### ON THE COHOMOLOGY OF STOVER SURFACE

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ABSTRACT. We study a surface discovered by Stover which is the surface with minimal Euler number and maximal automorphism group among smooth arithmetic ball quotient surfaces. We study the natural map  $\wedge^2 H^1(S,\mathbb{C}) \to H^2(S,\mathbb{C})$  and we discuss the problem related to the so-called Lagrangian surfaces. We obtain that this surface S has maximal Picard number and has no higher genus fibrations. We compute that its Albanese variety S is isomorphic to  $(\mathbb{C}/\mathbb{Z}[\alpha])^7$ , for S is isomorphic to S is isomorphic.

# 1. Introduction

By the recent work of M. Stover [14], the number of automorphisms of a smooth compact arithmetic ball quotient surface  $X = \Gamma \backslash \mathbb{B}_2$  is bounded by  $288 \cdot e(X)$ , where e(X) denotes the topological Euler number of X.

Furthermore, Stover characterizes the arithmetic ball quotient surfaces X whose automorphism groups attain this bound, which by analogy with Hurwitz curves, he calls  $Hurwitz\ ball\ quotients$ ; all such surfaces are finite Galois coverings of the Deligne-Mostow orbifold  $\Lambda\backslash\mathbb{B}_2$  corresponding to the quintuple (2/12,2/12,2/12,7/12,11/12) (see [12, 14]).

Stover constructs also a Hurwitz ball quotient S with Euler number e(S)=63 and automorphism group  $\operatorname{Aut}(S)$  isomorphic to  $U_3(3)\times\mathbb{Z}/3\mathbb{Z}$ , of order  $18144=2^53^47$ . He shows that S is the unique Hurwitz ball quotient with Euler number e=63, and moreover that e=63 is the minimal possible value for the Euler number of a Hurwitz ball quotient. Having this property the surface S can be seen as the 2-dimensional analog of the Klein's quartic which is the unique curve uniformized by the ball  $\mathbb{B}_1$  with minimal genus and maximal possible automorphism group.

Our aim is to study more closely the cohomology of this particular surface S, which we will call *Stover surface* in the following. This surface S has the following numerical invariants (see [14]):

e(S)	$H_1(S,\mathbb{Z})$	q	$p_g = h^{2,0}$	$h^{1,1}$	$b_2(S)$
63	$\mathbb{Z}^{14}$	7	27	35	89

Let V be a vector space. Let us recall that a 2-vector  $w \in \wedge^2 V$  has  $rank\ 1$  or is decomposable if there are vectors  $w_1, w_2 \in V$  with  $w = w_1 \wedge w_2$ . A vector  $w \in \wedge^2 V$  has  $rank\ 2$  if there exist linearly independent vectors  $w_i \in V$ , i = 1, ..., 4 such that  $w = w_1 \wedge w_2 + w_3 \wedge w_4$ .

Let B be an Abelian fourfold and let  $p: S \to B$  be a map such that p(S) generates B. We say that S is Lagrangian with respect to p if there exists a basis  $w_1, \ldots, w_4$  of  $p^*H^0(B, \Omega_B)$  such that the rank 2 vector  $w = w_1 \wedge w_2 + w_3 \wedge w_4$  is in the kernel of the natural map  $\phi^{2,0}: \wedge^2 H^0(S,\Omega_S) \to H^0(S,K_S)$ .

**Theorem 1.** The surface S has maximal Picard number. The natural map

$$\phi^{1,1}: H^0(S,\Omega_S) \otimes H^1(S,\mathcal{O}_S) \to H^1(S,\Omega_S)$$

is surjective with a 14-dimensional kernel. The kernel of the map

$$\phi^{2,0}: \wedge^2 H^0(S,\Omega_S) \to H^0(S,K_S)$$

is 7-dimensional and contains no decomposable elements. The set of rank 2 vectors in  $Ker(\phi^{2,0})$  is a quadric hypersurface.

There exists an infinite number (up to isogeny) of maps  $p: S \to B$  (where B is an Abelian fourfold) such that S is Lagrangian with respect to p.

The Albanese variety of S is isomorphic to  $(\mathbb{C}/\mathbb{Z}[\alpha])^7$ , for  $\alpha = e^{2i\pi/3}$ .

By the Castelnuovo - De Franchis Theorem, the fact that there are no decomposable elements in  $\wedge^2 H^0(S,\Omega_S)$  means that S has no fibration  $f:S\to C$  onto a curve of genus g>1. Moreover Theorem 1 implies that S has the remarkable feature that both maps

$$\phi^{2,0}: \wedge^2 H^{1,0}(S) \to H^{2,0}(S)$$
  
$$\phi^{1,1}: H^{1,0}(S) \otimes H^{0,1}(S) \to H^{1,1}(S)$$

have a non-trivial kernel. With Schoen surfaces (see [10, Remark 2.6]), this is the second example of surfaces enjoying such properties. For more on this subject, see e.g. [1, 5, 2, 8, 9].

We obtain these results using Sullivan's theory on the second lower quotient of the fundamental group  $\pi_1(S)$  of S (see [4]).

For the motivation and a historic account of surfaces with maximal Picard number we refer to [3].

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#### 2. The Second lower central quotient of the fundamental group of S

Let  $\Pi := \pi_1(X)$  be the fundamental group of a manifold X. The group  $H_1(X,\mathbb{Z})$  is the abelianization of  $\Pi$ :  $H_1(X,\mathbb{Z}) = \Pi/\Delta$  where  $\Delta := [\Pi,\Pi]$  is the derived subgroup of  $\Pi$ , that is, the subgroup generated by all elements  $[h,g] = g^{-1}h^{-1}gh$ ,  $h,g \in \Pi$ .

The second group in the lower central series  $[\Delta, \Pi]$  is the group generated by commutators [h, g], with  $h \in \Delta$ ,  $g \in \Pi$ . It is a normal subgroup of the commutator group  $\Delta$ . According to [4], we have the following results:

**Proposition 2.** (Sullivan) Let X be a compact connected Kähler manifold. There exists an exact sequence

$$0 \to Hom(\Delta/[\Delta,\Pi],\mathbb{R}) \to \wedge^2 H^1(X,\mathbb{R}) \to H^2(X,\mathbb{R}).$$

(Beauville) Suppose  $H_1(X,\mathbb{Z})$  is torsion free. Then the group  $\Delta/[\Delta,\Pi]$  is canonically isomorphic to the cokernel of the map

$$\mu: H_2(X,\mathbb{Z}) \to Alt^2(H^1(X,\mathbb{Z}))$$
 given by  $\mu(\sigma)(a,b) = \sigma \cap (a \wedge b)$ ,

where  $Alt^2(H^1(X,\mathbb{Z}))$  is the group of skew-symmetric integral bilinear forms on  $H^1(X,\mathbb{Z})$ .

In the case of the Stover surface, computer calculations give us the following result:

**Theorem 3.** Let  $\Pi = \pi_1(S)$  be the fundamental group of the Stover surface and  $\Delta = [\Pi, \Pi]$ . The group  $\Delta/[\Delta, \Pi]$  is isomorphic to  $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$ .

*Proof.* By the construction of S [14], the fundamental group  $\Pi$  is isomorphic to the kernel  $ker(\varphi)$  of the unique epimorphism  $\varphi: \Lambda \longrightarrow G$  from the Deligne-Mostow lattice  $\Lambda$  corresponding to the quintuple (2/12, 2/12, 2/12, 7/12, 11/12) onto the finite group  $G = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$ . The lattice  $\Lambda$  is described by Mostow in [12] as a complex reflection group, and by generators and relation by Cartwright and Steger in [7]. This lattice has presentation

$$\Lambda = \langle j, u, v, b | u^4, v^8, [u, j], [v, j], j^{-3}v^2, uvuv^{-1}uv^{-1}, (bj)^2(vu^2)^{-1}, [b, vu^2], b^3, (bvu^3)^3 \rangle.$$

MAGMA command LowIndexSubgroups is used to identify the unique subgroup  $\Gamma \triangleleft \Lambda$  of index 3, which is  $\Gamma = \langle u, jb, bj \rangle$ . Using the primitive permutation representation of  $U_3(3)$  of degree 28, MAGMA is able to identify an homomorphism  $\varphi$  from  $\Gamma$  onto  $U_3(3)$  induced from the assignment

$$\begin{aligned} u \mapsto & (3,8,23,20)(4,24,6,12)(7,9,14,22)(10,19,11,13)(15,16,21,18)(17,26,27,25) \\ jb \mapsto & (1,9,20,12,19,23,6,16)(2,27,14,17,13,26,15,25)(3,24)(4,5,10,21,7,11,28,8) \\ bj \mapsto & (1,13,20,15,19,2,6,14)(4,9,10,12,7,23,28,16)(5,27,21,17,11,26,8,25)(22,24). \end{aligned}$$

This homomorphism extends to an homomorphism  $\varphi$  from  $\Lambda$  onto G such that  $\Pi = ker(\varphi)$  is a torsion-free normal subgroup in  $\Lambda$ , it is the fundamental group of S (see [14]). Let be  $\Delta = [\Pi, \Pi]$  and  $\Delta_2 = [\Delta, \Pi]$ . It is easy to check that that  $\Delta_2$  is distinguished into  $\Pi$ . The image of  $\Delta$  under the quotient map  $\Pi \longrightarrow \Pi/\Delta_2$  is  $\Delta/\Delta_2$ , but we observe that it is also equal the commutator subgroup  $[\Pi/\Delta_2, \Pi/\Delta_2]$ , and therefore, the computation of  $\Delta/\Delta_2$  is reduced to the one of the derived group  $[\Pi/\Delta_2, \Pi/\Delta_2]$ .

The MAGMA command g:=Rewrite(G,g) is used to have generators and relations of both subgroups  $\Gamma < \Lambda$  and  $\Pi < \Gamma$ . The command NilpotentQuotient(.,2) applied to  $\Pi$  describes  $\Pi/\Delta_2$  in terms of a polycyclic presentation. The derived subgroup  $[\Pi/\Delta_2, \Pi/\Delta_2]$  is obtained with DerivedGroup(.) applied to  $\Pi/\Delta_2$ . Finally, applying the MAGMA function AQInvariants to  $[\Pi/\Delta_2, \Pi/\Delta_2]$ , MAGMA computes that the structure of  $\Delta/\Delta_2$  is  $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^{28}$ .

**Corollary 4.** The dimension of the kernel of  $\wedge^2 H^1(S,\mathbb{R}) \to H^2(S,\mathbb{R})$  is 28.

3. Computation of the map 
$$\wedge^2 H^1(S,\mathbb{C}) \to H^2(S,\mathbb{C})$$

Let A be the Albanese variety of the Stover surface S. The invariants are:

$$H_1(A,\mathbb{Z}) = H_1(S,\mathbb{Z}) = \mathbb{Z}^{14}, \ H_2(A,\mathbb{Z}) = \wedge^2 H_1(A,\mathbb{Z}), \ H^{2,0}(A) = \wedge^2 H^{1,0}(S)$$
  
 $H^{1,1}(A) = H^{1,0}(S) \otimes H^{0,1}(S), \ H^{0,2}(A) = \wedge^2 H^{0,1}(S),$ 

and

$H_1(A,\mathbb{Z})$	q	$h^{2,0}(A)$	$h^{1,1}(A)$	$b_2(A)$
$\mathbb{Z}^{14}$	7	21	49	91

We have a map respecting Hodge decomposition

$$H^{2,0}(A) \oplus H^{1,1}(A) \oplus H^{0,2}(A)$$
  
 $\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$   
 $H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$ 

which is an equivariant map of Aut(S)-modules. By Corollary 4, the kernel of that map is 28 dimensional; it is moreover a Aut(S)-module.

According to the Atlas tables [11], the group  $U_3(3)$  has 14 irreducible representations  $\chi_i$ ,  $1 \le i \le 14$  of respective dimension 1, 6, 7, 7, 7, 14, 21, 21, 27, 28, 28, 32, 32.

The irreducible representations of Aut $(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$  are the  $\chi_i^t$ , i = 1, ..., 14, t = 0, 1, 2 where  $(g, s) \in U_3(3) \times \mathbb{Z}/3\mathbb{Z}$  acts on the same space as  $\chi_i$  with action  $(g, s) \cdot v = \alpha^s g(v)$  with  $\alpha = e^{2i\pi/3}$  a primitive third root of unity.

**Theorem 5.** The image of S by the Albanese map  $\vartheta: S \to A$  is 2-dimensional.

The map  $H^{1,1}(A) \to H^{1,1}(S)$  is surjective, with a 14 dimensional kernel isomorphic to  $\chi_6^0$  as an  $\operatorname{Aut}(S)$ -module. We have  $H^1(S,\mathbb{Z}) = \chi_3^1 \oplus \chi_3^2$  and  $H^{1,1}(S) = \chi_1^0 \oplus \chi_3^0 \oplus \chi_{10}^0$ , as  $\operatorname{Aut}(S)$ -modules.

The kernel of the natural map  $\wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$  is 7-dimensional, isomorphic to  $\chi^0_3$  as a  $\operatorname{Aut}(S)$ -module.

The surface S has maximal Picard number.

The Albanese variety A of S is isomorphic to  $(\mathbb{C}/\mathbb{Z}[\alpha])^7$ , for  $\alpha = e^{2i\pi/3}$ .

Since A is CM, it follows that S is Albanese standard, that is, the class of its image inside its Albanese variety A sits in the subring of  $H^*(A, \mathbb{Q})$  generated by the divisor classes. That contrasts with the above mentioned Schoen surfaces, see [10].

*Proof.* Suppose that the image of S in A is 1-dimensional. Then there exists a smooth curve C of genus 7 with a fibration  $f: S \to C$  and the map  $\wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S)$  is the 0 map and the kernel of  $\wedge^2 H^1(S, \mathbb{C}) \to H^2(S, \mathbb{C})$  is at least 42 dimensional, which is impossible. Thus the image of S by the Albanese map  $\vartheta: S \to A$  is 2-dimensional.

According to the Atlas character table [11], the possibilities for the  $U_3(3)$ -module  $H_1(S, \mathbb{Z}) = H_1(A, \mathbb{Z}) = \mathbb{Z}^{14}$  are:

$$\chi_3^{\oplus 2}, \mathcal{R}_{\mathbb{Z}}(\chi_4) = \mathcal{R}_{\mathbb{Z}}(\chi_5) = \chi_4 \oplus \chi_5, \, \chi_4^{\oplus 2}, \, \chi_5^{\oplus 2} \text{ or } \chi_6$$

where  $\mathcal{R}_{\mathbb{Z}}(\chi_j)$  is the restriction to  $\mathbb{Z}$  of the 7-dimensional complex representation  $\chi_j$  defined over  $\mathbb{Z}[i]$ . It cannot be  $\chi_4^{\oplus 2}$  nor  $\chi_5^{\oplus 2}$  because these are not is not defined over  $\mathbb{Z}$  (some traces of elements are in  $\mathbb{Z}[i] \setminus \mathbb{Z}$ ). We cannot have  $H^1(S,\mathbb{Z}) = \chi_6$  since  $\chi_6$  remains irreducible, but  $H^1(S,\mathbb{Z}) \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$  is a Hodge decomposition on which the representation of  $U_3(3)$  splits.

By duality, the kernel of  $H^{2,0}(A) \to H^{2,0}(S)$  has same dimension d as the kernel of  $H^{0,2}(A) \to H^{0,2}(S)$ . Let k be the dimension of the kernel of the  $U_3(3)$ -equivariant map  $H^{1,1}(A) \to H^{1,1}(S)$ . We have 28 = k + 2d, moreover since  $h^{1,1}(S) = 35$  and  $h^{1,1}(A) = 49$ , we get 28 > k > 14.

Let us suppose that  $H^1(S,\mathbb{Z}) = \chi_4 \oplus \chi_5$ . Then the representation  $H^{1,1}(A)$  equals to  $\chi_4 \otimes \chi_5 = \chi_1 + \chi_7 + \chi_{10}$  (of dimension 1 + 21 + 27). An Abelian variety on which a finite group G acts possesses a G-invariant polarization (for example  $\sum_{g \in G} g^*L$ , where L is some polarization). Therefore the one dimensional  $\operatorname{Aut}(S)$ -invariant space of  $H^{1,1}(A)$  is generated by the class of an ample divisor and the natural map  $\vartheta^*: H^{1,1}(A) \to H^{1,1}(S)$  is injective on that subspace. Therefore the map  $\vartheta^*$  has a kernel of dimension k = 21, 27 or 48. This is impossible because k + 2d equals 28.

Hence, we have  $H^1(S,\mathbb{Z}) = \chi_3^{\oplus 2}$  and moreover

$$H^{2,0}(A) = \wedge^2 \chi_3 = \chi_3 \oplus \chi_6$$

(the dimensions are 21 = 7 + 14) and

$$H^{1,1}(A) = \chi_3^{\otimes 2} = \chi_1 \oplus \chi_3 \oplus \chi_6 \oplus \chi_{10}$$

(49 = 1 + 7 + 14 + 27). By checking the possibilities, we obtain k = 14,  $H^{1,1}(S) = \chi_1 \oplus \chi_3 \oplus \chi_{10}$ , and the map  $H^{1,1}(A) \to H^{1,1}(S)$  is surjective. The kernel of the map  $H^{2,0}(A) \to H^{2,0}(S)$  is isomorphic to  $\chi_3$ , of dimension 7, the action of  $U_3(3)$  on  $H^{2,0}(S)$  is then  $H^{2,0}(S) = \chi_6 \oplus \chi$ , where  $\chi$  is a 13 dimensional representation.

Let  $\sigma \in Aut(S) = U_3(3) \times \mathbb{Z}/3\mathbb{Z}$  be the order 3 automorphism commuting with every other element. It corresponds to an element  $\sigma' \in \Lambda$  normalizing  $\Pi$  in  $\Lambda$  and such that the group  $\Pi'$  generated by  $\Pi$  and  $\sigma'$  contains  $\Pi$  with index 3. Using MAGMA, one find that we can choose  $\sigma' = j^4$ , where j is the order 12 element described in the proof of Theorem 3.

The quotient surface  $S/\sigma$  of S by  $\sigma$  is equal to  $\mathbb{B}_2/\Pi'$ . The fundamental group of S' is  $\Pi'/\Pi'_{tors}$  where  $\Pi'_{tors}$  is the subgroup of  $\Pi'$  generated by torsion elements. Using MAGMA, one find that  $\Pi'$  has a set of 8 generators with 7 of them which are torsion elements. Using these elements, we readily compute that  $\Pi'/\Pi'_{tors}$  is trivial. Therefore the space of one-forms on S that are invariant by  $\sigma$  is 0. Using the symmetries of  $U_3(3)$ , one see that  $\sigma$  acts on the tangent space  $H^0(S,\Omega_S)^*$  as the multiplication by  $\alpha$  or  $\alpha^2$ . After possible permutation of  $\sigma$  and  $\sigma^2$ , we can suppose it is  $\alpha$ .

We see that the representation of Aut(S) on  $H_1(S,\mathbb{Z})$  is  $\chi_3^1 \oplus \chi_3^2$ . The lattice  $H_1(S,\mathbb{Z}) \subset H^0(S,\Omega_S)^*$  is moreover a  $\mathbb{Z}[\alpha]$ -module. The ring  $\mathbb{Z}[\alpha]$  is a principal ideal domain, therefore  $H_1(S,\mathbb{Z}) = \mathbb{Z}[\alpha]^7$  (for the choice of a certain basis) and A is isomorphic to  $(\mathbb{C}/\mathbb{Z}[\alpha])^7$ .

Therefore A has maximal Picard number and all the classes of  $H^{1,1}(A)$  are algebraic. These classes remain of course algebraic under the map  $H^{1,1}(A) \to H^{1,1}(S)$ , which is surjective. Thus S is a surface with maximal Picard number.

# 4. Lagrangian surfaces and Stover surface

Let B be an Abelian fourfold and let  $p: S \to B$  be a map such that p(S) generates B. Let us recall that S is Lagrangian with respect to p if there exists a basis  $w_1, \ldots, w_4$  of  $p^*H^0(B,\Omega_B)$  such that the rank 2 vector  $w = w_1 \wedge w_2 + w_3 \wedge w_4$  is in the kernel of the natural map  $\phi^{2,0}: \wedge^2 H^0(S,\Omega_S) \to H^0(S,K_S)$ . Let us now prove

**Theorem 6.** The 7 dimensional space  $Ker(\phi^{2,0})$  contains no decomposable elements. The algebraic set of rank 2 vectors in  $Ker(\phi^{2,0})$  is a quadric  $\tilde{Q} \subset Ker(\phi^{2,0})$ .

There exists an infinite number (up to isogeny) of maps  $p: S \to B$  where B is an Abelian fourfold such that S is Lagrangian with respect to p.

There exists an infinite number (up to isogeny) of maps  $p: S \to B$  where B is an Abelian fourfold such that

$$\tilde{Q} \cap p^* H^0(B, \wedge^2 \Omega_B) = \{0\},\$$

and for some of them we even have  $Ker(\phi^{2,0}) \cap p^*H^0(B, \wedge^2\Omega_B) = \{0\}.$ 

The generic rank 2 element w in  $\tilde{Q} \subset Ker(\phi^{2,0})$  does not correspond to any morphism to an Abelian fourfold.

Proof. We proved in Theorem 5 that

$$H^{2,0}(A) = \wedge^2 \chi_3 = \chi_3 \oplus \chi_6$$

and the kernel of  $\phi^{2,0}: H^{2,0}(A) \to H^{2,0}(S)$  is the 7-dimensional subspace with representation  $\chi_3$ . In a basis  $\gamma = (e_1, \ldots, e_7)$  of  $\chi_3 = H^0(S, \Omega_S) = H^{1,0}(S)$ , the two following matrices A, B are generators of the group  $U_3(3)$ :

Using the basis  $\beta = (e_{ij})_{1 \leq i < j \leq 7}$  of  $\wedge^2 \chi_3$   $(e_{ij} = e_i \wedge e_j)$  with order  $e_{ij} \leq e_{st}$  if i < s or i = s and  $j \leq t$ , one computes that the subspace  $Ker(\phi^{2,0}) = \chi_3 \subset \wedge^2 \chi_3$  is generated by the columns of the matrix  $M \in M_{21,7}$ , where  ${}^t M = (N, 2I_7)$ , for

and  $I_7$  the  $7 \times 7$  identity matrix. Knowing that, we obtain the ideal  $I_V$  of the algebraic set V of couples  $(w_1, w_2) \in \chi_3 \oplus \chi_3$  such that  $w_1 \wedge w_2 \in Ker(\phi^{2,0}) \subset \wedge^2 \chi_3$ . That ideal is generated by 14 homogeneous quadratic polynomials in the variables  $x_1, \ldots, x_{14}$ . Let W be

the algebraic set of couples  $(w_1, w_2) \in \chi_3 \oplus \chi_3$  such that  $w_1 \wedge w_2 = 0 \in \wedge^2 \chi_3$ . The ideal  $I_W$  of W is generated by the 2 by 2 minors of the matrix

$$L = \left(\begin{array}{ccc} x_1 & \dots & x_7 \\ x_8 & \dots & x_{14} \end{array}\right).$$

Since  $W \subset V$ , we have  $Rad(I_V) \subset Rad(I_W)$  where Rad(I) is the radical of an ideal I. On the other hand, using Maple, one can check that the 21 minors of L are in  $Rad(I_V)$ , hence  $Rad(I_W) \subset Rad(I_V)$ , thus V = W.

We therefore conclude that the kernel of  $\phi^{2,0}$  contains no decomposable elements.

A 2-vector w over a characteristic 0 field can be expressed uniquely as  $w = \sum_{i,j} a_{ij} e_i \wedge e_j$  where  $a_{ij} = -a_{ji}$ . The rank of the vector w is half the rank of the (skew-symmetric) coefficient matrix  $A_w := (a_{ij})_{1 \leq i,j \leq 7}$  of w [6, Thm 1.7 & Remark p. 13]. Thus the 2-vector  $w = a_1v_1 + \cdots + a_7v_7$  in  $Ker(\phi^{2,0})$  (where the  $v_i$ , i = 1...7 are the vectors corresponding to the columns of the matrix M) is a rank 2 vector if and only if the 49 6 × 6 minors of the matrix  $A_w$  are 0. The radical of the ideal generated by these minors is principal, generated by a homogeneous quadric in  $a_1, \ldots, a_7$  whose associated symmetric matrix is

Therefore  $w \in Ker(\phi^{2,0})$  has rank 2 if and only if  $(a_1, \ldots, a_7)Q^t(a_1, \ldots, a_7) = 0$ .

The point  $(10 + 8\alpha, -7, 0, 0, 7, 0, 0)$  lies on the associated smooth quadric  $\tilde{Q}$ , therefore  $\tilde{Q}(\mathbb{Q}[\alpha])$  is infinite. Let be w be a 2-vector in  $\tilde{Q}(\mathbb{Q}[\alpha])$ . The decomposable vector  $\wedge^2 w \neq 0$  has coordinates in  $\mathbb{Q}[\alpha]$  in the basis  $(e_{i1} \wedge \cdots \wedge e_{i4})$  of  $\wedge^4 H^0(S, \Omega_S)$ . The corresponding 4-dimensional vector space W is therefore generated by 4 vectors  $w_1, \ldots, w_4$  with coordinates over  $\mathbb{Q}[\alpha]$  in the basis  $\gamma = (e_1, \ldots, e_7)$  of  $H^0(S, \Omega_S)$ .

One computes that the image of  $\mathbb{Q}[\alpha][U_3(3) \times \mathbb{Z}/3\mathbb{Z}]$  in  $M_7(\mathbb{Q}[\alpha])$  is 49 dimensional over  $\mathbb{Q}[\alpha]$ , thus

$$\mathbb{Q}[U_3(3) \times \mathbb{Z}/3\mathbb{Z}] = M_7(\mathbb{Q}(\alpha)) (= End(A) \otimes \mathbb{Q})$$

in the basis  $\gamma$ ,  $(H_1(S, \mathbb{Q}[\alpha]))$  is the  $\mathbb{Q}[\alpha]$ -vector space generated by  $e_1, \ldots, e_k)$  and therefore there exists a morphism  $p: S \to E^4 = B$  (where  $E = \mathbb{C}/\mathbb{Z}[\alpha]$ ) such that  $W = p^*H^0(B, \Omega_B)$ . By hypothesis the image p(S) generates B. By construction

$$\wedge^2 p^* H^0(B, \Omega_B) \cap Ker(\phi^{2,0})$$

is at least one dimensional since it contains w, and therefore S is Lagrangian for p.

A contrario, the trace of an order 2 automorphism  $\sigma \in \operatorname{Aut}(S) \subset \operatorname{Aut}(A)$  acting on the tangent space of A at 0 equals to -1, therefore the image B' of the endomorphism  $p : \sigma - 1_A$ , where  $1_A$  is the identity of A is an Abelian fourfold. Using Maple, one computes that

$$\wedge^2 p^* H^0(B, \Omega_B) \cap Ker(f) = \{0\}.$$

Let  $\vartheta: S \to A$  be the Albanese map of S, and let  $q: A \to A$  be an endomorphism with a 4 dimensional image and a representation in  $M_7(\mathbb{Q}) \subset M_7(\mathbb{Q}(\alpha))$  in the basis  $\gamma$ . Since the matrix Q is positive definite, we have

$$\wedge^2 p^* H^0(B, \Omega_B) \cap \tilde{Q} = \{0\},\$$

where p is the map  $p = q \circ \vartheta : S \to B$ . Therefore S is not Lagrangian with respect to p.  $\square$ 

Remark 7. Let X be a surface and let  $\phi^{2,0}: \wedge^2 H^0(X,\Omega_X) \to H^0(X,K_X)$  be the natural map. Let be  $d=\dim Ker(\phi^{2,0})$  and  $q=\dim H^0(X,\Omega_X)$ . In the proof of Theorem 6, we saw that the set of rank k vectors in  $Ker(\phi^{2,0})$  is a determinantal variety: the intersection of minors of size  $\geq 2k+1$  of some anti-symmetric matrix of size  $q\times q$  with linear entries in d variables. It seems to the authors quite remarkable that for Stover's surface the set of rank 2 vectors (obtained as the zero set of 49 6 × 6 minors of a size q=7 matrix) is an hypersurface in  $Ker(\phi^{2,0})$ . That hypersurface is the only  $U_3(3)$ -invariant quadric of  $U_3(3)$  acting on  $Ker(\phi^{2,0})$ .

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