

Grid peeling and the affine curve-shortening flow*

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Abstract

In this paper we study an experimentally-observed connection between two seemingly unrelated processes, one from computational geometry and the other from differential geometry. The first one (which we call *grid peeling*) is the convex-layer decomposition of subsets $G \subset \mathbb{Z}^2$ of the integer grid, previously studied for the particular case $G = \{1, \dots, m\}^2$ by Har-Peled and Lidický (2013). The second one is the affine curve-shortening flow (ACSF), first studied by Alvarez *et al.* (1993) and Sapiro and Tannenbaum (1993). We present empirical evidence that, in a certain well-defined sense, grid peeling behaves at the limit like ACSF on convex curves. We offer some theoretical arguments in favor of this conjecture.

We also pay closer attention to the simple case where $G = \mathbb{N}^2$ is a quarter-infinite grid. This case corresponds to ACSF starting with an infinite L-shaped curve, which when transformed using the ACSF becomes a hyperbola for all times $t > 0$. We prove that, in the grid peeling of \mathbb{N}^2 , (1) the number of grid points removed up to iteration n is $\Theta(n^{3/2} \log n)$; and (2) the boundary at iteration n is sandwiched between two hyperbolas that are separated from each other by a constant factor.

1 Introduction

Let G be a planar point set. The *convex-layer decomposition* (or *onion decomposition*) of G [6, 8, 12, 14, 16] is a discrete algorithmic process in which points of G are iteratively removed, as follows: Let $G_0 = G$. Then, for each $n \geq 1$ such that $G_{n-1} \neq \emptyset$, let $H_n = \mathcal{CH}(G_{n-1})$ (the convex hull of the current set), let L_n be the set of vertices of H_n , and remove L_n from the current set by setting $G_n = G_{n-1} \setminus L_n$.¹ We call H_n the *n*th convex layer of G . This decomposition has applications in range-searching data structures [9] and as a measure of depth in robust statistics [6, 14].

Motivated by the question of whether grid points behave similarly to random points, Har-Peled and Lidický [16] studied the convex-layer decomposition of the $m \times m$ integer grid $G = \{1, \dots, m\}^2$. They proved that this point set has $\Theta(m^{4/3})$ convex layers. They also briefly noted that the convex layers of this point set appear to converge to circles as the process advances.

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¹Note that G_{n-1} might contain points which lie on the boundary of H_n but are not vertices. These points will still be present in G_n .

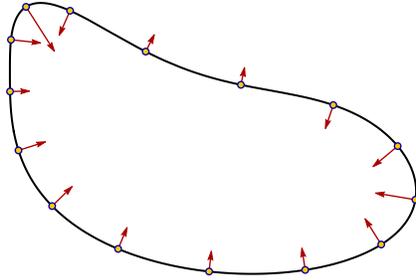


Figure 1: Affine curve-shortening flow. The arrows indicate the instantaneous velocity of different points along the curve at the shown time moment.

In this paper we explore an experimentally-observed connection between the convex-layer decomposition of more general subsets $G \subset \mathbb{Z}^2$ of the integer grid (which we call *grid peeling*), and a continuous process on smooth curves known as the *affine curve-shortening flow* (ACSF). Our conjectural connection between these two processes, if true, would show that in the square case studied by Har-Peled and Lidický, peeling indeed converges to a circular shape. More generally, it would show that for any convex shape, in the limit as the grid density becomes arbitrarily fine, the result of peeling the intersection of that shape with a grid converges to an ellipse.

1.1 The affine curve-shortening flow

In the affine curve-shortening flow, a smooth curve $\gamma \subset \mathbb{R}^2$ varies with time in the following way. At each moment in time, each point of γ moves perpendicularly to the curve, towards its local center of curvature, with instantaneous velocity $r^{-1/3}$, where r is that point's radius of curvature at that time. Thus, for a smooth convex curve, all points move inwards, possibly at different velocities. For non-convex curves, points of local non-convexity move outwards. See Figure 1.

The ACSF was first studied by Alvarez *et al.* [1] and Sapiro and Tannenbaum [24]. It differs from the more usual *curve-shortening flow* (CSF) [7, 11], in which each point moves with instantaneous velocity r^{-1} . Unlike the CSF, the ACSF is invariant under affine transformations: Applying an affine transformation to a curve, and then performing the ACSF, gives the same results (after rescaling the time parameter appropriately) as performing the ACSF and then applying the affine transformation to the shortened curves. Moreover, if the affine transformation preserves area, then the time scale is unaffected. For more on the ACSF see [7, 10, 18] and references cited there.

For the CSF, every smooth Jordan curve eventually becomes convex and then converges to a circle as it collapses to a point, without ever crossing itself. Angenent *et al.* [3] proved that, correspondingly, under the ACSF, every smooth Jordan curve becomes convex and then converges to an ellipse as it collapses to a point, without self-crossings.

Even if the initial curve γ is not smooth (e.g. it has sharp corners), as long as it satisfies certain natural conditions, there exists a unique time-dependent curve $\gamma(t)$ which satisfies the ACSF (or the CSF) condition for all $t > 0$, and which converges to γ as $t \rightarrow 0^+$. See [7], Theorems 3.26 and 3.28.

The ACSF was originally applied in computer vision, as a way of smoothing object boundaries [7] and of computing shape descriptors that are insensitive to the distortions caused by changes of viewpoint. Because peeling can be computed quickly and efficiently, by a purely combinatorial

algorithm [8], our conjectural connection between peeling and the ACSF could potentially provide an efficient way of performing these computations. However, to fully realize this potential application, it would be helpful to prove rigorous bounds on the accuracy of approximation, and to find a way to generalize the approximation so that it can handle non-convex curves as well. In the other direction, our conjecture would allow us to apply results on the well-understood behavior of the ACSF to the less well-understood algorithmic process of grid peeling. For instance, it would explain the circular layer shapes observed by Har-Peled and Lidický.

1.2 Organization of this paper

This paper is organized as follows. In [Section 2](#) we formalize our conjectured connection between peeling and the ACSF as [Conjecture 1](#), and provide a non-rigorous justification for the conjecture. In [Section 3](#) we describe our implementation details, and report on more detailed experiments that quantify the similarity between peeling and the ACSF. In [Section 4](#) we prove [Theorem 2](#), which shows that for bounded regions, the rates of peeling and the ACSF are within a constant factor of each other, a weaker form of our conjecture. In [Section 5](#) we examine more closely a special case of our conjecture on a quarter-infinite grid, and prove more precise results for that case.

2 The connection

Empirical evidence points to a connection between grid peeling and the ACSF. For a curve γ , let $\gamma(t)$, $t \geq 0$, be the result of applying ACSF on γ for time duration t . Given a positive integer n , let $(\mathbb{Z}/n)^2$ be the uniform grid with spacing $1/n$. Given a convex region $R \subset \mathbb{R}^2$, let $G_{[n]}(R) = R \cap (\mathbb{Z}/n)^2$ be the set of grid points of $(\mathbb{Z}/n)^2$ contained in R . Informally, for a convex curve γ , we have that peeling $G_{[n]}(\mathcal{CH}(\gamma))$ appears to approximate the ACSF on γ as $n \rightarrow \infty$.

This connection is illustrated in [Figure 2](#). [Figure 2](#) (left) shows the ACSF evolution of a sample convex curve γ , given by $\gamma = \{(x(a), y(a)) : 0 \leq a < 2\pi\}$ for $x(a) = ((1 - \sin a)/2)^2$ and $y(a) = ((1 - \sin(a + 2))/2)^{1.3}$. Specifically, the figure shows $\gamma(0.02t)$ for $t = 0, 1, 2, \dots, 14$. [Figure 2](#) (center) shows every fifth layer of the convex-layer decomposition of $G_{[30]}(R)$ for $R = \mathcal{CH}(\gamma)$. The similarity to [Figure 2](#) (left) is immediately evident. Finally, [Figure 2](#) (right) shows every 2714th layer of the convex-layer decomposition of $G_{[5000]}(R)$. [Figure 2](#) (left) and (right) are virtually indistinguishable to the naked eye.

We can formalize this resemblance by the following conjecture.

Conjecture 1. *There exists a constant $c \approx 1.6$ such that the following is true: Let $R \subset \mathbb{R}^2$ be a convex region, and let $\gamma = \partial R$ be its boundary. Let t^* be the time it takes for γ to collapse to a point under the ACSF (or $t^* = \infty$ for unbounded sets that never collapse). Fix a time $0 \leq t < t^*$, and let $\gamma' = \gamma(t)$ under the ACSF. For a fixed n , let G' be the m th convex layer of $G_{[n]}(R)$ for*

$$m = ctn^{4/3}. \tag{1}$$

Then, as $n \rightarrow \infty$, the boundary of the convex hull of G' converges pointwise to γ' .

In particular, the ACSF is known to converge to an ellipse for any closed initial boundary γ , in the limit as $t \rightarrow t^*$, when its shape is rescaled to have constant area. Correspondingly, by the conjecture, the convex layers of $G_{[n]}(R)$ should also converge to ellipses as $t \rightarrow t^*$ and $n \rightarrow \infty$. By symmetry, the convex layers of a square grid should indeed converge to circles.

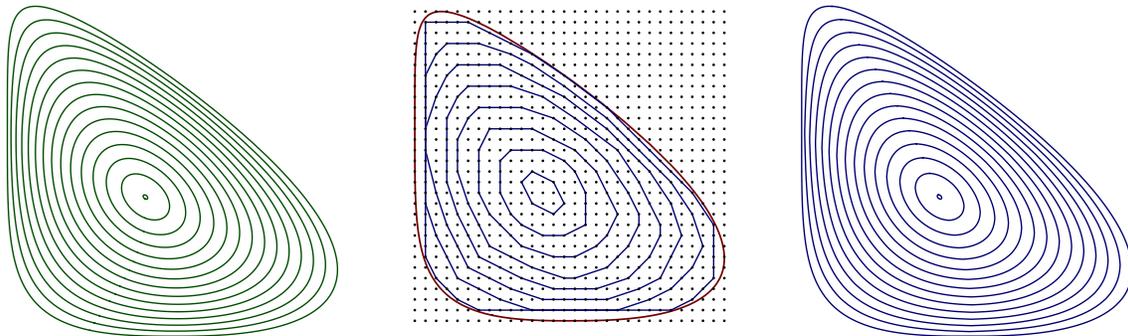


Figure 2: Left: ACSF evolution of a convex curve. Center and right: Convex-layer decomposition of the set of grid points inside the same convex curve, for different grid spacings.

2.1 Justification for Conjecture 1

One intuitive but somewhat vague justification for [Conjecture 1](#) is that the ACSF is invariant under affine transformations (in fact, it is the unique affine-invariant flow of least order [7]), and grid peeling is also invariant under a subgroup of affine transformations, namely the ones that preserve the unit grid.

A more detailed justification is as follows. Balog and Bárány [5] proved that, if R is the unit disk, then $\mathcal{CH}(G_{[n]}(R))$ has $\Theta(n^{2/3})$ vertices. Equivalently, if R is a disk of radius r , then $C = \mathcal{CH}(G_{[n]}(R))$ has $\Theta((nr)^{2/3})$ vertices. Let us assume these vertices are uniformly distributed along the boundary of C ,² so a portion of ∂C of length d contains $\Theta(dn^{2/3}r^{-1/3})$ vertices.

Now, let $R \subset \mathbb{R}^2$ be an arbitrary convex region with smooth boundary $\gamma = \partial R$, and fix a small portion δ of γ , of almost constant radius of curvature r . Let d be the length of δ . Let $C = \mathcal{CH}(G_{[n]}(R))$ for large n . Then the portion δ' of ∂C that is close to δ contains $\Theta(dn^{2/3}r^{-1/3})$ vertices. Let $\varepsilon > 0$ be much smaller than d . In order for δ' to advance inwards by distance ε , a total of εdn^2 grid points must be removed. This should take $\Theta(\varepsilon n^{4/3}r^{1/3})$ iterations. Therefore, δ' should move inwards at speed $\Theta(n^{-4/3}r^{-1/3})$. This is $\Theta(n^{4/3})$ times slower than ACSF, independently of r .

3 Implementation and experiments

We first implemented a simple front-tracking ACSF approximation method that works as follows. We sample a number m of points p_1, \dots, p_m along the given curve γ . For each point p_i , we estimate the normal vector and the radius of curvature at p_i by the normal vector v_i and radius r_i of the unique circle passing through points p_{i-1}, p_i, p_{i+1} . We simultaneously let all points move at the appropriate speeds for a short time interval $t = c \cdot (d_{\min})^{4/3}$, where d_{\min} is the minimum distance between two consecutive points, and c is a fixed parameter. Then we repeat the process. Hence, as the sample points get closer and closer, we take smaller and smaller time steps. Here the exponent $4/3$ was chosen in order for the simulation to be scale-independent.

A disadvantage of this method is that, as the curve becomes elliptical, the sample points tend to bunch together at the sharp ends of the ellipse, causing the time step to decrease very drastically.

²This seems to be the case empirically.

In order to overcome this problem, we then implemented a more sophisticated approach, in which each point is also given a tangential velocity component w_i (i.e. $w_i \perp v_i$). (Tangential velocities should not affect the evolution of a flow, since they only cause curve points to move within the curve.) We make the length of w_i proportional to $\|v_i\| \cdot \log(\|p_i - p_{i-1}\|/\|p_i - p_{i+1}\|)$. Hence, if p_i is equidistant from p_{i-1} and p_{i+1} , then $\|w_i\| = 0$. Otherwise, if p_i is closer to p_{i-1} than to p_{i+1} , say, then w_i points in the direction of p_{i+1} .

This simple approach was enough for our purposes. For more advanced flow simulation methods, see e.g. [7, 15, 21] and references cited there.

Our ACSF C++ program may be found at `ACSF.cpp`, in the ancillary files of this paper.

For the grid peeling simulations, we represent the grid subset as a one-dimensional array that stores, for each row, the x -coordinates of the leftmost and rightmost grid points in that row. We compute the convex hull at each iteration using Andrew’s modification of Graham’s scan [2, 13]. Thus, to find the $\Theta(n^{4/3})$ layers of an $n \times n$ grid we take $O(n)$ time per layer and $O(n^{7/3})$ time overall. Faster $O(n^2 \log n)$ -time algorithms are possible [8, 22, 23] but were unnecessary for our experiments. We implemented this peeling algorithm in two C++ programs, “`peel N2.cpp`” (for peeling \mathbb{N}^2) and “`peel shape.cpp`” (for peeling general shapes).

3.1 Experiments on bounded regions

In order to test [Conjecture 1](#), we ran both ACSF and grid peeling on several bounded convex regions, and compared the results. The regions we used are: $R_1 = \mathcal{CH}(\gamma)$ for the curve γ of [Figure 2](#); R_2 , a square of side 1; R_3 , a triangle with vertices $(0, 0)$, $(1, 3/4)$, $(2/5, 1)$; R_4 , a half-disk of diameter 1; and R_5 , a disk of diameter 1.

[Figure 3](#) shows the results for R_2, \dots, R_5 . Just as in [Figure 2](#), here each left figure shows ACSF with time steps of 0.02, and each right figure shows every 2714th convex layer, starting with a grid of spacing $1/5000$. Each left figure is barely distinguishable to the naked eye from the corresponding right figure.

In order to further test [Conjecture 1](#), we took the same regions R_1, \dots, R_5 , and measured the Hausdorff distance between the results of the two processes, for increasing values of the grid density n . For each R_i , we first ran our ACSF simulation with higher-precision parameters, until the times t_1, \dots, t_5 at which the area enclosed by the curve decreased to 95%, 90%, \dots , 75% of its original area.³ Then we ran grid peeling using a variety of grid spacings; specifically, $1/n$ for $n = 1000, 3000, 10000, 30000, 100000$. In each case, we ran the process until the times m_1, \dots, m_5 at which the number of grid points decreased to 95%, \dots , 75% of its original value.

For each case, we then computed the Hausdorff distance between the ACSF curve and the grid-peeling curve, both represented as polygonal chains.⁴ (For comparison, we also computed the initial Hausdorff distance, which reflects the inherent inaccuracy in approximating the given smooth curve by grid points.) [Figure 4](#) shows the results.

As we can see, the Hausdorff distance decreases with increasing n . Furthermore, for large values of n , the length of time has no major effect on the Hausdorff distance.

³Actually, for R_5 we did not simulate ACSF. We simply used the closed-form solution given by $r(t) = (r(0)^{4/3} - 4t/3)^{3/4}$, where $r(t)$ is the radius of the circle at time t .

⁴We computed the Hausdorff distances using a simple brute-force approach, using the fact that for convex polygonal chains the maximum distance is attained by a vertex (Atallah [4]). (Atallah [4] also presents a more efficient Hausdorff-distance algorithm for convex polygonal chains, which we did not use.)

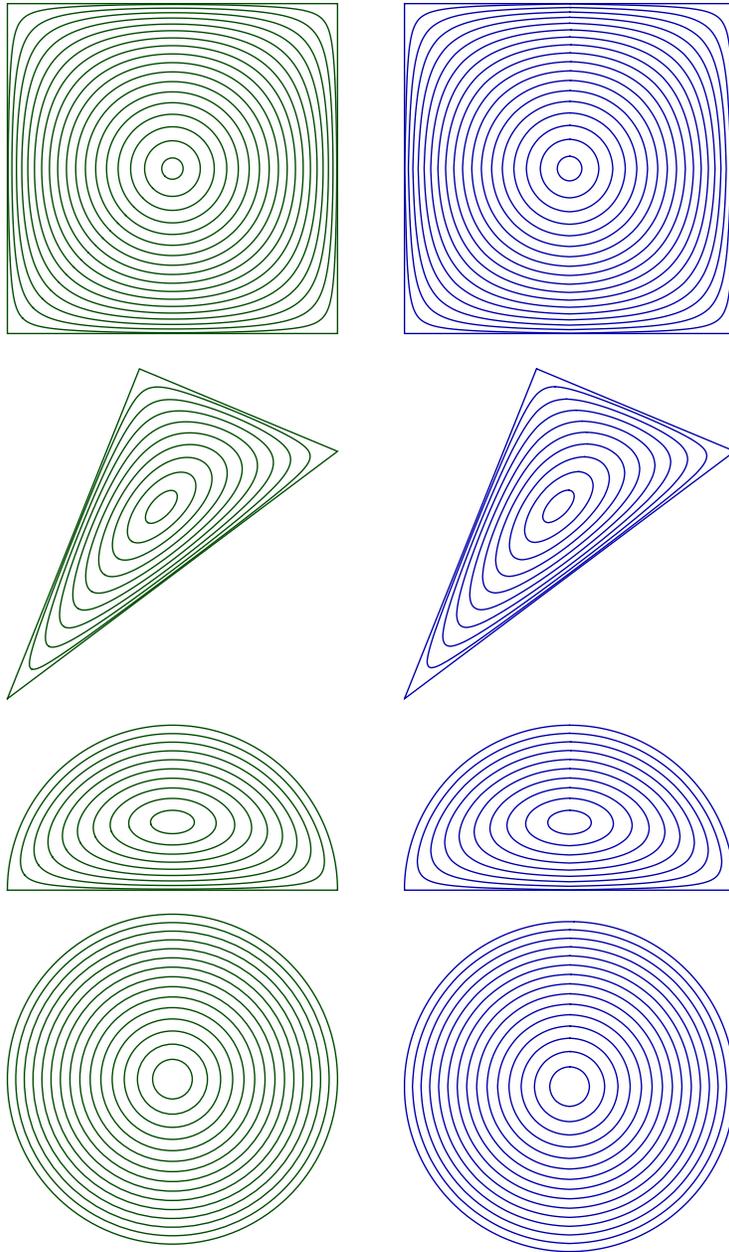


Figure 3: Comparison between ACSF (left) and grid peeling (right) on several test shapes.

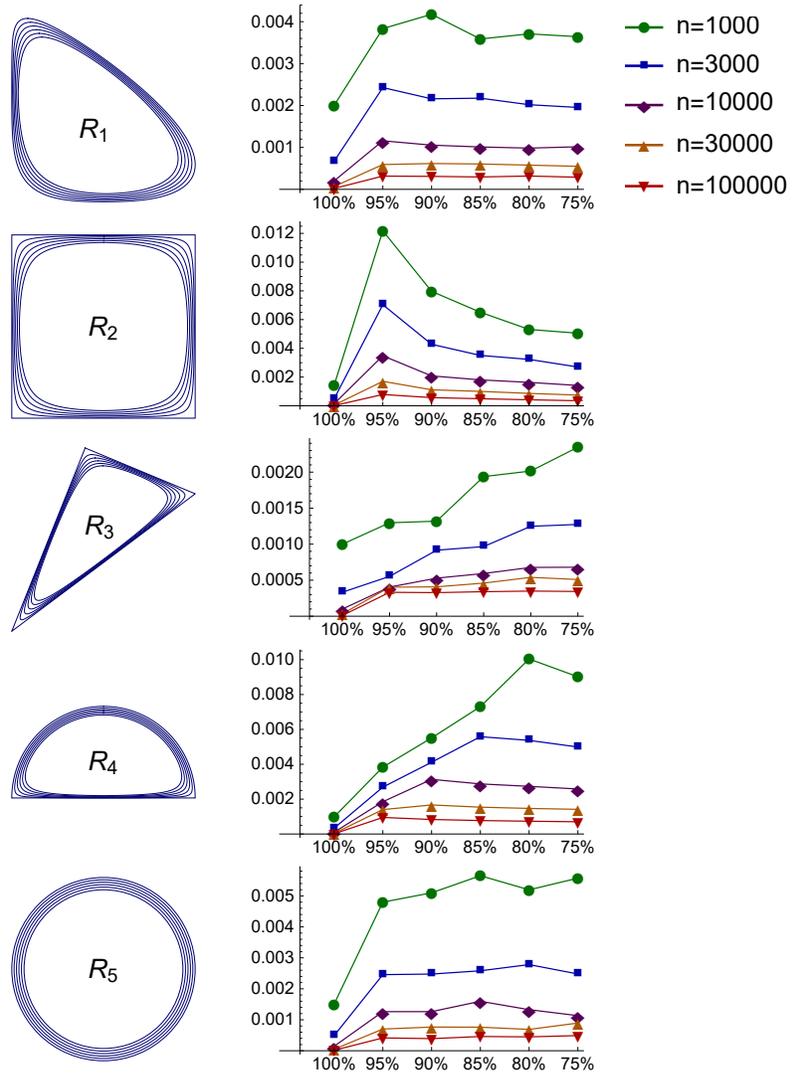


Figure 4: Hausdorff distance between the results of ACSF and grid peeling, for a variety of test shapes, times, and grid spacings.

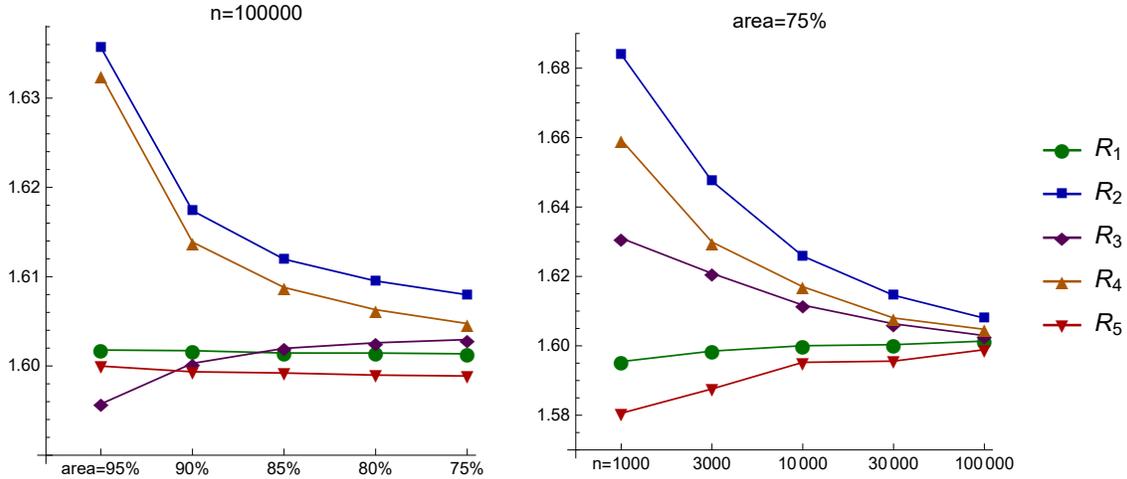


Figure 5: Approximations of the constant c given by the experiments. The left plot uses the data for $n = 100000$ and various values of the final area. The right plot uses the data for a final area of 75% and various values of n .

Finally, we checked whether the ACSF times t_1, \dots, t_5 are related to the grid peeling times m_1, \dots, m_5 in the manner predicted by [Conjecture 1](#). To do this, we solved for the constant c in (1), and computed the approximations $c \approx m_i / (t_i n^{4/3})$. The results are shown in [Figure 5](#). As can be seen, in all cases we obtained values close to 1.6; furthermore, the approximations get closer to each other as either n or t increases.

4 Number of convex layers for bounded regions

In this section we prove that for bounded regions R , [Conjecture 1](#) is asymptotically correct as far as the number of convex layers is concerned:

Theorem 2. *Let $R \subset \mathbb{R}^2$ be a bounded convex region. Then the number of convex layers of $G_{[n]}(R)$ is $\Theta(n^{4/3})$, with a constant of proportionality that depends on R .*

Correspondingly, in ACSF, if the given curve γ is dilated by a factor of k , then its evolution is dilated in time by a factor of $k^{4/3}$.

[Theorem 2](#) follows from the result of [16] on the number of convex layers of square grids. First, we note that the result of [16] can be readily generalized to rectangular grids using the same argument:⁵

Lemma 3. *Let m, n be integers satisfying $\sqrt{m} \leq n \leq m^2$. Then the number of convex layers of $G = \{1, \dots, m\} \times \{1, \dots, n\}$ is $\Theta(m^{2/3}n^{2/3})$.*

Now, let $R \subset \mathbb{R}^2$ be a given bounded convex region. By John's ellipsoid theorem [20], there exist two ellipses that satisfy $E_1 \subseteq R \subseteq E_2$, such that the ratio between their areas is at most 4.

Let n be an integer, and let $G = G_{[n]}(R)$. Scale up all these sets by a factor of n , obtaining R', E'_1, E'_2 , and $G' \subset \mathbb{Z}^2$. The grid peeling process is clearly invariant to linear transformations.

⁵The techniques of [16] are also presented in [Section 5](#) below.

We now focus on *grid-preserving* linear transformations, that is, linear transformations that map \mathbb{Z}^2 bijectively into \mathbb{Z}^2 . A linear transformation $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is grid-preserving if and only if it is of the form $f(p) = Mp$, where M is a 2×2 integer matrix with determinant ± 1 .

Lemma 4. *Let $v_1, v_2 \in \mathbb{Z}^2$ be linearly independent vectors. Then there exists a grid-preserving linear transformation f that maps v_1 into the x -axis, and such that $f(v_2)$ has slope at least 2 in absolute value.*

Proof. We first apply a grid-preserving linear transformation f_1 that maps v_1 to the x -axis. Denote $f_1(v_2) = (x_2, y_2)$. Then we apply a horizontal shear $f_2 : (x, y) \mapsto (x - my, y)$, for an appropriate $m \in \mathbb{Z}$. The appropriate m is either $\lfloor y_2/x_2 \rfloor$ or $\lceil y_2/x_2 \rceil$. \square

Now, the ellipse E'_1 contains a rectangle T_1 whose area is at least a constant fraction of the area of E_1 . Applying **Lemma 4**, we turn T_1 into a parallelogram T'_1 with two horizontal sides and two shorter, close-to-vertical sides. Hence, T'_1 contains an axis-parallel rectangle T''_1 whose area is at least a constant fraction of the area of T'_1 . If n is large enough, then the side lengths m_1 and m_2 of T''_1 will satisfy $\sqrt{m_1} \leq m_2 \leq m_1^2$. Therefore, we can apply the lower bound of **Lemma 3** on T''_1 .

The upper bound proceeds similarly, using ellipse E'_2 . This completes the proof of **Theorem 2**.

5 Peeling a quarter-infinite grid

A simple test case for **Conjecture 1** is the region $R = \{(x, y) : x \geq 0, y \geq 0\}$ (the first quadrant of the plane). In this case, the grid spacing n is irrelevant, so we can simply take $G = \mathbb{N}^2$ (where $\mathbb{N} = \{0, 1, 2, \dots\}$). The boundary of R is the L-shaped curve $\gamma_L = \{(x, 0) : x \geq 0\} \cup \{(0, y) : y \geq 0\}$.

The time-dependent hyperbola

$$\gamma_L(t) = \{(x, y) : y = (4/3^{3/2})t^{3/2}/x\}. \quad (2)$$

satisfies the ACSF condition for all $t > 0$ (as can be verified by a simple calculation), and it converges to γ_L as $t \rightarrow 0^+$. The hyperbola (2) is the only solution satisfying this property.⁶

Hence, by **Conjecture 1**, we would expect the convex layers of \mathbb{N}^2 to approach hyperbolas as the process goes on. Indeed, this is what occurs experimentally. In the next subsections we present our experimental and theoretical results regarding the convex layers of \mathbb{N}^2 . But let us first introduce some notation.

Notation. Throughout this section, we will define the sets G_n, L_n, H_n as in the Introduction for our choice of $G = \mathbb{N}^2$. Hence, $G_0 = \mathbb{N}^2$, $H_n = \mathcal{CH}(G_{n-1})$, L_n is the set of vertices of H_n , and $G_n = G_{n-1} \setminus L_n$. Let $B_n = \partial H_n$. Let (K_n, K_n) be the point of intersection of B_n with the line $y = x$, so the point (K_n, K_n) splits B_n into two congruent “arms”. Let $s(n) = |L_1| + \dots + |L_n|$ be the number of grid points removed up to iteration n . Given integers $x, n \in \mathbb{N}$, let $a_x(n)$ be the number of points of the x th column of G that have been removed up to iteration n ; i.e., let

⁶The existence of a unique solution was proven for doubly-differentiable curves, without the assumption of closedness, by Angenent *et al.* [3, Section 6] and stated for closed curves without the assumption of smoothness in [7, Theorem 3.28]. In the case here, the uniqueness of the solution can be proven by applying the result for closed curves to the boundary of a large square: If the quarterplane had multiple solutions, they could be approximated arbitrarily well by the solution near the corner of a large enough square, which would necessarily also have multiple solutions, violating [7, Theorem 3.28].

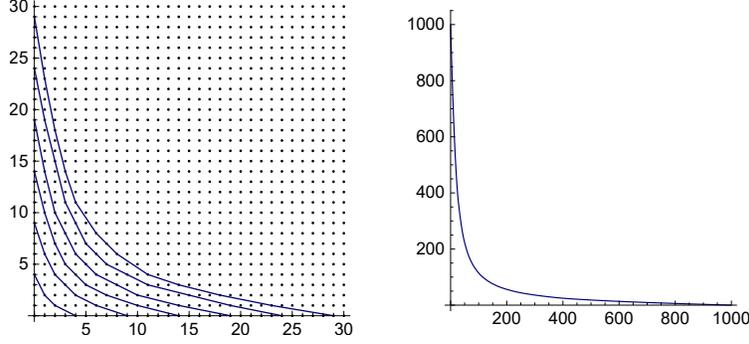


Figure 6: Left: Convex layers 5, 10, 15, \dots , 30 of \mathbb{N}^2 . Right: 1000th convex layer of \mathbb{N}^2 .

$a_x(n) = |\{(\{x\} \times \mathbb{N}) \cap (L_1 \cup \dots \cup L_n)\}|$. Note that in a fixed column, the points are removed in increasing order of y -coordinate; furthermore, for every fixed n the sequence $a_0(n), a_1(n), a_2(n), \dots$ is nonincreasing, with $a_0(n) = n$, $a_{n-1}(n) = 1$, and $a_n(n) = 0$. See [Figure 6](#).

5.1 Experiments

Given n , let B'_n be the result of scaling down the n th layer boundary B_n by a factor of $n^{3/4}$. According to [Conjecture 1](#), as $n \rightarrow \infty$ we would expect B'_n to converge to the hyperbola $\gamma_L(1/c)$ of (2). We would like to measure to what extent this happens. However, since we do not know the constant c to high precision, we performed these measurements as follows:

Given n , let $f_n(x) = K_n^2/x$ define the hyperbola that passes through the point (K_n, K_n) . Given a small real number $0 < \alpha < 1$, let $x_\alpha(n)$ denote the smallest integer $x > K_n$ for which $|a_x(n) - f(x)| > \alpha f(x)$. Hence, the portion of B_n that is between x -coordinates $x = K_n$ and $x = x_\alpha(n)$ is within an α -fraction of the hyperbola $f(x)$. By symmetry, the same can be said about the portion of B_n that is between y -coordinates $y = K_n$ and $y = x_\alpha(n)$. The ratio $x_\alpha(n)/K_n$ provides a scale-independent measure of the extent to which B_n is α -close to a hyperbola.

[Figure 7](#) illustrates the results of these measurements for increasing values of n and decreasing values of α . Specifically, we took $n = 10000, 30000, 100000, 300000$ and $\alpha = 0.1, 0.03, 0.01, 0.003$. As can be seen, for each fixed α , the portion of the hyperbola that is within an α -fraction of B_n grows as n increases.

Measuring the time relationship. [Conjecture 1](#) predicts that $K_n \approx 2(n/(3c))^{3/4}$ for large n . We tested this prediction by solving for c and seeing whether it approaches a constant close to 1.6. The results are shown in [Figure 8](#).

Layer sizes. The sequence $\{|L_n|\}$ represents the number of vertices in successive layers of the convex-layer decomposition of \mathbb{N}^2 . This sequence is now cataloged in the Online Encyclopedia of Integer Sequences as A293596. It starts as

$$1, 2, 2, 3, 4, 4, 3, 4, 6, 6, 5, 4, 6, 6, 8, 7, 6, 6, 6, 8, 9, 10, 10, 8, 8, 7, 8, 10, 10, 12, 13, 12, \dots$$

[Figure 9](#) plots this sequence at different scales. As can be seen, this sequence has regular waves that slowly increase in both length and amplitude. Nevertheless, the relative amplitude of the

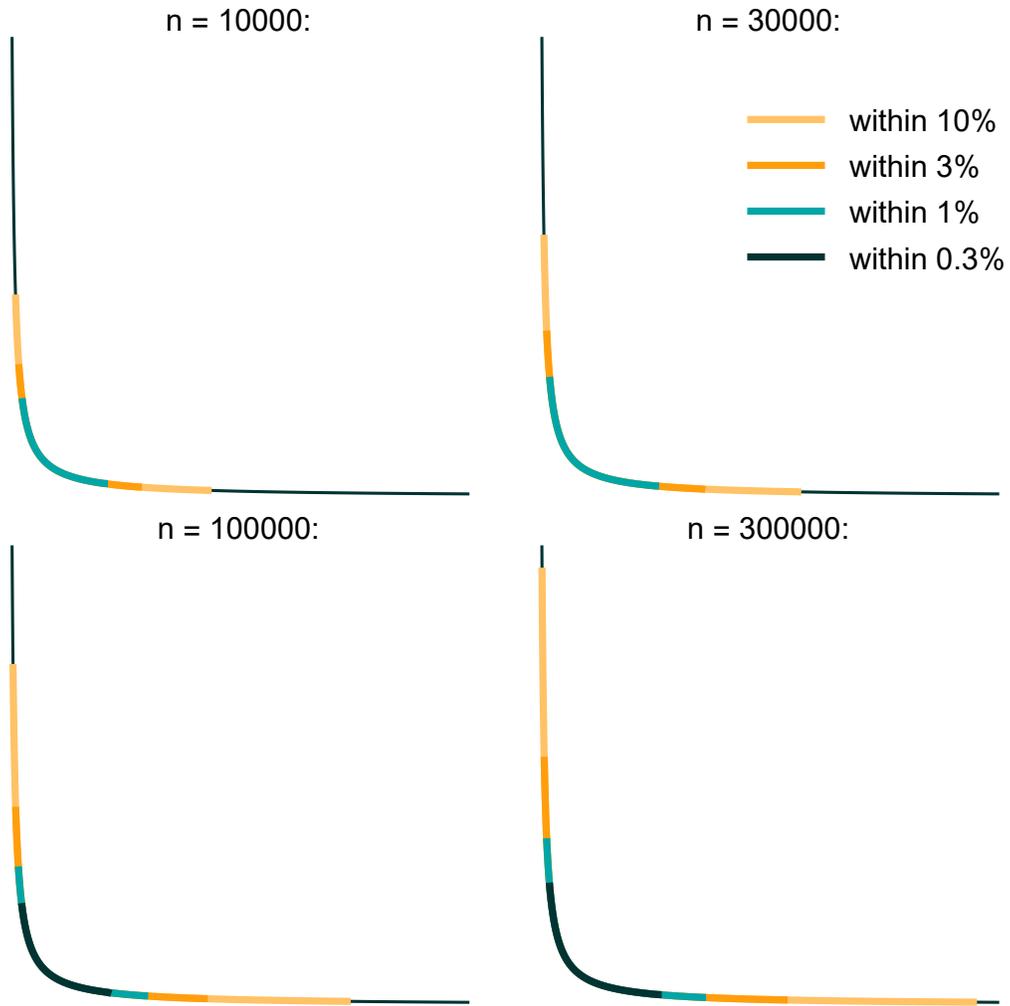


Figure 7: Measurements indicating that as n increases, an increasingly long portion of B_n is close to a hyperbola.

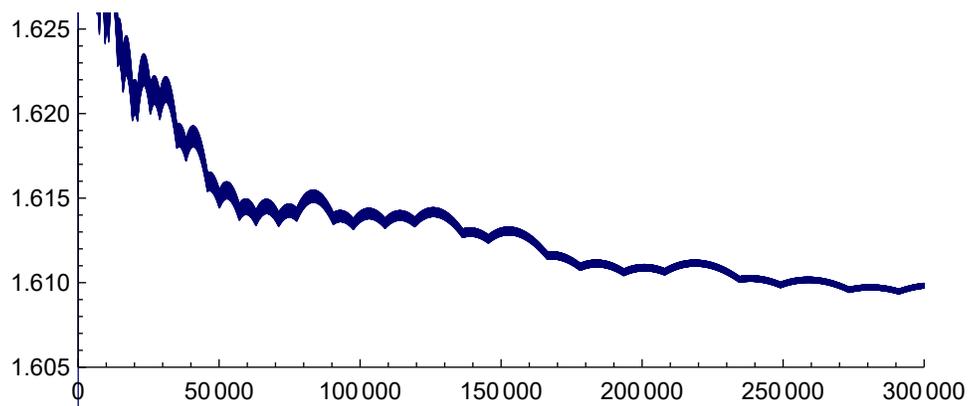


Figure 8: Approximations of the constant c given by experimental measurements of the point (K_n, K_n) .

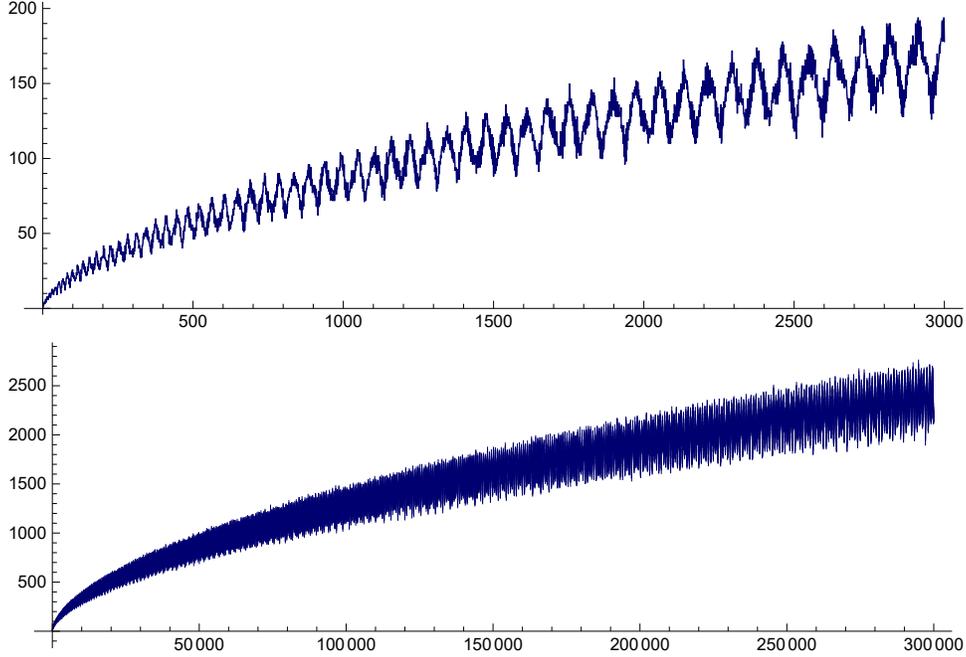


Figure 9: Number of vertices of the n th convex layer of \mathbb{N}^2 as a function of n .

waves (the ratio between their amplitude and their height above the x -axis) seems to decrease, perhaps tending to zero.

5.2 Rigorous results

We now obtain some rigorous results for the grid peeling of $G = \mathbb{N}^2$.

Theorem 5. *The convex-layer decomposition of the quarter-infinite grid $G = \mathbb{N}^2$ satisfies the following properties:*

1. *The number of grid points removed up to iteration n satisfies $s(n) = \Theta(n^{3/2} \log n)$.*
2. *$a_x(n) = O(n^{3/2}/x)$ for all n , and $a_x(n) = \Omega(n^{3/2}/x)$ for $c_1\sqrt{n} \leq x \leq c_1n$, where $0 < c_1 < 1$ is some constant.*
3. *$K_n = \Theta(n^{3/4})$.*
4. *$|L_n| = O(n^{1/2} \log n)$ and $|L_n| = \Omega(\log n)$.*

In other words, the boundary B_n is sandwiched between two hyperbolas that are separated from each other by a constant factor, where the upper hyperbola bounds B_n for all x , while the lower hyperbola bounds B_n only up to $x = c_1n$ (and symmetrically in the y -axis). Put differently, the scaled boundary B'_n is sandwiched between two hyperbolas $y \geq c_1x$ and $y \leq c_2x$ that are independent of n , where the lower hyperbola bounds B'_n only up to $x = c_1n^{1/4}$ (and symmetrically in the y -axis). See [Figure 10](#).

Regarding $|L_n|$, we would expect it to behave like $\Theta(n^{1/2} \log n)$, or even like $c'n^{1/2} \log n \pm o(n^{1/2} \log n)$ for some constant c' . However, our rigorous lower bound for $|L_n|$ is very weak.

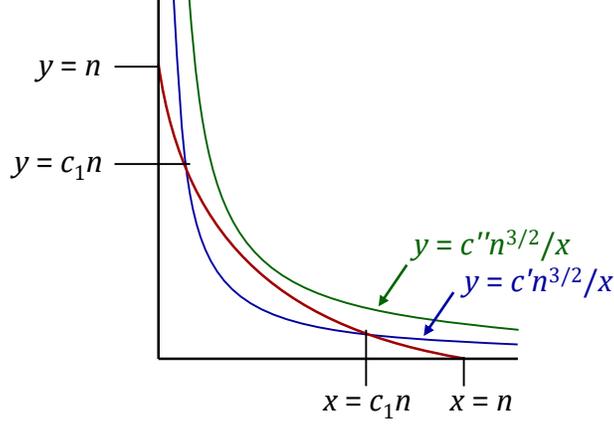


Figure 10: Schematic drawing of the upper and lower bounds for the n th convex layer of \mathbb{N}^2 .

5.3 Proof of Theorem 5

The proof of Theorem 5 is mainly based on the techniques of [16].

Lemma 6 (Jarník [19]). *Let $P \subset \{1, \dots, m\} \times \{1, \dots, n\}$ be in convex position. Then $|P| = O((mn)^{1/3})$.*

Proof. Let $p_0, p_1, p_2, \dots, p_{k-1}$ be the points of P listed in circular order around the boundary of $\mathcal{CH}(P)$, and let $v_i = p_{(i+1) \bmod k} - p_i$ be the vectors corresponding to the edges of $\mathcal{CH}(P)$. Note that these vectors are pairwise distinct. Let $x_{\max} = m^{2/3}n^{-1/3}$ and $y_{\max} = n^{2/3}m^{-1/3}$. Classify the vectors $v_i = (x_i, y_i)$ into three types as follows: (1) Those satisfying $|x_i| \leq x_{\max}$ and $|y_i| \leq y_{\max}$; (2) those satisfying $|x_i| > x_{\max}$; (3) the remaining ones (which satisfy $|y_i| > y_{\max}$). The number of vectors of type (1) is at most $4x_{\max}y_{\max} = O((mn)^{1/3})$. The number of vectors of type (2) is at most $2m/x_{\max} = O((mn)^{1/3})$. And the number of vectors of type (3) is at most $2n/y_{\max} = O((mn)^{1/3})$. \square

A vector $v = (x, y) \in \mathbb{Z}^2$ is said to be *primitive* if x and y are relatively prime.

Lemma 7. *Let $M = \{a + 1, \dots, a + m\} \times \{b + 1, \dots, b + n\} \subseteq \{1, \dots, N\}^2$. Then the number of primitive vectors in M is $(6/\pi^2)mn \pm O(N \log N)$.*

Proof. We start with the following classical number-theoretical result.

Lemma 8. *Let m, n be positive integers with $m \leq n$. Let $\rho(m, n)$ be the number of primitive vectors (x, y) in $\{1, \dots, m\} \times \{1, \dots, n\}$. Then $\rho(m, n) = (6/\pi^2)mn \pm O(n \log n)$.*

Proof. (Following Hardy and Wright [17], Theorem 332.) Let μ be the Möbius function, which sets $\mu(x) = -1$ if x is square-free and has an odd number of prime factors, $\mu(x) = 1$ if x is square-free and has an even number of prime factors, and $\mu(x) = 0$ if x is not square-free. Let $D(x, y)$ be the set of all common divisors of x and y . Then $\sum_{d \in D(x, y)} \mu(d)$ equals 1 if x and y are relatively prime, and 0 otherwise.

Clearly, $\sum_{x=1}^{\infty} \mu(x)/x^2$ converges to some positive real number smaller than 1. In fact, it converges to $6/\pi^2$ [17].

Therefore,

$$\begin{aligned}\rho(m, n) &= \sum_{x=1}^m \sum_{y=1}^n \sum_{d \in D(x, y)} \mu(d) = \sum_{d=1}^{mn} \mu(d) \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=1}^{mn} \mu(d) \left(\frac{mn}{d^2} - O\left(\frac{n}{d}\right) \right) \\ &= mn \sum_{d=1}^{mn} \frac{\mu(d)}{d^2} \pm O(n \log n) = mn \left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \pm O\left(\frac{1}{mn}\right) \right) \pm O(n \log n),\end{aligned}$$

and the claim follows. \square

Now, consider $M = \{a+1, \dots, a+m\} \times \{b+1, \dots, b+n\} \subseteq \{1, \dots, N\}^2$. The number of primitive vectors in M equals $\rho(a+m, b+n) - \rho(a+m, b) - \rho(a, b+n) + \rho(a, b)$, so [Lemma 7](#) follows by [Lemma 8](#). \square

5.3.1 Upper bounds

Lemma 9. *We have $a_x(n) \leq c_0 n^{3/2}/x$ for some constant c_0 .*

Proof. Given x , let $y = a_x(n)$. By iteration n , the entire corner subgrid $G' = \{0, \dots, x-1\} \times \{0, \dots, y-1\}$ has been removed. By [Lemma 6](#), each L_i contains $O((xy)^{1/3})$ points of G' . Hence, we must have $xy \leq O(n(xy)^{1/3})$, which implies $y = O(n^{3/2}/x)$. \square

Corollary 10. *We have $s(n) = O(n^{3/2} \log n)$.*

Corollary 11. *We have $K_n \leq \sqrt{c_0} n^{3/4}$ for the constant c_0 of [Lemma 9](#).*

Proof. Take $x_0 = \sqrt{c_0} n^{3/4}$, and note that $a_{x_0}(n) \leq x_0$. \square

By [Corollary 11](#), each “arm” of L_n is contained in an $O(n^{3/4}) \times n$ box. Hence, a hasty application of [Lemma 6](#) would yield $|L_n| = O(n^{7/12})$. However, we can do better: We can cover each arm of L_n by logarithmically many boxes of small area, and apply [Lemma 6](#) on each box.

Lemma 12. *We have $|L_n| = O(n^{1/2} \log n)$.*

Proof. Let $x_0 = \sqrt{c_0} n^{3/4}$ for the constant c_0 of [Lemma 9](#), and recall that $K_n \leq x_0$. Define the axis-parallel boxes T_i , $0 \leq i \leq \lceil \log_2(n/x_0) \rceil$, by $T_0 = [0, x_0]^2$ and $T_i = [2^{i-1}x_0, 2^i x_0] \times [0, x_0/2^i]$ for $i \geq 1$. By [Lemma 9](#), the right arm of L_n is contained in the union of these boxes. Furthermore, the area of T_0 is $c_0 n^{3/2}$, and the area of each T_i , $i \geq 1$, is $c_0 n^{3/2}/2$. Hence, by [Lemma 6](#), each T_i contains $O(\sqrt{n})$ points of L_n . Finally, the number of boxes is $O(\log n)$. \square

5.3.2 Lower bounds

Let $v = (x_v, -y_v)$ be a primitive vector with $x_v, y_v > 0$. Following [\[16\]](#), we say that v is *active* at iteration n if the unique line ℓ_v parallel to v that is tangent to H_n contains an edge of H_n (and so ℓ_v contains two points of L_n). Otherwise, if ℓ_v contains a single vertex of H_n (and a single point of L_n), then we say that v is *inactive* at iteration n . See [Figure 11](#).

Given such a vector v , let \mathcal{L}_v be the set of lines parallel to v that pass through points of \mathbb{N}^2 . We say that a line $\ell \in \mathcal{L}_v$ is *alive* at iteration n if ℓ intersects H_n ; otherwise, we say that ℓ is *dead* at iteration n . Note that, at a given iteration n , all the dead lines of \mathcal{L}_v lie below all its live lines.

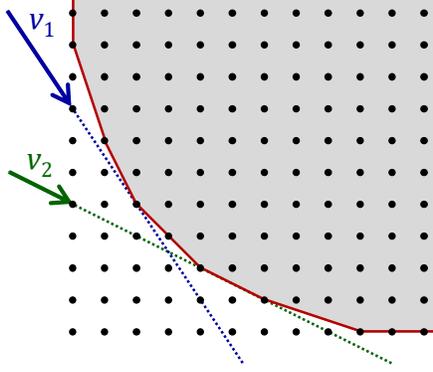


Figure 11: At the shown iteration, vector $v_1 = (2, -3)$ is inactive, while vector $v_2 = (2, -1)$ is active.

Lemma 13. *Let $v = (x_v, -y_v)$ be a primitive vector with $x_v, y_v > 0$. Then the number of lines of \mathcal{L}_v that pass below the point (x, y) is at most $xy_v + yx_v$.*

Proof. Each of the said lines passes through a grid point in $\{0, \dots, x_v - 1\} \times \{0, \dots, y + \lceil xy_v/x_v \rceil\}$. \square

Observation 14. *If v is inactive at iteration n , then the number of dead lines of \mathcal{L}_v strictly increases from iteration n to iteration $n + 1$; specifically, the tangent line ℓ_v dies.*

Observation 15. *The number of active vectors at iteration n equals $|L_n| - 1$.*

Given n , let $m = n^{1/2}/(16c_0)$ for the constant c_0 in [Lemma 9](#). Let \mathcal{V} be the set of all primitive vectors $(x_v, -y_v)$ with $x_v, y_v > 0$ and $x_v y_v \leq m$. By applying [Lemma 7](#) on the rectangles $\{2^{i-1}\sqrt{m}, \dots, 2^i\sqrt{m}\} \times \{0, \dots, 2^{-i}\sqrt{m}\}$ for $-\log_5 m \leq i \leq \log_5 m$, we obtain $|\mathcal{V}| = \Theta(m \log m) = \Theta(n^{1/2} \log n)$.

Lemma 16. *Up to iteration n , each vector of \mathcal{V} is active at least $n/2$ times.*

Proof. Consider a vector $v = (x_v, -y_v) \in \mathcal{V}$. Set $x = c_0^{1/2} n^{3/4} \sqrt{x_v/y_v}$ and $y = 1 + a_x(n) \leq 1 + c_0^{1/2} n^{3/4} \sqrt{y_v/x_v}$ (by [Lemma 9](#)). Since the grid point (x, y) has not been removed by iteration n , by [Lemma 13](#) the number of dead lines parallel to v is at most

$$xy_v + yx_v = 2c_0^{1/2} n^{3/4} \sqrt{x_v y_v} \leq 2c_0^{1/2} n^{3/4} \sqrt{m} = n/2.$$

Hence, the claim follows from [Observation 14](#). \square

Hence, by [Observation 15](#):

Corollary 17. *We have $s(n) = \Omega(n^{3/2} \log n)$.*

Lemma 18. *There exist constants $c_1, c_2 > 0$ such that, for every n and every x in the range $c_1\sqrt{n} \leq x \leq c_1 n$, we have $a_x(n) \geq c_2 n^{3/2}/x$.*

Proof. Given a slope μ in the range $1/m \leq \mu \leq m$, we will derive a lower bound for the distance between the origin and B_n in the direction μ . (So for example, taking $\mu = 1$ will yield a lower bound for K_n .)

Using [Lemma 9](#), take a grid point (x, y) with $y/x \approx \mu$ that has not been removed by iteration n . Specifically, let $x = (\sqrt{c_0/\mu})n^{3/4}$ and $y = (\sqrt{\mu c_0})n^{3/4}$. Define the rectangle $T = \{0, \dots, 3x - 1\} \times \{0, \dots, 3y - 1\}$, so $|T| = 9c_0n^{3/2}$. We claim that at least a constant fraction of the points of T have been removed by iteration n .

Indeed, let $\mathcal{V}' \subseteq \mathcal{V}$ be the set of all vectors $v = (x_v, -y_v) \in \mathcal{V}$ with $\mu/2 \leq y_v/x_v \leq 2\mu$. By applying [Lemma 7](#) on the rectangle whose opposite corners are $q/2$ and q for $q = (\sqrt{m/\mu}, \sqrt{\mu m})$, we have $|\mathcal{V}'| = \Theta(m) = \Theta(\sqrt{n})$.

Let $v \in \mathcal{V}'$, and let $i \leq n$ be an iteration in which v is active. Let $\ell \in \mathcal{L}_v$ be the line tangent to H_i . Line ℓ passes below point (x, y) , so by the construction of T , all the grid points in ℓ belong to T . Two of these grid points belong to L_i . Let us charge the pair (v, i) to the leftmost of these two points.

Doing this over all choices of v and i , we make a total of $\Theta(n^{3/2})$ charges to points of T . Furthermore, each point of T is charged at most once. Therefore, at least a constant fraction (say, a c' -fraction) of the points of T are deleted by iteration n .

Choose a constant $0 < c'' < 1 - \sqrt{1 - c'}$. Let $x' = 3c''x$ and $y' = 3c''y$ (so $y'/x' = \mu$). We claim that the grid point (x', y') has been removed by iteration n . Indeed, otherwise, all the points behind (x', y') (i.e. all the points $(x'', y'') \in T$ with $x'' \geq x'$ and $y'' > y'$) would also be present, and they constitute more than a $(1 - c')$ -fraction of T (by the choice of c'').

Rephrasing, given n and given x' in the range $3c''\sqrt{c_0}\sqrt{n} \leq x' \leq 3c''\sqrt{c_0}n$, we have

$$a_{x'}(n) \geq y' = 3c''y = 3c''c_0n^{3/2}/x = 9(c'')^2c_0n^{3/2}/x'.$$

□

Corollary 19. *We have $|L_n| = \Omega(\log n)$.*

Proof. The idea is that, since B_n is confined between two hyperbolas for a long stretch, it must make at least a certain number of turns. That number is $\Omega(\log n)$, by the following calculation:

For simplicity, let us scale down B_n by a factor of $n^{3/4}$, obtaining B'_n . Let $y = c_1/x$ and $y = c_2/x$ be the two bounding hyperbolas, where $c_1 < c_2$, and where the lower-hyperbola bound applies up to $x = c_3n^{1/4}$. The number of edges of B'_n is minimized if each edge starts and ends at the lower hyperbola and is tangent to the upper hyperbola. In such a case, an edge that starts at x -coordinate x_0 ends at x -coordinate kx_0 , for the constant $k = (\sqrt{c_2} + \sqrt{c_2 - c_1})/(\sqrt{c_2} - \sqrt{c_2 - c_1})$.

Hence, the number of edges is at least $\log_k n^{1/4} - O(1)$. □

6 Concluding remarks

The main open problem is to prove [Conjecture 1](#). Additionally, if the conjecture can be confirmed, it would be of interest to generalize the approximation to the ACSF that it yields, from convex curves to more general curves. Also, grid peeling for higher dimensions has not been studied at all, as far as we know.

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