MINIMAL MAHLER MEASURE IN REAL QUADRATIC FIELDS

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ABSTRACT. We consider upper and lower bounds on the minimal height of an irrational number lying in a particular real quadratic field.

1. INTRODUCTION

For a polynomial $F(x) = a_n \prod_{i=1}^n (x - \alpha_i)$ in $\mathbb{C}[x]$ one defines its Mahler measure M(F) as

$$M(F) = |a_n| \prod_{i=1}^n \max\{1, |\alpha_i|\}.$$

For an algebraic number α we use $M(\alpha)$ to denote the Mahler measure of an irreducible integer polynomial with root α . Thus the logarithmic Weil height of α can be written

$$h(\alpha) = \frac{\log M(\alpha)}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}.$$

Of course $M(\alpha) = 1$ iff α is a root of unity and the well known problem of Lehmer [3] is to determine whether there is a constant C > 1 such that $M(\alpha) > C$ otherwise. Schinzel [4] showed that for α in a Kroneckerian field (a totally real field or a quadratic extension of such a field) the value of $M(\alpha)$ must in fact grow with its degree, with the absolute minimum $M(\alpha) > 1$ achieved for the golden ratio

$$M\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2}.$$

Amoroso & Dvornicich [1] further extended this to cyclotomic fields. These of course include the quadratic fields $\mathbb{Q}(\sqrt{d})$, where d is a square-free positive integer. Since the golden ratio is not in all these fields we are interested in how

$$L(d) := \min\left\{M(\alpha) : \alpha \in \mathbb{Q}(\sqrt{d}) \setminus \mathbb{Q}\right\}$$

varies with d. We recall the discriminant of the field $\mathbb{Q}(\sqrt{d})$

$$D := \begin{cases} d, & \text{if } d \equiv 1 \mod 4, \\ 4d, & \text{if } d \equiv 2 \text{ or } 3 \mod 4. \end{cases}$$

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Since $M(\alpha) = M(-\alpha) = M(\overline{\alpha})$ we assume that our $\alpha \in \mathbb{Q}(\sqrt{d}) \setminus \mathbb{Q}$ takes the form (1) $\alpha = \frac{a + b\sqrt{d}}{c}, \ a, b, c \in \mathbb{Z}, \ a \ge 0, \ b > 0, \ c > 0, \ \gcd(a, b, c) = 1,$

with conjugate

$$\overline{\alpha} = \frac{a - b\sqrt{d}}{c},$$

and

(2)
$$M(\alpha) = k \max\{1, |\alpha|\} \max\{1, |\overline{\alpha}|\}$$

where k is the smallest positive integer such that

(3)
$$k(x-\alpha)(x-\overline{\alpha}) = k\left(x^2 - \frac{2a}{c}x + \frac{a^2 - b^2d}{c^2}\right) \in \mathbb{Z}[x].$$

We show that the minimal measure must grow with d:

Theorem 1.1. For square-free d in \mathbb{N}

$$\frac{1}{2}\sqrt{D} < L(d) < \sqrt{D}.$$

2. Proof of Theorem 1.1

The upper bound follows at once from the following constructive examples:

Lemma 2.1. Suppose that $d \ge 2$ is a square-free positive integer and let m be the integer in $(\sqrt{d} - 2, \sqrt{d})$ with the same parity as d. Then

$$M\left(\frac{m+\sqrt{d}}{2}\right) = \begin{cases} 2, & \text{if } d = 2, \\ \frac{1}{2}(\sqrt{d}+m), & \text{if } d \equiv 1 \mod 4, \\ \sqrt{d}+m, & \text{otherwise.} \end{cases}$$

Proof. Observe that $\alpha = (m + \sqrt{d})/2$ has $-1 < \overline{\alpha} < 0$, with $\alpha > 1$ for $d \ge 3$ (and $0 < \alpha < 1$ for d = 2). The minimal k to make $k(x - \alpha)(x - \overline{\alpha}) = k(x^2 - mx + \frac{1}{4}(m^2 - d))$ an integer polynomial is plainly k = 1 if $d \equiv 1 \mod 4$ and k = 2 for d = 2 or $3 \mod 4$, and the claim is clear from (2).

For the lower bound we first observe that c or c/2 must divide the lead coefficient k.

Lemma 2.2. Suppose that $d \ge 2$ is squarefree and α is of the form (1). Suppose that $k(x - \alpha)(x - \overline{\alpha})$ is in $\mathbb{Z}[x]$.

If c is even and $d \equiv 1 \mod 4$ then $c/2 \mid k$ with k = c/2 iff a, b are odd with $2c \mid a^2 - db^2$. If c is odd or $d \equiv 2$ or $3 \mod 4$ then $c \mid k$ with k = c iff $c \mid a^2 - db^2$.

Proof. Suppose that $p^t \parallel c$ with $t \ge 1$.

For $k(a^2 - db^2)/c^2$ to be in \mathbb{Z} we must have $p^{t+1} \mid k$ unless $p^t \mid a^2 - db^2$ and $p^t \mid k$ unless $p^{t+1} \mid a^2 - db^2$.

Hence we can assume that $p^{t+1} | a^2 - db^2$. Notice that in this case $p \nmid a$; since p | a and $p^2 | a^2 - db^2$ would imply $p^2 | db^2$, but gcd(a, b, c) = 1 means $p \nmid b$ and d is squarefree. In particular this case can not happen when p = 2 and $d \equiv 2$ or 3 mod 4 (since $a^2 - db^2 \not\equiv 0 \mod 4$), and a, b must be odd if $d \equiv 1 \mod 4$. Hence 2ka/c in \mathbb{Z} forces $p^t \mid k$ when p is odd and $2^{t-1} \mid k$ when p = 2 and $d \equiv 1 \mod 4$.

The following lemma completes the proof of the lower bound:

Lemma 2.3. Suppose that $d \ge 2$ is squarefree and α is of the form (1). Then

$$M(\alpha) > \frac{1}{2}\sqrt{D}.$$

Moreover

$$M(\alpha) > \sqrt{D}$$

unless b = 1 and $a < \sqrt{d}$, with $c \mid a^2 - d$ if $d \equiv 2$ or 3 mod 4 and with c even and $2c \mid a^2 - d$ if $d \equiv 1 \mod 4$.

Proof. Observing that

$$2\frac{b\sqrt{d}}{c} = \alpha - \overline{\alpha} \le \alpha + |\overline{\alpha}| < 2\alpha,$$

we have

$$M(\alpha) \ge k\alpha > k\frac{b\sqrt{d}}{c},$$

and the bound follows from $k \ge c$ if c is odd or $d \equiv 2$ or $3 \mod 4$ (with $k \ge 2c$ if $c \nmid a^2 - db^2$), and $k \ge c/2$ if c is even and $d \equiv 1 \mod 4$ (with $k \ge c$ if $2c \nmid a^2 - db^2$). If $a \ge \sqrt{d}$ then $M(\alpha) \ge k\alpha > \sqrt{D}$.

3. Computations

Hence $\frac{1}{2}\sqrt{D} < L(d) < \sqrt{D}$, and an α of the form (1) with $\frac{1}{2}\sqrt{D} < M(\alpha) < \sqrt{D}$ must be of the form

$$\alpha = \frac{a + \sqrt{d}}{c}, \ a < \sqrt{d},$$

with $c \mid a^2 - d$ if $d \equiv 2$ or $3 \mod 4$, and c even with $2c \mid a^2 - d$ if $d \equiv 1 \mod 4$. Since $|\overline{\alpha}| \leq \alpha$ we have $M(\alpha) = k \max\{1, \alpha, \alpha |\overline{\alpha}|\}$, and in these cases we have

(4)
$$M(\alpha) = \varepsilon \max\left\{c, a + \sqrt{d}, \frac{d - a^2}{c}\right\},$$

where

(5)
$$\varepsilon := \begin{cases} 1, & \text{if } d \equiv 2 \text{ or } 3 \mod 4, \\ \frac{1}{2}, & \text{if } d \equiv 1 \mod 4. \end{cases}$$

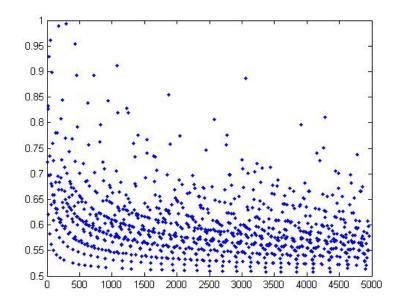


FIGURE 1. $L(d)/\sqrt{D}$ for $d \equiv 1 \mod 4$ less than five thousand.

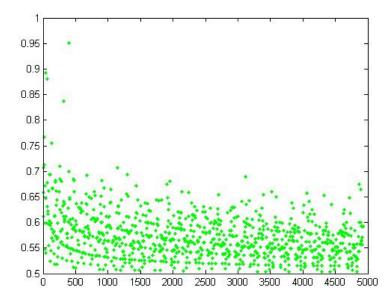


FIGURE 2. $L(d)/\sqrt{D}$ for $d \equiv 2 \mod 4$ less than five thousand.

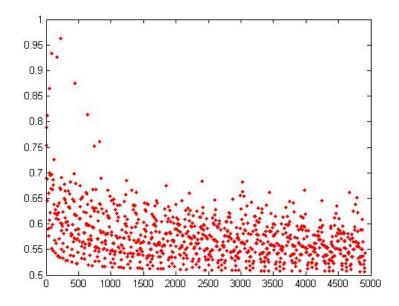


FIGURE 3. $L(d)/\sqrt{D}$ for $d \equiv 3 \mod 4$ less than five thousand.

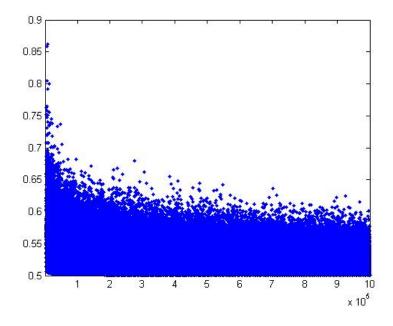


FIGURE 4. $L(d)/\sqrt{D}$ for $d \equiv 1 \mod 4$ between five thousand and one million.

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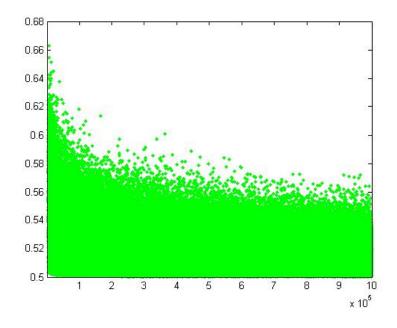


FIGURE 5. $L(d)/\sqrt{D}$ for $d\equiv 2 \mbox{ mod } 4$ between five thousand and one million.

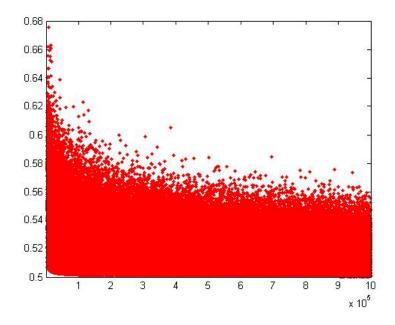


FIGURE 6. $L(d)/\sqrt{D}$ for $d \equiv 3 \mod 4$ between five thousand and one million.

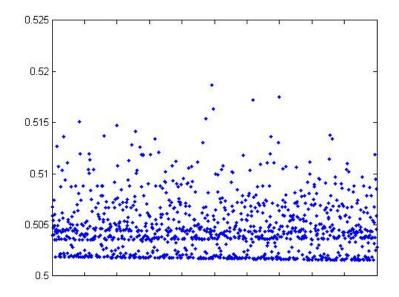


FIGURE 7. $L(d)/\sqrt{D}$ for d between one billion and one billion five thousand.

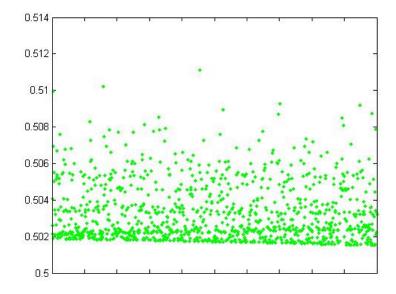


FIGURE 8. $L(d)/\sqrt{D}$ for $d \equiv 2 \mod 4$ between one billion and one billion five thousand.

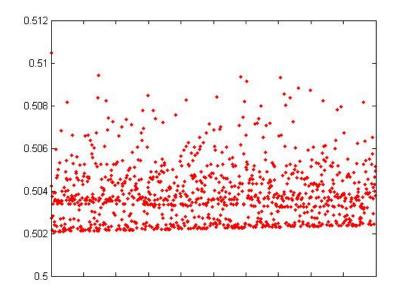


FIGURE 9. $L(d)/\sqrt{D}$ for $d \equiv 3 \mod 4$ between one billion and one billion five thousand.

4. How good are our bounds?

Theorem 1.1 tells us that

$$\frac{1}{2} < \frac{L(d)}{\sqrt{D}} < 1$$

Figures 7,8 & 9 make it seem reasonable to make the following conjecture:

Conjecture 4.1.

$$\lim_{d \to \infty} \frac{L(d)}{\sqrt{D}} = \frac{1}{2}$$

In view of (4) this can be equivalently written:

Conjecture 4.2. For any square-free positive integer d there exists an a and c with

$$a = o(\sqrt{d}), \ \ c = (1 + o(1))\sqrt{d},$$

and $c \mid d - a^2$ when $d \equiv 2$ or 3 mod 4, c even and $2c \mid d - a^2$ when $d \equiv 1 \mod 4$.

Checking computationally, pairs a and c satisfying

$$a < d^{2/5}, \quad d^{1/2} - d^{2/5} < c < d^{1/2} + d^{2/5}$$

and $c \mid d - a^2$ exist for all 827 < d < 2,000,000,000, and even c with $2c \mid d - a^2$ for all $d \equiv 1 \mod 4$ with 1,902,773 < d < 2,000,000,000.

The $\frac{1}{2}$ in the lower bound is the optimal absolute constant.

Theorem 4.1.

$$\liminf_{d \to \infty} \frac{L(d)}{\sqrt{D}} = \frac{1}{2}.$$

This follows at once from the following examples:

Small Examples. If $d = m^2 + 1$ then

$$M\left(\frac{\sqrt{d}+1}{m}\right) = \begin{cases} \sqrt{d}+1, & \text{if } m \text{ is odd,} \\ \frac{1}{2}\left(\sqrt{d}+1\right), & \text{if } m \text{ is even.} \end{cases}$$

It seems likely that the upper bound can be slightly reduced. The computations suggest that the largest value occurs at d = 293.

Conjecture 4.3.

$$\sup_{d} \frac{L(d)}{\sqrt{D}} = \frac{L(293)}{\sqrt{293}} = \frac{M\left(\frac{\sqrt{293}+15}{2}\right)}{\sqrt{293}} = \frac{17}{\sqrt{293}} = 0.993150\dots$$

If we separate out the residue classes mod 4:

Conjecture 4.4.

$$\sup_{d\equiv 2 \mod 4} \frac{L(d)}{\sqrt{D}} = \frac{L(398)}{2\sqrt{398}} = \frac{M\left(\frac{\sqrt{398}+18}{2}\right)}{2\sqrt{398}} = \frac{\sqrt{398}+18}{2\sqrt{398}} = 0.951129\dots$$
$$\sup_{d\equiv 3 \mod 4} \frac{L(d)}{\sqrt{D}} = \frac{L(227)}{2\sqrt{227}} = \frac{M\left(\frac{\sqrt{227}+13}{2}\right)}{2\sqrt{227}} = \frac{29}{2\sqrt{227}} = 0.962398\dots$$

We found only ten values of d, namely d = 293, 173, 227, 53, 437, 398, 83, 29, 167, 1077, with $L(d)/\sqrt{D} > 0.9$. As can be seen in the Appendix, the large values on each of the Figures 1, 2 & 3 noticeably seem to correspond to d with the property that d is a quadratic non-residues for all small primes $p \nmid d$ (specifically all $p < \sqrt{d}$ for d = 2or 3 mod 4 and $p < \sqrt{d}/2$ for $d = 1 \mod 4$). Most of these d (with the exception of 437) are of the form $d = \ell p$ with p prime and ℓ small, and all have $d \not\equiv 1 \mod 8$. The following lemma shows why such d have large L(d) values.

Lemma 4.1. Suppose that d is a squarefree positive integer with $d \equiv 2 \pmod{4}$ or 3 (mod 4) or 5 (mod 8), and that $\left(\frac{d}{p}\right) = -1$ for all primes $p \nmid d$ with

$$p < \begin{cases} \sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4}, \\ \sqrt{d}/2, & \text{if } d \equiv 5 \pmod{8}. \end{cases}$$

For each odd $A \mid d$, with $A < \sqrt{d}$, let m_A denote the integer in $\left(\sqrt{d}/A - 2, \sqrt{d}/A\right)$ with the same parity as d. Then, with ε as in (5),

$$L(d) = \varepsilon \min_{\substack{A|d, A < \sqrt{d} \ odd}} \min\left\{ M\left(\frac{\sqrt{d} + m_A A}{2A}\right), \ M\left(\frac{\sqrt{d} + (m_A - 2)A}{2A}\right) \right\}$$
$$= \varepsilon \min_{\substack{A|d, A < \sqrt{d} \ odd}} \min\left\{\sqrt{d} + m_A A, \ (d - (m_A - 2)^2 A^2)/2A \right\}$$
$$\geq \sqrt{D} - 2\varepsilon \max_{\substack{A|d, A < \sqrt{d} \ odd}} A.$$

Proof. Suppose that $d \equiv 2$ or 3 (mod 4). Since $\left(\frac{d}{p}\right) = -1$ for all $p < \sqrt{d}$, $p \nmid d$ we have $d - a^2 = A_1$ or $2A_1$ or A_1p or $2A_1p$ with $A_1 \mid d$ odd and $p > \sqrt{d}$ prime. Hence

we can assume that $c \mid d - a^2$ is of the form $c = A_2$ or $2A_2$ or A_2p or $2A_2p$ where $A_2 \mid A_1$ and $0 \le a < \sqrt{d}$, and

$$M\left(\frac{a+\sqrt{d}}{c}\right) = \max\left\{c, \ a+\sqrt{d}, \ (d-a^2)/c\right\}$$

Hence with $A = A_1$ or A_1/A_2 we can assume $A \mid d$ odd, a = kA, and it is enough to consider

$$M\left(\frac{\sqrt{d}+kA}{A}\right) = \max\left\{A,\sqrt{d}+kA,(d-k^2A^2)/A\right\}$$

with $A < \sqrt{d}$ or

$$M\left(\frac{\sqrt{d}+kA}{2A}\right) = \max\left\{2A,\sqrt{d}+kA,(d-k^2A^2)/2A\right\}$$

with $2A < \sqrt{d}$ and k and d the same parity. Hence

(6)
$$M\left(\frac{\sqrt{d}+kA}{A}\right) = \begin{cases} \sqrt{d}+kA, & \text{if } \sqrt{d}/A - 1 < k < \sqrt{d}/A, \\ (d-k^2A^2)/A, & \text{if } k < \sqrt{d}/A - 1, \\ \ge 2\sqrt{d}-A, \end{cases}$$

and for k and d the same parity

(7)
$$M\left(\frac{\sqrt{d}+kA}{2A}\right) = \begin{cases} \sqrt{d}+kA, & \text{if } \sqrt{d}/A - 2 < k < \sqrt{d}/A, \\ (d-k^2A^2)/2A, & \text{if } k < \sqrt{d}/A - 2, \end{cases}$$
$$\geq 2\sqrt{d} - 2A.$$

For $k \ge m_A$ the minimum of both is plainly

$$M\left(\frac{\sqrt{d} + m_A A}{2A}\right) = \sqrt{d} + m_A A$$

In (7) the smallest for $k \leq m_A - 2$ is

$$M\left(\frac{\sqrt{d} + (m_A - 2)A}{2A}\right) = (d - (m_A - 2)^2 A^2)/2A,$$

and for (6) the smallest for $k \leq m_A - 1$ is

$$M\left(\frac{\sqrt{d} + (m_A - 1)A}{A}\right) = (d - (m_A - 1)^2 A^2)/A.$$

Writing $m_A = \sqrt{d}/A - \delta$, $0 < \delta < 2$ and observing that

$$M\left(\frac{\sqrt{d} + (m_A - 1)A}{A}\right) - M\left(\frac{\sqrt{d} + (m_A - 2)A}{2A}\right) = \delta\sqrt{d} + A\left(1 - \frac{\delta^2}{2}\right) > \delta\left(\sqrt{d} - \frac{1}{2}A\right) > 0$$

the result follows.

Similarly for $d \equiv 5 \mod 8$ we must have $d - a^2 = 2^2 A_1$ or $2^2 A_1 p$, and our even c with $2c \mid d - a^2$ must be of the form $c = 2A_2$ or $2A_2p$. Thus we again reduce to

$$M\left(\frac{\sqrt{d}+kA}{2A}\right) = \frac{1}{2}\max\left\{2A,\sqrt{d}+kA,(d-k^2A^2)/2A\right\}$$

with $A \mid d$ odd, $2A < \sqrt{d}$, k odd, and the minimum occurs for $k = m_A$ or $m_A - 2$ as before.

Plainly $d \equiv 2$ or 3 (mod 4) or 5 (mod 8) with no divisors in $(o(\sqrt{d}, \sqrt{d})$ that are quadratic non-residues for all $p < \sqrt{d}$ would have $L(d) \ge \sqrt{D} - o(\sqrt{d})$. In particular infinitely many would immediately give

$$\limsup_{d \to \infty} \frac{L(d)}{\sqrt{D}} = 1$$

in contradiction to Conjecture 4.1, but this seems unlikely:

Conjecture 4.5. All but finitely many squarefree d have $\left(\frac{d}{p}\right) = 1$ for some odd prime $p < \frac{1}{2}\sqrt{D}$.

Assuming GRH for the mod 4d character

$$\chi(n) := \begin{cases} \left(\frac{d}{n}\right), & \text{if } \gcd(n, 4d) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

(where $\left(\frac{d}{n}\right)$ denotes the Jacobi symbol), we have the bound

(8)
$$\left|\sum_{n \le x} \chi(n) \Lambda(n)\right| \ll x^{\frac{1}{2}} \log^2(Dx)$$

(see, for example, [2, Chapter 20]), and so we should in fact have $\left(\frac{d}{p}\right) = 1$ for some prime $p \ll \log^4 D$.

Note, a squarefree $d \not\equiv 1 \mod 8$ with $\left(\frac{d}{p}\right) = -1$ for all odd primes $p \nmid d$ with $p < \frac{1}{2}\sqrt{D}$ must be of the form $d = (kA)^2 \pm 2A$ or $((2k-1)A)^2 \pm 4A$, for some k and squarefree odd $A \mid d$ with $A < \frac{1}{2}\sqrt{D}$. To see this, write $d = N^2 + r$, $N = [\sqrt{d}]$, $1 \leq r \leq 2N$. If r is even then A = r/2 is odd if $d \equiv 2, 3 \mod 4$ and A = r/4 is odd if $d \equiv 5 \mod 8$. Since d is a square mod A we must have $p \mid A \Rightarrow p \mid d$. As d is squarefree, $A < \frac{1}{2}\sqrt{D}$ must be squarefree with $A \mid d$, giving $d = N^2 + 2A$ or $N^2 + 4A$ with $A \mid N$. Similarly for r odd

$$d = \left(\frac{r+1}{2}\right)^2 + \left(N - \frac{1}{2}(r-1)\right)\left(N + \frac{1}{2}(r-1)\right),\,$$

with $A = N - \frac{1}{2}(r-1)$ odd for d = 2 or $3 \mod 4$ and $A = \frac{1}{2}\left(N - \frac{1}{2}(r-1)\right)$ odd for $d \equiv 5 \mod 8$. Since d is a square mod A, $A < \frac{1}{2}\sqrt{D}$ is squarefree with $A \mid d$, giving $d = (N+1)^2 - 2A$ or $(N+1)^2 - 4A$ with $A \mid N+1$.

Conversely if d is a quadratic residue mod p for a suitably sized p or if $d \equiv 1 \mod 8$ then we can obtain a bound less than one for $L(d)/\sqrt{D}$:

Lemma 4.2. Suppose that d is a square mod q, where q is odd or $4 \mid q$ and λ defined by

$$\lambda\sqrt{d} = \begin{cases} q, & \text{if } q \text{ is odd,} \\ \frac{1}{4}q, & \text{if } q \text{ is even,} \end{cases}$$

has $0 < \lambda < 1$. Then

$$\frac{L(d)}{\sqrt{D}} \leq \begin{cases} \frac{1}{2}(1+\lambda+\sqrt{(1-\lambda)^2-4\lambda^2}), & \text{ if } 0 < \lambda < \frac{1}{4}(\sqrt{5}-1), \\ \frac{1}{4\lambda}, & \text{ if } \frac{1}{4}(\sqrt{5}-1) < \lambda < \frac{1}{2}(\sqrt{3}-1), \\ \frac{1}{2}(1+\lambda), & \text{ if } \frac{1}{2}(\sqrt{3}-1) < \lambda < 1. \end{cases}$$

Notice that if we assume GRH then estimate (8), with

$$\sum_{x-y \le n \le x} \Lambda(n) = y + O(x^{\frac{1}{2}} \log^2 x)$$

from assuming RH, guarantees that $\left(\frac{d}{p}\right) = 1$ for some prime p in

$$\left(\frac{1}{2}(\sqrt{3}-1)d^{\frac{1}{2}},\frac{1}{2}(\sqrt{3}-1)d^{\frac{1}{2}}+cd^{\frac{1}{4}}\log^2 d\right)$$

for suitably large c, and Lemma 4.2 gives

$$\frac{L(d)}{\sqrt{D}} \le \frac{1}{4}(\sqrt{3}+1) + O\left(\frac{\log^2 d}{d^{\frac{1}{4}}}\right) = 0.683012\ldots + o(1).$$

Proof. Suppose that r_0 has $r_0^2 \equiv d \mod q$. If q is odd we take r to be the integer in $(\sqrt{d} - 2q, \sqrt{d})$ with the same parity as d and $r \equiv r_0 \mod q$, write $r = \sqrt{d} - \delta q$, and set

$$\alpha_1 = \frac{\sqrt{d}+r}{2q}, \ \alpha_2 = \frac{\sqrt{d}+r-2q}{2q}$$

Notice that c = 2q and a = r or r - 2q will have $c \mid (d - a^2)$ with $2c \mid d - a^2$ when $d = 1 \mod 4$.

For $4 \mid q$ (which of course only occurs when $d = 1 \mod 4$) we write $q = 2^l q_1$ with q_1 odd and $l \ge 2$ and take r to be the integer in $(\sqrt{d} - 2^{l-1}q_1, \sqrt{d})$ with $r \equiv r_0 \mod 1$ $2^{l-1}q_1$ and set

$$\alpha_1 = \frac{\sqrt{d} + r}{2^{l-1}q_1}, \ \ \alpha_2 = \frac{\sqrt{d} + r - 2^{l-1}q_1}{2^{l-1}q_1}.$$

Again $c = 2^{l-1}q_1$ and a = r or $r - 2^{l-1}q_1$ will have $2c \mid (d - a^2)$.

Writing $r = \sqrt{d} - \delta q$ for q odd, and $r = \sqrt{d} - \delta 2^{l-2}q_1$ for q even, we have $r = (1 - \lambda \delta) \sqrt{d}$ with $0 < \delta < 2$ and

$$\alpha_1 = \frac{(2-\delta\lambda)}{2\lambda}, \ \overline{\alpha}_1 = -\frac{\delta}{2}, \ \alpha_2 = \frac{(2-2\lambda-\delta\lambda)}{2\lambda}, \ \overline{\alpha}_2 = -\frac{\delta}{2}-1$$

For α_1 and α_2 we also plainly have $k = \epsilon c = 2\epsilon \lambda \sqrt{d} = \lambda \sqrt{D}$.

Clearly $\alpha_1 > 0$, $\alpha_2 > -1$, $-1 < \overline{\alpha}_1 < 0$ and $-2 < \overline{\alpha}_2 < -1$. If $\alpha_1 < 1$ then $M(\alpha_1) = \lambda \sqrt{D} < \frac{1}{2}(1+\lambda)\sqrt{D}$. Hence we can assume that $\alpha_1 > 1$ (this is automatic for $\lambda < \frac{1}{2}$).

So

$$M(\alpha_1) = \lambda \sqrt{D} \alpha_1 = \sqrt{D} \left(1 - \frac{\delta}{2} \lambda \right).$$

If $\alpha_2 < 1$ then

$$M(\alpha_2) = \lambda \sqrt{D} |\overline{\alpha}_2| = \sqrt{D} \lambda \left(1 + \frac{\delta}{2}\right),$$

and plainly

$$\min\{M(\alpha_1), M(\alpha_2)\} \le \frac{1}{2} \left(M(\alpha_1) + M(\alpha_2)\right) = \frac{1}{2} (1+\lambda)\sqrt{D}.$$

So we can assume that $\alpha_2 > 1$ and

$$M(\alpha_2) = \lambda \sqrt{D} |\overline{\alpha}_2| \alpha_2 = \sqrt{D} \left(1 + \frac{\delta}{2}\right) \left(1 - \lambda - \frac{\delta}{2}\lambda\right).$$

Observing that the quadratic is maximized for $\frac{\delta}{2} = \frac{1}{2\lambda} - 1$ we plainly have

$$M(\alpha_2) \le \sqrt{D} \frac{1}{4\lambda}$$

with this less than $\frac{1}{2}(1+\lambda)\sqrt{D}$ for $\frac{1}{2}(\sqrt{3}-1) < \lambda < 1$. For $\lambda < \frac{1}{4}(\sqrt{5}-1)$ the value $\frac{\delta}{2} = \frac{1}{2\lambda}\left(1-\lambda-\sqrt{(1-\lambda)^2-4\lambda^2}\right)$ equating $M(\alpha_1)$ and $M(\alpha_2)$ is less than $\frac{1}{2\lambda}-1$ and the minimum of the two is at most the value at that point:

$$\min\{M(\alpha_1), M(\alpha_2)\} \le \sqrt{D} \left(\frac{1}{2}(1+\lambda) + \frac{1}{2}\sqrt{(1-\lambda)^2 - 4\lambda^2}\right).$$

In particular from the lemma we immediately obtain a bound away from 1 for the $d \equiv 1 \mod 8$.

Corollary 4.1. If $d \equiv 1 \pmod{8}$ then

$$\frac{L(d)}{\sqrt{D}} \le \frac{1}{4}(\sqrt{5}+1) = 0.809016\dots$$

If $d \equiv 1 \pmod{3}$ then

$$\frac{L(d)}{\sqrt{D}} \le \frac{1}{7}(2+3\sqrt{2}) = 0.891805\dots$$

Computations indicate room for improvement in these bounds.

Conjecture 4.6.

$$\sup_{d \equiv 1 \mod 8} \frac{L(d)}{\sqrt{D}} = \frac{L(41)}{\sqrt{41}} = \frac{M\left(\frac{\sqrt{41+27}}{4}\right)}{\sqrt{41}} = \frac{\sqrt{41}+3}{2\sqrt{41}} = 0.734261\dots$$
$$\sup_{d \equiv 1 \mod 3} \frac{L(d)}{\sqrt{D}} = \frac{L(13)}{\sqrt{13}} = \frac{M\left(\frac{\sqrt{13}+1}{2}\right)}{\sqrt{13}} = \frac{4}{\sqrt{13}} = 0.832050\dots$$

Proof. If $d \equiv 1 \pmod{8}$ then we can solve $r^2 \equiv d \mod 2^l$ for any l. Hence if we pick l such that $\frac{1}{4}(\sqrt{5}-1)\sqrt{d} \leq 2^{l-2} \leq \frac{1}{2}(\sqrt{5}-1)\sqrt{d}$ and we can apply the lemma with $\frac{1}{4}(\sqrt{5}-1) \leq \lambda \leq \frac{1}{2}(\sqrt{5}-1)$. Likewise, for an odd prime p, if $p \nmid d$ and $\left(\frac{d}{p}\right) = 1$ then we can solve $r^2 \equiv d \mod p^l$ for any l. Choosing l so that

$$\frac{1}{1+\sqrt{(p-1)^2+4}}\sqrt{d} \le p^l \le \frac{p}{1+\sqrt{(p-1)^2+4}}\sqrt{d},$$

and applying the lemma with $q = p^l$ gives

$$\frac{L(d)}{\sqrt{D}} \le \frac{1}{2} \left(1 + \frac{p}{1 + \sqrt{(p-1)^2 + 4}} \right).$$

Taking p = 3 gives the result claimed for $d \equiv 1 \mod 3$.

Likewise, for $d \equiv 1, 4 \mod 5$ we get the upper bound 0.956859... (from d = 29 we know 0.928476... would be best possible).

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For $d \equiv 1, 2, 4 \mod 7$ we get 0.977844... and for $d \equiv 1, 3, 4, 5, 9 \mod 11$ the bound 0.991157... (from d = 53 these can not be reduced below 0.961523...).

For $d \equiv 1, 3, 4, 9, 10, 12 \mod 13$ we get the bound 0.993713... (the optimal bound is likely 0.988371... from d = 173).

For $d \equiv 1, 2, 4, 8, 9, 13, 15, 16 \mod 17$ our bound gives 0.996364... (optimal is probably 0.993150... at d = 293).

5. Appendix of Large Values

We give the largest values found in Figure 1, Figure 2 & Figure 3 down to the first value not satisfying the quadratic non-residue conditions of Lemma 4.1.

Largest values for $d \equiv 1 \mod 4$.

$$\begin{split} \frac{L(293)}{\sqrt{293}} &= \frac{M\left(\frac{\sqrt{293}+15}{\sqrt{293}}\right)}{\sqrt{293}} = \frac{17}{\sqrt{293}} = 0.993150\ldots, \quad \left(\frac{293}{p}\right) = -1, p = 3, 5, 7, 11, 13, \\ \frac{L(173)}{\sqrt{173}} &= \frac{M\left(\frac{\sqrt{173}+11}{2}\right)}{\sqrt{173}} = \frac{13}{\sqrt{173}} = 0.988371\ldots, \quad \left(\frac{173}{p}\right) = -1, p = 3, 5, 7, 11, 13, \\ \frac{L(53)}{\sqrt{53}} &= \frac{M\left(\frac{\sqrt{53}+5}{2}\right)}{\sqrt{53}} = \frac{7}{\sqrt{53}} = 0.961523\ldots, \quad \left(\frac{53}{p}\right) = -1, p = 3, 5, \\ \frac{L(437)}{\sqrt{437}} &= \frac{M\left(\frac{\sqrt{437}+19}{2}\right)}{\sqrt{437}} = \frac{\sqrt{437}+19}{2\sqrt{437}} = 0.954446\ldots, \quad \left(\frac{437}{p}\right) = -1, p = 3, 5, 7, 11, 13, 17, 29, \\ \frac{L(29)}{\sqrt{29}} &= \frac{M\left(\frac{\sqrt{29}+3}{\sqrt{29}}\right)}{\sqrt{29}} = \frac{5}{\sqrt{29}} = 0.928476\ldots, \quad \left(\frac{29}{p}\right) = -1, p = 3, \\ \frac{L(1077)}{\sqrt{1077}} &= \frac{M\left(\frac{\sqrt{1077}+27}{6}\right)}{\sqrt{1077}} = \frac{\sqrt{1077}+27}{2\sqrt{1077}} = 0.911363\ldots, \quad \left(\frac{1077}{p}\right) = -1, p = 5, 7, 11, 13, 17, 19, 23, \\ \frac{L(453)}{\sqrt{453}} &= \frac{M\left(\frac{\sqrt{453}+15}{\sqrt{453}}\right)}{\sqrt{453}} = \frac{19}{\sqrt{453}} = 0.892697\ldots, \quad \left(\frac{453}{p}\right) = -1, p = 5, 7, 11, 13, 17, 19, \\ \frac{L(3053)}{\sqrt{3053}} &= \frac{M\left(\frac{\sqrt{3053}+41}{14}\right)}{\sqrt{3053}} = \frac{49}{\sqrt{3053}} = 0.886814\ldots, \quad \left(\frac{3053}{7}\right) = 1. \\ \text{Note other α may achieve the minimum, for example $M\left(\frac{\sqrt{437}+19}{2}\right) = M\left(\frac{\sqrt{437}+19}{38}\right). \end{split}$$$

Largest values for $d \equiv 2 \mod 4$.

$$\begin{split} \frac{L(398)}{2\sqrt{398}} &= \frac{M\left(\frac{\sqrt{398}+18}{2}\right)}{2\sqrt{398}} = \frac{\sqrt{398}+18}{2\sqrt{398}} = 0.951129\ldots, \quad \left(\frac{398}{p}\right) = -1, \ p = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \\ \frac{L(38)}{2\sqrt{38}} &= \frac{M\left(\frac{\sqrt{38}+4}{2}\right)}{2\sqrt{38}} = \frac{11}{2\sqrt{38}} = 0.892217\ldots, \quad \left(\frac{38}{p}\right) = -1, \ p = 3, 5, \\ \frac{L(62)}{2\sqrt{62}} &= \frac{M\left(\frac{\sqrt{62}+6}{2\sqrt{62}}\right)}{2\sqrt{62}} = \frac{\sqrt{62}+6}{2\sqrt{62}} = 0.881000\ldots, \quad \left(\frac{62}{p}\right) = -1, \ p = 3, 5, 7, 11, \\ \frac{L(318)}{2\sqrt{318}} &= \frac{M\left(\frac{\sqrt{318}+12}{6}\right)}{2\sqrt{318}} = \frac{\sqrt{318}+12}{2\sqrt{318}} = 0.836463\ldots, \quad \left(\frac{318}{p}\right) = -1, \ p = 5, 7, 11, 13, 17, 19, 23, \\ \frac{L(14)}{2\sqrt{14}} &= \frac{M\left(\frac{\sqrt{14}+2}{2}\right)}{2\sqrt{14}} = \frac{\sqrt{14}+2}{2\sqrt{14}} = 0.767261\ldots, \quad \left(\frac{14}{p}\right) = -1, \ p = 3, \\ \frac{L(138)}{2\sqrt{138}} &= \frac{M\left(\frac{\sqrt{138}+6}{6}\right)}{2\sqrt{138}} = \frac{\sqrt{138}+6}{2\sqrt{138}} = 0.755376\ldots, \quad \left(\frac{138}{p}\right) = -1, \ p = 5, 7, 11, 13, \\ \frac{L(22)}{2\sqrt{22}} &= \frac{M\left(\frac{\sqrt{22}+2}{3}\right)}{2\sqrt{22}} = \frac{\sqrt{22}+2}{2\sqrt{22}} = 0.713200\ldots, \quad \left(\frac{22}{3}\right) = 1. \end{split}$$

Largest values for $d \equiv 3 \mod 4$.

$$\begin{split} \frac{L(227)}{2\sqrt{227}} &= \frac{M\left(\frac{\sqrt{227}+13}{2}\right)}{2\sqrt{227}} = \frac{29}{2\sqrt{227}} = 0.962398\ldots, \quad \left(\frac{227}{p}\right) = -1, \ p = 3, 5, 7, 11, 13, 17, 19, 23, \\ \frac{L(83)}{2\sqrt{33}} &= \frac{M\left(\frac{\sqrt{38}+7}{2}\right)}{2\sqrt{83}} = \frac{17}{2\sqrt{83}} = 0.932966\ldots, \quad \left(\frac{83}{p}\right) = -1, \ p = 3, 5, 7, 11, 13, 17, 19, 23, \\ \frac{L(167)}{2\sqrt{167}} &= \frac{M\left(\frac{\sqrt{167}+11}{2}\right)}{2\sqrt{167}} = \frac{\sqrt{167}+11}{2\sqrt{167}} = 0.925602\ldots, \quad \left(\frac{167}{p}\right) = -1, \ p = 3, 5, 7, 11, 13, 17, 19, \\ \frac{L(447)}{2\sqrt{447}} &= \frac{M\left(\frac{\sqrt{447}+15}{2}\right)}{2\sqrt{447}} = \frac{37}{2\sqrt{447}} = 0.875019\ldots, \quad \left(\frac{447}{p}\right) = -1, \ p = 3, 5, 7, 11, 13, 17, 19, \\ \frac{L(47)}{2\sqrt{447}} &= \frac{M\left(\frac{\sqrt{47}+5}{2\sqrt{47}}\right)}{2\sqrt{47}} = \frac{\sqrt{47}+5}{2\sqrt{47}} = 0.864662\ldots, \quad \left(\frac{47}{p}\right) = -1, \ p = 3, 5, 7, \\ \frac{L(635)}{2\sqrt{635}} &= \frac{M\left(\frac{\sqrt{635}+15}{10}\right)}{2\sqrt{635}} = \frac{41}{2\sqrt{635}} = 0.813517\ldots, \quad \left(\frac{635}{p}\right) = -1, \ p = 3, 7, 11, 13, 17, 19, 23, 29, 31, 37, \\ \frac{L(23)}{2\sqrt{23}} &= \frac{M\left(\frac{\sqrt{22}+3}{2\sqrt{23}}\right)}{2\sqrt{23}} = \frac{\sqrt{23}+3}{2\sqrt{23}} = 0.812771\ldots, \quad \left(\frac{23}{p}\right) = -1, \ p = 3, 7, 11, 13, 17, 19, 23, 29, 31, 37, \\ \frac{L(3)}{2\sqrt{3}} &= \frac{M\left(\frac{\sqrt{33}+1}{2\sqrt{3}}\right)}{2\sqrt{3}} = \frac{\sqrt{3}+1}{2\sqrt{3}} = 0.788675\ldots, \quad \left(\frac{3}{p}\right) = -1, \ p = 3, 7, 11, 13, 17, 19, 23, 29, 31, 37, \\ \frac{L(827)}{2\sqrt{827}} &= \frac{M\left(\frac{\sqrt{837}+15}{2\sqrt{827}}\right)}{2\sqrt{827}} = 0.760800\ldots, \quad \left(\frac{827}{7}\right) = 1. \end{split}$$

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