algorithm, we have the following Proposition 2.8, which can be shown by using Theorem 3.7 in a way similar to Perron Perron 07. We denote by $\Delta_{K}^{\mathcal{P} e r}=\Delta_{K, r}^{\mathcal{P} e r}$ the set of the periodic points of the transformation $T=T_{K, r}$, i.e.,

$$
\Delta_{K}^{\mathcal{P} e r}=\Delta_{K, r}^{\mathcal{P} e r}:=\left\{\begin{array}{l|l}
(\alpha, \beta) \in \Delta_{K} & \begin{array}{c}
\text { there exist } m, n \in \mathbb{Z}_{>0} \\
\text { such that } m \neq n \text { and } \\
T_{K, r}^{m}(\alpha, \beta)=T_{K, r}^{n}(\alpha, \beta)
\end{array} \tag{22}
\end{array}\right\} .
$$

Proposition 2.8. Let $(\alpha, \beta) \in \Delta_{K}^{\mathcal{P e r}}$. Then, there exists a constant $c(\alpha, \beta)>0$ and $\eta(\alpha, \beta)>0$ such that $\eta(\alpha, \beta) \leq \frac{3}{2}$ and

$$
\begin{aligned}
& \left|\alpha-\frac{q_{n}}{p_{n}}\right| \leq \frac{c(\alpha, \beta)}{p_{n}^{\eta(\alpha, \beta)}}, \quad\left|\beta-\frac{r_{n}}{p_{n}}\right| \leq \frac{c(\alpha, \beta)}{p_{n}^{\eta(\alpha, \beta)}}, \\
& \left|\alpha-\frac{q_{n}^{\prime}}{p_{n}^{\prime}}\right| \leq \frac{c(\alpha, \beta)}{\left(p_{n}^{\prime}\right)^{\eta(\alpha, \beta)}}, \quad\left|\beta-\frac{r_{n}^{\prime}}{p_{n}^{\prime}}\right| \leq \frac{c(\alpha, \beta)}{\left(p_{n}^{\prime}\right)^{\eta(\alpha, \beta)}}, \\
& \left|\alpha-\frac{q_{n}^{\prime \prime}}{p_{n}^{\prime \prime}}\right| \leq \frac{c(\alpha, \beta)}{\left(p_{n}^{\prime \prime}\right)^{\eta(\alpha, \beta)}}, \quad\left|\beta-\frac{r_{n}^{\prime \prime}}{p_{n}^{\prime \prime}}\right| \leq \frac{c(\alpha, \beta)}{\left(p_{n}^{\prime \prime}\right)^{\eta(\alpha, \beta)}}
\end{aligned}
$$

hold. Furthermore, $\eta(\alpha, \beta)=\frac{3}{2}$ holds if and only if $K$ is not a totally real cubic field.
We can also give some examples of periodic expansions in the similar manner as in [Tamura and Yasutomi 09].

Theorem 2.9. Let $m \in \mathbb{Z}_{>0}$. Let $\lambda$ be the real root of $x^{3}-m x^{2}-1$. Let $K=\mathbb{Q}(\lambda)$. Then,

$$
\left(\frac{1}{1+\lambda+\lambda^{2}}, \frac{\lambda}{1+\lambda+\lambda^{2}}\right) \in \mathcal{P} e r_{K}^{5 / 2}
$$

Proof. For $n \in \mathbb{Z}_{\geq 0}$ let $\left(\alpha_{n}, \beta_{n}\right)=T_{K, 5 / 2}^{n}\left(\frac{1}{1+\lambda+\lambda^{2}}, \frac{\lambda}{1+\lambda+\lambda^{2}}\right)$. Then, we will prove for $0 \leq$ $k \leq m-1$

$$
\begin{aligned}
& \alpha_{k}=\frac{\lambda}{(m-k+1) \lambda^{2}+\lambda+1}, \beta_{k}=\frac{\lambda^{2}}{(m-k+1) \lambda^{2}+\lambda+1} \\
& \varepsilon\left(\alpha_{k}, \beta_{k}\right)=(0,2), \\
& \alpha_{m+k}=\frac{\lambda^{2}}{(m-k+1) \lambda^{2}+\lambda+1}, \beta_{m+k}=\frac{(m-k) \lambda^{2}+1}{(m-k+1) \lambda^{2}+\lambda+1} \\
& \varepsilon\left(\alpha_{m+k}, \beta_{m+k}\right)=(2,1), \\
& \alpha_{2 m+k}=\frac{(m-k) \lambda^{2}+1}{(m-k+1) \lambda^{2}+\lambda+1}, \beta_{2 m+k}=\frac{\lambda}{(m-k+1) \lambda^{2}+\lambda+1} \\
& \varepsilon\left(\alpha_{2 m+k}, \beta_{2 m+k}\right)=(1,0)
\end{aligned}
$$

In what follows, we suppose that $k \in \mathbb{Z}$. First, let $\zeta_{k}, \eta_{k}$ and $\xi_{k}$ be

$$
\begin{aligned}
\zeta_{k} & =\frac{\lambda}{(m-k+1) \lambda^{2}+\lambda+1}, \quad \eta_{k}=\frac{\lambda^{2}}{(m-k+1) \lambda^{2}+\lambda+1} \\
\xi_{k} & =\frac{(m-k) \lambda^{2}+1}{(m-k+1) \lambda^{2}+\lambda+1}
\end{aligned}
$$

Simple calculations show the following:
(1) $\zeta_{k}+\eta_{k}+\xi_{k}=1$,
(2) $\zeta_{0}=\alpha_{0}$ and $\eta_{0}=\beta_{0}$,
(3) $\left(\zeta_{k}, \eta_{k}\right) \in \Delta_{K}$ holds,
(4) $T_{(0,2)}\left(\zeta_{k}, \eta_{k}\right)=\left(\zeta_{k+1}, \eta_{k+1}\right)$ holds,
(5) $\left(\zeta_{m}, \eta_{m}\right)=\left(\eta_{0}, \xi_{0}\right)$ holds.

We prove that $\varepsilon\left(\zeta_{k}, \eta_{k}\right)=(0,2)$ for $0 \leq k<m$. It is easy to see that

$$
\begin{aligned}
& N\left(\zeta_{k}\right)=\frac{1}{m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}}, \\
& N\left(\eta_{k}\right)=\frac{1}{m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}}, \\
& N\left(\xi_{k}\right)=\frac{k^{2} m-k^{3}+1}{m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}} .
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{\zeta_{k}^{5 / 2}}{N\left(\zeta_{k}\right)} & =\frac{\lambda^{5 / 2}\left(m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}\right)}{\left((m-k+1) \lambda^{2}+\lambda+1\right)^{5 / 2}} \\
\frac{\eta_{k}^{5 / 2}}{N\left(\eta_{k}\right)} & =\frac{\lambda^{5}\left(m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}\right)}{\left((m-k+1) \lambda^{2}+\lambda+1\right)^{5 / 2}} \\
\frac{\xi_{k}^{5 / 2}}{N\left(\xi_{k}\right)} & =\frac{\left((m-k) \lambda^{2}+1\right)^{5 / 2}\left(m^{2}+\left(k^{2}-3 k\right) m-k^{3}+3 k^{2}\right)}{\left(k^{2} m-k^{3}+1\right)\left((m-k+1) \lambda^{2}+\lambda+1\right)^{5 / 2}}
\end{aligned}
$$

Since $\lambda>1$, we have $\frac{\zeta_{k}^{5 / 2}}{N\left(\zeta_{k}\right)}<\frac{\eta_{k}^{5 / 2}}{N\left(\eta_{k}\right)}$. We easily see that $m^{2}(m-k)^{2} \geq k^{2} m-k^{3}+1$. Since $m<\lambda<m+1$, we see that

$$
\left((m-k) \lambda+\frac{1}{\lambda}\right)^{5 / 2}>m^{2}(m-k)^{2} \geq k^{2} m-k^{3}+1
$$

which implies

$$
\frac{\zeta_{k}^{5 / 2}}{N\left(\zeta_{k}\right)}<\frac{\xi_{k}^{5 / 2}}{N\left(\xi_{k}\right)}
$$

Thus, we have $\varepsilon\left(\zeta_{k}, \eta_{k}\right)=(0,2)$. We can easily prove that $\left(\alpha_{k}, \beta_{k}\right)=\left(\zeta_{k}, \eta_{k}\right)$ for $0 \leq k \leq$ $m-1,\left(\alpha_{m+k}, \beta_{m+k}\right)=\left(\eta_{k}, \xi_{k}\right)$ and $\left(\alpha_{2 m+k}, \beta_{2 m+k}\right)=\left(\xi_{k}, \zeta_{k}\right)$ on induction of $k$.

## 3 Continued Fraction Expansion and Acceleration of Continued Fraction

As we have already seen that for any given $(x, y) \in \Delta_{K}$ for any given real cubic field $K$, we can consider a sequence $\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ defined by

$$
\varepsilon\left(\alpha_{n}, \beta_{n}\right):=\varepsilon\left(T^{n}(\alpha, \beta)\right) \in \operatorname{Ind}:=\{(i, j) \mid i, j \in\{0,1,2\}, i \neq j\}
$$

obtained by the algorithm given in Section 2,
In this section, we shall describe the continued fraction expansion of $\frac{1}{1-\alpha-\beta}(\alpha, \beta)$ according to the "expansion" $\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ of $(\alpha, \beta)$. In what follows of this section, we use column vectors instead of row vectors. We denote by $\overline{\bar{x}}={ }^{t}\left(x^{(0)}, x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{3}$ (resp., $\bar{x}={ }^{t}\left(x^{(1)}, x^{(2)}\right) \in \mathbb{R}^{2}$ ) an vector of dimension 3 (resp., of dimension 2), where $t$ indicates the transpose.

We need some definitions. For $n \in \mathbb{Z}_{>0}$ and a set $S$ we denote by $M(n, S) n \times n$ matrices with entries in $S$. We put

$$
\begin{align*}
C(\bar{a}) & :=\left(\begin{array}{cc}
{ }^{\overline{0}} & 1 \\
E_{2} & \bar{a}
\end{array}\right), \bar{a} \in \mathbb{Z}_{\geq 0}^{2},  \tag{23}\\
P_{n}=\left(\overline{\bar{p}}_{n-2} \overline{\bar{p}}_{n-1} \overline{\bar{p}}_{n}\right) & :=C\left(\bar{a}_{0}\right) C\left(\bar{a}_{1}\right) \cdots C\left(\bar{a}_{n}\right), \bar{a}_{n} \in \mathbb{Z}_{\geq 0}^{2}, \\
\left(n \geq-1, P_{-1}\right. & \left.:=E_{3}, \overline{\bar{p}}_{n}={ }^{t}\left(p_{n}^{(0)}, p_{n}^{(1)}, p_{n}^{(2)}\right)\right),
\end{align*}
$$

where $E_{m}$ is the unit matrix of size $m \times m$.
We write

$$
\begin{aligned}
& \frac{1}{\binom{x}{y}}:=\binom{1 / y}{x / y} \quad(x, y \in \mathbb{R}, y \neq 0) \\
& {\left[\bar{a}_{0} ; \bar{a}_{1}, \ldots, \bar{a}_{n}\right] }:=\bar{a}_{0}+\frac{1}{\bar{a}_{1}+\frac{1}{\bar{a}_{n}}} \\
& \ddots \frac{1}{\bar{a}^{2}}
\end{aligned}
$$

and

$$
\left[\bar{a}_{0} ; \bar{a}_{1}, \bar{a}_{2}, \ldots\right]:=\lim _{n \rightarrow \infty}\left[\bar{a}_{0} ; \bar{a}_{1}, \bar{a}_{2}, \ldots, \bar{a}_{n}\right]
$$

as far as the limit exists.
Using Lemma 2.3 and

$$
C(\bar{a})^{f r a c}\binom{x}{y}=\pi \kappa\left(\begin{array}{c}
y \\
1+a y \\
x+b y
\end{array}\right)=\binom{a}{b}+\frac{1}{\binom{x}{y}},
$$

where $\bar{a}=\binom{a}{b}$, we get the following formula, see for example, Nikishin and Sorokin 91, Tamura 95.
Formula 3.1. (1) $\left[\bar{a}_{0} ; \bar{a}_{1}, \ldots, \bar{a}_{n}, \bar{x}\right]=P_{n}^{f r a c}(\bar{x})$;
(2) $\left[\bar{a}_{0} ; \bar{a}_{1}, \ldots, \bar{a}_{n}\right]=\frac{1}{p_{n}^{(0)}}\binom{p_{n}^{(1)}}{p_{n}^{(2)}}$ holds provided $p_{n}^{(0)} \neq 0$.

It is convenient to write two dimensional continued fraction $\left[\bar{a}_{0} ; \bar{a}_{1}, \ldots, \bar{a}_{n}\right]$ (resp., $\left[\bar{a}_{0} ; \bar{a}_{1}, \bar{a}_{2}, \ldots\right]$ ) as a finite word (resp., an infinite word) over $\mathbb{Z}_{\geq 0}^{2}$ :

$$
\begin{aligned}
(\mathrm{CF}) \bar{a}_{0} \bar{a}_{1} \ldots \bar{a}_{n} & :=\left[\bar{a}_{0} ; \bar{a}_{1}, \ldots, \bar{a}_{n}\right], \\
(\mathrm{CF}) \bar{a}_{0} \bar{a}_{1} \bar{a}_{2} \ldots & :=\left[\bar{a}_{0} ; \bar{a}_{1}, \bar{a}_{2}, \ldots\right] .
\end{aligned}
$$

For $\bar{x}={ }^{t}(x, y) \in \triangle$, we denote by $\bar{x}^{*}={ }^{t}\left(x^{*}, y^{*}\right)$ a vector defined by

$$
\bar{x}^{*}=\binom{x^{*}}{y^{*}}=\frac{1}{1-x-y}\binom{x}{y} .
$$

Notice that ${ }^{t}\left(x^{*}, y^{*}\right) \in \mathbb{R}_{>0}^{2}$ always holds for any $\bar{x}={ }^{t}(x, y) \in \triangle$.
Now we can state our theorem.
Theorem 3.2. Let $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}=\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ be the expansion of $(\alpha, \beta) \in \Delta_{K}$. Then

$$
\begin{equation*}
\binom{\alpha^{*}}{\beta^{*}}=(C F) W_{\varepsilon_{0}} W_{\varepsilon_{1}} \ldots W_{\varepsilon_{n-1}} \ldots \tag{24}
\end{equation*}
$$

where

$$
\begin{array}{ll}
W_{(1,2)}=\binom{0}{0}\binom{0}{0}\binom{0}{1}, & W_{(0,1)}=\binom{0}{0}\binom{0}{1}\binom{0}{0},
\end{array} W_{(0,2)}=\binom{0}{0}\binom{0}{0}\binom{1}{0}, ~\binom{0}{0}\binom{1}{0}\binom{0}{0}, \quad W_{(1,0)}=\binom{1}{0}\binom{0}{0}\binom{0}{0}, \quad W_{(2,0)}=\binom{0}{1}\binom{0}{0}\binom{0}{0} . . ~ \$ W_{(2,1)}=\left(\begin{array}{l}
0
\end{array}\right)
$$

One can check the following lemmas by direct calculation.
Lemma 3.3. Let $A_{(i, j)}^{*}=S^{-1} A_{(i, j)} S \quad((i, j) \in$ Ind $)$, where $A_{(i, j)}$ and $S$ are matrices as in Section 圆 Then,

$$
A_{(i, j)}^{*}=M_{(i, j)} \quad(\forall(i, j) \in \text { Ind }),
$$

where $M_{(i, j)}=\left(m_{k \ell}\right)_{0 \leq k \leq 2,0 \leq \ell \leq 2} \in G L_{3}(\mathbb{Z})$ defined by

$$
m_{k \ell}:=\left\{\begin{array}{cc}
1 & k=\ell \text { or }(k, \ell)=(i, j) \\
0 & \text { otherwise }
\end{array}\right.
$$

Lemma 3.4. Let $R, U, V$ be matrices defined by

$$
R=C(\overline{0}), \quad U=C(\bar{u}), \quad V=C(\bar{v}),
$$

where $\overline{0}={ }^{t}(0,0), \bar{u}={ }^{t}(1,0), \bar{v}={ }^{t}(0,1)$, and $C(\bar{a})\left(\bar{a} \in \mathbb{Z}_{\geq 0}^{2}\right)$ is the matrix (23). Then,

$$
\begin{array}{lll}
M_{(1,2)}=R R V, & M_{(0,1)}=R V R, & M_{(0,2)}=R R U, \\
M_{(2,1)}=R U R, & M_{(1,0)}=U R R, & M_{(2,0)}=V R R .
\end{array}
$$

Proof of Theorem 3.2. One can see that $\varepsilon_{0}=(i, j) \in$ Ind implies

$$
A_{(i, j)}^{-1}\left(\begin{array}{l}
1 \\
\alpha \\
\beta
\end{array}\right) \sim\left(\begin{array}{c}
1 \\
\alpha_{1} \\
\beta_{1}
\end{array}\right)
$$

Hence, by induction, we have

$$
\left(\begin{array}{l}
1 \\
\alpha \\
\beta
\end{array}\right) \sim A_{\varepsilon_{0}} A_{\varepsilon_{1}} \cdots A_{\varepsilon_{n-1}}\left(\begin{array}{c}
1 \\
\alpha_{n} \\
\beta_{n}
\end{array}\right),\left(\varepsilon_{k}=\varepsilon\left(\alpha_{k}, \beta_{k}\right)\right)
$$

so that

$$
S^{-1}\left(\begin{array}{l}
1 \\
\alpha \\
\beta
\end{array}\right) \sim S^{-1} A_{\varepsilon_{0}} S S^{-1} A_{\varepsilon_{1}} S \cdots S^{-1} A_{\varepsilon_{n-1}} S S^{-1}\left(\begin{array}{c}
1 \\
\alpha_{0} \\
\beta_{0}
\end{array}\right)=A_{\varepsilon_{0}}^{*} A_{\varepsilon_{1}}^{*} \cdots A_{\varepsilon_{n-1}}^{*} S^{-1}\left(\begin{array}{c}
1 \\
\alpha_{n} \\
\beta_{n}
\end{array}\right)
$$

which implies

$$
\begin{gathered}
\binom{\alpha^{*}}{\beta^{*}}=A_{\varepsilon_{0}}^{* f r a c} A_{\varepsilon_{1}}^{* f r a c} \ldots A_{\varepsilon_{n-1}}^{* \operatorname{frac}}\binom{\alpha_{n}^{*}}{\beta_{n}^{*}}, \\
\binom{\alpha^{*}}{\beta^{*}}:=\frac{1}{1-\alpha-\beta}\binom{\alpha}{\beta}, \quad\binom{\alpha_{n}^{*}}{\beta_{n}^{*}}:=\frac{1}{1-\alpha_{n}-\beta_{n}}\binom{\alpha_{n}}{\beta_{n}}
\end{gathered}
$$

which together with Lemma 3.3 and Lemma 3.4, we get Theorem 3.2.

We can make reduction of the continued fraction of the form (24) in Theorem 3.2 by applying the following reduction rule:

$$
(\mathrm{CF}) \cdots\binom{a}{b}\binom{0}{0}\binom{0}{0}\binom{c}{d} \cdots=(\mathrm{CF}) \cdots\binom{a+c}{b+d} \cdots,
$$

in particular

$$
(\mathrm{CF}) \cdots\binom{a}{b}\binom{0}{0}\binom{0}{0}\binom{0}{0}\binom{c}{d} \cdots=(\mathrm{CF}) \cdots\binom{a}{b}\binom{c}{d} \cdots .
$$

We give an example. Let $\lambda$ be the real root of $x^{3}-m x^{2}-1,\left(m \in \mathbb{Z}_{>0}\right)$ as in Theorem 2.9. Then, Theorem 2.9 says that

$$
\begin{align*}
\frac{1}{1-\xi-\eta}\binom{\xi}{\eta} & =(\mathrm{CF}) W_{(0,2)}^{m} W_{(2,1)}^{m} W_{(1,0)}^{m} W_{(0,2)}^{m} W_{(2,1)}^{m} W_{(1,0)}^{m} \cdots  \tag{25}\\
\xi & =\frac{1}{1+\lambda+\lambda^{2}}, \quad \eta=\frac{\lambda}{1+\lambda+\lambda^{2}}
\end{align*}
$$

Hence, applying the reduction rule repeatedly, we get periodic continued fractions:

$$
\begin{align*}
\frac{1}{1-\xi-\eta}\binom{\xi}{\eta} & =(\mathrm{CF})\left(\begin{array}{l}
* \\
0 \\
0
\end{array}\right)\binom{0}{0}\binom{m}{0}\binom{0}{0}\binom{m}{0}\binom{0}{0}\binom{m}{0}\binom{0}{0}\left(\begin{array}{l}
* \\
0 \\
0
\end{array}\right) \\
& =(\mathrm{CF})\binom{0}{0}\binom{*}{0}\binom{m}{0}\binom{0}{0}\binom{m}{0}\binom{0}{0}\left(\begin{array}{c}
* \\
0 \\
0
\end{array}\right) \\
& =(\mathrm{CF})\binom{0}{0}\binom{0}{0}\binom{m}{0}, \tag{26}
\end{align*}
$$

which is an accelerated continued fraction of (25). In other words, if we say (26) is a canonical continued fraction, then the expression (24) given in Theorem 3.2 can be considered as a (slow or additive) continued fraction expansion of a canonical continued fraction. For $n \in \mathbb{Z}_{>0} M \in M\left(n, \mathbb{Z}_{\geq 0}\right)$ is called primitive, if there exists an positive integer $m$ such that $M^{m} \in M\left(n, \mathbb{Z}_{>0}\right)$.

Lemma 3.5. Let $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}=\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ be the expansion of $(\alpha, \beta) \in \Delta_{K}$. We suppose that $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}$ is purely periodic and the length of the period is $l$. Then, every eigenvalue of the matrix $M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$ is in $K \backslash \mathbb{Q}$.

Proof. We suppose that an eigenvalue $\lambda$ of the matrix $M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$ denoted by $C=\left(c_{i, j}\right)$ associated with the eigenvector ${ }^{t}(1-\alpha-\beta, \alpha, \beta)$ is a rational number. Then, we have

$$
C\left(\begin{array}{c}
1-\alpha-\beta  \tag{27}\\
\alpha \\
\beta
\end{array}\right)=\lambda\left(\begin{array}{c}
1-\alpha-\beta \\
\alpha \\
\beta
\end{array}\right)
$$

Since $1-\alpha-\beta, \alpha, \beta$ are linearly independent over $\mathbb{Q}, C=\lambda E$, where $E$ is the unit matrix. Let $\left(i^{\prime}, j^{\prime}\right)=\varepsilon_{0}$. Then, $C=M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$ implies that $c_{i^{\prime}, j^{\prime}} \geq 1$, which leads to the contradiction that $C=\lambda E$.

Remark 3.6. The irreducibility of the matrix $M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$ follows from Lemma 3.5 as far as we are concentrated with cubic field $K$.

Theorem 3.7. Let $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}=\left\{\varepsilon\left(\alpha_{n}, \beta_{n}\right)\right\}_{n=0}^{\infty}$ be the expansion of $(\alpha, \beta) \in \Delta_{K}$. We suppose that $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}$ is purely periodic and the length of the period is l. Then, the matrix $M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$ is primitive.

Proof. For each $n \in \mathbb{Z}_{\geq 0}$ we put $C_{n}$ by

$$
C_{n}=\left({ }_{n} c_{i, j}\right)=M_{\varepsilon_{0}} \cdots M_{\varepsilon_{n}} .
$$

We see easily that ${ }_{n} c_{i, j} \leq{ }_{n+1} c_{i, j}$ for each $n \in \mathbb{Z}_{\geq 0}$ and $i, j$ with $0 \leq i, j \leq 2$. We suppose that there exists $i_{0}$, $j_{0}$ with $0 \leq i_{0}, j_{0} \leq 2$ such that $\lim _{n \rightarrow \infty}{ }_{n} c_{i_{0}, j_{0}}=0$, which is equivalent to that for every $n \in \mathbb{Z}_{\geq 0}{ }_{n} c_{i_{0}, j_{0}}=0$. Since we see ${ }_{0} c_{i, i}=1$ for every $i$ with $0 \leq i \leq 2$, we have $i_{0} \neq j_{0}$. For simplicity, we consider the case where $\left(i_{0}, j_{0}\right)=(0,2)$. First, we suppose that $\lim _{n \rightarrow \infty}{ }_{n} c_{0,1}=0$, which is equivalent to that for every $n \in \mathbb{Z}_{\geq 0}{ }_{n} c_{0,1}=0$. Since $\left|C_{n}\right|=1$ for every $n \in \mathbb{Z}_{\geq 0}$, it follows that ${ }_{n} c_{0,0}=1$ for every $n \in \mathbb{Z}_{\geq 0}$. Then, we have

$$
{ }^{t} C_{l-1}\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)
$$

which implies that 1 is an eigenvalue of $M_{\varepsilon_{0}} \cdots M_{\varepsilon_{l-1}}$, which contradicts Lemma 3.5. Next, we suppose that $\lim _{n \rightarrow \infty} c_{0,1}>0$, which is equivalent to that there exists $n_{0} \in \mathbb{Z}_{\geq 0}$ such that ${ }_{n_{0}} c_{0,1}>0$. We see that if $n>n_{0}$, then $\varepsilon_{n} \notin\{(0,2),(1,2)\}$. Therefore, by the pure periodicity of $\left\{\varepsilon_{n}\right\}_{\geq 0}$ we get $\varepsilon_{n} \notin\{(0,2),(1,2)\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Hence, we can write

$$
C_{l-1}=\left(\begin{array}{ccc}
* & * & 0 \\
* & * & 0 \\
* & * & 1
\end{array}\right)
$$

so that

$$
C_{l-1}\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

which contradicts Lemma 3.5. Thus, we have proved the theorem for the case where $\left(i_{0}, j_{0}\right)=(0,2)$. We can do the same for the other cases.

## 4 Numerical experiments

We put

$$
\operatorname{dh}\left(\frac{p}{q}\right):=\max \left\{\left\lfloor\log _{10}|p|+1\right\rfloor,\left\lfloor\log _{10}|q|+1\right\rfloor\right\}, \quad \operatorname{dh}(0):=0
$$

for $\frac{p}{q}(p, q \in \mathbb{Z}$ are coprime $)$. The function dh can be extended to $\mathbb{Q}[x]$ :

$$
\operatorname{dh}(g):=\max _{0 \leq i \leq n}\left\{\operatorname{dh}\left(a_{i}\right)\right\}
$$

for $g(x)=\sum_{i=0}^{n} a_{i} x^{i} \in \mathbb{Q}[x]$. We define $\overline{\mathrm{dh}}, \mathrm{dh}_{\mathrm{F}}$ and $\operatorname{rdh}_{\mathrm{F}}$ by

$$
\begin{aligned}
& \overline{\operatorname{dh}}(\alpha):=\max _{i \in\{1,2\}}\left\{\operatorname{dh}\left(\phi_{\alpha_{i}}\right)\right\}, \\
& \operatorname{dh}_{\mathrm{F}}(n ; \alpha):=\overline{\operatorname{dh}}\left(T_{K}^{n}(\alpha)\right), \\
& \operatorname{rdh}_{\mathrm{F}}(n ; \alpha):=\frac{\overline{\operatorname{dh}}\left(T_{K}^{n}(\alpha)\right)}{\overline{\operatorname{dh}}(\alpha)}, \\
& \operatorname{dh}_{\mathrm{F}}(\alpha):=\max _{n \in \mathbb{Z} \geq 0}\left\{\operatorname{dh}_{\mathrm{F}}(n ; \alpha)\right\}, \\
& \operatorname{rdh}_{\mathrm{F}}(\alpha):=\max _{n \in \mathbb{Z} \geq 0}\left\{\operatorname{rdh}_{\mathrm{F}}(n ; \alpha)\right\},
\end{aligned}
$$

for $\alpha=\left(\alpha_{1}, \alpha_{2}\right) \in \Delta_{K}$ and $n \in \mathbb{Z}_{\geq 0}$, where $\phi_{\alpha_{i}} \in \mathbb{Q}[x](i \in\{1,2\})$ is the monic minimal polynomial of $\alpha_{i}$. The function $\operatorname{dh}_{\mathrm{F}}(n ; \alpha)$ (resp., $\left.\operatorname{rdh}_{\mathrm{F}}(n ; \alpha)\right)$ is referred to as the $n$th decimal height of $\alpha$ (resp., the nth relative decimal height of $\alpha$ ). We computed the length of the periods of $\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right)$ for $T_{K, r}$ with $K=\mathbb{Q}(\sqrt[3]{m}), r=5 / 2$ for all $m \in \mathbb{Z}$ with $2 \leq m \leq 10000(\sqrt[3]{m} \notin \mathbb{Q})$ and these decimal heights, cf. Table 4 given below, where $\langle x\rangle$ is the fractional part of $x$. For the calculation of the tables, we used a computer equipped with GiNaC [GiNaC 13] on GNU C ++1 . We confirmed that $\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right) \in \Delta_{K, 5 / 2}^{\mathcal{P} e r}$ for all $m$ with $2 \leq m \leq 10000(\sqrt[3]{m} \notin \mathbb{Q})$.

Table 4: The result of periodicity test for $\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right)$ for all noncubic positive integers $2 \leq m \leq 10000$ with $r=5 / 2$.

| Range of $m\left(m_{1} \leq m \leq m_{2}\right)$ | $L_{A}\left(m_{1}, m_{2}\right)$ | $H_{A}\left(m_{1}, m_{2}\right)$ | $R_{A}\left(m_{1}, m_{2}\right)$ |
| :---: | ---: | ---: | ---: |
| $2 \leq m \leq 200$ | 4494 | 7 | 3 |
| $201 \leq m \leq 400$ | 13641 | 8 | $7 / 3$ |
| $401 \leq m \leq 600$ | 13578 | 8 | 2 |
| $601 \leq m \leq 800$ | 30447 | 8 | 2 |
| $801 \leq m \leq 1000$ | 36963 | 8 | 2 |
| $1001 \leq m \leq 1200$ | 31119 | 9 | $9 / 4$ |
| $1201 \leq m \leq 1400$ | 68529 | 9 | $9 / 5$ |
| $1401 \leq m \leq 1600$ | 65310 | 9 | $9 / 5$ |
| $1601 \leq m \leq 1800$ | 62598 | 9 | $9 / 5$ |
| $1801 \leq m \leq 2000$ | 52551 | 9 | $9 / 5$ |
| $2001 \leq m \leq 2200$ | 74931 | 10 | 2 |
| Continued on next page |  |  |  |

[^1]Table 4 - continued from previous page

| Range of $m\left(m_{1} \leq m \leq m_{2}\right)$ | $L_{A}\left(m_{1}, m_{2}\right)$ | $H_{A}\left(m_{1}, m_{2}\right)$ | $R_{A}\left(m_{1}, m_{2}\right)$ |
| :---: | :---: | :---: | :---: |
| $2201 \leq m \leq 2400$ | 177570 | 9 | 9/5 |
| $2401 \leq m \leq 2600$ | 97446 | 9 | 9/5 |
| $2601 \leq m \leq 2800$ | 79923 | 9 | 9/5 |
| $2801 \leq m \leq 3000$ | 121134 | 10 | 9/5 |
| $3001 \leq m \leq 3200$ | 107577 | 9 | 9/5 |
| $3201 \leq m \leq 3400$ | 107919 | 10 | 2 |
| $3401 \leq m \leq 3600$ | 95388 | 10 | 9/5 |
| $3601 \leq m \leq 3800$ | 150393 | 10 | 9/5 |
| $3801 \leq m \leq 4000$ | 133650 | 10 | 9/5 |
| $4001 \leq m \leq 4200$ | 137787 | 10 | 2 |
| $4201 \leq m \leq 4400$ | 242391 | 10 | 2 |
| $4401 \leq m \leq 4600$ | 322374 | 10 | 2 |
| $4601 \leq m \leq 4800$ | 180246 | 10 | 2 |
| $4801 \leq m \leq 5000$ | 124335 | 10 | 2 |
| $5001 \leq m \leq 5200$ | 282870 | 10 | 2 |
| $5201 \leq m \leq 5400$ | 169845 | 10 | 2 |
| $5401 \leq m \leq 5600$ | 134589 | 10 | 2 |
| $5601 \leq m \leq 5800$ | 236004 | 10 | 2 |
| $5801 \leq m \leq 6000$ | 298266 | 10 | 2 |
| $6001 \leq m \leq 6200$ | 439470 | 10 | 2 |
| $6201 \leq m \leq 6400$ | 249141 | 10 | 2 |
| $6401 \leq m \leq 6600$ | 188673 | 10 | 2 |
| $6601 \leq m \leq 6800$ | 176733 | 10 | 2 |
| $6801 \leq m \leq 7000$ | 462093 | 11 | 2 |
| $7001 \leq m \leq 7200$ | 160650 | 10 | 5/3 |
| $7201 \leq m \leq 7400$ | 619809 | 10 | 5/3 |
| $7401 \leq m \leq 7600$ | 241893 | 10 | 5/3 |
| $7601 \leq m \leq 7800$ | 254790 | 10 | 5/3 |
| $7801 \leq m \leq 8000$ | 232170 | 10 | 5/3 |
| $8001 \leq m \leq 8200$ | 398433 | 11 | 11/6 |
| $8201 \leq m \leq 8400$ | 211460 | 11 | 11/6 |
| $8401 \leq m \leq 8600$ | 264786 | 10 | 5/3 |
| $8601 \leq m \leq 8800$ | 293934 | 11 | 11/6 |
| $8801 \leq m \leq 9000$ | 785715 | 10 | 5/3 |
| $9001 \leq m \leq 9200$ | 265377 | 11 | 11/6 |
| $9201 \leq m \leq 9400$ | 377157 | 11 | 11/6 |
| $9401 \leq m \leq 9600$ | 258939 | 10 | 5/3 |
| $9601 \leq m \leq 9800$ | 269877 | 10 | 5/3 |
| $9801 \leq m \leq 10000$ | 276768 | 10 | 5/3 |

In Table 4. $L_{A}\left(m_{1}, m_{2}\right), H_{A}\left(m_{1}, m_{2}\right)$ and $R_{A}\left(m_{1}, m_{2}\right)$ are numbers defined by $L_{A}\left(m_{1}, m_{2}\right):=$ the maximum value of the length of the shortest period of the expansion of $\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right)$ for $m_{1} \leq m \leq m_{2}$ with $\sqrt[3]{m} \notin \mathbb{Q}$,

$$
\begin{aligned}
& H_{A}\left(m_{1}, m_{2}\right):=\max _{m_{1} \leq m \leq m_{2}, \sqrt[3]{m} \notin \mathbb{Q}} \operatorname{dh}_{F}\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right), \\
& R_{A}\left(m_{1}, m_{2}\right):=\max _{m_{1} \leq m \leq m_{2}, \sqrt[3]{m} \notin \mathbb{Q},} \operatorname{rdh}_{F}\left(\langle\sqrt[3]{m}\rangle / 2,\left\langle\sqrt[3]{m^{2}}\right\rangle / 2\right),
\end{aligned}
$$

which are well-defined by the periodicity. This Table 4 together with following numerical experiments by PCs for some totally real cubic fields etc. support Conjecture 7.1 given at the end of our paper, which says that $\Delta_{K}=\Delta_{K, r}^{\mathcal{P e r}}$ holds, cf. (22). On the other hand, the explosion phenomenon takes place if we apply classical algorithms (the Jacobi-Perron algorithm, the modified Jacobi-Perron algorithm etc.), cf. Tamura and Yasutomi 09, Tamura and Yasutomi 11, Tamura and Yasutomi 10.

Let $K$ be a real cubic field and let $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}$ be its positive $\mathbb{Q}$-basis with

$$
\alpha=\frac{\alpha^{(1)}}{\alpha^{(0)}+\alpha^{(1)}+\alpha^{(2)}}, \beta=\frac{\alpha^{(2)}}{\alpha^{(0)}+\alpha^{(1)}+\alpha^{(2)}} .
$$

Let $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}$ be the expansion of the $(\alpha, \beta)$. Suppose that $\varepsilon_{k}, \ldots, \varepsilon_{k+l-1}$ is the period of the expansion. Then $\left(1-\alpha_{k}-\beta_{k}, \alpha_{k}, \beta_{k}\right)$ becomes an eigenvector with respect to an eigenvlaue of $M_{\varepsilon_{k}} \cdots M_{\varepsilon_{k+l-1}}$ which will be denoted by $\lambda(\alpha, \beta)$. The eigenvalue is important for the Diophantine approximation to $(\alpha, \beta)$. We denote by $N_{t}$ a set

$$
\begin{aligned}
& \left\{\left(\frac{1}{1+\alpha+\alpha^{2}}, \frac{\alpha}{1+\alpha+\alpha^{2}}\right) \left\lvert\, \begin{array}{l}
\alpha \text { is the positive maximal root of } \\
\text { some irreducible } p \in P_{t}
\end{array}\right.\right\}, \\
& P_{t}:=\left\{x^{3}+a_{2} x^{2}+a_{1} x+a_{0}\left|a_{i} \in \mathbb{Z},\left|a_{i}\right| \leq t \text { for } i=0,1,2\right\} \quad(t>0)\right.
\end{aligned}
$$

We put $n_{t}, p_{t}, c_{t}, r_{t}, s_{t}, r h_{t}$ as follows:

$$
\begin{aligned}
n_{t} & =\sharp N_{t}, \\
p_{t} & =\sharp\left\{(\alpha, \beta) \in N_{t} \mid(\alpha, \beta) \text { is periodic by our algorithm }\right\}, \\
c_{t} & =\sharp\left\{(\alpha, \beta) \in N_{t} \mid \mathbb{Q}(\alpha, \beta) \text { has a complex embedding }\right\} \\
r_{t} & =\sharp\left\{(\alpha, \beta) \in N_{t} \mid \mathbb{Q}(\alpha, \beta) \text { is a totally real cubic field }\right\}, \\
s_{t} & =\sharp\left\{(\alpha, \beta) \in N_{t} \mid \lambda(\alpha, \beta) \text { is the Pisot number }\right\} \\
r h_{t} & =\max \left\{\operatorname{rdh}_{F}(\alpha, \beta) \mid(\alpha, \beta) \in N_{t}\right\} .
\end{aligned}
$$

Then, we get $n_{15}=18797, p_{15}=18797, c_{15}=7689, r_{15}=11108, s_{15}=18797$ and $r h_{15}=7 / 3$, i.e., every $(\alpha, \beta) \in N_{15}$ is periodic by this algorithm, and $\lambda(\alpha, \beta)$ becomes

Pisot number for all the element $(\alpha, \beta) \in N_{15}$ without any exceptions (cf. Conjecture 7.2 given in Section 7 ). For this calculation, we used a computer equipped with GiNaC GiNaC 13] on GNU C $++{ }^{2}$. We note that the maximal eigenvalue of $\prod_{\lambda \in \Lambda, M_{\lambda} \in B M} M_{\lambda}$ $(\Lambda<\infty)$ is not always a Pisot number, where $B M:=\left\{M_{(i, j)} \mid 0 \leq i, j \leq 2, i \neq j\right\}$. For example, $M_{(1,0)} M_{(0,1)} M_{(2,0)} M_{(0,2)}$ has a non-Pisot maximal eigenvalue.

## 5 Stepped surfaces and Substitutions

We shortly prepare the geometric tools. Let us denote by $\overline{\bar{e}}_{i}(i=0,1,2)$ the canonical basis of $\mathbb{R}^{3}$, i.e.,

$$
\overline{\bar{e}}_{0}:={ }^{t}(1,0,0), \overline{\bar{e}}_{1}:={ }^{t}(0,1,0), \overline{\bar{e}}_{2}:={ }^{t}(0,0,1) .
$$

For $\overline{\bar{x}} \in \mathbb{Z}^{3}, i=0,1,2$, we mean by $\left(\overline{\bar{x}}, i^{*}\right)$ a unit square defined by

$$
\left(\overline{\bar{x}}, i^{*}\right):=\left\{\overline{\bar{x}}+t \overline{\bar{e}}_{j}+u \overline{\bar{e}}_{k} \mid t, u \in[0,1],\{i, j, k\}=\{0,1,2\}\right\}
$$

(see Figure 3).


Figure 3: $\left(\overline{\overline{0}}, i^{*}\right)$ and $\sum_{i=0}^{2}\left(\overline{\overline{0}}, i^{*}\right)$.
Let $\overline{\bar{\alpha}}={ }^{t}\left(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}\right) \in \mathbb{R}_{>0}^{3}$ and $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}$ be linearly independent over $\mathbb{Q}$. Notice that without loss of generality, we may assume $\alpha^{(0)}+\alpha^{(1)}+\alpha^{(2)}=1$. We consider the sets $\mathcal{P}(\overline{\bar{\alpha}}), \mathcal{P}^{>}(\overline{\bar{\alpha}})$, and $\mathcal{P} \geq(\overline{\bar{\alpha}})$ :

$$
\begin{aligned}
\mathcal{P}(\overline{\bar{\alpha}}) & :=\left\{\overline{\bar{x}} \in \mathbb{R}^{3} \mid\langle\overline{\bar{x}}, \overline{\bar{\alpha}}\rangle=0\right\}, \\
\mathcal{P}^{>}(\overline{\bar{\alpha}}) & :=\left\{\overline{\bar{x}} \in \mathbb{R}^{3} \mid\langle\overline{\bar{x}}, \overline{\bar{\alpha}}\rangle>0\right\}, \\
\mathcal{P}^{\geq}(\overline{\bar{\alpha}}) & :=\left\{\overline{\bar{x}} \in \mathbb{R}^{3} \mid\langle\overline{\bar{x}}, \overline{\bar{\alpha}}\rangle \geq 0\right\} .
\end{aligned}
$$

[^2]where $\langle\cdot, \cdot\rangle$ means the inner product. We put
\[

$$
\begin{aligned}
\mathscr{S}(\overline{\bar{\alpha}}) & :=\left\{\left(\overline{\bar{x}}, i^{*}\right) \mid i=0,1,2,\langle\overline{\bar{x}}, \overline{\bar{\alpha}}\rangle>0,\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{i}, \overline{\bar{\alpha}}\right\rangle \leq 0\right\}, \\
\mathscr{S}^{\prime}(\overline{\bar{\alpha}}) & :=\left\{\left(\overline{\bar{x}}, i^{*}\right) \mid i=0,1,2,\langle\overline{\bar{x}}, \overline{\bar{\alpha}}\rangle \geq 0,\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{i}, \overline{\bar{\alpha}}\right\rangle<0\right\},
\end{aligned}
$$
\]

which are subsets of $\mathbb{Z}^{3} \times\left\{0^{*}, 1^{*}, 2^{*}\right\}$. By the definition of $\mathscr{S}(\overline{\bar{\alpha}})$ (and $\mathscr{S}^{\prime}(\overline{\bar{\alpha}})$ ), we see that $\mathscr{S}(\overline{\bar{\alpha}})$ (and $\left.\mathscr{S}^{\prime}(\overline{\bar{\alpha}})\right)$ consists of the nearest unit squares clinging to the plane $\mathcal{P}(\overline{\bar{\alpha}})$. $\mathscr{S}(\overline{\bar{\alpha}})$ (and $\mathscr{S}^{\prime}(\overline{\bar{\alpha}})$ ) will be referred to as the stepped surface with respect to the direction $\overline{\bar{\alpha}}$. We also says that $\mathscr{S}(\overline{\bar{\alpha}})$ (and $\mathscr{S}^{\prime}(\overline{\bar{\alpha}})$ ) is the stepped surface of the plane $\mathcal{P}(\overline{\bar{\alpha}})$.

Remark 5.1. The difference between $\mathscr{S}(\overline{\bar{\alpha}})$ and $\mathscr{S}^{\prime}(\overline{\bar{\alpha}})$ is that $\mathscr{S}(\overline{\bar{\alpha}}) \backslash \mathscr{S}^{\prime}(\overline{\bar{\alpha}})=\left\{\left(\overline{\bar{e}}_{i}, i^{*}\right)\right\}_{i=0,1,2}$, $\mathscr{S}^{\prime}(\overline{\bar{\alpha}}) \backslash \mathscr{S}(\overline{\bar{\alpha}})=\left\{\left(\overline{\overline{0}}, i^{*}\right)\right\}_{i=0,1,2}$.

Moreover, we define $\mathcal{S}(\overline{\bar{\alpha}})$ :

$$
\mathcal{S}(\overline{\bar{\alpha}}):=\{\Lambda \mid \# \Lambda<+\infty, \Lambda \subset \mathscr{S}(\overline{\bar{\alpha}})\}
$$

We denote by $\mathcal{G}(\overline{\bar{\alpha}})$ the $\mathbb{Z}$-free module generated by all the finite squares:

Remark that the sum on the right-hand side of (28) is a formal sum. Hence, we have $\mathcal{G}(\overline{\bar{\alpha}})=\left\{\sum_{\lambda \in \Lambda} m_{\lambda} \lambda \mid \Lambda \subset \mathscr{S}(\overline{\bar{\alpha}}), \# \Lambda<+\infty, m_{\lambda} \in \mathbb{Z}\right\}$. In what follows, we only consider the case $m_{\lambda}=1$. When $m_{\lambda}=1$, we call the element $\sum_{\lambda \in \Lambda} \lambda$ of $\mathcal{G}(\overline{\bar{\alpha}})$ a patch of $\mathscr{S}(\overline{\bar{\alpha}})$. In some cases, $\{\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{S}(\overline{\bar{\alpha}})$ is also called as a patch, but an element of $\mathcal{S}(\overline{\bar{\alpha}})$ and $\mathcal{G}(\overline{\bar{\alpha}})$ should be distinguished. It will be convenient to define two maps, ${ }_{s} \Psi_{g}: \mathcal{S}(\overline{\bar{\alpha}}) \rightarrow \mathcal{G}(\overline{\bar{\alpha}})$ and ${ }_{g} \Psi_{s}: \mathcal{G}(\overline{\bar{\alpha}}) \rightarrow \mathcal{S}(\overline{\bar{\alpha}})$ as follows: for $\Lambda \in \mathcal{S}(\overline{\bar{\alpha}})$

$$
\begin{aligned}
& { }_{s} \Psi_{g}\left(\{\lambda\}_{\lambda \in \Lambda}\right):=\sum_{\lambda \in \Lambda} \lambda \in \mathcal{G}(\overline{\bar{\alpha}}) \quad\left(\text { for }\{\lambda\}_{\lambda \in \Lambda} \in \mathcal{S}(\overline{\bar{\alpha}})\right), \\
& { }_{g} \Psi_{s}\left(\sum_{\lambda \in \Lambda} \lambda\right):=\{\lambda\}_{\lambda \in \Lambda} \in \mathcal{S}(\overline{\bar{\alpha}}) \quad\left(\text { for } \sum_{\lambda \in \Lambda} \lambda \in \mathcal{G}(\overline{\bar{\alpha}})\right) .
\end{aligned}
$$

For example, for $\sum_{i=0}^{2}\left(\boldsymbol{e}_{i}, i^{*}\right)$ and $\left\{\left(\boldsymbol{e}_{i}, i^{*}\right)\right\}_{i=0,1,2},{ }_{g} \Psi_{s}\left(\sum_{i=0}^{2}\left(\boldsymbol{e}_{i}, i^{*}\right)\right)=$ $\left\{\left(\boldsymbol{e}_{i}, i^{*}\right)\right\}_{i=0,1,2},{ }_{s} \Psi_{g}\left(\left\{\left(\boldsymbol{e}_{i}, i^{*}\right)\right\}_{i=0,1,2}\right)=\sum_{i=0}^{2}\left(\boldsymbol{e}_{i}, i^{*}\right)$. For $\gamma, \delta \in \mathcal{G}(\overline{\bar{\alpha}})$, we denote $\gamma \prec \delta$ if ${ }_{g} \Psi_{s}(\gamma) \subset{ }_{g} \Psi_{s}(\delta)$. Taking $\mathscr{S}^{\prime}(\overline{\bar{\alpha}})$ instead of $\mathscr{S}(\overline{\bar{\alpha}})$, we can define $\mathcal{S}^{\prime}(\overline{\bar{\alpha}})$ and $\mathcal{G}^{\prime}(\overline{\bar{\alpha}})$ similarly. For each $(i, j) \in I n d$, we consider the substitution $\sigma_{(i, j)}$ as follows:

$$
\sigma_{(i, j)}:\left\{\begin{array}{rlll}
i & \mapsto & j i \\
k & \mapsto & k
\end{array} \quad(k \neq i)\right.
$$

And the so-called incidence matrix $L_{(i, j)}$ of $\sigma_{(i, j)}$ is the square matrix of size $3 \times 3$ defined by $L_{(i, j)}=\left(l_{i^{\prime} j^{\prime}}\right)$ where $l_{i^{\prime} j^{\prime}}$ is the number of occurrences of a letter $i^{\prime}$ appearing in $\sigma_{(i, j)}\left(j^{\prime}\right)$. Notice that $L_{(i, j)}=M_{(j, i)}$ for each $(i, j) \in I n d$ where $M_{(j, i)}$ is the matrix given in Lemma 3.3. For $(\alpha, \beta) \in \Delta_{K}$, we put $\overline{\bar{\nu}}(\alpha, \beta):={ }^{t}(1-\alpha-\beta, \alpha, \beta)$. In the sequel we fix an arbitrary $(\alpha, \beta) \in \Delta_{K}$ and an arbitrary $n \in \mathbb{Z}_{\geq 0}$. Then, the dual substitution $\Theta_{\varepsilon_{n}}$ of $\sigma_{\varepsilon_{n}}$, which is an endomorphism from $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ to $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$ introduced in Arnoux and Ito 01, can be defined by

$$
\begin{gather*}
\Theta_{\varepsilon_{n}}\left(\overline{\bar{x}}, i^{*}\right):=L_{\varepsilon_{n}}^{-1} \overline{\bar{x}}+\sum_{j=0}^{2} \sum_{\substack{S: \\
\sigma_{\varepsilon_{n}}(j)=P i S}}\left(L_{\varepsilon_{n}}^{-1}(f(S)), j^{*}\right),  \tag{29}\\
\Theta_{\varepsilon_{n}}\left(\sum_{\lambda \in \Lambda}\left(\overline{\bar{x}}, i^{*}\right)_{\lambda}\right):=\sum_{\lambda \in \Lambda}\left(\Theta_{\varepsilon_{n}}\left(\overline{\bar{x}}, i^{*}\right)_{\lambda}\right)
\end{gather*}
$$

for $i=0,1,2$, where $f(w):={ }^{t}\left(|w|_{0},|w|_{1},|w|_{2}\right)\left(|w|_{i}\right.$ is the number of occurrences of a symbol $i$ appearing in a finite word $w \in\{0,1,2\}^{*}$ ), and $P$ (resp., $S$ ) means that the prefix (resp., suffix) of $i$ of $\sigma(j)$ with $\sigma(j)=P i S$ and

$$
\overline{\bar{y}}+\sum_{\lambda \in \Lambda}\left(\overline{\bar{x}}_{\lambda}, i_{\lambda}^{*}\right):=\sum_{\lambda \in \Lambda}\left(\overline{\bar{y}}+\overline{\bar{x}}_{\lambda}, i_{\lambda}^{*}\right) .
$$

In particular,

$$
\Theta_{(i, j)}:\left\{\begin{align*}
\left(\overline{\bar{x}}, j^{*}\right) & \mapsto\left(L_{(i, j)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{i}\right), i^{*}\right)+\left(L_{(i, j)}^{-1} \overline{\bar{x}}, j^{*}\right),  \tag{30}\\
\left(\overline{\bar{x}}, k^{*}\right) & \mapsto\left(L_{(i, j)}^{-1} \overline{\bar{x}}, k^{*}\right) \quad(k \neq j) .
\end{align*}\right.
$$

(see Figure (4).


Figure 4: $\quad \Theta_{\varepsilon}(\mathcal{U}), \mathcal{U}=\sum_{i=0}^{2}\left(\boldsymbol{e}_{i}, i^{*}\right), \varepsilon \in \operatorname{Ind}$.

In view of definitions, we have

Lemma 5.2. There exists a constant $c_{n}$ satisfying

$$
\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)=c_{n}{ }^{t} L_{\varepsilon_{n}}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)
$$

where

$$
c_{n}:=\left\{\begin{array}{lll}
\frac{1}{1-\beta_{n}} & \text { if } & \varepsilon_{n} \in\{(1,2),(0,2)\}, \\
\frac{1}{1-\alpha_{n}} & \text { if } & \varepsilon_{n} \in\{(2,1),(0,1)\}, \\
\frac{1}{\alpha_{n}+\beta_{n}} & \text { if } & \varepsilon_{n} \in\{(1,0),(2,0)\} .
\end{array}\right.
$$

Proof. Let $\varepsilon_{n}=(1,2)$, then from the fact that

$$
\left(\alpha_{n+1}, \beta_{n+1}\right)=T_{(1,2)}\left(\alpha_{n}, \beta_{n}\right)=\left(\frac{\alpha_{n}-\beta_{n}}{1-\beta_{n}}, \frac{\beta_{n}}{1-\beta_{n}}\right),
$$

we have

$$
\begin{aligned}
\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right) & ={\frac{1}{1-\beta_{n}}}^{t}\left(1-\alpha_{n}-\beta_{n}, \alpha_{n}-\beta_{n}, \beta_{n}\right) \\
& =\frac{1}{1-\beta_{n}}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
1-\alpha_{n}-\beta_{n} \\
\alpha_{n} \\
\beta_{n}
\end{array}\right)=c_{n}^{t} L_{(1,2)}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right) .
\end{aligned}
$$

The proof for the other cases is analogously done.

We define $\varphi_{n}: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ by

$$
\overline{\bar{x}}_{n}={ }^{t}\left(x_{n}, y_{n}, z_{n}\right)=\varphi_{n}\left(\overline{\bar{x}}_{n+1}\right)=\varphi_{n}{ }^{t}\left(x_{n+1}, y_{n+1}, z_{n+1}\right):=L_{\varepsilon_{n}}^{-1 t}\left(x_{n+1}, y_{n+1}, z_{n+1}\right) .
$$

Then, we have
Lemma 5.3. Let $c_{n}$ be numbers given in Lemma 5.2. Then,

$$
\left\langle\varphi_{n} \overline{\bar{x}}_{n+1}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\frac{1}{c_{n}}\left\langle\overline{\bar{x}}_{n+1}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle, \quad \overline{\bar{x}}_{n+1} \in \mathbb{R}^{3}
$$

holds.

Proof. By Lemma 5.2, we get

$$
\begin{aligned}
\left\langle\varphi_{n} \overline{\bar{x}}_{n+1}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle & =\left\langle L_{\varepsilon_{n}}^{-1} \overline{\bar{x}}_{n+1}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\left\langle\overline{\bar{x}}_{n+1},{ }^{t} L_{\varepsilon_{n}}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \\
& =\frac{1}{c_{n}}\left\langle\overline{\bar{x}}_{n+1}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle .
\end{aligned}
$$

From Lemma 5.3, it follows
Corollary 5.4. We have

$$
\begin{aligned}
& \varphi_{n}\left(\mathcal{P}^{>}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)\right)=\mathcal{P}^{>}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right), \\
& \varphi_{n}\left(\mathcal{P}^{\geq}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)\right)=\mathcal{P}^{\geq}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right) .
\end{aligned}
$$

The following theorem is important related to the stepped surface.
Theorem 5.5 ('Bijectivity' of $\Theta$ ). Let us assume that $\left(\alpha_{n}, \beta_{n}\right) \in \Delta_{K}$, then

$$
\Theta_{\varepsilon_{n}}: \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right) \rightarrow \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)
$$

satisfies the following:
(1) if $\left(\overline{\bar{x}}, i^{*}\right)$ is a unit square of $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$, then the image $\Theta_{\varepsilon_{n}}\left(\overline{\bar{x}}, i^{*}\right)$ belongs to $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right) ;$
(2) two distinct unit squares are sent to disjoint images (which are patches of squares) except for their boundaries;
(3) for any $\left(\overline{\bar{z}}, k^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$, there exists a square $\left(\overline{\bar{x}}, i^{*}\right) \in$ $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ such that $\Theta_{\varepsilon_{n}}\left(\overline{\bar{x}}, i^{*}\right) \succ\left(\overline{\bar{z}}, j^{*}\right)$.

Proof. For instance, we consider the case where $\varepsilon_{n}=(2,0)$. We shall show the following three properties:
(1) if $\left(\overline{\bar{x}}, i^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$, then $\Theta_{(2,0)}\left(\overline{\bar{x}}, i^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$;
(2) if $\left(\overline{\bar{x}}, i^{*}\right) \neq\left(\overline{\bar{y}}, j^{*}\right)\left(\left(\overline{\bar{x}}, i^{*}\right),\left(\overline{\bar{y}}, j^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)\right)$, then ${ }_{g} \Psi_{s}\left(\Theta_{(2,0)}\left(\overline{\bar{x}}, i^{*}\right)\right) \cap_{g} \Psi_{s}\left(\Theta_{(2,0)}\left(\overline{\bar{y}}, j^{*}\right)\right)=\emptyset ;$
(3) if for any $\left(\overline{\bar{z}}, k^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$, then there exists $\left(\overline{\bar{x}}, i^{*}\right) \in$ $\mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ such that $\Theta_{(2,0)}\left(\overline{\bar{x}}, i^{*}\right) \succ\left(\overline{\bar{z}}, k^{*}\right)$.

About (1): If $\left(\overline{\bar{x}}, i^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right), i=0,1,2$, then

$$
\begin{align*}
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right), \\
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 1^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 1^{*}\right),  \tag{31}\\
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 2^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 2^{*}\right)
\end{align*}
$$

follow from (30). Hence, it suffices to show

$$
\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right),\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right),\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 1^{*}\right),\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right) .
$$

From the definition of $\left(\overline{\bar{x}}, i^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$, we know that

$$
\begin{align*}
\left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & >0,  \tag{32}\\
\left\langle\overline{\bar{x}}-\overline{\bar{e}} i, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & \leq 0, \tag{33}
\end{align*}
$$

where the equality of (33)) holds only for $\overline{\bar{x}}=\overline{\bar{e}}_{i}$. We suppose that $\left(\overline{\bar{x}}, 0^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$. By Lemma 5.2 and (32),

$$
\left\langle L_{(2,0)}^{-1} \overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\left\langle\overline{\bar{x}},{ }^{t} L_{(2,0)}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\frac{1}{c_{n}}\left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle>0,
$$

and by Lemma 5.2 and (33), $\left\langle L_{(2,0)}^{-1} \overline{\bar{x}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=$ $\left\langle L_{(2,0)}^{-1}\left(\overline{\bar{x}}-\overline{\bar{e}}_{0}\right), \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0},{ }^{t} L_{(2,0)}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=$
$\frac{1}{c_{n}}\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \leq 0$. Hence, we obtain $\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$. By Lemma 5.2 and (32), $\left\langle L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=$
$\left\langle\overline{\bar{x}}+\overline{\bar{e}}_{2},{ }^{t} L_{(2,0)}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\frac{1}{c_{n}}\left\langle\overline{\bar{x}}+\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle>0$ and by Lemma 5.2 and (33),
$\left\langle L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right)-\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\left\langle L_{(2,0)}^{-1}\left(\overline{\bar{x}}-\overline{\bar{e}}_{0}\right), \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle=\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0},{ }^{t} L_{(2,0)}^{-1} \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle$
$=\frac{1}{c_{n}}\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \leq 0$. Therefore, we get $\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$.
We can show $\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 1^{*}\right),\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$ in a similar manner.
About (2): For

$$
\begin{equation*}
\left(\overline{\bar{x}}, i^{*}\right),\left(\overline{\bar{y}}, j^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right) \text { satisfying }\left(\overline{\bar{x}}, i^{*}\right) \neq\left(\overline{\bar{y}}, j^{*}\right) \tag{34}
\end{equation*}
$$

we will check whether there exist a common unit square between $\Theta_{(2,0)}\left(\overline{\bar{x}}, i^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, j^{*}\right)$. There are six patterns of the combinations of $i^{*}$ and $j^{*}$ as $\left\{\left(0^{*}, 0^{*}\right),\left(0^{*}, 1^{*}\right),\left(0^{*}, 2^{*}\right),\left(1^{*}, 1^{*}\right),\left(1^{*}, 2^{*}\right),\left(2^{*}, 2^{*}\right)\right\}$.
(i) Case of $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 0^{*}\right)$ : By (30),

$$
\begin{aligned}
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right) \\
& \Theta_{(2,0)}\left(\overline{\bar{y}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{y}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{y}}+\overline{\bar{e}}_{2}\right), 2^{*}\right) .
\end{aligned}
$$

Thus, if there exist a common square between $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 0^{*}\right)$, then

$$
\begin{aligned}
& \left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{y}}, 0^{*}\right) \text { or } \\
& \left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right)=\left(L_{(2,0)}^{-1}\left(\overline{\bar{y}}+\overline{\bar{e}}_{2}\right), 2^{*}\right)
\end{aligned}
$$

holds which implies $\overline{\bar{x}}=\overline{\bar{y}}$. This contradicts (34). Therefore, there are no common unit squares between $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 0^{*}\right)$.
(ii) Case of $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 1^{*}\right)$ : from (30),

$$
\begin{aligned}
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right), \\
& \Theta_{(2,0)}\left(\overline{\bar{y}}, 1^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{y}}, 1^{*}\right)
\end{aligned}
$$

It is clear that there does not exist a common square between $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 1^{*}\right)$.
(iii) Case of $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 2^{*}\right)$ : By (30),

$$
\begin{aligned}
& \Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right), \\
& \Theta_{(2,0)}\left(\overline{\bar{y}}, 2^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{y}}, 2^{*}\right) .
\end{aligned}
$$

Thus, if there exist a common square between $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 2^{*}\right)$, then

$$
\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{y}}, 2^{*}\right),
$$

i.e.,

$$
\begin{equation*}
\overline{\bar{x}}=\overline{\bar{y}}-\overline{\bar{e}}_{2} . \tag{35}
\end{equation*}
$$

On the other hand, from $\left(\overline{\bar{x}}, 0^{*}\right),\left(\overline{\bar{y}}, 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$, we have

$$
\begin{array}{r}
\left\langle\overline{\bar{x}}, \overline{\bar{v}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle>0 \\
\left\langle\overline{\bar{y}}-\overline{\bar{e}}_{2}, \overline{\bar{v}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \leq 0 . \tag{37}
\end{array}
$$

However, using (35) and (37), we get

$$
\left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=\left\langle\overline{\bar{y}}-\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \leq 0
$$

which contradicts (36). Therefore, there are no common unit squares between $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right)$ and $\Theta_{(2,0)}\left(\overline{\bar{y}}, 2^{*}\right)$.

The other cases can be proved analogously.
About (3): $\operatorname{For}\left(\overline{\bar{z}}, i^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$, we have

$$
\begin{array}{r}
\left\langle\overline{\bar{z}}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 \\
\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{i}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0 . \tag{39}
\end{array}
$$

(i) For $\left(\overline{\bar{z}}, 0^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$, there exists $\left(\overline{\bar{x}}, 0^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ satisfying

$$
\begin{equation*}
\overline{\bar{x}}=L_{(2,0)} \overline{\bar{z}} \tag{40}
\end{equation*}
$$

such that $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right) \succ\left(\overline{\bar{z}}, 0^{*}\right)$. In fact, from (30) and (40), it follows

$$
\begin{aligned}
\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right) & =\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right) \\
& =\left(\overline{\bar{z}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}}_{2}\right), 2^{*}\right) \succ\left(\overline{\bar{z}}, 0^{*}\right) .
\end{aligned}
$$

By (40), Lemma 5.2, and (38), we get

$$
\begin{aligned}
\left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & =\left\langle L_{(2,0)} \overline{\bar{z}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle\overline{\bar{z}},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =c_{n}\left\langle\overline{\bar{z}}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 .
\end{aligned}
$$

By (40), Lemma 5.2, and (39), we get

$$
\begin{aligned}
& \left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=\left\langle L_{(2,0)}\left(\overline{\bar{z}}-\overline{\bar{e}}_{0}\right), \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{0},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=c_{n}\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{0}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0 .
\end{aligned}
$$

(ii) For $\left(\overline{\bar{z}}, 1^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$, there exists $\left(\overline{\bar{x}}, 1^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ satisfying

$$
\begin{equation*}
\overline{\bar{x}}=L_{(2,0)} \overline{\bar{z}} \tag{41}
\end{equation*}
$$

such that $\Theta_{(2,0)}\left(\overline{\bar{x}}, 1^{*}\right) \succ\left(\overline{\bar{z}}, 1^{*}\right)$. In fact, from (30) and (41), it follows

$$
\Theta_{(2,0)}\left(\overline{\bar{x}}, 1^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 1^{*}\right)=\left(\overline{\bar{z}}, 1^{*}\right) \succ\left(\overline{\bar{z}}, 1^{*}\right)
$$

By (41), Lemma 5.2, and (38), we get

$$
\begin{aligned}
& \left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle L_{(2,0)} \overline{\bar{z}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=\left\langle\overline{\bar{z}},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=c_{n}\left\langle\overline{\bar{z}}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 .
\end{aligned}
$$

By (41), Lemma 5.2, and (39), we get

$$
\begin{aligned}
& \left\langle\overline{\bar{x}}-\overline{\bar{e}}_{1}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{1}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=\left\langle L_{(2,0)}\left(\overline{\bar{z}}-\overline{\bar{e}}_{1}\right), \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{1},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=c_{n}\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{1}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0 .
\end{aligned}
$$

(iii) $\operatorname{For}\left(\overline{\bar{z}}, 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right)$,
(a) if

$$
\begin{equation*}
\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 \tag{42}
\end{equation*}
$$

then there exists $\left(\overline{\bar{x}}, 0^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ satisfying

$$
\begin{equation*}
\overline{\bar{x}}=L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{2} \tag{43}
\end{equation*}
$$

such that $\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right) \succ\left(\overline{\bar{z}}, 2^{*}\right)$;
(b) if

$$
\begin{equation*}
\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0, \tag{44}
\end{equation*}
$$

then there exists $\left(\overline{\bar{x}}, 2^{*}\right) \in \mathcal{G}\left(\overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right)$ satisfying

$$
\begin{equation*}
\overline{\bar{x}}=L_{(2,0)} \overline{\bar{z}} \tag{45}
\end{equation*}
$$

such that $\Theta_{(2,0)}\left(\overline{\bar{x}}, 2^{*}\right) \succ\left(\overline{\bar{z}}, 2^{*}\right)$.
About (a): From (30) together with (43), it follows

$$
\begin{aligned}
\Theta_{(2,0)}\left(\overline{\bar{x}}, 0^{*}\right) & =\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(L_{(2,0)}^{-1}\left(\overline{\bar{x}}+\overline{\bar{e}_{2}}\right), 2^{*}\right) \\
& =\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 0^{*}\right)+\left(\overline{\bar{z}}, 2^{*}\right) \succ\left(\overline{\bar{z}}, 2^{*}\right) .
\end{aligned}
$$

By (43), Lemma 5.2, and (42), we get,

$$
\begin{aligned}
\left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & =\left\langle L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle L_{(2,0)}\left(\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0}\right), \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =c_{n}\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 .
\end{aligned}
$$

By (41), Lemma 5.2, and (39),

$$
\begin{aligned}
\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & =\left\langle L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{2}-\overline{\bar{e}}_{0}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle L_{(2,0)}\left(\overline{\bar{z}}-\overline{\bar{e}}_{2}\right), \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =c_{n}\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0 .
\end{aligned}
$$

About (b): From (30) together with (45), it follows

$$
\Theta_{(2,0)}\left(\overline{\bar{x}}, 2^{*}\right)=\left(L_{(2,0)}^{-1} \overline{\bar{x}}, 2^{*}\right)=\left(\overline{\bar{z}}, 2^{*}\right) \succ\left(\overline{\bar{z}}, 2^{*}\right)
$$

By (45), Lemma 5.2, and (38), we get,

$$
\begin{aligned}
& \left\langle\overline{\bar{x}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=\left\langle L_{(2,0)}^{\overline{\bar{z}}}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle=\left\langle\overline{\bar{z}},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& \quad=c_{n}\left\langle\overline{\bar{z}}, \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle>0 .
\end{aligned}
$$

By (45), Lemma 5.2, and (44), we get,

$$
\begin{aligned}
\left\langle\overline{\bar{x}}-\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle & =\left\langle L_{(2,0)} \overline{\bar{z}}-\overline{\bar{e}}_{2}, \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle L_{(2,0)}\left(\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0}\right), \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0},{ }^{t} L_{(2,0)} \overline{\bar{\nu}}\left(\alpha_{n+1}, \beta_{n+1}\right)\right\rangle \\
& =\left\langle\overline{\bar{z}}-\overline{\bar{e}}_{2}+\overline{\bar{e}}_{0},, c_{n} \overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right\rangle \leq 0 .
\end{aligned}
$$

By the arguments (i)-(iii), we obtain the assertion (3).
Other cases can be proved analogously.
Corollary 5.6. Let $K$ be a real cubic field and $\left(\alpha_{0}, \beta_{0}\right) \in \Delta_{K}$. Moreover, we assume that the sequence $\left\{\left(\alpha_{n}, \beta_{n}, \varepsilon_{n}\right)\right\}_{n=0,1,2, \ldots}$ is periodic with a period of length p, i.e.,

$$
\begin{equation*}
\exists m \geq 0, \exists p \geq 1:\left(\alpha_{m}, \beta_{m}, \varepsilon_{m}\right)=\left(\alpha_{m+p}, \beta_{m+p}, \varepsilon_{m+p}\right) \tag{46}
\end{equation*}
$$

Then, the stepped surface $\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{0}, \beta_{0}\right)\right)$ can be presented by

$$
\begin{equation*}
{ }_{s} \Psi_{g}\left(\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{0}, \beta_{0}\right)\right)\right)=\Theta_{\varepsilon_{0}} \circ \Theta_{\varepsilon_{1}} \circ \cdots \circ \Theta_{\varepsilon_{m-1}}\left({ }_{s} \Psi_{g}\left(\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{m}, \beta_{m}\right)\right)\right)\right) \tag{47}
\end{equation*}
$$

and the stepped surface $\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{m}, \beta_{m}\right)\right)$ can be characterized as the fixed point of the tiling substitution $\Theta=\Theta_{\varepsilon_{m}} \Theta_{\varepsilon_{m+1}} \cdots \Theta_{\varepsilon_{m+p-1}}$ by

$$
\begin{equation*}
\Theta\left({ }_{s} \Psi_{g}\left(\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{m}, \beta_{m}\right)\right)\right)\right)={ }_{s} \Psi_{g}\left(\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{m}, \beta_{m}\right)\right)\right) . \tag{48}
\end{equation*}
$$

Remark 5.7. Notice that by the bijectivity of $\Theta_{\varepsilon_{n}}$, the right-hand side (resp., left-hand side) of (47) (resp., (48)) can be defined by extending the finite sum of squares to an infinite sum. We can do the same for ${ }_{s} \Psi_{g}$ and ${ }_{g} \Psi_{s}$.

Proof. It is clear that (48) is valid by (46). Moreover, by the bijectivity of $\Theta_{\varepsilon_{n}}, n=$ $0,1, \ldots, m-1$, we get (47).

Let $\mathcal{U}$ and $\mathcal{U}^{\prime}$ be fundamental patches

$$
\mathcal{U}:=\sum_{i=0}^{2}\left(\overline{\bar{e}}_{i}, i^{*}\right), \mathcal{U}^{\prime}:=\sum_{i=0}^{2}\left(\overline{\overline{0}}, i^{*}\right)
$$

We put

$$
\begin{aligned}
\gamma_{n} & :=\Theta_{\varepsilon_{0}} \ldots \Theta_{\varepsilon_{n-2}} \Theta_{\varepsilon_{n-1}}(\mathcal{U}) \quad \text { for } \mathcal{U} \in \mathcal{G}\left(\overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right), \\
\left(\text { resp., } \gamma_{n}^{\prime}\right. & \left.:=\Theta_{\varepsilon_{0}} \ldots \Theta_{\varepsilon_{n-2}} \Theta_{\varepsilon_{n-1}}\left(\mathcal{U}^{\prime}\right) \quad \text { for } \mathcal{U}^{\prime} \in \mathcal{G}^{\prime}\left(\overline{\bar{v}}\left(\alpha_{n}, \beta_{n}\right)\right)\right),
\end{aligned}
$$

which is a sequence of patches of $\mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{0}, \beta_{0}\right)\right)$ (resp., $\left.\mathscr{S}^{\prime}\left(\overline{\bar{\nu}}\left(\alpha_{0}, \beta_{0}\right)\right)\right)$. Then we have the following.

Corollary 5.8. (1) The difference of $\gamma_{n}$ and $\gamma_{n}^{\prime}$ is that

$$
{ }_{g} \Psi_{s}\left(\gamma_{n}\right) \backslash_{g} \Psi_{s}\left(\gamma_{n}^{\prime}\right)={ }_{g} \Psi_{s}(\mathcal{U}), \quad{ }_{g} \Psi_{s}\left(\gamma_{n}^{\prime}\right) \backslash_{g} \Psi_{s}\left(\gamma_{n}\right)={ }_{g} \Psi_{s}\left(\mathcal{U}^{\prime}\right)
$$

(2) $\gamma_{n} \prec \gamma_{n+1}$ for all $n$.

$$
\begin{equation*}
\bigcup_{n=0}^{\infty} \Psi_{s}\left(\gamma_{n}\right) \subset \mathscr{S}\left(\overline{\bar{\nu}}\left(\alpha_{0}, \beta_{0}\right)\right) . \tag{3}
\end{equation*}
$$

## 6 Examples

We give some examples.
Example 6.1 ( $K$ is not totally real). Let $\delta$ be the real root of $x^{3}-2, K=\mathbb{Q}(\delta)$ and $\alpha=2 / 3-2 \delta / 3+\delta^{2} / 6, \beta=2 / 3+\delta / 3-\delta^{2} / 3$. Then $(\alpha, \beta) \in \Delta_{K}$ and the expansion $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}$ of $(\alpha, \beta)$ obtained by our continued fraction algorithm is given by

$$
\begin{aligned}
\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}= & (2,0),(0,2),(2,1),(2,1),(0,1),(1,0),(0,2),(0,2),(1,2),(2,1) \\
& (1,0),(1,0), \ldots
\end{aligned}
$$

and the eigenvalue $\lambda(\alpha, \beta)>1$ coming from the period of the expansions is a Pisot number with $x^{3}-57 x^{2}+3 x-1$ as its minimal polynomial (see Figure 5). On the other hand, we can observe the explosion phenomenon (as in the example in (Tamura and Yasutomi 09]) related to the expansions obtained by the Jacobi-Perron and the modified Jacobi-Perron algorithms; consequently, we can not expect the periodicity of the expansions.

Example 6.2 ( $K$ is totally real). (1) Let $\delta$ be the root of $x^{3}-6 x^{2}+7 x-1$ with $\delta>4$, $K=\mathbb{Q}(\delta)$ and $\alpha=-1 / 3-4 \delta / 3+\delta^{2} / 3, \beta=-2+5 \delta-\delta^{2}$. Then, $(\alpha, \beta) \in \Delta_{K}$ and ${ }^{t}(1-\alpha-\beta, \alpha, \beta)$ is an eigenvector of $M_{(1,0)} M_{(0,1)} M_{(2,0)} M_{(0,2)} M_{(0,1)}$ with respect to its eigenvalue $\delta$. We note that $\delta$ is not a Pisot number (see Figure [6).
(2) On the other hand, we have the expansion of $(\alpha, \beta)$ as follows:

$$
\begin{aligned}
\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}= & (1,0),(0,2),(2,1),(2,1),(2,1),(1,0),(1,0), \\
& { }^{*}(1,0),(0,2),(2,0),(1,0),(2,0),(0,1),(1,0),(2,0), \ldots,
\end{aligned}
$$

and $\lambda(\alpha, \beta)$ is a Pisot number having its minimal polynomial $x^{3}-13 x^{2}+10 x-1$ (see Figure 7).

Example 6.3 ( $K$ is not totally real). (1) (Completely non-admissible)
Let $\delta$ be the real root of $x^{3}-5 x^{2}-2 x-1, K=\mathbb{Q}(\delta)$ and $\alpha=11 / 5+9 \delta / 5-2 \delta^{2} / 5$, $\beta=-7 / 5+7 \delta / 5-\delta^{2} / 5$. Then, $(\alpha, \beta) \in \Delta_{K}$ and $^{t}(1-\alpha-\beta, \alpha, \beta)$ is an eigenvector of $M_{(0,1)} M_{(2,0)} M_{(1,2)} M_{(0,1)} M_{(2,0)} M_{(1,2)}$ with respect to its eivenvalue $\delta$. We note that $(0,1)(2,0),(2,0)(1,2),(1,2)(0,1),(2,0)(1,2),(1,2)(0,1)$ are forbidden words given in Table 1 and $\delta$ is a Pisot number (see Figure 8).
(2) We have the expansion of $(\alpha, \beta)$ as follows:

$$
\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}=\left(0^{*}, 2\right),(2,1),(1,0),(1,0),(0,2),(2,1),(2,1),(1,0),\left(0,{ }^{*} 2\right), \ldots
$$

and $\lambda(\alpha, \beta)$ is a Pisot number having $x^{3}-29 x^{2}-6 x-1$ as its minimal polynomial (see Figure (1).


$$
n=1
$$



Figure 5: $\left(\Theta_{(2,0)} \Theta_{(0,2)} \Theta_{(2,1)} \Theta_{(2,1)} \Theta_{(0,1)} \Theta_{(1,0)} \Theta_{(0,2)} \Theta_{(0,2)} \Theta_{(1,2)} \Theta_{(2,1)} \Theta_{(1,0)} \Theta_{(1,0)}\right)^{n}$ ( $\left.\mathcal{U}\right)$ in Example 6.1 where the point is located at $(1,1,1)$.

$$
n=0
$$



$$
n=1
$$

$$
n=2
$$

Figure 6: $\left(\Theta_{(1,0)} \Theta_{(0,1)} \Theta_{(2,0)} \Theta_{(0,2)} \Theta_{(0,1)}\right)^{n}(\mathcal{U})$ in Example 6.2 (1).


$$
n=2
$$

Figure 7: $\Theta_{(1,0)} \Theta_{(0,2)} \Theta_{(2,1)} \Theta_{(2,1)} \Theta_{(2,1)} \Theta_{(1,0)} \Theta_{(1,0)}\left(\Theta_{(1,0)} \Theta_{(0,2)} \Theta_{(2,0)} \Theta_{(1,0)} \Theta_{(2,0)} \Theta_{(0,1)} \Theta_{(1,0)} \Theta_{(2,0)}\right)^{n}(\mathcal{U})$ in Example 6.2 (2).


Figure 8: $\left(\Theta_{(0,1)} \Theta_{(2,0)} \Theta_{(1,2)} \Theta_{(0,1)} \Theta_{(2,0)} \Theta_{(1,2)}\right)^{n}(\mathcal{U})$ in Example 6.3 (1).


$$
n=1
$$



Figure 9: $\left(\Theta_{(0,2)} \Theta_{(2,1)} \Theta_{(1,0)} \Theta_{(1,0)} \Theta_{(0,2)} \Theta_{(2,1)} \Theta_{(2,1)} \Theta_{(1,0)} \Theta_{(0,2)}\right)^{n}(\mathcal{U})$ in Example 6.3 (2).

## 7 Conjectures

We give following conjectures which are supported by the numerical experiments.
Conjecture 7.1. Let $K$ be a real cubic field and $r=5 / 2 . \Delta_{K}=\Delta_{K, r}^{\mathcal{P e r}}$ holds for any real cubic field $K$.

Conjecture 7.2. Let $K$ be a real cubic field and let $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}$ be its positive $\mathbb{Q}$-basis with

$$
\alpha=\frac{\alpha^{(1)}}{\alpha^{(0)}+\alpha^{(1)}+\alpha^{(2)}}, \beta=\frac{\alpha^{(2)}}{\alpha^{(0)}+\alpha^{(1)}+\alpha^{(2)}} .
$$

Let $\left\{\varepsilon_{n}\right\}_{n=0}^{\infty}$ be the expansion of the $(\alpha, \beta)$. Suppose that $\varepsilon_{k+1}, \ldots, \varepsilon_{k+l}$ is the period of the expansion. Then, $M_{\varepsilon_{k+1}} \ldots M_{\varepsilon_{k+l}}$ has a Pisot number as its eigenvalue.

The following theorem (Theorem 5 in Fernique Fernique 05]) together with Conjectures 7.1, 7.2 implies Conjecture 7.5,

Theorem 7.3 (Fernique). Let $\mathscr{S}(\overline{\bar{\alpha}})$ be a stepped surface of such that there exist two generalized substitutions $\Theta_{\text {Prep }}$ and $\Theta_{\mathcal{P} e r}$ verifying:

$$
\mathscr{S}(\overline{\bar{\alpha}})=\Theta_{\text {Prep }}(\mathscr{S}) \text { with } \mathscr{S}=\Theta_{\mathcal{P e r}}(\mathscr{S}) \text {. }
$$

If $\Theta_{\mathcal{P} e r}$ is of Pisot type and bijective on $\mathscr{S}$, then there exists a finite patch $P$ of $\mathscr{S}$ such that

$$
\mathscr{S}(\overline{\bar{\alpha}})=\Theta_{\text {Prep }}\left(\lim _{n \rightarrow \infty} \Theta_{\mathcal{P} \text { er }}(P)\right) .
$$

Remark 7.4. Theorem 7.3 gives an effective generation of any stepped surface under the Pisot condition.

Conjecture 7.5. Let $\overline{\bar{\alpha}}=\left(\alpha_{0}, \alpha_{1}, \alpha_{2}\right)$ be any $\mathbb{Q}$-basis of arbitrarily given real cubic number field. Then the stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ is finitely descriptive, i.e., there exist a finite word $\varepsilon_{0} \varepsilon_{1} \cdots \varepsilon_{k-1} \in$ Ind $^{*}=\bigcup_{n=0}^{\infty}$ Ind ${ }^{n}$, and a nonempty word $\varepsilon_{k} \varepsilon_{k+1} \cdots \varepsilon_{k+l-1} \in$ $\operatorname{Ind}^{*}(k \geq 0, l>1)$, and a patch $P$ consisting of finite squares such that

$$
\mathscr{S}(\overline{\bar{\alpha}})={ }_{g} \Psi_{s}\left(\Theta_{\varepsilon_{0}} \Theta_{\varepsilon_{1}} \cdots \Theta_{\varepsilon_{k-1}}\left(\lim _{n \rightarrow \infty}\left(\Theta_{\varepsilon_{k}} \Theta_{\varepsilon_{k+1}} \cdots \Theta_{\varepsilon_{k+l-1}}\right)^{n}(P)\right)\right) .
$$

Remark 7.6. Conjecture 7.5 says that any stepped surface $\mathscr{S}(\overline{\bar{\alpha}})$ for any $\mathbb{Q}$-basis $\overline{\bar{\alpha}} \in K^{3}$ for any given real cubic number field $K$ is finitely descriptive and generated only by 6 substitutions. For notation of $\Theta_{\varepsilon}(\varepsilon \in$ Ind $)$, see Section 5. Notice that without loss of generality, we may assume $\alpha_{0}, \alpha_{1}, \alpha_{2}>0$ and $\alpha_{0}+\alpha_{1}+\alpha_{2}=1$ by the symmetry of the lattice $\mathbb{Z}^{3}$.

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[^1]:    ${ }^{1}$ The routine that was written for this purpose can be downloaded from the web site http://www.lab2.toho-u.ac.jp/sci/c/math/yasutomi/mfarey.html

[^2]:    ${ }^{2}$ The routine that was written for this purpose can be downloaded from the web site http://www.lab2.toho-u.ac.jp/sci/c/math/yasutomi/mfarey.html

