# MINIMAL MAHLER MEASURE IN REAL QUADRATIC FIELDS 

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#### Abstract

We consider upper and lower bounds on the minimal height of an irrational number lying in a particular real quadratic field.


## 1. Introduction

For a polynomial $F(x)=a_{n} \prod_{i=1}^{n}\left(x-\alpha_{i}\right)$ in $\mathbb{C}[x]$ one defines its Mahler measure $M(F)$ as

$$
M(F)=\left|a_{n}\right| \prod_{i=1}^{n} \max \left\{1,\left|\alpha_{i}\right|\right\}
$$

For an algebraic number $\alpha$ we use $M(\alpha)$ to denote the Mahler measure of an irreducible integer polynomial with root $\alpha$. Thus the logarithmic Weil height of $\alpha$ can be written

$$
h(\alpha)=\frac{\log M(\alpha)}{[\mathbb{Q}(\alpha): \mathbb{Q}]}
$$

Of course $M(\alpha)=1$ iff $\alpha$ is a root of unity and the well known problem of Lehmer 3] is to determine whether there is a constant $C>1$ such that $M(\alpha)>C$ otherwise. Schinzel [4] showed that for $\alpha$ in a Kroneckerian field (a totally real field or a quadratic extension of such a field) the value of $M(\alpha)$ must in fact grow with its degree, with the absolute minimum $M(\alpha)>1$ achieved for the golden ratio

$$
M\left(\frac{1+\sqrt{5}}{2}\right)=\frac{1+\sqrt{5}}{2}
$$

Amoroso \& Dvornicich [1] further extended this to cyclotomic fields. These of course include the quadratic fields $\mathbb{Q}(\sqrt{d})$, where $d$ is a square-free positive integer. Since the golden ratio is not in all these fields we are interested in how

$$
L(d):=\min \{M(\alpha): \alpha \in \mathbb{Q}(\sqrt{d}) \backslash \mathbb{Q}\}
$$

varies with $d$. We recall the discriminant of the field $\mathbb{Q}(\sqrt{d})$

$$
D:=\left\{\begin{array}{lc}
d, & \text { if } d \equiv 1 \bmod 4 \\
4 d, & \text { if } d \equiv 2 \text { or } 3 \bmod 4
\end{array}\right.
$$

[^0]Since $M(\alpha)=M(-\alpha)=M(\bar{\alpha})$ we assume that our $\alpha \in \mathbb{Q}(\sqrt{d}) \backslash \mathbb{Q}$ takes the form

$$
\begin{equation*}
\alpha=\frac{a+b \sqrt{d}}{c}, a, b, c \in \mathbb{Z}, a \geq 0, b>0, c>0, \operatorname{gcd}(a, b, c)=1 \tag{1}
\end{equation*}
$$

with conjugate

$$
\bar{\alpha}=\frac{a-b \sqrt{d}}{c},
$$

and

$$
\begin{equation*}
M(\alpha)=k \max \{1,|\alpha|\} \max \{1,|\bar{\alpha}|\} \tag{2}
\end{equation*}
$$

where $k$ is the smallest positive integer such that

$$
\begin{equation*}
k(x-\alpha)(x-\bar{\alpha})=k\left(x^{2}-\frac{2 a}{c} x+\frac{a^{2}-b^{2} d}{c^{2}}\right) \in \mathbb{Z}[x] . \tag{3}
\end{equation*}
$$

We show that the minimal measure must grow with $d$ :
Theorem 1.1. For square-free $d$ in $\mathbb{N}$

$$
\frac{1}{2} \sqrt{D}<L(d)<\sqrt{D}
$$

## 2. Proof of Theorem 1.1

The upper bound follows at once from the following constructive examples:
Lemma 2.1. Suppose that $d \geq 2$ is a square-free positive integer and let $m$ be the integer in $(\sqrt{d}-2, \sqrt{d})$ with the same parity as $d$. Then

$$
M\left(\frac{m+\sqrt{d}}{2}\right)= \begin{cases}2, & \text { if } d=2 \\ \frac{1}{2}(\sqrt{d}+m), & \text { if } d \equiv 1 \bmod 4 \\ \sqrt{d}+m, & \text { otherwise }\end{cases}
$$

Proof. Observe that $\alpha=(m+\sqrt{d}) / 2$ has $-1<\bar{\alpha}<0$, with $\alpha>1$ for $d \geq 3$ (and $0<\alpha<1$ for $d=2$ ). The minimal $k$ to make $k(x-\alpha)(x-\bar{\alpha})=$ $k\left(x^{2}-m x+\frac{1}{4}\left(m^{2}-d\right)\right)$ an integer polynomial is plainly $k=1$ if $d \equiv 1 \bmod$ 4 and $k=2$ for $d=2$ or $3 \bmod 4$, and the claim is clear from $(2)$.

For the lower bound we first observe that $c$ or $c / 2$ must divide the lead coefficient $k$.

Lemma 2.2. Suppose that $d \geq 2$ is squarefree and $\alpha$ is of the form (1). Suppose that $k(x-\alpha)(x-\bar{\alpha})$ is in $\mathbb{Z}[x]$.

If $c$ is even and $d \equiv 1$ mod 4 then $c / 2 \mid k$ with $k=c / 2$ iff $a, b$ are odd with $2 c \mid a^{2}-d b^{2}$. If $c$ is odd or $d \equiv 2$ or 3 mod 4 then $c \mid k$ with $k=c$ iff $c \mid a^{2}-d b^{2}$.
Proof. Suppose that $p^{t} \| c$ with $t \geq 1$.
For $k\left(a^{2}-d b^{2}\right) / c^{2}$ to be in $\mathbb{Z}$ we must have $p^{t+1} \mid k$ unless $p^{t} \mid a^{2}-d b^{2}$ and $p^{t} \mid k$ unless $p^{t+1} \mid a^{2}-d b^{2}$.

Hence we can assume that $p^{t+1} \mid a^{2}-d b^{2}$. Notice that in this case $p \nmid a$; since $p \mid a$ and $p^{2} \mid a^{2}-d b^{2}$ would imply $p^{2} \mid d b^{2}$, but $\operatorname{gcd}(a, b, c)=1$ means $p \nmid b$ and $d$ is squarefree. In particular this case can not happen when $p=2$ and $d \equiv 2$ or $3 \bmod$ $4\left(\right.$ since $\left.a^{2}-d b^{2} \not \equiv 0 \bmod 4\right)$, and $a, b$ must be odd if $d \equiv 1 \bmod 4$. Hence $2 k a / c$ in $\mathbb{Z}$ forces $p^{t} \mid k$ when $p$ is odd and $2^{t-1} \mid k$ when $p=2$ and $d \equiv 1 \bmod 4$.

The following lemma completes the proof of the lower bound:
Lemma 2.3. Suppose that $d \geq 2$ is squarefree and $\alpha$ is of the form 11. Then

$$
M(\alpha)>\frac{1}{2} \sqrt{D}
$$

Moreover

$$
M(\alpha)>\sqrt{D}
$$

unless $b=1$ and $a<\sqrt{d}$, with $c \mid a^{2}-d$ if $d \equiv 2$ or 3 mod 4 and with $c$ even and $2 c \mid a^{2}-d$ if $d \equiv 1 \bmod 4$.

Proof. Observing that

$$
2 \frac{b \sqrt{d}}{c}=\alpha-\bar{\alpha} \leq \alpha+|\bar{\alpha}|<2 \alpha
$$

we have

$$
M(\alpha) \geq k \alpha>k \frac{b \sqrt{d}}{c}
$$

and the bound follows from $k \geq c$ if $c$ is odd or $d \equiv 2$ or $3 \bmod 4($ with $k \geq 2 c$ if $\left.c \nmid a^{2}-d b^{2}\right)$, and $k \geq c / 2$ if $c$ is even and $d \equiv 1 \bmod 4\left(\right.$ with $k \geq c$ if $\left.2 c \nmid a^{2}-d b^{2}\right)$. If $a \geq \sqrt{d}$ then $M(\alpha) \geq k \alpha>\sqrt{D}$.

## 3. Computations

Hence $\frac{1}{2} \sqrt{D}<L(d)<\sqrt{D}$, and an $\alpha$ of the form (1) with $\frac{1}{2} \sqrt{D}<M(\alpha)<\sqrt{D}$ must be of the form

$$
\alpha=\frac{a+\sqrt{d}}{c}, a<\sqrt{d},
$$

with $c \mid a^{2}-d$ if $d \equiv 2$ or $3 \bmod 4$, and $c$ even with $2 c \mid a^{2}-d$ if $d \equiv 1 \bmod 4$. Since $|\bar{\alpha}| \leq \alpha$ we have $M(\alpha)=k \max \{1, \alpha, \alpha|\bar{\alpha}|\}$, and in these cases we have

$$
\begin{equation*}
M(\alpha)=\varepsilon \max \left\{c, a+\sqrt{d}, \frac{d-a^{2}}{c}\right\} \tag{4}
\end{equation*}
$$

where

$$
\varepsilon:= \begin{cases}1, & \text { if } d \equiv 2 \text { or } 3 \bmod 4  \tag{5}\\ \frac{1}{2}, & \text { if } d \equiv 1 \bmod 4\end{cases}
$$



Figure 1. $L(d) / \sqrt{D}$ for $d \equiv 1 \bmod 4$ less than five thousand.


Figure 2. $L(d) / \sqrt{D}$ for $d \equiv 2 \bmod 4$ less than five thousand.


Figure 3. $L(d) / \sqrt{D}$ for $d \equiv 3 \bmod 4$ less than five thousand.


Figure 4. $L(d) / \sqrt{D}$ for $d \equiv 1 \bmod 4$ between five thousand and one million.


Figure 5. $L(d) / \sqrt{D}$ for $d \equiv 2 \bmod 4$ between five thousand and one million.


Figure 6. $L(d) / \sqrt{D}$ for $d \equiv 3 \bmod 4$ between five thousand and one million.


Figure 7. $L(d) / \sqrt{D}$ for $d$ between one billion and one billion five thousand.


Figure 8. $L(d) / \sqrt{D}$ for $d \equiv 2 \bmod 4$ between one billion and one billion five thousand.


Figure 9. $L(d) / \sqrt{D}$ for $d \equiv 3 \bmod 4$ between one billion and one billion five thousand.

> 4. How GOOD ARE OUR BOUNDS?

Theorem 1.1 tells us that

$$
\frac{1}{2}<\frac{L(d)}{\sqrt{D}}<1
$$

Figures $78 \& 9$ make it seem reasonable to make the following conjecture:

## Conjecture 4.1.

$$
\lim _{d \rightarrow \infty} \frac{L(d)}{\sqrt{D}}=\frac{1}{2}
$$

In view of (4) this can be equivalently written:
Conjecture 4.2. For any square-free positive integer $d$ there exists an and $c$ with

$$
a=o(\sqrt{d}), \quad c=(1+o(1)) \sqrt{d}
$$

and $c \mid d-a^{2}$ when $d \equiv 2$ or $3 \bmod 4, c$ even and $2 c \mid d-a^{2}$ when $d \equiv 1 \bmod 4$.
Checking computationally, pairs $a$ and $c$ satisfying

$$
a<d^{2 / 5}, \quad d^{1 / 2}-d^{2 / 5}<c<d^{1 / 2}+d^{2 / 5}
$$

and $c \mid d-a^{2}$ exist for all $827<d<2,000,000,000$, and even $c$ with $2 c \mid d-a^{2}$ for all $d \equiv 1 \bmod 4$ with $1,902,773<d<2,000,000,000$.

The $\frac{1}{2}$ in the lower bound is the optimal absolute constant.
Theorem 4.1.

$$
\liminf _{d \rightarrow \infty} \frac{L(d)}{\sqrt{D}}=\frac{1}{2}
$$

This follows at once from the following examples:
Small Examples. If $d=m^{2}+1$ then

$$
M\left(\frac{\sqrt{d}+1}{m}\right)= \begin{cases}\sqrt{d}+1, & \text { if } m \text { is odd } \\ \frac{1}{2}(\sqrt{d}+1), & \text { if } m \text { is even }\end{cases}
$$

It seems likely that the upper bound can be slightly reduced. The computations suggest that the largest value occurs at $d=293$.

## Conjecture 4.3.

$$
\sup _{d} \frac{L(d)}{\sqrt{D}}=\frac{L(293)}{\sqrt{293}}=\frac{M\left(\frac{\sqrt{293}+15}{2}\right)}{\sqrt{293}}=\frac{17}{\sqrt{293}}=0.993150 \ldots
$$

If we separate out the residue classes $\bmod 4$ :

## Conjecture 4.4.

$$
\begin{aligned}
& \sup _{d \equiv 2 \bmod 4} \frac{L(d)}{\sqrt{D}}=\frac{L(398)}{2 \sqrt{398}}=\frac{M\left(\frac{\sqrt{398}+18}{2}\right)}{2 \sqrt{398}}=\frac{\sqrt{398}+18}{2 \sqrt{398}}=0.951129 \ldots \\
& \sup _{d \equiv 3 \bmod 4} \frac{L(d)}{\sqrt{D}}=\frac{L(227)}{2 \sqrt{227}}=\frac{M\left(\frac{\sqrt{227}+13}{2}\right)}{2 \sqrt{227}}=\frac{29}{2 \sqrt{227}}=0.962398 \ldots
\end{aligned}
$$

We found only ten values of $d$, namely $d=293,173,227,53,437,398,83,29,167,1077$, with $L(d) / \sqrt{D}>0.9$. As can be seen in the Appendix, the large values on each of the Figures $1,2 \& 3$ noticeably seem to correspond to $d$ with the property that $d$ is a quadratic non-residues for all small primes $p \nmid d$ (specifically all $p<\sqrt{d}$ for $d=2$ or $3 \bmod 4$ and $p<\sqrt{d} / 2$ for $d=1 \bmod 4$ ). Most of these $d$ (with the exception of 437) are of the form $d=\ell p$ with $p$ prime and $\ell$ small, and all have $d \not \equiv 1 \bmod 8$. The following lemma shows why such $d$ have large $L(d)$ values.

Lemma 4.1. Suppose that $d$ is a squarefree positive integer with $d \equiv 2(\bmod 4)$ or $3(\bmod 4)$ or $5(\bmod 8)$, and that $\left(\frac{d}{p}\right)=-1$ for all primes $p \nmid d$ with

$$
p< \begin{cases}\sqrt{d}, & \text { if } d \equiv 2 \operatorname{or} 3(\bmod 4) \\ \sqrt{d} / 2, & \text { if } d \equiv 5(\bmod 8)\end{cases}
$$

For each odd $A \mid d$, with $A<\sqrt{d}$, let $m_{A}$ denote the integer in $(\sqrt{d} / A-2, \sqrt{d} / A)$ with the same parity as $d$. Then, with $\varepsilon$ as in (5),

$$
\begin{aligned}
L(d) & =\varepsilon \min _{A \mid d, A<\sqrt{d} \text { odd }} \min \left\{M\left(\frac{\sqrt{d}+m_{A} A}{2 A}\right), M\left(\frac{\sqrt{d}+\left(m_{A}-2\right) A}{2 A}\right)\right\} \\
& =\varepsilon \min _{A \mid d, A<\sqrt{d} \text { odd }} \min \left\{\sqrt{d}+m_{A} A,\left(d-\left(m_{A}-2\right)^{2} A^{2}\right) / 2 A\right\} \\
& \geq \sqrt{D}-2 \varepsilon \varepsilon_{A \mid d, A<\sqrt{d} \text { odd }} A .
\end{aligned}
$$

Proof. Suppose that $d \equiv 2$ or $3(\bmod 4)$. Since $\left(\frac{d}{p}\right)=-1$ for all $p<\sqrt{d}, p \nmid d$ we have $d-a^{2}=A_{1}$ or $2 A_{1}$ or $A_{1} p$ or $2 A_{1} p$ with $A_{1} \mid d$ odd and $p>\sqrt{d}$ prime. Hence
we can assume that $c \mid d-a^{2}$ is of the form $c=A_{2}$ or $2 A_{2}$ or $A_{2} p$ or $2 A_{2} p$ where $A_{2} \mid A_{1}$ and $0 \leq a<\sqrt{d}$, and

$$
M\left(\frac{a+\sqrt{d}}{c}\right)=\max \left\{c, a+\sqrt{d},\left(d-a^{2}\right) / c\right\}
$$

Hence with $A=A_{1}$ or $A_{1} / A_{2}$ we can assume $A \mid d$ odd, $a=k A$, and it is enough to consider

$$
M\left(\frac{\sqrt{d}+k A}{A}\right)=\max \left\{A, \sqrt{d}+k A,\left(d-k^{2} A^{2}\right) / A\right\}
$$

with $A<\sqrt{d}$ or

$$
M\left(\frac{\sqrt{d}+k A}{2 A}\right)=\max \left\{2 A, \sqrt{d}+k A,\left(d-k^{2} A^{2}\right) / 2 A\right\}
$$

with $2 A<\sqrt{d}$ and $k$ and $d$ the same parity. Hence

$$
\begin{align*}
M\left(\frac{\sqrt{d}+k A}{A}\right) & = \begin{cases}\sqrt{d}+k A, & \text { if } \sqrt{d} / A-1<k<\sqrt{d} / A \\
\left(d-k^{2} A^{2}\right) / A, & \text { if } k<\sqrt{d} / A-1\end{cases}  \tag{6}\\
& \geq 2 \sqrt{d}-A
\end{align*}
$$

and for $k$ and $d$ the same parity

$$
\begin{align*}
M\left(\frac{\sqrt{d}+k A}{2 A}\right) & = \begin{cases}\sqrt{d}+k A, & \text { if } \sqrt{d} / A-2<k<\sqrt{d} / A \\
\left(d-k^{2} A^{2}\right) / 2 A, & \text { if } k<\sqrt{d} / A-2,\end{cases}  \tag{7}\\
& \geq 2 \sqrt{d}-2 A .
\end{align*}
$$

For $k \geq m_{A}$ the minimum of both is plainly

$$
M\left(\frac{\sqrt{d}+m_{A} A}{2 A}\right)=\sqrt{d}+m_{A} A
$$

In (7) the smallest for $k \leq m_{A}-2$ is

$$
M\left(\frac{\sqrt{d}+\left(m_{A}-2\right) A}{2 A}\right)=\left(d-\left(m_{A}-2\right)^{2} A^{2}\right) / 2 A
$$

and for (6) the smallest for $k \leq m_{A}-1$ is

$$
M\left(\frac{\sqrt{d}+\left(m_{A}-1\right) A}{A}\right)=\left(d-\left(m_{A}-1\right)^{2} A^{2}\right) / A
$$

Writing $m_{A}=\sqrt{d} / A-\delta, 0<\delta<2$ and observing that
$M\left(\frac{\sqrt{d}+\left(m_{A}-1\right) A}{A}\right)-M\left(\frac{\sqrt{d}+\left(m_{A}-2\right) A}{2 A}\right)=\delta \sqrt{d}+A\left(1-\frac{\delta^{2}}{2}\right)>\delta\left(\sqrt{d}-\frac{1}{2} A\right)>0$
the result follows.
Similarly for $d \equiv 5 \bmod 8$ we must have $d-a^{2}=2^{2} A_{1}$ or $2^{2} A_{1} p$, and our even $c$ with $2 c \mid d-a^{2}$ must be of the form $c=2 A_{2}$ or $2 A_{2} p$. Thus we again reduce to

$$
M\left(\frac{\sqrt{d}+k A}{2 A}\right)=\frac{1}{2} \max \left\{2 A, \sqrt{d}+k A,\left(d-k^{2} A^{2}\right) / 2 A\right\}
$$

with $A \mid d$ odd, $2 A<\sqrt{d}, k$ odd, and the minimum occurs for $k=m_{A}$ or $m_{A}-2$ as before.

Plainly $d \equiv 2$ or $3(\bmod 4)$ or $5(\bmod 8)$ with no divisors in $(o(\sqrt{d}, \sqrt{d})$ that are quadratic non-residues for all $p<\sqrt{d}$ would have $L(d) \geq \sqrt{D}-o(\sqrt{d})$. In particular infinitely many would immediately give

$$
\limsup _{d \rightarrow \infty} \frac{L(d)}{\sqrt{D}}=1
$$

in contradiction to Conjecture 4.1, but this seems unlikely:
Conjecture 4.5. All but finitely many squarefree d have $\left(\frac{d}{p}\right)=1$ for some odd prime $p<\frac{1}{2} \sqrt{D}$.

Assuming GRH for the mod $4 d$ character

$$
\chi(n):= \begin{cases}\left(\frac{d}{n}\right), & \text { if } \operatorname{gcd}(n, 4 d)=1 \\ 0, & \text { otherwise }\end{cases}
$$

(where $\left(\frac{d}{n}\right)$ denotes the Jacobi symbol), we have the bound

$$
\begin{equation*}
\left|\sum_{n \leq x} \chi(n) \Lambda(n)\right| \ll x^{\frac{1}{2}} \log ^{2}(D x) \tag{8}
\end{equation*}
$$

(see, for example, [2, Chapter 20]), and so we should in fact have $\left(\frac{d}{p}\right)=1$ for some prime $p \ll \log ^{4} D$.

Note, a squarefree $d \not \equiv 1 \bmod 8$ with $\left(\frac{d}{p}\right)=-1$ for all odd primes $p \nmid d$ with $p<\frac{1}{2} \sqrt{D}$ must be of the form $d=(k A)^{2} \pm 2 A$ or $((2 k-1) A)^{2} \pm 4 A$, for some $k$ and squarefree odd $A \mid d$ with $A<\frac{1}{2} \sqrt{D}$. To see this, write $d=N^{2}+r, N=[\sqrt{d}]$, $1 \leq r \leq 2 N$. If $r$ is even then $A=r / 2$ is odd if $d \equiv 2,3 \bmod 4$ and $A=r / 4$ is odd if $d \equiv 5 \bmod 8$. Since $d$ is a square $\bmod A$ we must have $p|A \Rightarrow p| d$. As $d$ is squarefree, $A<\frac{1}{2} \sqrt{D}$ must be squarefree with $A \mid d$, giving $d=N^{2}+2 A$ or $N^{2}+4 A$ with $A \mid N$. Similarly for $r$ odd

$$
d=\left(\frac{r+1}{2}\right)^{2}+\left(N-\frac{1}{2}(r-1)\right)\left(N+\frac{1}{2}(r-1)\right)
$$

with $A=N-\frac{1}{2}(r-1)$ odd for $d=2$ or $3 \bmod 4$ and $A=\frac{1}{2}\left(N-\frac{1}{2}(r-1)\right)$ odd for $d \equiv 5 \bmod 8$. Since $d$ is a square $\bmod \mathrm{A}, A<\frac{1}{2} \sqrt{D}$ is squarefree with $A \mid d$, giving $d=(N+1)^{2}-2 A$ or $(N+1)^{2}-4 A$ with $A \mid N+1$.

Conversely if $d$ is a quadratic residue $\bmod p$ for a suitably sized $p$ or if $d \equiv 1$ $\bmod 8$ then we can obtain a bound less than one for $L(d) / \sqrt{D}$ :

Lemma 4.2. Suppose that $d$ is a square $\bmod q$, where $q$ is odd or $4 \mid q$ and $\lambda$ defined by

$$
\lambda \sqrt{d}= \begin{cases}q, & \text { if } q \text { is odd } \\ \frac{1}{4} q, & \text { if } q \text { is even }\end{cases}
$$

has $0<\lambda<1$. Then

$$
\frac{L(d)}{\sqrt{D}} \leq \begin{cases}\frac{1}{2}\left(1+\lambda+\sqrt{(1-\lambda)^{2}-4 \lambda^{2}}\right), & \text { if } 0<\lambda<\frac{1}{4}(\sqrt{5}-1) \\ \frac{1}{4 \lambda}, & \text { if } \frac{1}{4}(\sqrt{5}-1)<\lambda<\frac{1}{2}(\sqrt{3}-1) \\ \frac{1}{2}(1+\lambda), & \text { if } \frac{1}{2}(\sqrt{3}-1)<\lambda<1\end{cases}
$$

Notice that if we assume GRH then estimate (8), with

$$
\sum_{x-y \leq n \leq x} \Lambda(n)=y+O\left(x^{\frac{1}{2}} \log ^{2} x\right)
$$

from assuming RH, guarantees that $\left(\frac{d}{p}\right)=1$ for some prime $p$ in

$$
\left(\frac{1}{2}(\sqrt{3}-1) d^{\frac{1}{2}}, \frac{1}{2}(\sqrt{3}-1) d^{\frac{1}{2}}+c d^{\frac{1}{4}} \log ^{2} d\right)
$$

for suitably large $c$, and Lemma 4.2 gives

$$
\frac{L(d)}{\sqrt{D}} \leq \frac{1}{4}(\sqrt{3}+1)+O\left(\frac{\log ^{2} d}{d^{\frac{1}{4}}}\right)=0.683012 \ldots+o(1)
$$

Proof. Suppose that $r_{0}$ has $r_{0}^{2} \equiv d \bmod q$. If $q$ is odd we take $r$ to be the integer in $(\sqrt{d}-2 q, \sqrt{d})$ with the same parity as $d$ and $r \equiv r_{0} \bmod q$, write $r=\sqrt{d}-\delta q$, and set

$$
\alpha_{1}=\frac{\sqrt{d}+r}{2 q}, \quad \alpha_{2}=\frac{\sqrt{d}+r-2 q}{2 q} .
$$

Notice that $c=2 q$ and $a=r$ or $r-2 q$ will have $c \mid\left(d-a^{2}\right)$ with $2 c \mid d-a^{2}$ when $d=1 \bmod 4$.

For $4 \mid q$ (which of course only occurs when $d=1 \bmod 4$ ) we write $q=2^{l} q_{1}$ with $q_{1}$ odd and $l \geq 2$ and take $r$ to be the integer in $\left(\sqrt{d}-2^{l-1} q_{1}, \sqrt{d}\right)$ with $r \equiv r_{0} \bmod$ $2^{l-1} q_{1}$ and set

$$
\alpha_{1}=\frac{\sqrt{d}+r}{2^{l-1} q_{1}}, \quad \alpha_{2}=\frac{\sqrt{d}+r-2^{l-1} q_{1}}{2^{l-1} q_{1}}
$$

Again $c=2^{l-1} q_{1}$ and $a=r$ or $r-2^{l-1} q_{1}$ will have $2 c \mid\left(d-a^{2}\right)$.
Writing $r=\sqrt{d}-\delta q$ for $q$ odd, and $r=\sqrt{d}-\delta 2^{l-2} q_{1}$ for $q$ even, we have $r=(1-\lambda \delta) \sqrt{d}$ with $0<\delta<2$ and

$$
\alpha_{1}=\frac{(2-\delta \lambda)}{2 \lambda}, \quad \bar{\alpha}_{1}=-\frac{\delta}{2}, \quad \alpha_{2}=\frac{(2-2 \lambda-\delta \lambda)}{2 \lambda}, \quad \bar{\alpha}_{2}=-\frac{\delta}{2}-1 .
$$

For $\alpha_{1}$ and $\alpha_{2}$ we also plainly have $k=\epsilon c=2 \epsilon \lambda \sqrt{d}=\lambda \sqrt{D}$.
Clearly $\alpha_{1}>0, \alpha_{2}>-1,-1<\bar{\alpha}_{1}<0$ and $-2<\bar{\alpha}_{2}<-1$.
If $\alpha_{1}<1$ then $M\left(\alpha_{1}\right)=\lambda \sqrt{D}<\frac{1}{2}(1+\lambda) \sqrt{D}$. Hence we can assume that $\alpha_{1}>1$ (this is automatic for $\lambda<\frac{1}{2}$ ).

So

$$
M\left(\alpha_{1}\right)=\lambda \sqrt{D} \alpha_{1}=\sqrt{D}\left(1-\frac{\delta}{2} \lambda\right)
$$

If $\alpha_{2}<1$ then

$$
M\left(\alpha_{2}\right)=\lambda \sqrt{D}\left|\bar{\alpha}_{2}\right|=\sqrt{D} \lambda\left(1+\frac{\delta}{2}\right)
$$

and plainly

$$
\min \left\{M\left(\alpha_{1}\right), M\left(\alpha_{2}\right)\right\} \leq \frac{1}{2}\left(M\left(\alpha_{1}\right)+M\left(\alpha_{2}\right)\right)=\frac{1}{2}(1+\lambda) \sqrt{D}
$$

So we can assume that $\alpha_{2}>1$ and

$$
M\left(\alpha_{2}\right)=\lambda \sqrt{D}\left|\bar{\alpha}_{2}\right| \alpha_{2}=\sqrt{D}\left(1+\frac{\delta}{2}\right)\left(1-\lambda-\frac{\delta}{2} \lambda\right)
$$

Observing that the quadratic is maximized for $\frac{\delta}{2}=\frac{1}{2 \lambda}-1$ we plainly have

$$
M\left(\alpha_{2}\right) \leq \sqrt{D} \frac{1}{4 \lambda}
$$

with this less than $\frac{1}{2}(1+\lambda) \sqrt{D}$ for $\frac{1}{2}(\sqrt{3}-1)<\lambda<1$. For $\lambda<\frac{1}{4}(\sqrt{5}-1)$ the value $\frac{\delta}{2}=\frac{1}{2 \lambda}\left(1-\lambda-\sqrt{(1-\lambda)^{2}-4 \lambda^{2}}\right)$ equating $M\left(\alpha_{1}\right)$ and $M\left(\alpha_{2}\right)$ is less than $\frac{1}{2 \lambda}-1$ and the minimum of the two is at most the value at that point:

$$
\min \left\{M\left(\alpha_{1}\right), M\left(\alpha_{2}\right)\right\} \leq \sqrt{D}\left(\frac{1}{2}(1+\lambda)+\frac{1}{2} \sqrt{(1-\lambda)^{2}-4 \lambda^{2}}\right)
$$

In particular from the lemma we immediately obtain a bound away from 1 for the $d \equiv 1 \bmod 8$.

Corollary 4.1. If $d \equiv 1(\bmod 8)$ then

$$
\frac{L(d)}{\sqrt{D}} \leq \frac{1}{4}(\sqrt{5}+1)=0.809016 \ldots
$$

If $d \equiv 1(\bmod 3)$ then

$$
\frac{L(d)}{\sqrt{D}} \leq \frac{1}{7}(2+3 \sqrt{2})=0.891805 \ldots
$$

Computations indicate room for improvement in these bounds.

## Conjecture 4.6.

$$
\begin{aligned}
& \sup _{d \equiv 1 \bmod 8} \frac{L(d)}{\sqrt{D}}=\frac{L(41)}{\sqrt{41}}=\frac{M\left(\frac{\sqrt{41}+27}{4}\right)}{\sqrt{41}}=\frac{\sqrt{41}+3}{2 \sqrt{41}}=0.734261 \ldots, \\
& \sup _{d \equiv 1 \bmod 3} \frac{L(d)}{\sqrt{D}}=\frac{L(13)}{\sqrt{13}}=\frac{M\left(\frac{\sqrt{13}+1}{2}\right)}{\sqrt{13}}=\frac{4}{\sqrt{13}}=0.832050 \ldots
\end{aligned}
$$

Proof. If $d \equiv 1(\bmod 8)$ then we can solve $r^{2} \equiv d \bmod 2^{l}$ for any $l$. Hence if we pick $l$ such that $\frac{1}{4}(\sqrt{5}-1) \sqrt{d} \leq 2^{l-2} \leq \frac{1}{2}(\sqrt{5}-1) \sqrt{d}$ and we can apply the lemma with $\frac{1}{4}(\sqrt{5}-1) \leq \lambda \leq \frac{1}{2}(\sqrt{5}-1)$. Likewise, for an odd prime $p$, if $p \nmid d$ and $\left(\frac{d}{p}\right)=1$ then we can solve $r^{2} \equiv d \bmod p^{l}$ for any $l$. Choosing $l$ so that

$$
\frac{1}{1+\sqrt{(p-1)^{2}+4}} \sqrt{d} \leq p^{l} \leq \frac{p}{1+\sqrt{(p-1)^{2}+4}} \sqrt{d}
$$

and applying the lemma with $q=p^{l}$ gives

$$
\frac{L(d)}{\sqrt{D}} \leq \frac{1}{2}\left(1+\frac{p}{1+\sqrt{(p-1)^{2}+4}}\right)
$$

Taking $p=3$ gives the result claimed for $d \equiv 1 \bmod 3$.
Likewise, for $d \equiv 1,4 \bmod 5$ we get the upper bound $0.956859 \ldots($ from $d=29$ we know $0.928476 \ldots$ would be best possible).

For $d \equiv 1,2,4 \bmod 7$ we get $0.977844 \ldots$ and for $d \equiv 1,3,4,5,9 \bmod 11$ the bound $0.991157 \ldots$ (from $d=53$ these can not be reduced below $0.961523 \ldots$...

For $d \equiv 1,3,4,9,10,12 \bmod 13$ we get the bound $0.993713 \ldots$ (the optimal bound is likely $0.988371 \ldots$ from $d=173$ ).

For $d \equiv 1,2,4,8,9,13,15,16 \bmod 17$ our bound gives $0.996364 \ldots$ (optimal is probably $0.993150 \ldots$ at $d=293$ ).

## 5. Appendix of Large Values

We give the largest values found in Figure 1, Figure 2 \& Figure 3 down to the first value not satisfying the quadratic non-residue conditions of Lemma 4.1.

Largest values for $d \equiv 1 \bmod 4$.

$$
\begin{aligned}
& \frac{L(293)}{\sqrt{293}}=\frac{M\left(\frac{\sqrt{293}+15}{2}\right)}{\sqrt{293}}=\frac{17}{\sqrt{293}}=0.993150 \ldots, \quad\left(\frac{293}{p}\right)=-1, p=3,5,7,11,13, \\
& \frac{L(173)}{\sqrt{173}}=\frac{M\left(\frac{\sqrt{173}+11}{2}\right)}{\sqrt{173}}=\frac{13}{\sqrt{173}}=0.988371 \ldots, \quad\left(\frac{173}{p}\right)=-1, p=3,5,7,11, \\
& \frac{L(53)}{\sqrt{53}}=\frac{M\left(\frac{\sqrt{53}+5}{2}\right)}{\sqrt{53}}=\frac{7}{\sqrt{53}}=0.961523 \ldots, \quad\left(\frac{53}{p}\right)=-1, p=3,5, \\
& \frac{L(437)}{\sqrt{437}}=\frac{M\left(\frac{\sqrt{437}+19}{2}\right)}{\sqrt{437}}=\frac{\sqrt{437}+19}{2 \sqrt{437}}=0.954446 \ldots, \quad\left(\frac{437}{p}\right)=-1, p=3,5,7,11,13,17,29, \\
& \frac{L(29)}{\sqrt{29}}=\frac{M\left(\frac{\sqrt{29}+3}{2}\right)}{\sqrt{29}}=\frac{5}{\sqrt{29}}=0.928476 \ldots, \quad\left(\frac{29}{p}\right)=-1, p=3, \\
& \frac{L(1077)}{\sqrt{1077}}=\frac{M\left(\frac{\sqrt{1077}+27}{6}\right)}{\sqrt{1077}}=\frac{\sqrt{1077}+27}{2 \sqrt{1077}}=0.911363 \ldots, \quad\left(\frac{1077}{p}\right)=-1, p=5,7,11,13,17,19,23, \\
& \frac{L(77)}{\sqrt{77}}=\frac{M\left(\frac{\sqrt{77}+7}{2}\right)}{\sqrt{77}}=\frac{\sqrt{77}+7}{\sqrt{77}}=0.898862 \ldots, \quad\left(\frac{77}{p}\right)=-1, p=3,5, \\
& \frac{L(453)}{\sqrt{453}}=\frac{M\left(\frac{\sqrt{453}+15}{6}\right)}{\sqrt{453}}=\frac{19}{\sqrt{453}}=0.892697 \ldots, \quad\left(\frac{453}{p}\right)=-1, p=5,7,11,13,17, \\
& \frac{L(717)}{\sqrt{717}}=\frac{M\left(\frac{\sqrt{717}+21}{6}\right)}{\sqrt{717}}=\frac{\sqrt{717}+21}{\sqrt{717}}=0.892129 \ldots, \quad\left(\frac{717}{p}\right)=-1, p=5,7,11,13,17,19, \\
& \frac{L(3053)}{\sqrt{3053}}=\frac{M\left(\frac{\sqrt{3053}+41}{14}\right)}{\sqrt{3053}}=\frac{49}{\sqrt{3053}}=0.886814 \ldots, \quad\left(\frac{3053}{7}\right)=1 .
\end{aligned}
$$

Note other $\alpha$ may achieve the minimum, for example $M\left(\frac{\sqrt{437}+19}{2}\right)=M\left(\frac{\sqrt{437}+19}{38}\right)$.

Largest values for $d \equiv 2 \bmod 4$.

$$
\begin{aligned}
\frac{L(398)}{2 \sqrt{398}} & =\frac{M\left(\frac{\sqrt{398}+18}{2}\right)}{2 \sqrt{398}}=\frac{\sqrt{398}+18}{2 \sqrt{398}}=0.951129 \ldots, \quad\left(\frac{398}{p}\right)=-1, p=3,5,7,11,13,17,19,23,29,31, \\
\frac{L(38)}{2 \sqrt{38}} & =\frac{M\left(\frac{\sqrt{38}+4}{2}\right)}{2 \sqrt{38}}=\frac{11}{2 \sqrt{38}}=0.892217 \ldots, \quad\left(\frac{38}{p}\right)=-1, p=3,5, \\
\frac{L(62)}{2 \sqrt{62}} & =\frac{M\left(\frac{\sqrt{62}+6}{2}\right)}{2 \sqrt{62}}=\frac{\sqrt{62}+6}{2 \sqrt{62}}=0.881000 \ldots, \quad\left(\frac{62}{p}\right)=-1, p=3,5,7,11, \\
\frac{L(318)}{2 \sqrt{318}} & =\frac{M\left(\frac{\sqrt{318}+12}{6}\right)}{2 \sqrt{318}}=\frac{\sqrt{318}+12}{2 \sqrt{318}}=0.836463 \ldots, \quad\left(\frac{318}{p}\right)=-1, p=5,7,11,13,17,19,23, \\
\frac{L(14)}{2 \sqrt{14}} & =\frac{M\left(\frac{\sqrt{14}+2}{2}\right)}{2 \sqrt{14}}=\frac{\sqrt{14}+2}{2 \sqrt{14}}=0.767261 \ldots, \quad\left(\frac{14}{p}\right)=-1, p=3, \\
\frac{L(138)}{2 \sqrt{138}} & =\frac{M\left(\frac{\sqrt{138}+6}{6}\right)}{2 \sqrt{138}}=\frac{\sqrt{138}+6}{2 \sqrt{138}}=0.755376 \ldots, \quad\left(\frac{138}{p}\right)=-1, p=5,7,11,13, \\
\frac{L(22)}{2 \sqrt{22}} & =\frac{M\left(\frac{\sqrt{22}+2}{3}\right)}{2 \sqrt{22}}=\frac{\sqrt{22}+2}{2 \sqrt{22}}=0.713200 \ldots, \quad\left(\frac{22}{3}\right)=1 .
\end{aligned}
$$

Largest values for $d \equiv 3 \bmod 4$.

$$
\begin{aligned}
& \frac{L(227)}{2 \sqrt{227}}=\frac{M\left(\frac{\sqrt{227}+13}{2}\right)}{2 \sqrt{227}}=\frac{29}{2 \sqrt{227}}=0.962398 \ldots, \quad\left(\frac{227}{p}\right)=-1, p=3,5,7,11,13,17,19,23, \\
& \frac{L(83)}{2 \sqrt{83}}=\frac{M\left(\frac{\sqrt{83}+7}{2}\right)}{2 \sqrt{83}}=\frac{17}{2 \sqrt{83}}=0.932966 \ldots, \quad\left(\frac{83}{p}\right)=-1, p=3,5,7,11,13, \\
& \frac{L(167)}{2 \sqrt{167}}=\frac{M\left(\frac{\sqrt{167}+11}{2}\right)}{2 \sqrt{167}}=\frac{\sqrt{167}+11}{2 \sqrt{167}}=0.925602 \ldots, \quad\left(\frac{167}{p}\right)=-1, p=3,5,7,11,13,17,19, \\
& \frac{L(447)}{2 \sqrt{447}}=\frac{M\left(\frac{\sqrt{447}+15}{6}\right)}{2 \sqrt{447}}=\frac{37}{2 \sqrt{447}}=0.875019 \ldots, \quad\left(\frac{447}{p}\right)=-1, p=5,7,11,13,17, \\
& \frac{L(47)}{2 \sqrt{47}}=\frac{M\left(\frac{\sqrt{47}+5}{2}\right)}{2 \sqrt{47}}=\frac{\sqrt{47}+5}{2 \sqrt{47}}=0.864662 \ldots, \quad\left(\frac{47}{p}\right)=-1, p=3,5,7, \\
& \frac{L(635)}{2 \sqrt{635}}=\frac{M\left(\frac{\sqrt{635}+15}{10}\right)}{2 \sqrt{635}}=\frac{41}{2 \sqrt{635}}=0.813517 \ldots, \quad\left(\frac{635}{p}\right)=-1, p=3,7,11,13,17,19,23,29,31,37, \\
& \frac{L(23)}{2 \sqrt{23}}=\frac{M\left(\frac{\sqrt{23}+3}{2}\right)}{2 \sqrt{23}}=\frac{\sqrt{23}+3}{2 \sqrt{23}}=0.812771 \ldots, \quad\left(\frac{23}{p}\right)=-1, p=3,5, \\
& \frac{L(3)}{2 \sqrt{3}}=\frac{M\left(\frac{\sqrt{3}+1}{2}\right)}{2 \sqrt{3}}=\frac{\sqrt{3}+1}{2 \sqrt{3}}=0.788675 \ldots, \quad\left(\frac{3}{p}\right)=-1, p=3,7,11,13,17,19,23,29,31,37, \\
& \frac{L(827)}{2 \sqrt{827}}=\frac{M\left(\frac{\sqrt{827}+15}{14}\right)}{2 \sqrt{827}}=\frac{\sqrt{827}+15}{2 \sqrt{827}}=0.760800 \ldots, \quad\left(\frac{827}{7}\right)=1 .
\end{aligned}
$$

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