# ON THE COHOMOLOGY OF STOVER SURFACE 

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#### Abstract

We study a surface discovered by Stover which is the surface with minimal Euler number and maximal automorphism group among smooth arithmetic ball quotient surfaces. We study the natural map $\wedge^{2} H^{1}(S, \mathbb{C}) \rightarrow H^{2}(S, \mathbb{C})$ and we discuss the problem related to the so-called Lagrangian surfaces. We obtain that this surface $S$ has maximal Picard number and has no higher genus fibrations. We compute that its Albanese variety $A$ is isomorphic to $(\mathbb{C} / \mathbb{Z}[\alpha])^{7}$, for $\alpha=e^{2 i \pi / 3}$.


## 1. Introduction

By the recent work of M. Stover [14, the number of automorphisms of a smooth compact arithmetic ball quotient surface $X=\Gamma \backslash \mathbb{B}_{2}$ is bounded by $288 \cdot e(X)$, where $e(X)$ denotes the topological Euler number of $X$.
Furthermore, Stover characterizes the arithmetic ball quotient surfaces $X$ whose automorphism groups attain this bound, which by analogy with Hurwitz curves, he calls Hurwitz ball quotients; all such surfaces are finite Galois coverings of the Deligne-Mostow orbifold $\Lambda \backslash \mathbb{B}_{2}$ corresponding to the quintuple $(2 / 12,2 / 12,2 / 12,7 / 12,11 / 12)$ (see [12, 14]).
Stover constructs also a Hurwitz ball quotient $S$ with Euler number $e(S)=63$ and automorphism group $\operatorname{Aut}(S)$ isomorphic to $U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}$, of order $18144=2^{5} 3^{4} 7$. He shows that $S$ is the unique Hurwitz ball quotient with Euler number $e=63$, and moreover that $e=63$ is the minimal possible value for the Euler number of a Hurwitz ball quotient. Having this property the surface $S$ can be seen as the 2-dimensional analog of the Klein's quartic which is the unique curve uniformized by the ball $\mathbb{B}_{1}$ with minimal genus and maximal possible automorphism group.
Our aim is to study more closely the cohomology of this particular surface $S$, which we will call Stover surface in the following. This surface $S$ has the following numerical invariants (see [14]):

| $e(S)$ | $H_{1}(S, \mathbb{Z})$ | $q$ | $p_{g}=h^{2,0}$ | $h^{1,1}$ | $b_{2}(S)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 63 | $\mathbb{Z}^{14}$ | 7 | 27 | 35 | 89 |

Let $V$ be a vector space. Let us recall that a 2-vector $w \in \wedge^{2} V$ has rank 1 or is decomposable if there are vectors $w_{1}, w_{2} \in V$ with $w=w_{1} \wedge w_{2}$. A vector $w \in \wedge^{2} V$ has rank 2 if there exist linearly independent vectors $w_{i} \in V, i=1, . ., 4$ such that $w=w_{1} \wedge w_{2}+w_{3} \wedge w_{4}$.

Let $B$ be an Abelian fourfold and let $p: S \rightarrow B$ be a map such that $p(S)$ generates $B$. We say that $S$ is Lagrangian with respect to $p$ if there exists a basis $w_{1}, \ldots, w_{4}$ of $p^{*} H^{0}\left(B, \Omega_{B}\right)$ such that the rank 2 vector $w=w_{1} \wedge w_{2}+w_{3} \wedge w_{4}$ is in the kernel of the natural map $\phi^{2,0}: \wedge^{2} H^{0}\left(S, \Omega_{S}\right) \rightarrow H^{0}\left(S, K_{S}\right)$.

Theorem 1. The surface $S$ has maximal Picard number. The natural map

$$
\phi^{1,1}: H^{0}\left(S, \Omega_{S}\right) \otimes H^{1}\left(S, \mathcal{O}_{S}\right) \rightarrow H^{1}\left(S, \Omega_{S}\right)
$$

is surjective with a 14-dimensional kernel. The kernel of the map

$$
\phi^{2,0}: \wedge^{2} H^{0}\left(S, \Omega_{S}\right) \rightarrow H^{0}\left(S, K_{S}\right)
$$

is 7-dimensional and contains no decomposable elements. The set of rank 2 vectors in $\operatorname{Ker}\left(\phi^{2,0}\right)$ is a quadric hypersurface.
There exists an infinite number (up to isogeny) of maps $p: S \rightarrow B$ (where $B$ is an Abelian fourfold) such that $S$ is Lagrangian with respect to $p$.
The Albanese variety of $S$ is isomorphic to $(\mathbb{C} / \mathbb{Z}[\alpha])^{7}$, for $\alpha=e^{2 i \pi / 3}$.
By the Castelnuovo - De Franchis Theorem, the fact that there are no decomposable elements in $\wedge^{2} H^{0}\left(S, \Omega_{S}\right)$ means that $S$ has no fibration $f: S \rightarrow C$ onto a curve of genus $g>1$. Moreover Theorem 1 implies that $S$ has the remarkable feature that both maps

$$
\begin{gathered}
\phi^{2,0}: \wedge^{2} H^{1,0}(S) \rightarrow H^{2,0}(S) \\
\phi^{1,1}: H^{1,0}(S) \otimes H^{0,1}(S) \rightarrow H^{1,1}(S)
\end{gathered}
$$

have a non-trivial kernel. With Schoen surfaces (see [10, Remark 2.6]), this is the second example of surfaces enjoying such properties. For more on this subject, see e.g. [1, 5, 2, 8, 8].

We obtain these results using Sullivan's theory on the second lower quotient of the fundamental group $\pi_{1}(S)$ of $S$ (see [4]).

For the motivation and a historic account of surfaces with maximal Picard number we refer to [3].

Aknowledgements We are grateful to Marston Conder and Derek Holt for their help in the computations of Theorem 3.

## 2. The Second lower central quotient of the fundamental group of $S$

Let $\Pi:=\pi_{1}(X)$ be the fundamental group of a manifold $X$. The group $H_{1}(X, \mathbb{Z})$ is the abelianization of $\Pi: H_{1}(X, \mathbb{Z})=\Pi / \Delta$ where $\Delta:=[\Pi, \Pi]$ is the derived subgroup of $\Pi$, that is, the subgroup generated by all elements $[h, g]=g^{-1} h^{-1} g h, h, g \in \Pi$.
The second group in the lower central series $[\Delta, \Pi]$ is the group generated by commutators $[h, g]$, with $h \in \Delta, g \in \Pi$. It is a normal subgroup of the commutator group $\Delta$. According to [4], we have the following results:

Proposition 2. (Sullivan) Let $X$ be a compact connected Kähler manifold. There exists an exact sequence

$$
0 \rightarrow \operatorname{Hom}(\Delta /[\Delta, \Pi], \mathbb{R}) \rightarrow \wedge^{2} H^{1}(X, \mathbb{R}) \rightarrow H^{2}(X, \mathbb{R})
$$

(Beauville) Suppose $H_{1}(X, \mathbb{Z})$ is torsion free. Then the group $\Delta /[\Delta, \Pi]$ is canonically isomorphic to the cokernel of the map

$$
\mu: H_{2}(X, \mathbb{Z}) \rightarrow \operatorname{Alt}^{2}\left(H^{1}(X, \mathbb{Z})\right) \text { given by } \mu(\sigma)(a, b)=\sigma \cap(a \wedge b)
$$

where $A l t^{2}\left(H^{1}(X, \mathbb{Z})\right)$ is the group of skew-symmetric integral bilinear forms on $H^{1}(X, \mathbb{Z})$.
In the case of the Stover surface, computer calculations give us the following result:
Theorem 3. Let $\Pi=\pi_{1}(S)$ be the fundamental group of the Stover surface and $\Delta=[\Pi, \Pi]$. The group $\Delta /[\Delta, \Pi]$ is isomorphic to $\mathbb{Z} / 4 \mathbb{Z} \times \mathbb{Z}^{28}$.

Proof. By the construction of $S$ [14], the fundamental group $\Pi$ is isomorphic to the kernel $\operatorname{ker}(\varphi)$ of the unique epimorphism $\varphi: \Lambda \longrightarrow G$ from the Deligne-Mostow lattice $\Lambda$ corresponding to the quintuple $(2 / 12,2 / 12,2 / 12,7 / 12,11 / 12)$ onto the finite group $G=U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}$. The lattice $\Lambda$ is described by Mostow in [12] as a complex reflection group, and by generators and relation by Cartwright and Steger in [7]. This lattice has presentation

$$
\Lambda=\left\langle j, u, v, b \mid u^{4}, v^{8},[u, j],[v, j], j^{-3} v^{2}, u v u v^{-1} u v^{-1},(b j)^{2}\left(v u^{2}\right)^{-1},\left[b, v u^{2}\right], b^{3},\left(b v u^{3}\right)^{3}\right\rangle
$$

MAGMA command LowIndexSubgroups is used to identify the unique subgroup $\Gamma \triangleleft \Lambda$ of index 3 , which is $\Gamma=\langle u, j b, b j\rangle$. Using the primitive permutation representation of $U_{3}(3)$ of degree 28 , MAGMA is able to identify an homomorphism $\varphi$ from $\Gamma$ onto $U_{3}(3)$ induced from the assignment

$$
\begin{aligned}
u & \mapsto(3,8,23,20)(4,24,6,12)(7,9,14,22)(10,19,11,13)(15,16,21,18)(17,26,27,25) \\
j b & \mapsto(1,9,20,12,19,23,6,16)(2,27,14,17,13,26,15,25)(3,24)(4,5,10,21,7,11,28,8) \\
b j & \mapsto(1,13,20,15,19,2,6,14)(4,9,10,12,7,23,28,16)(5,27,21,17,11,26,8,25)(22,24) .
\end{aligned}
$$

This homomorphism extends to an homomorphism $\varphi$ from $\Lambda$ onto $G$ such that $\Pi=\operatorname{ker}(\varphi)$ is a torsion-free normal subgroup in $\Lambda$, it is the fundamental group of $S$ (see [14). Let be $\Delta=[\Pi, \Pi]$ and $\Delta_{2}=[\Delta, \Pi]$. It is easy to check that that $\Delta_{2}$ is distinguished into $\Pi$. The image of $\Delta$ under the quotient map $\Pi \longrightarrow \Pi / \Delta_{2}$ is $\Delta / \Delta_{2}$, but we observe that it is also equal the commutator subgroup $\left[\Pi / \Delta_{2}, \Pi / \Delta_{2}\right]$, and therefore, the computation of $\Delta / \Delta_{2}$ is reduced to the one of the derived group $\left[\Pi / \Delta_{2}, \Pi / \Delta_{2}\right]$.
The MAGMA command $\mathrm{g}:=\operatorname{Rewrite}(\mathrm{G}, \mathrm{g})$ is used to have generators and relations of both subgroups $\Gamma<\Lambda$ and $\Pi<\Gamma$. The command NilpotentQuotient (.,2) applied to $\Pi$ describes $\Pi / \Delta_{2}$ in terms of a polycyclic presentation. The derived subgroup $\left[\Pi / \Delta_{2}, \Pi / \Delta_{2}\right]$ is obtained with DerivedGroup (.) applied to $\Pi / \Delta_{2}$. Finally, applying the MAGMA function AQInvariants to $\left[\Pi / \Delta_{2}, \Pi / \Delta_{2}\right]$, MAGMA computes that the structure of $\Delta / \Delta_{2}$ is $\mathbb{Z} / 4 \mathbb{Z} \times \mathbb{Z}^{28}$ 。

Corollary 4. The dimension of the kernel of $\wedge^{2} H^{1}(S, \mathbb{R}) \rightarrow H^{2}(S, \mathbb{R})$ is 28.

$$
\text { 3. Computation of the map } \wedge^{2} H^{1}(S, \mathbb{C}) \rightarrow H^{2}(S, \mathbb{C})
$$

Let $A$ be the Albanese variety of the Stover surface $S$. The invariants are:

$$
\begin{gathered}
H_{1}(A, \mathbb{Z})=H_{1}(S, \mathbb{Z})=\mathbb{Z}^{14}, H_{2}(A, \mathbb{Z})=\wedge^{2} H_{1}(A, \mathbb{Z}), H^{2,0}(A)=\wedge^{2} H^{1,0}(S) \\
\\
H^{1,1}(A)=H^{1,0}(S) \otimes H^{0,1}(S), H^{0,2}(A)=\wedge^{2} H^{0,1}(S)
\end{gathered}
$$

and

| $H_{1}(A, \mathbb{Z})$ | $q$ | $h^{2,0}(A)$ | $h^{1,1}(A)$ | $b_{2}(A)$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{Z}^{14}$ | 7 | 21 | 49 | 91 |

We have a map respecting Hodge decomposition

$$
\begin{gathered}
H^{2,0}(A) \oplus H^{1,1}(A) \oplus H^{0,2}(A) \\
\downarrow \\
\downarrow \\
H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)
\end{gathered}
$$

which is an equivariant map of $\operatorname{Aut}(S)$-modules. By Corollary 4 the kernel of that map is 28 dimensional ; it is moreover a $\operatorname{Aut}(S)$-module.

According to the Atlas tables [11, the group $U_{3}(3)$ has 14 irreducible representations $\chi_{i}, 1 \leq i \leq 14$ of respective dimension $1,6,7,7,7,14,21,21,21,27,28,28,32,32$.

The irreducible representations of $\operatorname{Aut}(S)=U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}$ are the $\chi_{i}^{t}, i=1, \ldots, 14, t=$ $0,1,2$ where $(g, s) \in U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}$ acts on the same space as $\chi_{i}$ with action $(g, s) \cdot v=\alpha^{s} g(v)$ with $\alpha=e^{2 i \pi / 3}$ a primitive third root of unity.

Theorem 5. The image of $S$ by the Albanese map $\vartheta: S \rightarrow A$ is 2-dimensional.
The map $H^{1,1}(A) \rightarrow H^{1,1}(S)$ is surjective, with a 14 dimensional kernel isomorphic to $\chi_{6}^{0}$ as an $\operatorname{Aut}(S)$-module. We have $H^{1}(S, \mathbb{Z})=\chi_{3}^{1} \oplus \chi_{3}^{2}$ and $H^{1,1}(S)=\chi_{1}^{0} \oplus \chi_{3}^{0} \oplus \chi_{10}^{0}$, as $\operatorname{Aut}(S)$ modules.
The kernel of the natural map $\wedge^{2} H^{0}\left(S, \Omega_{S}\right) \rightarrow H^{0}\left(S, K_{S}\right)$ is 7-dimensional, isomorphic to $\chi_{3}^{0}$ as a $\operatorname{Aut}(S)$-module.
The surface $S$ has maximal Picard number.
The Albanese variety $A$ of $S$ is isomorphic to $(\mathbb{C} / \mathbb{Z}[\alpha])^{7}$, for $\alpha=e^{2 i \pi / 3}$.

Since $A$ is CM, it follows that $S$ is Albanese standard, that is, the class of its image inside its Albanese variety A sits in the subring of $H^{*}(A, \mathbb{Q})$ generated by the divisor classes. That contrasts with the above mentioned Schoen surfaces, see [10].

Proof. Suppose that the image of $S$ in $A$ is 1-dimensional. Then there exists a smooth curve $C$ of genus 7 with a fibration $f: S \rightarrow C$ and the map $\wedge^{2} H^{0}\left(S, \Omega_{S}\right) \rightarrow H^{0}\left(S, K_{S}\right)$ is the 0 map and the kernel of $\wedge^{2} H^{1}(S, \mathbb{C}) \rightarrow H^{2}(S, \mathbb{C})$ is at least 42 dimensional, which is impossible. Thus the image of $S$ by the Albanese map $\vartheta: S \rightarrow A$ is 2-dimensional.

According to the Atlas character table [11], the possibilities for the $U_{3}(3)$-module $H_{1}(S, \mathbb{Z})=$ $H_{1}(A, \mathbb{Z})=\mathbb{Z}^{14}$ are:

$$
\chi_{3}^{\oplus 2}, \mathcal{R}_{\mathbb{Z}}\left(\chi_{4}\right)=\mathcal{R}_{\mathbb{Z}}\left(\chi_{5}\right)=\chi_{4} \oplus \chi_{5}, \chi_{4}^{\oplus 2}, \chi_{5}^{\oplus 2} \text { or } \chi_{6}
$$

where $\mathcal{R}_{\mathbb{Z}}\left(\chi_{j}\right)$ is the restriction to $\mathbb{Z}$ of the 7 -dimensional complex representation $\chi_{j}$ defined over $\mathbb{Z}[i]$. It cannot be $\chi_{4}^{\oplus 2}$ nor $\chi_{5}^{\oplus 2}$ because these are not is not defined over $\mathbb{Z}$ (some traces of elements are in $\mathbb{Z}[i] \backslash \mathbb{Z})$. We cannot have $H^{1}(S, \mathbb{Z})=\chi_{6}$ since $\chi_{6}$ remains irreducible, but $H^{1}(S, \mathbb{Z}) \otimes \mathbb{C}=H^{1,0} \oplus H^{0,1}$ is a Hodge decomposition on which the representation of $U_{3}(3)$ splits.

By duality, the kernel of $H^{2,0}(A) \rightarrow H^{2,0}(S)$ has same dimension $d$ as the kernel of $H^{0,2}(A) \rightarrow H^{0,2}(S)$. Let $k$ be the dimension of the kernel of the $U_{3}(3)$-equivariant map $H^{1,1}(A) \rightarrow H^{1,1}(S)$. We have $28=k+2 d$, moreover since $h^{1,1}(S)=35$ and $h^{1,1}(A)=49$, we get $28 \geq k \geq 14$.

Let us suppose that $H^{1}(S, \mathbb{Z})=\chi_{4} \oplus \chi_{5}$. Then the representation $H^{1,1}(A)$ equals to $\chi_{4} \otimes \chi_{5}=\chi_{1}+\chi_{7}+\chi_{10}$ (of dimension $1+21+27$ ). An Abelian variety on which a finite group $G$ acts possesses a $G$-invariant polarization (for example $\sum_{g \in G} g^{*} L$, where $L$ is some polarization). Therefore the one dimensional Aut $(S)$-invariant space of $H^{1,1}(A)$ is generated by the class of an ample divisor and the natural map $\vartheta^{*}: H^{1,1}(A) \rightarrow H^{1,1}(S)$ is injective on that subspace. Therefore the map $\vartheta^{*}$ has a kernel of dimension $k=21,27$ or 48 . This is impossible because $k+2 d$ equals 28 .

Hence, we have $H^{1}(S, \mathbb{Z})=\chi_{3}^{\oplus 2}$ and moreover

$$
H^{2,0}(A)=\wedge^{2} \chi_{3}=\chi_{3} \oplus \chi_{6}
$$

(the dimensions are $21=7+14$ ) and

$$
H^{1,1}(A)=\chi_{3}^{\otimes 2}=\chi_{1} \oplus \chi_{3} \oplus \chi_{6} \oplus \chi_{10}
$$

$(49=1+7+14+27)$. By checking the possibilities, we obtain $k=14, H^{1,1}(S)=\chi_{1} \oplus \chi_{3} \oplus \chi_{10}$, and the map $H^{1,1}(A) \rightarrow H^{1,1}(S)$ is surjective. The kernel of the map $H^{2,0}(A) \rightarrow H^{2,0}(S)$ is isomorphic to $\chi_{3}$, of dimension 7 , the action of $U_{3}(3)$ on $H^{2,0}(S)$ is then $H^{2,0}(S)=\chi_{6} \oplus \chi$, where $\chi$ is a 13 dimensional representation.

Let $\sigma \in \operatorname{Aut}(S)=U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}$ be the order 3 automorphism commuting with every other element. It corresponds to an element $\sigma^{\prime} \in \Lambda$ normalizing $\Pi$ in $\Lambda$ and such that the group $\Pi^{\prime}$ generated by $\Pi$ and $\sigma^{\prime}$ contains $\Pi$ with index 3 . Using MAGMA, one find that we can choose $\sigma^{\prime}=j^{4}$, where $j$ is the order 12 element described in the proof of Theorem 3]
The quotient surface $S / \sigma$ of $S$ by $\sigma$ is equal to $\mathbb{B}_{2} / \Pi^{\prime}$. The fundamental group of $S^{\prime}$ is $\Pi^{\prime} / \Pi_{\text {tors }}^{\prime}$ where $\Pi_{\text {tors }}^{\prime}$ is the subgroup of $\Pi^{\prime}$ generated by torsion elements. Using MAGMA, one find that $\Pi^{\prime}$ has a set of 8 generators with 7 of them which are torsion elements. Using these elements, we readily compute that $\Pi^{\prime} / \Pi_{\text {tors }}^{\prime}$ is trivial. Therefore the space of one-forms on $S$ that are invariant by $\sigma$ is 0 . Using the symmetries of $U_{3}(3)$, one see that $\sigma$ acts on the tangent space $H^{0}\left(S, \Omega_{S}\right)^{*}$ as the multiplication by $\alpha$ or $\alpha^{2}$. After possible permutation of $\sigma$ and $\sigma^{2}$, we can suppose it is $\alpha$.

We see that the representation of $\operatorname{Aut}(S)$ on $H_{1}(S, \mathbb{Z})$ is $\chi_{3}^{1} \oplus \chi_{3}^{2}$. The lattice $H_{1}(S, \mathbb{Z}) \subset$ $H^{0}\left(S, \Omega_{S}\right)^{*}$ is moreover a $\mathbb{Z}[\alpha]$-module. The ring $\mathbb{Z}[\alpha]$ is a principal ideal domain, therefore $H_{1}(S, \mathbb{Z})=\mathbb{Z}[\alpha]^{7}$ (for the choice of a certain basis) and $A$ is isomorphic to $(\mathbb{C} / \mathbb{Z}[\alpha])^{7}$.

Therefore $A$ has maximal Picard number and all the classes of $H^{1,1}(A)$ are algebraic. These classes remain of course algebraic under the map $H^{1,1}(A) \rightarrow H^{1,1}(S)$, which is surjective. Thus $S$ is a surface with maximal Picard number.

## 4. Lagrangian surfaces and Stover surface

Let $B$ be an Abelian fourfold and let $p: S \rightarrow B$ be a map such that $p(S)$ generates $B$. Let us recall that $S$ is Lagrangian with respect to $p$ if there exists a basis $w_{1}, \ldots, w_{4}$ of $p^{*} H^{0}\left(B, \Omega_{B}\right)$ such that the rank 2 vector $w=w_{1} \wedge w_{2}+w_{3} \wedge w_{4}$ is in the kernel of the natural map $\phi^{2,0}: \wedge^{2} H^{0}\left(S, \Omega_{S}\right) \rightarrow H^{0}\left(S, K_{S}\right)$. Let us now prove
Theorem 6. The 7 dimensional space $\operatorname{Ker}\left(\phi^{2,0}\right)$ contains no decomposable elements. The algebraic set of rank 2 vectors in $\operatorname{Ker}\left(\phi^{2,0}\right)$ is a quadric $\tilde{Q} \subset \operatorname{Ker}\left(\phi^{2,0}\right)$.
There exists an infinite number (up to isogeny) of maps $p: S \rightarrow B$ where $B$ is an Abelian fourfold such that $S$ is Lagrangian with respect to $p$.
There exists an infinite number (up to isogeny) of maps $p: S \rightarrow B$ where $B$ is an Abelian fourfold such that

$$
\tilde{Q} \cap p^{*} H^{0}\left(B, \wedge^{2} \Omega_{B}\right)=\{0\}
$$

and for some of them we even have $\operatorname{Ker}\left(\phi^{2,0}\right) \cap p^{*} H^{0}\left(B, \wedge^{2} \Omega_{B}\right)=\{0\}$.
The generic rank 2 element $w$ in $\tilde{Q} \subset \operatorname{Ker}\left(\phi^{2,0}\right)$ does not correspond to any morphism to an Abelian fourfold.

Proof. We proved in Theorem 5 that

$$
H^{2,0}(A)=\wedge^{2} \chi_{3}=\chi_{3} \oplus \chi_{6}
$$

and the kernel of $\phi^{2,0}: H^{2,0}(A) \rightarrow H^{2,0}(S)$ is the 7-dimensional subspace with representation $\chi_{3}$. In a basis $\gamma=\left(e_{1}, \ldots, e_{7}\right)$ of $\chi_{3}=H^{0}\left(S, \Omega_{S}\right)=H^{1,0}(S)$, the two following matrices $A, B$ are generators of the group $U_{3}(3)$ :

$$
A=\left(\begin{array}{ccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right), B=\left(\begin{array}{ccccccc}
0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right) .
$$

Using the basis $\beta=\left(e_{i j}\right)_{1 \leq i<j \leq 7}$ of $\wedge^{2} \chi_{3}\left(e_{i j}=e_{i} \wedge e_{j}\right)$ with order $e_{i j} \leq e_{s t}$ if $i<s$ or $i=s$ and $j \leq t$, one computes that the subspace $\operatorname{Ker}\left(\phi^{2,0}\right)=\chi_{3} \subset \wedge^{2} \chi_{3}$ is generated by the columns of the matrix $M \in M_{21,7}$, where ${ }^{t} M=\left(N, 2 I_{7}\right)$, for

$$
N=\left(\begin{array}{cccccccccccccc}
0 & 0 & 2 & -2 & -2 & -2 & 0 & 2 & 2 & -2 & 2 & 2 & 2 & 2 \\
-1 & 0 & 0 & 2 & 4 & 0 & 1 & -3 & -3 & 1 & -3 & -4 & -2 & -4 \\
0 & -2 & 0 & -2 & -2 & 0 & -2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 \\
-1 & -2 & 2 & 0 & -2 & 0 & -1 & 1 & 3 & 1 & 1 & 0 & 0 & 2 \\
-1 & 1 & -1 & 3 & 1 & 3 & 0 & -4 & -2 & 2 & 0 & -4 & -2 & -2 \\
0 & 3 & -3 & 1 & -1 & 1 & 1 & -3 & -3 & -1 & 1 & 0 & -2 & -2 \\
1 & 1 & 1 & 3 & 3 & 1 & 2 & -2 & 0 & 0 & 0 & -2 & 0 & -2
\end{array}\right) \in M_{7,14}
$$

and $I_{7}$ the $7 \times 7$ identity matrix. Knowing that, we obtain the ideal $I_{V}$ of the algebraic set $V$ of couples $\left(w_{1}, w_{2}\right) \in \chi_{3} \oplus \chi_{3}$ such that $w_{1} \wedge w_{2} \in \operatorname{Ker}\left(\phi^{2,0}\right) \subset \wedge^{2} \chi_{3}$. That ideal is generated by 14 homogeneous quadratic polynomials in the variables $x_{1}, \ldots, x_{14}$. Let $W$ be
the algebraic set of couples $\left(w_{1}, w_{2}\right) \in \chi_{3} \oplus \chi_{3}$ such that $w_{1} \wedge w_{2}=0 \in \wedge^{2} \chi_{3}$. The ideal $I_{W}$ of $W$ is generated by the 2 by 2 minors of the matrix

$$
L=\left(\begin{array}{ccc}
x_{1} & \ldots & x_{7} \\
x_{8} & \ldots & x_{14}
\end{array}\right)
$$

Since $W \subset V$, we have $\operatorname{Rad}\left(I_{V}\right) \subset \operatorname{Rad}\left(I_{W}\right)$ where $\operatorname{Rad}(I)$ is the radical of an ideal $I$. On the other hand, using Maple, one can check that the 21 minors of $L$ are in $\operatorname{Rad}\left(I_{V}\right)$, hence $\operatorname{Rad}\left(I_{W}\right) \subset \operatorname{Rad}\left(I_{V}\right)$, thus $V=W$.
We therefore conclude that the kernel of $\phi^{2,0}$ contains no decomposable elements.
A 2 -vector $w$ over a characteristic 0 field can be expressed uniquely as $w=\sum_{i, j} a_{i j} e_{i} \wedge e_{j}$ where $a_{i j}=-a_{j i}$. The rank of the vector $w$ is half the rank of the (skew-symmetric) coefficient matrix $A_{w}:=\left(a_{i j}\right)_{1 \leq i, j \leq 7}$ of $w$ [6, Thm 1.7\& Remark p. 13]. Thus the 2-vector $w=a_{1} v_{1}+\cdots+a_{7} v_{7}$ in $\operatorname{Ker}\left(\phi^{2,0}\right)$ (where the $v_{i}, i=1 . .7$ are the vectors corresponding to the columns of the matrix $M$ ) is a rank 2 vector if and only if the $496 \times 6$ minors of the matrix $A_{w}$ are 0 . The radical of the ideal generated by these minors is principal, generated by a homogeneous quadric in $a_{1}, \ldots, a_{7}$ whose associated symmetric matrix is

$$
Q=\left(\begin{array}{ccccccc}
7 & 3 & 3 & 1 & -3 & -3 & -5 \\
3 & 7 & 3 & 3 & 1 & -3 & -3 \\
3 & 3 & 7 & 3 & 3 & 1 & -3 \\
1 & 3 & 3 & 7 & 3 & 3 & 1 \\
-3 & 1 & 3 & 3 & 7 & 3 & 3 \\
-3 & -3 & 1 & 3 & 3 & 7 & 3 \\
-5 & -3 & -3 & 1 & 3 & 3 & 7
\end{array}\right)
$$

Therefore $w \in \operatorname{Ker}\left(\phi^{2,0}\right)$ has rank 2 if and only if $\left(a_{1}, \ldots, a_{7}\right) Q^{t}\left(a_{1}, \ldots, a_{7}\right)=0$.
The point $(10+8 \alpha,-7,0,0,7,0,0)$ lies on the associated smooth quadric $\tilde{Q}$, therefore $\tilde{Q}(\mathbb{Q}[\alpha])$ is infinite. Let be $w$ be a 2 -vector in $\tilde{Q}(\mathbb{Q}[\alpha])$. The decomposable vector $\wedge^{2} w \neq 0$ has coordinates in $\mathbb{Q}[\alpha]$ in the basis $\left(e_{i 1} \wedge . \cdots \wedge e_{i 4}\right)$ of $\wedge^{4} H^{0}\left(S, \Omega_{S}\right)$. The corresponding 4dimensional vector space $W$ is therefore generated by 4 vectors $w_{1}, \ldots, w_{4}$ with coordinates over $\mathbb{Q}[\alpha]$ in the basis $\gamma=\left(e_{1}, \ldots, e_{7}\right)$ of $H^{0}\left(S, \Omega_{S}\right)$.
One computes that the image of $\mathbb{Q}[\alpha]\left[U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}\right]$ in $M_{7}(\mathbb{Q}[\alpha])$ is 49 dimensional over $\mathbb{Q}[\alpha]$, thus

$$
\mathbb{Q}\left[U_{3}(3) \times \mathbb{Z} / 3 \mathbb{Z}\right]=M_{7}(\mathbb{Q}(\alpha))(=\operatorname{End}(A) \otimes \mathbb{Q})
$$

in the basis $\gamma,\left(H_{1}(S, \mathbb{Q}[\alpha])\right.$ is the $\mathbb{Q}[\alpha]$-vector space generated by $\left.e_{1}, \ldots, e_{k}\right)$ and therefore there exists a morphism $p: S \rightarrow E^{4}=B$ (where $\left.E=\mathbb{C} / \mathbb{Z}[\alpha]\right)$ such that $W=p^{*} H^{0}\left(B, \Omega_{B}\right)$. By hypothesis the image $p(S)$ generates $B$. By construction

$$
\wedge^{2} p^{*} H^{0}\left(B, \Omega_{B}\right) \cap \operatorname{Ker}\left(\phi^{2,0}\right)
$$

is at least one dimensional since it contains $w$, and therefore $S$ is Lagrangian for $p$.
A contrario, the trace of an order 2 automorphism $\sigma \in \operatorname{Aut}(S) \subset \operatorname{Aut}(A)$ acting on the tangent space of $A$ at 0 equals to -1 , therefore the image $B^{\prime}$ of the endomorphism $p: \sigma-1_{A}$, where $1_{A}$ is the identity of $A$ is an Abelian fourfold. Using Maple, one computes that

$$
\wedge^{2} p^{*} H^{0}\left(B, \Omega_{B}\right) \cap \operatorname{Ker}(f)=\{0\}
$$

Let $\vartheta: S \rightarrow A$ be the Albanese map of $S$, and let $q: A \rightarrow A$ be an endomorphism with a 4 dimensional image and a representation in $M_{7}(\mathbb{Q}) \subset M_{7}(\mathbb{Q}(\alpha))$ in the basis $\gamma$. Since the matrix $Q$ is positive definite, we have

$$
\wedge^{2} p^{*} H^{0}\left(B, \Omega_{B}\right) \cap \tilde{Q}=\{0\}
$$

where $p$ is the map $p=q \circ \vartheta: S \rightarrow B$. Therefore $S$ is not Lagrangian with respect to $p$.

Remark 7. Let $X$ be a surface and let $\phi^{2,0}: \wedge^{2} H^{0}\left(X, \Omega_{X}\right) \rightarrow H^{0}\left(X, K_{X}\right)$ be the natural map. Let be $d=\operatorname{dim} \operatorname{Ker}\left(\phi^{2,0}\right)$ and $q=\operatorname{dim} H^{0}\left(X, \Omega_{X}\right)$. In the proof of Theorem 6, we saw that the set of rank $k$ vectors in $\operatorname{Ker}\left(\phi^{2,0}\right)$ is a determinantal variety: the intersection of minors of size $\geq 2 k+1$ of some anti-symmetric matrix of size $q \times q$ with linear entries in $d$ variables. It seems to the authors quite remarkable that for Stover's surface the set of rank 2 vectors (obtained as the zero set of $496 \times 6$ minors of a size $q=7$ matrix) is an hypersurface in $\operatorname{Ker}\left(\phi^{2,0}\right)$. That hypersurface is the only $U_{3}(3)$-invariant quadric of $U_{3}(3)$ acting on $\operatorname{Ker}\left(\phi^{2,0}\right)$.

## References

[1] Barja M.A., Naranjo J.C., Pirola G.P., On the topological index of irregular surfaces, J. Algebraic Geom. 16 (2007), no. 3, 435-458.
[2] Bastianelli F., Pirola G.P., Stoppino L., Galois closure and Lagrangian varieties, Advances in Math., 225 (2010), 3463-3501.
[3] Beauville A., Some surfaces with maximal Picard number, Journal de l'École Polytechnique 1 (2014), 101-116.
[4] Beauville A., On the second lower quotient of the fundamental group of a compact Kahler manifold, ArXiv, to appear in "Algebraic and Complex Geometry" in honor of K. Hulek (Springer).
[5] Bogomolov F., Tschinkel Y., Lagrangian subvarieties of abelian fourfolds, Asian J. Math. 4 (2000), no. 1, 19-36.
[6] Bryant R.L., Chern S.S., Gardner R.B., Goldschmidt H.H., Griffiths P.A. Exterior differential systems, (Mathematical Sciences Research Institute publications, 18), Springer-Verlag (1991).
[7] Cartwright D., Steger T., Enumeration of the 50 fake projective planes, Comptes Rendus Mathematique 348 (1): 11-13, doi:10.1016/j.crma.2009.11.016
[8] Campana, F. Remarques sur les groupes de Kähler nilpotents, Ann. Sci. École Norm. Sup. (4) 28 (1995), no. 3, 307-316.
[9] Causin A, Pirola G.P., Hermitian matrices and cohomology of Kähler varieties, Manuscripta Math. 121 (2006), no. 2, 157-168.
[10] Ciliberto C., Mendes Lopes M., Roulleau X., On Schoen surfaces, to appear in Comment. Math. Helvet.
[11] Conway, J., Curtis R., Parker R., Norton S., Wilson R., Atlas of finite groups. Oxford University Press, 1985.
[12] Mostow G.D., On a remarkable class of polyhedra in complex hyperbolic space, Pacific J. Math., 86(1) 1980, 171-276.
[13] Schoen C., Albanese standard and Albanese Exotic varieties, J. of London Math. Soc. (2) 74 (2006) 304-320.
[14] Stover M., Hurwitz ball quotients, Math. Z. 277 (2014), no. 1-2, 75-91
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