THE MODALITY OF A BOREL SUBGROUP IN A SIMPLE ALGEBRAIC GROUP OF TYPE E_8

SIMON M. GOODWIN, PETER MOSCH, AND GERHARD RÖHRLE

Abstract. Let G be a simple algebraic group over an algebraically closed field k, where $\operatorname{char} k$ is either 0 or a good prime for G. We consider the *modality* $\operatorname{mod}(B : \mathfrak{u})$ of the action of a Borel subgroup B of G on the Lie algebra $\mathfrak{u}$ of the unipotent radical of B, and report on computer calculations used to show that $\operatorname{mod}(B : \mathfrak{u}) = 20$, when G is of type E_8. This completes the determination of the values for $\operatorname{mod}(B : \mathfrak{u})$ for G of exceptional type.

Let G be a simple algebraic group over an algebraically closed field k, where $\operatorname{char} k$ is either 0 or a good prime for G. Let B be a Borel subgroup of G with unipotent radical U, and let $\mathfrak{u} = \operatorname{Lie} U$. The modality of the adjoint action of the Borel subgroup B on $\mathfrak{u}$ is defined by $\operatorname{mod}(B : \mathfrak{u}) := \max_{i \in \mathbb{Z}_{\geq 0}}(\dim(\mathfrak{u}(B,i)) - i)$, where $\mathfrak{u}(B,i) := \{x \in \mathfrak{u} \mid \dim B \cdot x = i\}$, and is intuitively the maximal number of parameters on which a family of B-orbits in $\mathfrak{u}$ depends. As $\operatorname{mod}(B : \mathfrak{u})$ is an important invariant of the action of B on $\mathfrak{u}$ it is of interest to determine its value. As a general reference for the modality of the action of an algebraic group on an algebraic variety, we refer to [Vi].

In [JR] the values of $\operatorname{mod}(B : \mathfrak{u})$ are determined for G up to rank 7 excluding type B_7 and E_7; they are found by combining lower bounds from [Rö1, Prop. 3.3] with upper bounds obtained by computer calculation, and are presented in [JR, Tables II and III]. We refer to the introduction of [JR] and the references therein for prior history of finding values of $\operatorname{mod}(B : \mathfrak{u})$, and for motivation. The known values of $\operatorname{mod}(B : \mathfrak{u})$ were extended to G up to rank 8 excluding type E_8 in [GMR2, §5]; this required computer calculations, which are explained below. We note also that, as is explained in [GG, §6], results in [PS] can be used to determine $\operatorname{mod}(B : \mathfrak{u})$ for G of type A_l for $l \leq 15$.

Our main theorem gives $\operatorname{mod}(B : \mathfrak{u})$ in the case G is of type E_8.

Theorem. *Let G be a simple algebraic group of type* E_8 *over the algebraically closed field k, where* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > 5$. *Let B be a Borel subgroup of G with unipotent radical U. Then* $\operatorname{mod}(B : \mathfrak{u}) = 20$.

Our theorem completes the list of values for $\operatorname{mod}(B : \mathfrak{u})$ for G of exceptional type as presented in the table below.

Type of G	G_2	F_4	E_6	E_7	E_8
$\operatorname{mod}(B : \mathfrak{u})$	1	4	5	10	20

Modality of the action of B on $\mathfrak{u}$

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We move on to review the computer programme from [GMR2] and explain how it was adapted to show that 20 is an upper bound for $\mathrm{mod}(B : \mathfrak{u})$ for G of type E_8. Thanks to [Rö1, Prop. 3.3] it is known that 20 is a lower bound for $\mathrm{mod}(B : \mathfrak{u})$, so combining these bounds proves our theorem. In the discussion below we refer to $\mathrm{mod}(U : \mathfrak{u})$ and $\mathrm{mod}(U : \mathfrak{u}^*)$, which are defined analogously to $\mathrm{mod}(B : \mathfrak{u})$.

It is shown in [Go, Prop. 5.4] that each U-orbit in $\mathfrak{u}$ admits a so called *minimal representative*. As explained in [GR1, §2], the minimal representatives are partitioned into certain locally closed subsets X_c of $\mathfrak{u}$ for c running over some index set C. This gives a parametrization of the U-orbits in $\mathfrak{u}$, so we can deduce that $\mathrm{mod}(U : \mathfrak{u}) = \max_{c \in C} \dim X_c$, and thus by [GMR2, Thm. 5.1] that $\mathrm{mod}(B : \mathfrak{u}) = \max_{c \in C} \dim X_c - \mathrm{rank}\, G$. An algorithm for determining all the varieties X_c for $c \in C$ is given in [GR1, §3]. This algorithm was programmed in GAP, [GAP], and subsequent developments were made in [GMR1] including calls to SINGULAR, [SIN]. The resulting programme was used to obtain the parametrization of the U-orbits in $\mathfrak{u}$ when G is of rank up to 7 except for type E_7; so this can also be used to determine $\mathrm{mod}(B : \mathfrak{u})$ in these cases.

The results in [GMR2, §3] show that a similar algorithm is valid for the coadjoint action of U on $\mathfrak{u}^*$. In particular, there is a parametrization of minimal representatives of U-orbits in $\mathfrak{u}^*$ by certain locally closed subsets $\mathcal{X}_c$ of $\mathfrak{u}^*$ for c running over an index set C. This algorithm was programmed and used to obtain a complete description of the varieties $\mathcal{X}_c$, when G has rank up to 8, with the exception of type E_8. Since $\mathrm{mod}(U : \mathfrak{u}) = \mathrm{mod}(U : \mathfrak{u}^*)$, see [Rö2, Thm 1.4], we have $\mathrm{mod}(B : \mathfrak{u}) = \max_{c \in C} \dim \mathcal{X}_c - \mathrm{rank}\, G$. Thus this allowed us to determine $\mathrm{mod}(B : \mathfrak{u})$ when G has rank up to 8, with the exception of type E_8.

The algorithm for determining the varieties $\mathcal{X}_c$ for c in C involves a certain polynomial-resolving subroutine, as explained in [GMR1, §3]. This is the most complicated and computationally expensive part of the programme. We adapted our algorithm, so that in cases where the programme is not able to resolve all the polynomial conditions in a specified amount of time it simply disregards these unresolved conditions. Thus the modified computation determines a variety $\mathcal{Y}_c \supseteq \mathcal{X}_c$, which we can view as an "upper bound" for a parametrization of the minimal representatives in $\mathcal{X}_c$, so that $\dim \mathcal{Y}_c \geq \dim \mathcal{X}_c$, for each $c \in C$. Consequently, $\mathrm{mod}(U : \mathfrak{u}) \leq \max_{c \in C} \dim \mathcal{Y}_c$.

We ran the programme for the case G of type E_8 and determined a variety $\mathcal{Y}_c$ for every c in C. From the output of the computation we obtain that $\mathrm{mod}(B : \mathfrak{u}) \leq 20$ as required to verify our theorem.

We move on to mention consequences of our calculations for the finite groups of rational points, when G is defined over a finite field. Suppose that G is defined and split over the field $\mathbb{F}_p$ where p is a good prime p for G. Let q be a power of p and denote by $G(q)$ the group of $\mathbb{F}_q$-rational points of G. Also assume that B is defined over $\mathbb{F}_q$, so U is defined over $\mathbb{F}_q$ and $U(q)$ is a Sylow p-subgroup of $G(q)$. We write $k(U(q))$ for the number of conjugacy classes of $U(q)$ (which is also the number of complex irreducible characters of $U(q)$). As explained in [GMR2, §4], the parametrization of the coadjoint U-orbits in $\mathfrak{u}^*$ by the varieties $\mathcal{X}_c$ for c in C, gives a method to calculate $k(U(q))$. In fact in the cases considered in [GMR2], there is a polynomial $g(t) \in \mathbb{Z}[t]$ such that $k(U(q)) = g(q)$; and, moreover, $g(t)$ does not depend on p. Our adapted programme calculates a polynomial $h(t) \in \mathbb{Z}[t]$ such that $k(U(q)) \leq h(q)$ and $h(t)$ does not depend on p. Moreover, an upper bound for $\mathrm{mod}(U : \mathfrak{u})$ can be easily read off as the degree of $h(t)$; we refer to [GMR2, §5] for further details. Note that we do not claim here that $k(U(q))$ is necessarily a polynomial in q for G of type E_8, and remark that [PS, Thm. 1.4] suggests that this might not be the case for general G.

We end by noting that our calculation of $\mathrm{mod}(B : \mathfrak{u})$ can be used to determine the dimension of the commuting varieties of $\mathfrak{u}$ and $\mathfrak{b}$ as are studied in [GR2] and [GG], respectively.

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School of Mathematics, University of Birmingham, Birmingham, B15 2TT, United Kingdom
E-mail address: s.m.goodwin@bham.ac.uk

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
E-mail address: peter.mosch@rub.de
E-mail address: gerhard.roehrle@rub.de