

COMPUTATIONAL EXPLORATIONS OF THE THOMPSON GROUP T FOR THE AMENABILITY PROBLEM OF F

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ABSTRACT. It is a long standing open problem whether the Thompson group F is an amenable group. In this paper we show that if A, B, C denote the standard generators of Thompson group T and $D := CBA^{-1}$ then

$$\sqrt{2} + \sqrt{3} < \frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \leq 2 + \sqrt{2}.$$

Moreover, the upper bound is attained if the Thompson group F is amenable. Here, the norm of an element in the group ring $\mathbb{C}T$ is computed in $B(\ell^2(T))$ via the regular representation of T . Using the “cyclic reduced” numbers $\tau(((C + C^2)(D + D^2 + D^3))^n)$, $n \in \mathbb{N}$, and some methods from our previous paper [10] we can obtain precise lower bounds as well as good estimates of the spectral distributions of $\frac{1}{12}((I + C + C^2)(I + D + D^2 + D^3))^*(I + C + C^2)(I + D + D^2 + D^3)$, where τ is the tracial state on the group von Neumann algebra $L(T)$. Our extensive numerical computations suggest that

$$\frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \approx 3.28,$$

and thus that F might be non-amenable. However, we can in no way rule out that $\frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| = 2 + \sqrt{2}$.

1. INTRODUCTION

Definition 1.1. The Thompson group T is the group of (cyclic) order preserving homeomorphisms $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ for which:

- f and f^{-1} are piecewise linear with finitely many breakpoints.
- all breakpoints of f and f^{-1} are in $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.
- all slopes of f are in the set $2^{\mathbb{Z}} := \{2^n \mid n \in \mathbb{Z}\}$.

T is a countable group. It is generated by the elements C, D , whose graphs are shown in Fig. 1. Moreover, it has the finite presentation

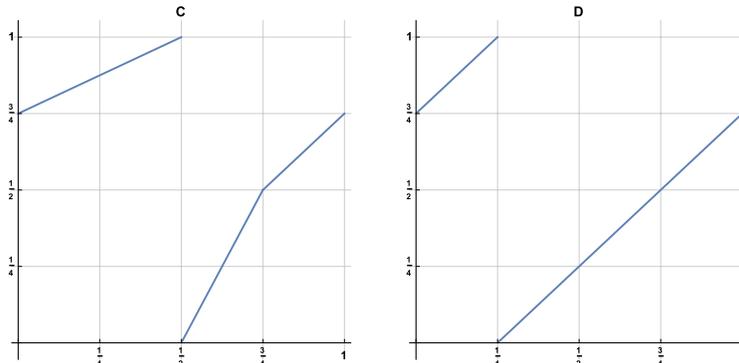
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FIGURE 1. The generators C, D of the Thompson group T .

$$(1) \quad T = \langle C, D \mid C^3 = D^4 = (CD)^5 = e, \\ [CDC, D^2(CDC)D^2] = [D^2(CDC)D^2, C(D^2CDCD^2)C^{-1}] = e \rangle,$$

where the group commutator is given by the definition $[g, h] = ghg^{-1}h^{-1}$. If A, B, C denote the standard generators of T [6], then $D = CBA^{-1}$.

Recall that F is the subgroup of T given by

$$F = \langle A, B \rangle = \{f \in T \mid f(0) = 0\}.$$

It is a long standing open problem whether the Thompson group F is amenable. The present paper is a continuation of the work started in [10] by the same authors. In that paper, we tested the amenability problem of F by estimating norms of certain elements in the group ring $\mathbb{C}F$ using computers. Thanks to the new algorithms we devised to compute words in C, D in polynomial time (see Section 2) and to some results by Haagerup and Olesen [12], we can now test the amenability problem of F by computing norms of certain elements in $\mathbb{C}T$. Extrapolations of our computational results suggest the same as our previous paper, namely, that F might not be amenable. A recent experimental work by Elder, Rechnitzer and Janse van Reusburg [7] on the amenability problem of F using statistical methods arrives also to the same conclusion that F might be non-amenable (see also [8]).

As in [10, Section 2], by the norm of $\|a\|$ of an element a in the group ring of a discrete group Γ we mean

$$\|a\| = \|\lambda(a)\|_{B(\ell^2(\Gamma))},$$

where λ is the left regular representation of Γ . We continue using the standard convention of writing $a \in B(\ell^2(\Gamma))$ instead of $\lambda(a) \in B(\ell^2(\Gamma))$ for any $a \in \mathbb{C}\Gamma$. Our starting point is the following theorem (see Section 2 for more details)

Theorem 1.2. *Let C, D be the generators of T , whose graphs are shown in Fig. 1, and let I denote the unit element of T . Then*

$$\sqrt{2} + \sqrt{3} < \frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \leq 2 + \sqrt{2}.$$

Moreover, the upper bound is attained if the Thompson group F is amenable.

In comparison, one gets for the free product $\mathbb{Z}_3 \star \mathbb{Z}_4$ on two generators $c \in \mathbb{Z}_3, d \in \mathbb{Z}_4$ that

$$(2) \quad \frac{1}{\sqrt{12}} \|(e + c + c^2)(e + d + d^2 + d^3)\| = \sqrt{2} + \sqrt{3}.$$

(cf. Section 4). Given a discrete group Γ , let $L(\Gamma)$ denote its von Neumann algebra. That is, $L(\Gamma)$ is the von Neumann algebra in $B(\ell^2(\Gamma))$ generated by Γ . Then we can define the normal faithful tracial state τ on $L(\Gamma)$ by

$$\tau(h) := \langle h\delta_e, \delta_e \rangle, \quad h \in L(\Gamma).$$

(see e.g. [14, Section 6.7]). Moreover, we can express the norm of h in terms of the moments

$$m_n(h^*h) := \tau((h^*h)^n), \quad n \in \mathbb{N}_0,$$

namely, (cf. [10, Section 4])

$$\|h\| = \|h^*h\|^{1/2} = \lim_{n \rightarrow \infty} \tau((h^*h)^n)^{\frac{1}{2n}}.$$

The challenge is to compute all the moments $m_n(h^*h)$, and in practice we can only compute a finite number of them. In this paper, we were able to compute the moments $m_n(h^*h)$, $n = 0, \dots, 28$ for the element

$$h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3),$$

using efficient methods, both mathematically and computationally. This procedure can be adapted to elements in a discrete group Γ that are expressed similarly. Define the self-adjoint operator

$$\tilde{h} := \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix} \in M_2(B(\ell^2(T))),$$

and let $\tilde{\tau} := \tau \otimes \tau_2$, where $\tau_2 := \frac{1}{2}Tr$ on $M_2(\mathbb{C})$. Then

$$\Lambda : C(\sigma(\tilde{h})) \rightarrow \mathbb{C}$$

given by $\Lambda(f) := \tilde{\tau}(f(\tilde{h}))$ is a positive functional on $C(\sigma(\tilde{h}))$ such that $\Lambda(1) = 1$. Hence there is a unique Borel probability measure for which

$$\tilde{\tau}(f(\tilde{h})) = \int_{\sigma(\tilde{h})} f d\mu, \quad \forall f \in C(\sigma(\tilde{h}))$$

Since $\tilde{\tau}$ is faithful, the support $\text{supp}(\mu) = \sigma(\tilde{h}) \subset [-\|h\|, \|h\|]$. Such measure μ is invariant under the reflection $t \mapsto -t$ because the odd moments of \tilde{h} are zero, and it satisfies

$$\int_{-\|h\|}^{\|h\|} t^{2n} d\mu(t) = m_{2n}(\tilde{h}) = m_n(h^*h), \quad n \in \mathbb{N}_0$$

(cf. [10, Section 2]). Moreover, $\pm\|h\| \in \text{supp}(\mu)$ by symmetry of μ . The theory of orthonormal polynomials applied to this measure (cf. [10, Section 4]) together with the moments yield an increasing sequence of numbers converging to $\|h\|$. We can calculate the first 28 of these numbers using our computed moments, and a suitable extrapolation of these numbers ($n = 0, \dots, 28$) gives

$$3.2016 \leq \frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \approx 3.28.$$

We found that this sequence of numbers actually gives much better lower bounds than the sequence of “roots” and “ratios” of the moments that also converge to the norm (cf. Section 5). Furthermore, we also estimate the Lebesgue density of μ with fairly high precision, which shows that the measure μ is very close to zero on the interval $[3.22, 2 + \sqrt{2}]$. However, we cannot rule out that the measure has very “thin tail” stretching all the way up to $2 + \sqrt{2}$.

In comparison, one gets Eq. (2) when one considers the free product $\mathbb{Z}_3 \star \mathbb{Z}_4$ on two generators $c \in \mathbb{Z}_3, d \in \mathbb{Z}_4$. The measure μ based on c, d instead of C, D will be denoted by μ_{free} , and it can be computed explicitly (See Section 4)

$$(3) \quad \mu_{\text{free}} = \frac{1}{2\pi} \frac{\sqrt{24 - (x^2 - 5)^2}}{x(12 - x^2)} \mathbf{1}_{[-\sqrt{2}-\sqrt{3}, \sqrt{2}-\sqrt{3}] \cup [\sqrt{3}-\sqrt{2}, \sqrt{2}+\sqrt{3}]}(x) dx + \frac{3}{4} \delta_0.$$

In our previous paper, we estimated the norm $3.60613 \leq \|A + B + A^{-1} + B^{-1}\| \approx 3.87$, where A, B are the standard generators of F , by using the first 24 moments of $(A + B + A^{-1} + B^{-1})^2$. Elvey-Price [9] succeeded in computing 7 more moments, which pass the test of [10, Theorem B.1] for possible computational errors. Using these 31 moments, the updated estimated norm remains unchanged, while the new lower bound is

$$3.64271 \leq \|A + B + A^{-1} + B^{-1}\| \approx 3.87,$$

with a very likely lower bound of 3.70211, based on the same list of moments.

In comparison,

$$3.7873 \leq \|C + D + C^{-1} + D^{-1}\| \approx 3.84,$$

which suggests that the actual norm is also much closer to 4, even though it cannot be 4 because the Thompson group T is not amenable [4]. This norm is estimated in the same way as we did with the norm $\|A + B + A^{-1} + B^{-1}\|$ (cf. Appendix B). The composition of words in C, D is given in Section 2, and it was inspired by the Belk and Brown forest algorithm in [3] (see also [2]).

2. ACTION OF C, D , THE CYCLIC REDUCED NUMBERS ζ_n , AND AMENABILITY.

Since the elements of the Thompson group T are piecewise linear functions with breakpoints on the dyadics, it makes sense to introduce the set of dyadic intervals

$$\mathcal{D} := \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right] \mid n \in \mathbb{N}_0, k = 0, \dots, 2^n - 1 \right\}.$$

Note that the unit interval is the largest dyadic interval, and cutting a dyadic interval into two equal halves gives two new dyadic intervals. Then we write the elements of T as functions $\mathcal{D} \rightarrow \mathcal{D}$. For instance, the element C whose graph is in Figure 1 is determined by the function

$$f_C : \{d_1, d_2, d_3\} \rightarrow \{r_1, r_2, r_3\}$$

given by

$$f_C(d_1) = r_3 \quad f_C(d_2) = r_1 \quad f_C(d_3) = r_2,$$

where the dyadic intervals $d_1 := [0, \frac{1}{2}]$, $d_2 := [\frac{1}{2}, \frac{3}{4}]$, $d_3 := [\frac{3}{4}, 1]$ in this case are obtained from the breakpoints of C and note that they only overlap on their boundaries. Similarly $r_1 := [0, \frac{1}{2}]$, $r_2 := [\frac{1}{2}, \frac{3}{4}]$, $r_3 := [\frac{3}{4}, 1]$ are obtained from the breakpoints of C^{-1} .

Observe that \mathcal{D} is not closed under union, e.g. the union of the dyadic intervals $[\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}] \in \mathcal{D}$ is the interval $[\frac{1}{4}, \frac{3}{4}] \notin \mathcal{D}$ which is not dyadic. This asymmetry

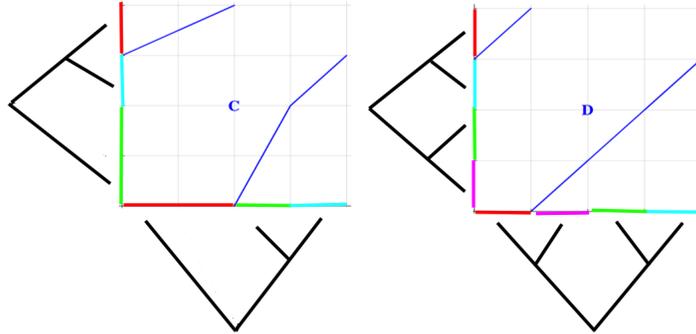


FIGURE 2. Construction of the doubletrees for the generators C, D of the Thompson group T .

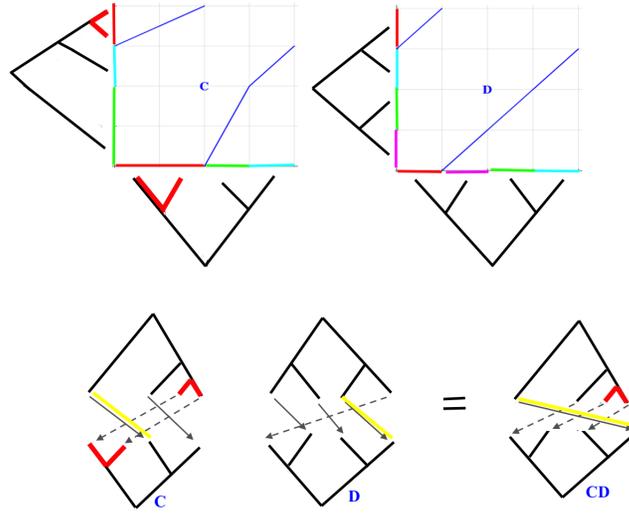


FIGURE 3. Doubletree composition.

is used to construct, for a given $x \in T$, a so-called domain tree and range tree with vertices in \mathcal{D} . Namely for the domain tree, we start with a subset L of diadic intervals of \mathcal{D} which only intersect on their boundaries and whose union is the unit interval and where the set of boundary points contains all the breakpoints of x . We then represent each diadic interval with a dot, and we join two of these dots with a caret whenever the union is also a diadic interval provided that the two intervals only overlap on their boundaries. The tip of the caret is the joined diadic interval. We do this recursively until we get the unit interval. For instance, in our example, for the domain of f_C , we can join d_2 and d_3 with a caret to obtain the diadic interval $[1/2, 1]$ and then we join this and d_1 with a caret to obtain the diadic interval $[0, 1]$. The domain tree of C is illustrated in Figure 2. To get the corresponding range tree we do the same construction for the diadic intervals $x(L) := \{x(d) \mid d \in L\}$.

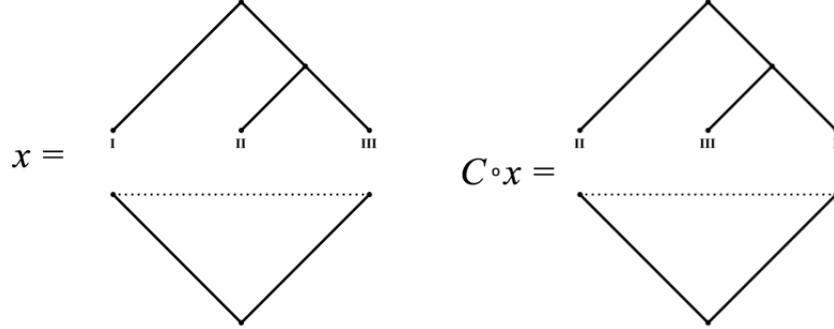


FIGURE 4. Case 1C: Trees I, II, III appear in the range tree of $x \in T$. The action of C simply rotates these trees (and each tree keeps its own leaves).

The domain tree (resp. range tree) of $x \in T$ will be denoted with $d(x)$ (resp. $r(x)$).

Definition 2.1 (Doubletree). The *doubletree* of $x \in T$ is the pair of trees $r(x)$, $d(x)$ together with the permutation (a rotation) $\sigma : \text{leaves}(r(x)) \rightarrow \text{leaves}(d(x))$ of its leaves.

For instance in our example, the doubletree of C is shown in Figure 3. Composition of doubletrees is done as if we were composing its corresponding piecewise linear maps. However, it might be necessary to subdivide the diadic intervals in domain and range to accomplish this task. (This corresponds to replace a domain leaf and its corresponding range leaf with a double caret). For instance, if we compose $C \circ D$ using the doubletrees shown in Figure 2, we see that the range diadic interval $[0, 1/4]$ of D is too small to be composed with the domain diadic interval $[0, 1/2]$ of C . But once we subdivide $[0, 1/2] \in \text{leaves}(d(C))$ into $[0, 1/4]$, $[1/4, 1/2]$ (and subdivide as well the image under C) then the composition can be done. See Figure 3.

A doubletree with no double-carets is said to be *reduced*.

2.1. The action of C and D on any element of T . We investigate the double-tree composition $C \circ x$, $x \in T$ in three cases, where the doubletrees of C and x are assumed to be reduced (i.e. no double-carets).

Case 1C: (non-degenerate)

Assume that the root of the range tree $r(x)$ has a left and a right node. The tree starting at the left node is called tree I. Assume that the right node has a left node and a right node. The tree starting at the right-left node is called tree II, and the one starting at the right-right node is called tree III. The trees I, II, III, can be 0-trees, i.e. they can be leaves.

Then the root of the range tree $r(C \circ x)$ has a left and a right node. The tree starting at the left node is tree II. The right node has a left node and a right node. The tree starting at the right-left node is tree III, and the one starting at the right-right node is tree I. The leaves of tree I, which map to leaves of the domain tree $d(x)$, stay with the tree, and the same holds for the other two trees II, III. See Figure 4. The rotation permutation starts with $\sigma(C \circ x)(1) = \sigma(x)(\sigma(C)(1))$.

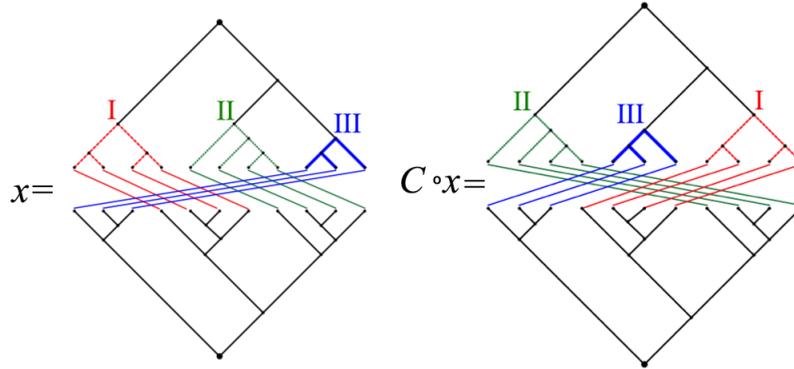


FIGURE 5. Composition $C \circ x$, where $x \in T$ satisfies the hypothesis of case 1C (non-degenerate).

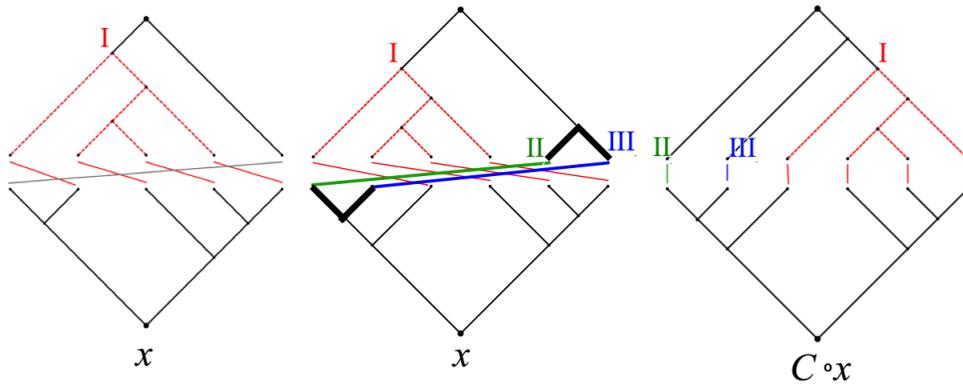


FIGURE 6. Composition $C \circ x$, where $x \in T$ satisfies the hypothesis of case 2C (degenerate).

Reduction occurs only if tree III and tree I are 0-trees and the leaves σ_{III} , σ_I on the domain form a caret. Reduction occurs again only if tree II is also a 0-tree; in such case we have the identity. An example of case 1C, is shown in Figure 5.

Case 2C:(degenerate, missing tree II)

Assume that the root of the range tree $r(x)$ has a left and a right node. Assume that the right node is a 0-tree (a leaf), which maps to another leaf of the domain tree $d(x)$. Insert a double caret at these two leaves and proceed as in case 1C. An example for this case is shown in Figure 6.

Case 3C:(degenerate, missing trees, I,II,III)

The element x is the identity. Thus $C \circ x = C$.

The composition $D \circ x$, $x \in T$ is done in 5 cases, where the trees of D and x are assumed to be reduced.

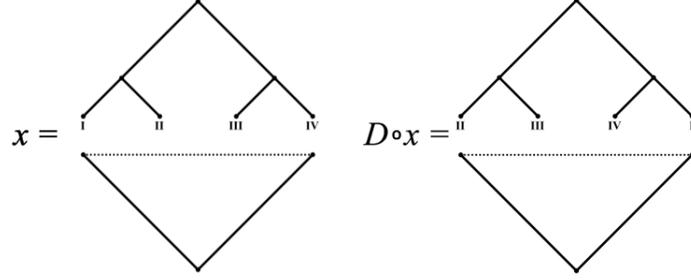


FIGURE 7. Case 1D: Trees I, II, III, IV appear in the range tree of $x \in T$. The action of D simply rotates these trees (and each tree keeps its own leaves).

Case 1D: (non-degenerate)

Assume that the root of the range tree $r(x)$ has a left and a right node, which both have also a left node and a right node. The tree starting at the left-left node is called tree I. The one starting at the left-right node is called tree II. The one starting at the right-left node is called tree III, and the one starting at the right-right node is called tree IV. The trees I, II, III, IV can be 0-trees, i.e. they can be leaves.

Then, the root of the range tree $r(D \circ x)$ has a left and a right node, which both have also left and right nodes. The tree starting at the left-left node is tree II. The tree starting at the left-right node is tree III. The tree starting at the right-left node is tree IV. The tree starting at the right-right node is tree I. The leaves of tree I, which map to leaves of the domain tree $d(x)$, stay with the tree, and the same holds for the other three trees II, III, IV. See Figure 7. The rotation permutation starts with $\sigma(D \circ x)(1) = \sigma(x)(\sigma(D)(1))$.

Reduction occurs only in the following two cases:

- (a) II and III are 0-trees and the leaves $\sigma_{II}, \sigma_{III}$ on the domain form a caret.
- (b) IV and I are 0-trees and the leaves σ_{IV}, σ_I on the domain form a caret.

Reduction occurs again, only if both (a) and (b) occur; in such case we have the identity. An example of case 1D with reduction (a) is shown in Figure 8.

Case 2D: (degenerate, missing tree II)

Assume that the root of the range tree $r(x)$ has a left and a right node. Assume that the left node is a leaf which we call v , while the right node has a left and a right node. The leaf v maps to a leaf of the domain tree $d(x)$. Insert a double caret at these two leaves. Then proceed as in case 1D. An example is shown in Figure 9.

Case 3D: (degenerate, missing tree III)

Assume that the root of the range tree $r(x)$ has a left and a right node. Assume that the right node is a leaf which we call v , while the left node has a left and a right node. The leaf v maps to a leaf of the domain tree $d(x)$. Insert a double caret at these two leaves. Then proceed as in case 1D. An example is shown in Figure 10.

Case 4D: (degenerate, missing trees II, III)

Assume that the root of the range tree $r(x)$ has a left and a right node, such that both trees are 0-trees. Since x is assumed to be reduced, $x = D^2$. Thus $D \circ x = D^3 = D^{-1}$.

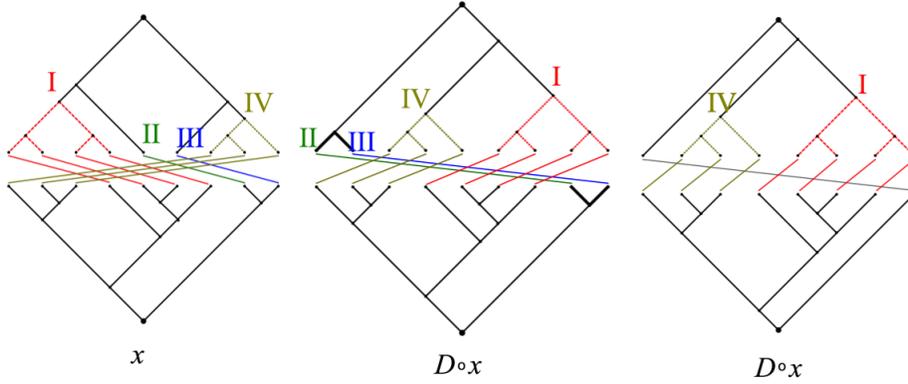


FIGURE 8. Composition $D \circ x$, where $x \in T$ satisfies the hypothesis of case 1D (non-degenerate).

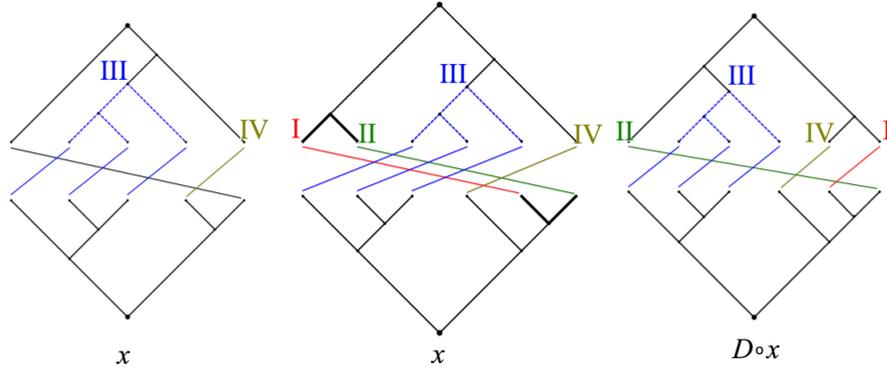


FIGURE 9. Composition $D \circ x$, where $x \in T$ satisfies the hypothesis of case 2D (degenerate).

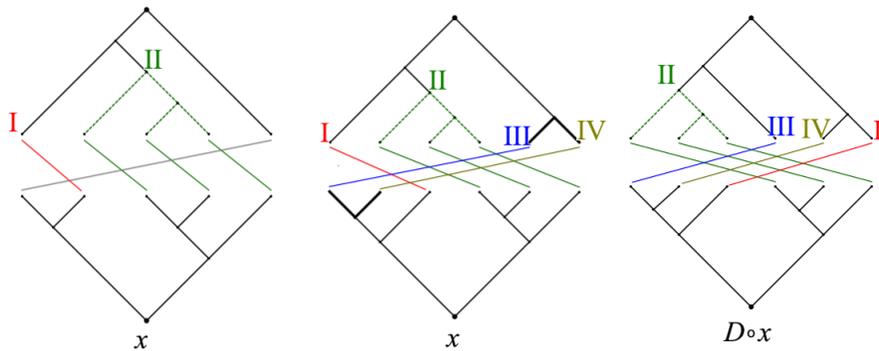


FIGURE 10. Composition $D \circ x$, where $x \in T$ satisfies the hypothesis of case 3D (degenerate).

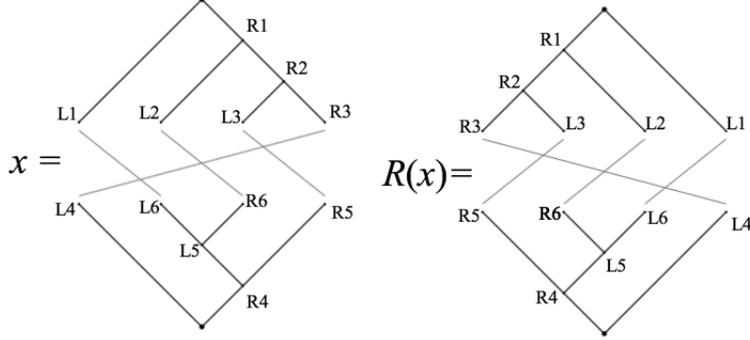


FIGURE 11. The rotation operator R on doubletrees swaps the nodes of every caret in the domain and range trees.

Case 5D: (degenerate, missing trees I, II, III, IV)

The element x is the identity. Thus $D \circ x = D$.

2.2. **The cyclic reduced numbers ζ_n .** Let $R : T \rightarrow T$, be defined by

$$R(x)(t) = 1 - x(1 - t), \quad x \in T, t \in \mathbb{R}/\mathbb{Z}.$$

Then the graph of $R(x)$ is the graph of x rotated by 180 degrees. The rotation map R is a group isomorphism, and

$$(4) \quad R \circ R = id, \quad C^{-1} = D^2 R(C) D^2, \quad D^{-1} = D^2 R(D) D^2.$$

The doubletree of $R(x)$, $x \in T$, is obtained by swapping all the nodes of every caret in the domain and range trees of x , i.e. left node (resp. right node) becomes right node (resp. left node). An example is shown in Figure 11.

Define

$$a := C + C^2 \quad b := D + D^2 + D^3,$$

and recall that $C^3 = I$, $D^4 = I$. Since $C, D \in L(T)$ are unitaries, $C^* = C^{-1}$ and $D^* = D^{-1}$. Moreover, a and b are self-adjoint, i.e. $a^* = a$ and $b^* = b$. Define the “cyclic reduced” numbers

$$\zeta_n := \tau((ab)^n), \quad n \in \mathbb{N}.$$

Define $N_0 := \{D, D^2, D^{-1}\}$. For $n \in \mathbb{N}$ let,

$$N_n := \{(t_0, s_1, t_1, \dots, s_n, t_n) \mid s_j \in \{C, C^{-1}\}, j = 1, \dots, n, \\ t_i \in \{D, D^2, D^{-1}\}, i = 0, \dots, n\},$$

and note that

$$b(ab)^n = \sum_{(t_0, s_1, t_1, \dots, s_n, t_n) \in N_n} t_0(s_1 t_1 \cdots s_n t_n), \quad n \in \mathbb{N}_0.$$

We define the product of a tuple as follows

$$(5) \quad \pi((t_1, t_2, \dots, t_n)) := t_1 t_2 \cdots t_n, \quad t_1, \dots, t_n \in \{C, D, C^{-1}, D^2, D^{-1}\},$$

and the inverse of each entry of a tuple as

$$(6) \quad J((t_1, t_2, \dots, t_n)) := (t_1^{-1}, t_2^{-1}, \dots, t_n^{-1}).$$

For $w = (t_1, \dots, t_n)$ it holds by Eq. (4)

$$(7) \quad \pi(J(w)) = D^2 R(\pi(w)) D^2.$$

We say that the word $\pi(J(w))$ is the “reverse-inverse” of the word $\pi(w)$. For example, if $w = (C, D, C^2)$ then $\pi(J(w)) = C^{-1} D^{-1} C^{-2}$, which corresponds to first reversing the letters of the word CDC^2 and then taking the inverse. Define for $n \in \mathbb{N}$,

$$\begin{aligned} S_n &:= \{w, J(w') \mid w, w' \in \{C\} \times N_{n-1}, \pi(w) < \pi(J(w)), \pi(J(w')) < \pi(w')\} \\ S'_n &:= \{w, J(w') \mid w, w' \in N_{n-1} \times \{C^{-1}\}, \pi(w) < \pi(J(w)), \pi(J(w')) < \pi(w')\} \\ E_n &:= \{w \in \{C\} \times N_{n-1} \mid \pi(w) = \pi(J(w))\}. \\ E'_n &:= \{w \in N_{n-1} \times \{C^{-1}\} \mid \pi(w) = \pi(J(w))\}. \end{aligned}$$

Here the comparison $\pi(w) < \pi(J(w))$ is done by comparing the lexicographic order of their corresponding serialized (e.g. preorder) reduced doubletrees. The letter S_n stands for smaller, e.g. that the word $\pi(w)$ is smaller than its reverse-inverse $\pi(J(w))$. Similarly, the letter E_n stands for equal. Since $b(ab)^n$ is self-adjoint, $w \in \{C\} \times N_n$ if and only if $w^{-1} \in N_n \times \{C^{-1}\}$, where w^{-1} is the inverse of each entry in the tuple w and in reverse order. Define for $n \in \mathbb{N}$,

$$(8) \quad \zeta_{2n}^s := \sum_{(x,y) \in S_n \times S'_n} \langle x, y \rangle, \quad \zeta_{2n}^e := \sum_{(x,y) \in E_n \times E'_n} \langle x, y \rangle$$

$$(9) \quad \zeta_{2n+1}^s := \sum_{(x,y) \in S_n \times S'_{n+1}} \langle x, y \rangle, \quad \zeta_{2n+1}^e := \sum_{(x,y) \in E_n \times E'_{n+1}} \langle x, y \rangle,$$

where $\langle x, y \rangle := \tau(y^*x)$ is the inner product on $L(T)$ associated with the trace τ . Then

Lemma 2.2.

$$\zeta_n = 2\zeta_n^s + 4\zeta_n^e, \quad n \geq 2.$$

Proof. For $n \in \mathbb{N}$, define

$$X_n := \{(s_1, t_1, \dots, s_n, t_n) \mid s_i \in \{C, C^{-1}\}, t_i \in \{D, D^2, D^{-1}\}, i = 1, \dots, n\}.$$

Then

$$(ab)^n = \sum_{x \in X_n} \pi(x).$$

Since $(ab)^n = Cb(ab)^{n-1} + C^{-1}b(ab)^{n-1}$, we can partition X_n as follows

$$\begin{aligned} X_n &= \{w, J(w) \mid w \in \{C\} \times N_{n-1}\} \\ &= S_n \sqcup L_n \sqcup E_n \sqcup J(E_n), \end{aligned}$$

where

$$L_n := \{w, J(w') \mid w, w' \in \{C\} \times N_{n-1}, \pi(w) > \pi(J(w)), \pi(J(w')) > \pi(w')\}.$$

Thus

$$(ab)^n = \sum_{x \in S_n} \pi(x) + \sum_{x \in L_n} \pi(x) + 2 \sum_{x \in E_n} \pi(x).$$

Similarly,

$$(ba)^n = \sum_{x \in S'_n} \pi(x) + \sum_{x \in L'_n} \pi(x) + 2 \sum_{x \in E'_n} \pi(x),$$

where

$$L'_n := \{w, J(w') \mid w, w' \in N_{n-1} \times \{C^{-1}\}, \pi(w) > \pi(J(w)), \pi(J(w')) > \pi(w')\}.$$

The inner product $\langle \sum_{S_n} \pi(x), \sum_{E'_n} \pi(x) \rangle = 0$ because if there is a $w \in S_n, w' \in E'_n$, such that $\pi(w) = \pi(w')$ then by Eq. (7), $\pi(J(w)) = \pi(J(w'))$, and thus $\pi(w) < \pi(J(w)) = \pi(J(w')) = \pi(w') = \pi(w)$, a contradiction.

Similarly, the inner product $\langle \sum_{S_n} \pi(x), \sum_{L'_n} \pi(x) \rangle = 0$ because if there is a $w \in S_n, w' \in L'_n$, such that $\pi(w) = \pi(w')$ then by Eq. (7) $\pi(w) < \pi(J(w)) = \pi(J(w')) < \pi(w') = \pi(w)$, a contradiction. The innerproducts

$$\left\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_n} \pi(x) \right\rangle = \left\langle \sum_{x \in L_n} \pi(x), \sum_{x \in L'_n} \pi(x) \right\rangle$$

coincide because $L_n = J(S_n), L'_n = J(S'_n)$ and because by Eq. (7)

$$\pi(x) = \pi(x') \iff \pi(J(x)) = \pi(J(x')).$$

It follows that for $n \in \mathbb{N}$,

$$\begin{aligned} \zeta_{2n} &= \langle (ab)^n, (ba)^n \rangle \\ &= \left\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_n} \pi(x) \right\rangle + \left\langle \sum_{x \in L_n} \pi(x), \sum_{x \in L'_n} \pi(x) \right\rangle + 4 \left\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_n} \pi(x) \right\rangle \\ &= 2 \left\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_n} \pi(x) \right\rangle + 4 \left\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_n} \pi(x) \right\rangle \\ &= 2\zeta_{2n}^s + 4\zeta_{2n}^e, \end{aligned}$$

and

$$\begin{aligned} \zeta_{2n+1} &= \langle (ab)^n, (ba)^{n+1} \rangle \\ &= 2 \left\langle \sum_{x \in S_n} \pi(x), \sum_{x \in S'_{n+1}} \pi(x) \right\rangle + 4 \left\langle \sum_{x \in E_n} \pi(x), \sum_{x \in E'_{n+1}} \pi(x) \right\rangle \\ &= 2\zeta_{2n+1}^s + 4\zeta_{2n+1}^e. \end{aligned}$$

□

2.3. Amenability.

Definition 2.3 (Amenability). A discrete group Γ is said to be amenable if there exists a finitely additive measure $\mu : P(\Gamma) \rightarrow [0, 1]$ with $\mu(\Gamma) = 1$ such that

$$\mu(xA) = \mu(A), \quad \text{for all } A \in P(\Gamma), x \in \Gamma,$$

where $P(\Gamma)$ denotes the power set of Γ .

Let $P, Q, R \in L(T)$ be the projections given by

$$(10) \quad P := \frac{I + S}{2} \quad Q := \frac{I + C + C^2}{3} \quad R := \frac{I + D + D^2 + D^3}{4},$$

where $S := D^2$.

Theorem 2.4. *Let $P, Q, R \in L(T)$ be the projections in Eq. (10), which are defined in terms of the generators C, D of Thompson group T . Then*

$$\|QR\| \leq \frac{1 + \sqrt{2}}{\sqrt{6}}.$$

Moreover, the upper bound is attained if the Thompson group F is amenable.

Proof. It is well-known that

$$T \supset \langle S, C \rangle = \mathbb{Z}_2 \star \mathbb{Z}_3 \cong PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) / \{\pm I\},$$

(cf. [13, p.11, Example 1.5.3]). By [5, Proposition 2.5.9] the reduced C^* -algebra of any subgroup H of a (discrete) group G is a subalgebra of $C_r^*(G)$ in a canonical way, namely, one takes the left regular representation of G and restricts it to H . Hence

$$C_r^*(\mathbb{Z}_2 \star \mathbb{Z}_3) = C_r^*(\langle S, C \rangle) \subset C_r^*(T).$$

By this and Example 4.2 (or [1, Remark 15]) we get

$$\|PQ\| = \frac{1 + \sqrt{2}}{\sqrt{6}}.$$

By spectral reasons

$$(11) \quad R \leq P.$$

Hence $0 \leq QRQ \leq QPQ$, and by the C^* -identity (i.e. $\|x^*x\| = \|x\|^2$) we get

$$\|QR\|^2 = \|QRQ\| \leq \|QPQ\|.$$

Since the norm of an element is the same as the norm of its adjoint (i.e. $\|x^*\| = \|x\|$),

$$\|QPQ\|^{1/2} = \|PQP\|^{1/2} = \|PQ\| = \frac{1 + \sqrt{2}}{\sqrt{6}}.$$

Thus

$$(12) \quad \|QR\| \leq \frac{1 + \sqrt{2}}{\sqrt{6}} = 0.97 \dots$$

We will now prove that $\|QR\| \geq \frac{1 + \sqrt{2}}{\sqrt{6}}$ if F is amenable. Let $H := \ell^2(\mathbb{Z}[1/2] \cap [0, 1))$, and let $\pi : T \rightarrow B(H)$ be the representation given by

$$\pi(x)\delta_t = \delta_{x(t)}, \quad x \in T, \quad t \in \mathbb{Z}[1/2] \cap [0, 1),$$

where the set $\{\delta_t \mid t \in \mathbb{Z}[1/2] \cap [0, 1)\}$ is an orthonormal basis for the Hilbert space H . Let $S_1, S_2 \in B(H)$ be the isometries defined by

$$S_1\delta_t := \delta_{t/2}, \quad S_2\delta_t := \delta_{(t+1)/2}, \quad t \in \mathbb{Z}[1/2] \cap [0, 1),$$

and note that they satisfy the relation $S_1S_1^* + S_2S_2^* = I$. Then by [12, Proposition 4.3], $\pi(\mathbb{C}T)$ is contained in the Cuntz algebra $\mathcal{O}_2 = C^*(S_1, S_2)$. If F is amenable then by [12, Proposition 4.4], π is weakly contained in the left regular representation of T . This is equivalent to the existence of a $*$ -homomorphism ρ from the reduced C^* -algebra $C_r^*(T)$ into the C^* -algebra generated by $\pi(T)$ such that $\pi = \rho \circ \lambda$. Hence

$$\|\pi(x)\|_{B(H)} = \|\rho(\lambda(x))\|_{B(H)} \leq \|\lambda(x)\|_{C_r^*(T)}, \quad x \in \mathbb{C}T.$$

Thus amenability of F implies that the norm $\|QR\|$ is greater or equal than the norm $\|\pi(QR)\|$. Thus, it suffices to show that $\|\pi(QR)\| = \frac{1+\sqrt{2}}{\sqrt{6}}$. We have $\|\pi(QR)\|^2 = \|\pi(RQR)\|$, and

$$\pi(RQR) = \frac{1}{48}(S_1 + S_2)^2(2I + S_1 + S_2)^*(2I + S_1 + S_2)(S_1 + S_2)^{*2}.$$

Substituting $x := \frac{1}{\sqrt{2}}(S_1 + S_2)$ gives

$$\pi(RQR) = \frac{1}{6}x^2(\sqrt{2}I + x)^*(\sqrt{2}I + x)x^{*2}.$$

Since x is a proper isometry (i.e. $x^*x = I \neq xx^*$),

$$\|x^2(\sqrt{2}I + x)^*(\sqrt{2}I + x)x^{*2}\| = \|(\sqrt{2}I + x)^*(\sqrt{2}I + x)\|.$$

Consequently,

$$\|\pi(RQR)\| = \frac{1}{6}\|(\sqrt{2}I + x)^*(\sqrt{2}I + x)\| = \frac{1}{6}\|\sqrt{2}I + x\|^2.$$

Now, the spectrum of x is the entire unit disc because x is a proper isometry. (One uses the Wold decomposition to show this). Thus the spectral radius of $\sqrt{2}I + x$ is $\sqrt{2} + 1$. Hence the norm is at least this number and by the triangle inequality we get

$$\|\sqrt{2}I + x\| = 1 + \sqrt{2},$$

as required. \square

3. TWO PROJECTIONS

Let $\Gamma := \langle c, d \rangle$ be a discrete group on two generators such that c is of order k and d of order ℓ , for some integers $\ell \geq k \geq 2$. Let $L(\Gamma)$ denote its von Neumann algebra, i.e. $L(\Gamma)$ is the von Neumann algebra in $B(\ell^2(\Gamma))$ generated by Γ . Then

$$\tau(h) = \langle h\delta_e, \delta_e \rangle, \quad h \in L(\Gamma)$$

defines a normal faithful tracial state on $L(\Gamma)$ (see e.g. [23, Section 6.7]). Let

$$P := \frac{e + c + \cdots + c^{k-1}}{k} \in \mathbb{C}\Gamma \subset C_r^*(\Gamma) \subset L(\Gamma) \subset B(\ell^2(\Gamma))$$

$$Q := \frac{1 + d + \cdots + d^{\ell-1}}{\ell} \in \mathbb{C}\Gamma \subset C_r^*(\Gamma) \subset L(\Gamma) \subset B(\ell^2(\Gamma))$$

be fixed projections such that

$$\|PQ\| < 1.$$

Let

$$\alpha := \tau(P) = \frac{1}{k}, \quad \beta := \tau(Q) = \frac{1}{\ell}$$

and notice that

$$0 < \beta \leq \alpha \leq \frac{1}{2}.$$

It follows by P. R. Halmos' work [11] that the C^* -algebra $A := C^*(P, Q, I)$ generated by 2 projections P, Q and the identity I is the direct sum of an abelian C^* -algebra and a C^* -algebra of type I_2 i.e. of the form $C(X) \otimes M_2(\mathbb{C})$ where X equals the spectrum of a certain positive contraction. The abelian part is of dimension at most 4 and its minimal projections are given as

$$e_{00} := I - P \vee Q = (I - P) \wedge (I - Q)$$

$$e_{11} := P \wedge Q \quad e_{10} := P \wedge (I - Q) \quad e_{01} := (I - P) \wedge Q,$$

where some, and even all may vanish. The central support for the I_2 part of A is denoted f and it is given by

$$(13) \quad f = I - e_{11} - e_{10} - e_{01} - e_{00}.$$

We define

$$(14) \quad R := P - e_{11} - e_{10}, \quad S := Q - e_{11} - e_{01}$$

and see that R and S are projections satisfying $R \leq f$, $S \leq f$ because $Pe_{1i} = e_{1i}P = e_{1i}$, $Pe_{0i} = e_{0i}P = 0$ and $Qe_{i1} = e_{i1}Q = e_{i1}$, $Qe_{i0} = e_{i0}Q = 0$, $i = 0, 1$. Moreover,

$$R \wedge S = 0 \quad R \vee S = f.$$

So inside fAf , the pair of projections R, S actually fit the description given by Halmos. Thus, there exists an isomorphism Φ of fAf onto $C(\sigma(RSR)) \otimes M_2(\mathbb{C}) = M_2(C(\sigma(RSR)))$, such that

$$\begin{aligned} \Phi(f) &= I_{C(\sigma(RSR))} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \Phi(R) &= I_{C(\sigma(RSR))} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\Phi(S) = \begin{pmatrix} RSR & (RSR)^{1/2}(f - RSR)^{1/2} \\ (RSR)^{1/2}(f - RSR)^{1/2} & f - RSR \end{pmatrix}.$$

Now,

$$\begin{aligned} P - \alpha I &= (R - \alpha f) + (1 - \alpha)e_{11} + (1 - \alpha)e_{10} - \alpha e_{01} - \alpha e_{00} \\ Q - \beta I &= (S - \beta f) + (1 - \beta)e_{11} + (1 - \beta)e_{01} - \beta e_{10} - \beta e_{00} \\ (P - \alpha I)(Q - \beta I) &= (R - \alpha f)(S - \beta f) + (1 - \alpha)(1 - \beta)e_{11} - \alpha(1 - \beta)e_{01} - \beta(1 - \alpha)e_{10} + \\ &\quad (-\alpha)(-\beta)e_{00}. \end{aligned}$$

Thus since $Re_{ij} = Se_{ij} = fe_{ij} = 0$ for $i, j \in \{0, 1\}$ we get

$$(15) \quad \begin{aligned} \tau\left(\left((P - \alpha I)(Q - \beta I)\right)^n\right) &= \tau\left(\left((R - \alpha f)(S - \beta f)\right)^n\right) + (\alpha\beta)^n \tau(e_{00}) + \\ &\quad (-\alpha(1 - \beta))^n \tau(e_{01}) + (-\beta(1 - \alpha))^n \tau(e_{10}) + \\ &\quad ((1 - \alpha)(1 - \beta))^n \tau(e_{11}). \end{aligned}$$

Similarly, since

$$\begin{aligned} PQ &= RS + e_{11} \\ (I - P)(I - Q) &= (f - R)(f - S) + e_{00} \\ P(I - Q) &= R(f - S) + e_{10} \\ (I - P)Q &= (f - R)S + e_{01} \end{aligned}$$

we get

$$\begin{aligned} \tau((PQ)^n) &= \tau((RS)^n) + \tau(e_{11}) \\ \tau\left(\left((I - P)(I - Q)\right)^n\right) &= \tau\left(\left((f - R)(f - S)\right)^n\right) + \tau(e_{00}) \\ \tau\left(\left(P(I - Q)\right)^n\right) &= \tau\left(\left(R(f - S)\right)^n\right) + \tau(e_{10}) \\ \tau\left(\left((I - P)Q\right)^n\right) &= \tau\left(\left((f - R)S\right)^n\right) + \tau(e_{01}). \end{aligned}$$

Since $\|PQ\| < 1$, $\|e_{11}\| = \|e_{11}PQ\| < 1$ because the norm is submultiplicative, and thus the minimal projection

$$e_{11} = 0,$$

and since $PQP = RSR + e_{11}$, the spectrum

$$\sigma(RSR) = \sigma(PQP) \subset [0, 1].$$

By the continuous functional calculus for the positive element $RSR \in fAf$ we can write

$$\begin{aligned} \Phi(f) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \Phi(R) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \\ \Phi(S) &= \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}, & & 0 < t < 1. \\ \Phi(RSR) &= \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, & & 0 < t < 1. \\ \Phi(R(f-S)R) &= \begin{pmatrix} 1-t & 0 \\ 0 & 0 \end{pmatrix}, & & 0 < t < 1. \\ \Phi((f-R)S(f-R)) &= \begin{pmatrix} 0 & 0 \\ 0 & 1-t \end{pmatrix}, & & 0 < t < 1. \\ \Phi((f-R)(f-S)(f-R)) &= \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, & & 0 < t < 1. \end{aligned}$$

The restriction of τ to fAf is a trace on this algebra, so by the spectral theorem to the positive element RSR , there must exist a positive measure μ on $\sigma(RSR)$ such that for any $x \in fAf$ we have

$$\begin{aligned} \tau(x) &= \int_{t \in \sigma(RSR)} \frac{1}{2} \Phi(x)_{11}(t) + \frac{1}{2} \Phi(x)_{22}(t) d\mu(t) \\ &= \int_0^1 \tau_t(x) d\mu(t) \end{aligned}$$

where

$$\tau_t(x) := \frac{1}{2} \text{Tr}(\Phi(x)(t)),$$

and $\frac{1}{2} \text{Tr}$ is the normalized trace of $M_2(\mathbb{C})$. Thus

$$1 = \tau(I) = \int_0^1 \tau_t(f) d\mu(t) + \tau(e_{11}) + \tau(e_{10}) + \tau(e_{01}) + \tau(e_{00}),$$

and since $\tau((pq)^n) = \tau((pqp)^n)$ for any two projections p, q and any $n \in \mathbb{N}$, we get

$$\begin{aligned} (16) \quad \tau((PQ)^n) &= \int_0^1 \tau_t((RS)^n) d\mu(t) + \tau(e_{11}) \\ \tau(((I-P)(I-Q))^n) &= \int_0^1 \tau_t(((f-R)(f-S))^n) d\mu(t) + \tau(e_{00}) \\ \tau((P(I-Q))^n) &= \int_0^1 \tau_t((R(f-S))^n) d\mu(t) + \tau(e_{10}) \\ \tau(((I-P)Q)^n) &= \int_0^1 \tau_t(((f-R)S)^n) d\mu(t) + \tau(e_{01}). \end{aligned}$$

It is not difficult to show that $\mu(\{0\}) = 0$. We extend μ to a probability measure on $[0, 1]$ by putting

$$\begin{aligned}\mu(\{0\}) &:= \mu_{10} + \mu_{01} \\ \mu(\{1\}) &:= \mu_{00} + \mu_{11},\end{aligned}$$

where

$$\mu_{ij} := \tau(e_{ij}), \quad i, j \in \{0, 1\}.$$

Lemma 3.1. *Define*

$$\nu := \mu - (\mu_{10} - \mu_{01})\delta_0 - (\mu_{00} - \mu_{11})\delta_1.$$

Then ν is a measure on $[0, 1]$ with total mass $\nu([0, 1]) = 2\beta$, and

$$\tau((PQ)^n) = \frac{1}{2} \int_{[0,1]} t^n d\nu(t), \quad n \geq 1.$$

Moreover,

$$\mu_{10} - \mu_{01} = \alpha - \beta \geq 0, \quad \mu_{11} = 0, \quad \mu_{00} = 1 - \alpha - \beta \geq 0, \quad \nu(\{1\}) = 0.$$

Proof. Letting $n = 1$ in Eq. (16), we get

$$\begin{aligned}1 = \tau(I) &= \mu_{00} + \mu_{01} + \mu_{10} + \mu_{11} + \mu((0, 1)). \\ \alpha = \tau(P) &= \tau(P(I - Q)) + \tau(PQ) = \mu_{10} + \mu_{11} + \frac{1}{2}\mu((0, 1)). \\ \beta = \tau(Q) &= \tau((I - P)Q) + \tau(PQ) = \mu_{01} + \mu_{11} + \frac{1}{2}\mu((0, 1)). \\ 1 - \alpha = \tau(I - P) &= \mu_{00} + \mu_{01} + \frac{1}{2}\mu((0, 1)). \\ 1 - \beta = \tau(I - Q) &= \mu_{00} + \mu_{10} + \frac{1}{2}\mu((0, 1)).\end{aligned}$$

Combining these equations we get

$$0 \leq \alpha - \beta = \tau(P) - \tau(Q) = \mu_{10} - \mu_{01}$$

$$0 \leq 1 - \alpha - \beta = \tau(I - P) - \tau(Q) = \mu_{00} - \mu_{11}.$$

Since $\|PQ\| < 1$, $\mu_{11} = 0$. Thus

$$\mu_{00} = 1 - \alpha - \beta.$$

We can write μ_{ij} , $i = 0, 1$ as follows

$$\begin{aligned}\mu_{01} &= \frac{1}{2}\mu(\{0\}) - \frac{\mu_{10} - \mu_{01}}{2} \\ \mu_{10} &= \frac{1}{2}\mu(\{0\}) + \frac{\mu_{10} - \mu_{01}}{2} \\ \mu_{00} &= \frac{1}{2}\mu(\{1\}) + \frac{\mu_{00} - \mu_{11}}{2} \\ \mu_{11} &= \frac{1}{2}\mu(\{1\}) - \frac{\mu_{00} - \mu_{11}}{2},\end{aligned}$$

and the measure ν on $[0, 1]$ defined in the lemma satisfies

$$\begin{aligned}
\nu(\{0\}) &= \mu(\{0\}) - (\mu_{10} - \mu_{01}) \\
\nu(\{1\}) &= \mu(\{1\}) - (\mu_{00} - \mu_{11}) \\
\mu_{01} &= \frac{\nu(\{0\}) + (\mu_{10} - \mu_{01})}{2} - \frac{\mu_{10} - \mu_{01}}{2} = \frac{1}{2}\nu(\{0\}) \\
(17) \quad \mu_{10} &= \frac{1}{2}\nu(\{0\}) + (\mu_{10} - \mu_{01}) \\
\mu_{00} &= \frac{\nu(\{1\}) + (\mu_{00} - \mu_{11})}{2} + \frac{\mu_{00} - \mu_{11}}{2} = \frac{1}{2}\nu(\{1\}) + (\mu_{00} - \mu_{11}) \\
\mu_{11} &= \frac{1}{2}\nu(\{1\}).
\end{aligned}$$

Moreover, the total mass of ν is

$$\begin{aligned}
\nu([0, 1]) &= \nu(\{0\}) + \nu(\{1\}) + \mu((0, 1)) \\
&= 1 - (\mu_{10} - \mu_{01} + \mu_{00} - \mu_{11}) = 1 - (\alpha - \beta + 1 - \alpha - \beta) \\
&= 2\beta,
\end{aligned}$$

and

$$\int_{[0,1]} \frac{1}{2} t^n d\nu(t) = \frac{1}{2} \int_{(0,1)} t^n d\mu(t) + \frac{1}{2} \nu(\{1\}) = \tau((PQ)^n), \quad n \in \mathbb{N}.$$

□

By the following proposition and Lemma 3.1 we can state the “cyclic numbers”

$$\tau(((P - \alpha I)(Q - \beta I))^n), \quad n \in \mathbb{N},$$

in terms of the moments $\tau((PQ)^n)$.

Theorem 3.2. *For $n \in \mathbb{N}$,*

$$\begin{aligned}
\tau(((P - \alpha I)(Q - \beta I))^n) &= (\alpha\beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + \\
&\quad (\alpha(1 - \alpha)\beta(1 - \beta))^{n/2} \int_{[0,1]} T_n\left(\frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1 - \alpha)\beta(1 - \beta)}}\right) d\nu(t),
\end{aligned}$$

where T_n is the Chebyshev polynomial of the first kind.

Proof. Let

$$E := P - \alpha I \quad F := Q - \beta I, \quad E' := R - \alpha f, \quad F' := S - \beta f$$

Then the spectrum $\sigma(E) = \{-\alpha, 1 - \alpha\}$, $\sigma(F) = \{-\beta, 1 - \beta\}$, and by Eq.(15) we get

$$\begin{aligned}
\tau((EF)^n) &= ((-\alpha)(-\beta))^n \mu_{00} + (-\alpha(1 - \beta))^n \mu_{01} + (-\beta(1 - \alpha))^n \mu_{10} + ((1 - \alpha)(1 - \beta))^n \mu_{11} + \\
&\quad \int_{(0,1)} \tau_t((E'F')^n) d\mu(t) \\
&= (\alpha\beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + \nu(\{0\}) \left(\frac{1}{2}(-\beta(1 - \alpha))^n + \frac{1}{2}(-\alpha(1 - \beta))^n\right) + \\
&\quad \frac{1}{2}\nu(\{1\})((1 - \alpha)(1 - \beta))^n + \int_{(0,1)} \tau_t((E'F')^n) d\mu(t) \\
&= (\alpha\beta)^n \mu_{00} + (-\beta(1 - \alpha))^n (\mu_{10} - \mu_{01}) + \int_{[0,1]} \tau_t((E'F')^n) d\nu(t),
\end{aligned}$$

where we used Eq. (17) in the second equality.

To complete the proof we just need to show that

$$\int_{[0,1]} \tau_t((E'F')^n) d\nu(t) = (\alpha(1-\alpha)\beta(1-\beta))^{n/2} \int_{[0,1]} T_n\left(\frac{t-\alpha-\beta+2\alpha\beta}{2\sqrt{\alpha(1-\alpha)\beta(1-\beta)}}\right) d\nu(t).$$

Since

$$\begin{aligned} \Phi(E') &= \Phi(R - \alpha f) = \begin{pmatrix} 1-\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \\ \Phi(F') &= \Phi(S - \beta f) = \begin{pmatrix} t-\beta & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t-\beta \end{pmatrix} \end{aligned}$$

we get

$$\Phi(E'F') = \begin{pmatrix} (1-\alpha)(t-\beta) & (1-\alpha)\sqrt{t(1-t)} \\ -\alpha\sqrt{t(1-t)} & -\alpha(1-t-\beta) \end{pmatrix}.$$

Since

$$\begin{aligned} \text{Tr}(\Phi(E'F')) &= (1-\alpha)(t-\beta) - \alpha(1-t-\beta) = t - \alpha - \beta + 2\alpha\beta \\ \det(\Phi(E'F')) &= \alpha(1-\alpha)(\beta(1-\beta)) \end{aligned}$$

the characteristic polynomial $\lambda^2 + \text{Tr}(\Phi(E'F'))\lambda + \det(\Phi(E'F'))$ of $\Phi(E'F')$ is

$$\lambda^2 - (t - \alpha - \beta + 2\alpha\beta)\lambda + \alpha(1-\alpha)(1-\beta).$$

The eigenvalues of $\Phi(E'F')$ are

$$\begin{aligned} \lambda_1(t) &:= \frac{t - \alpha - \beta + 2\alpha\beta + \sqrt{(t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1-\alpha)\beta(1-\beta)}}{2} \\ \lambda_2(t) &:= \frac{t - \alpha - \beta + 2\alpha\beta - \sqrt{(t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1-\alpha)\beta(1-\beta)}}{2}. \end{aligned}$$

If $|t - \alpha - \beta + 2\alpha\beta| \leq 2\sqrt{\alpha(1-\alpha)\beta(1-\beta)}$ then $\lambda_2 = \overline{\lambda_1}$ and

$$\lambda_1 = |\lambda_1|e^{i\theta} \quad \lambda_2 = |\lambda_2|e^{-i\theta} \quad |\lambda_1| = |\lambda_2| = \sqrt{\alpha(1-\alpha)\beta(1-\beta)},$$

where

$$\cos \theta = \frac{\text{Re } \lambda_1}{2|\lambda_1|} = \frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1-\alpha)\beta(1-\beta)}}.$$

Let t_1, t_2 be the roots of $(t - \alpha - \beta + 2\alpha\beta)^2 - 4\alpha(1-\alpha)\beta(1-\beta)$ with $t_1 < t_2$. Then for $t_1 < t < t_2$ it holds

$$\begin{aligned} \tau_t((E'F')^n) &= \frac{\lambda_1(t)^n + \lambda_2(t)^n}{2} \\ &= |\lambda_1|^n \frac{e^{in\theta} + e^{-in\theta}}{2} \\ &= |\alpha(1-\alpha)\beta(1-\beta)|^{n/2} \cos(n\theta) \\ &= (\alpha(1-\alpha)\beta(1-\beta))^{n/2} T_n\left(\frac{t - \alpha - \beta + 2\alpha\beta}{2\sqrt{\alpha(1-\alpha)\beta(1-\beta)}}\right), \end{aligned}$$

where T_n is the Chebyshev polynomials of the first kind. Since both sides of this equality are polynomials which agree on the interval $[t_1, t_2]$, they must be the same

polynomials. Hence

$$\begin{aligned} \int_{[0,1]} \tau_t((E'F')^n) d\nu(t) &= \int_{[0,1]} \frac{\lambda_1(t)^n + \lambda_2(t)^n}{2} d\nu(t). \\ &= \int_{[0,1]} (\alpha(1-\alpha)\beta(1-\beta))^{n/2} T_n\left(\frac{t-\alpha-\beta+2\alpha\beta}{2\sqrt{\alpha(1-\alpha)\beta(1-\beta)}}\right) d\nu(t). \end{aligned}$$

□

4. THE FREE CASE

Let $p \in L(\Gamma)$ be a nonzero projection different from the identity. Here projections mean self-adjoint idempotents ($p^2 = p = p^*$). Let μ be the spectral distribution measure of p with respect to the trace τ on $L(\Gamma)$. Then $\text{supp}(\mu) = \sigma(p) = \{0, 1\}$ because τ is faithful, and

$$\tau(f(p)) = \int_{\sigma(p)} f d\mu = \int_{\sigma(p)} (f(0)1_{\{0\}} + f(1)1_{\{1\}}) d\mu = f(0)\mu(\{0\}) + f(1)\mu(\{1\}),$$

for any $f \in C(\{0, 1\})$. In particular, by letting $f(t) = t$ and $f(t) = 1 - t$ we get

$$\mu(\{0\}) = 1 - \tau(p), \quad \mu(\{1\}) = \tau(p).$$

Moreover, the moments $m_n(p) = \tau(p^n) = \tau(p)$ for $n \in \mathbb{N}$.

Theorem 4.1. *Let $\Gamma := \mathbb{Z}_k \star \mathbb{Z}_\ell$ be the free product on two generators $c \in \mathbb{Z}_k$, $d \in \mathbb{Z}_\ell$ for some integers $k, \ell \geq 2$. Let*

$$p := \frac{e + c + \cdots + c^{k-1}}{k}, \quad q := \frac{e + d + \cdots + d^{\ell-1}}{\ell} \in \mathbb{C}\Gamma$$

be two nonzero projections different from the identity. Moreover, assume that $\alpha := \tau(p)$, $\beta := \tau(q)$ satisfy $0 < \beta \leq \alpha < 1$, $\alpha + \beta < 1$ and let

$$\begin{aligned} \lambda_1 &:= \sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} \\ \lambda_2 &:= \sqrt{\alpha(1-\beta)} - \sqrt{\beta(1-\alpha)}. \end{aligned}$$

Then the spectral distribution measure μ_{free} of $\begin{pmatrix} 0 & (pq)^ \\ pq & 0 \end{pmatrix} \in M_2(L(\Gamma))$ with respect to the trace $\tilde{\tau} := \tau \otimes \tau_2$ on $M_2(L(\Gamma))$ is*

$$\mu_{\text{free}} = \frac{1}{2\pi} \frac{\sqrt{(\lambda_1^2 - t^2)(t^2 - \lambda_2^2)}}{t(1-t^2)} 1_{[-\lambda_1, -\lambda_2] \cup [\lambda_2, \lambda_1]} dt + (1-\beta)\delta_0,$$

where dt is the Lebesgue measure. In particular,

$$\|pq\| = \sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)}.$$

Proof. Let μ (resp. ν) be the spectral distribution measure of p (resp. q) with respect to the trace τ of $L(\Gamma)$. We now compute the S-transform (cf. [17, p.30-32] or [16]):

$$\begin{aligned} \psi_\mu(s) &:= \int_0^\infty \frac{ts}{1-ts} d\mu(s) = \int_0^\infty (ts + t^2s^2 + t^3s^3 + \cdots) d\mu(t) \\ &= m_1(p)s + m_2(p)s^2 + m_3(p)s^3 + \cdots = \alpha s + \alpha s^2 + \alpha s^3 + \cdots \\ &= \frac{\alpha s}{1-s}. \end{aligned}$$

Similarly,

$$\psi_\nu(s) = \frac{\beta s}{1-s}.$$

The formal power series ψ_μ (resp. ψ_ν) has to satisfy the equations $\chi_\mu(\psi_\mu(s)) = s$, $\psi_\mu(\chi_\mu(z)) = z$ and $S_\mu(z) = \frac{z+1}{z}\chi_\mu(z)$ (resp. $\chi_\nu(\psi_\nu(s)) = s$, $\psi_\nu(\chi_\nu(z)) = z$ and $S_\nu(z) = \frac{z+1}{z}\chi_\nu(z)$). Solving for $\chi_\mu, S_\mu, \chi_\nu, S_\nu$, we get

$$z = \psi_\mu(s) \iff (1-s)z = \alpha s \implies \chi_\mu(z) = s = \frac{z}{\alpha+z}$$

$$z = \psi_\nu(s) \implies \chi_\nu(z) = s = \frac{z}{\beta+z}.$$

$$S_\mu(z) = \frac{z+1}{z}\chi_\mu(z) = \frac{z+1}{\alpha+z}$$

$$S_\nu(z) = \frac{z+1}{z}\chi_\nu(z) = \frac{z+1}{\beta+z}$$

By [17, Theorem 3.6.3], the multiplicative free convolution is

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z)S_\nu(z) = \frac{(z+1)^2}{(\alpha+z)(\beta+z)}.$$

Thus

$$\chi_{\mu \boxtimes \nu}(z) = \frac{z}{z+1}S_{\mu \boxtimes \nu}(z) = \frac{z(z+1)}{(\alpha+z)(\beta+z)},$$

and

$$\begin{aligned} s = \chi_{\mu \boxtimes \nu}(z) &\iff (\alpha+z)(\beta+z)s = z(z+1) \\ \implies \psi_{\mu \boxtimes \nu}(s) = z &= \frac{1}{2} \frac{1 - (\alpha + \beta)s \pm \sqrt{d(s)}}{s-1} \end{aligned}$$

where $d(s)$ is the discriminant

$$d(s) := (\alpha - \beta)^2 s^2 + (4\alpha\beta - 2\alpha - 2\beta)s + 1.$$

Letting $s = \frac{1}{\lambda}$, the discriminant $d(s)$ can be written as

$$d\left(\frac{1}{\lambda}\right) = \frac{(\lambda - \lambda_1^2)(\lambda - \lambda_2^2)}{\lambda^2},$$

where

$$\lambda_1 := \sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} \quad \lambda_2 := \sqrt{\alpha(1-\beta)} - \sqrt{\beta(1-\alpha)}.$$

Let $\psi := \psi_{\mu \boxtimes \nu}$. Then the Cauchy transform $G(\lambda)$ of $\mu \boxtimes \nu$ is

$$G(\lambda) - \frac{1}{\lambda} = \frac{1}{\lambda} \psi\left(\frac{1}{\lambda}\right) = \frac{1}{2\lambda} \frac{1 - (\alpha + \beta)\frac{1}{\lambda} + \sqrt{d\left(\frac{1}{\lambda}\right)}}{\frac{1}{\lambda} - 1} = \frac{\lambda - \alpha - \beta + \sqrt{(\lambda - \lambda_1^2)(\lambda - \lambda_2^2)}}{2\lambda(1-\lambda)}.$$

By Stieltjes inversion formula, $\mu \boxtimes \nu$ is given by

$$\begin{aligned} (18) \quad \mu \boxtimes \nu(x) &= \lim_{y \rightarrow 0^+} \left(-\frac{1}{\pi} \operatorname{Im} G(x + iy) \right) dx + c_0 \delta_0 \\ &= \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - x)(x - \lambda_2^2)}}{2x(1-x)} 1_{[\lambda_2^2, \lambda_1^2]}(x) dx + c_0 \delta_0. \end{aligned}$$

where $c_0 := 1 - \beta$, is the residue of G at the simple pole 0, and dx is the Lebesgue measure. Since $0 < \beta \leq \alpha < 1$ and $\alpha + \beta < 1$, the integral

$$(19) \quad \int_{\lambda_2^2}^{\lambda_1^2} \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - x)(x - \lambda_2^2)}}{2x(1 - x)} dx = \beta,$$

(cf. [15, Example 4.3.3]). By [17, Remarks 3.6.2(iii)], we get that $\mu \boxtimes \nu = \mu_{p^{\frac{1}{2}}qp^{\frac{1}{2}}}$, the spectral distribution of $p^{\frac{1}{2}}qp^{\frac{1}{2}}$ with respect to the (unique) trace τ on $L(\Gamma)$. Since $\tau((p^{\frac{1}{2}}qp^{\frac{1}{2}})^n) = \tau((pqp)^n)$ for every $n \in \mathbb{N}_0$, $\mu \boxtimes \nu = \mu_{pqp}$, the spectral distribution of pqp w.r.t. τ . Define the self-adjoint operator

$$S := \begin{pmatrix} 0 & (pq)^* \\ pq & 0 \end{pmatrix} \in M_2(L(\Gamma)),$$

and let $\mu_{\text{free}} := \mu_S$ be the spectral distribution of S with respect to the normal faithful trace $\tau \otimes \tau_2$, where $\tau_2 := \frac{1}{2}Tr$ is the normalized trace on $M_2(\mathbb{C})$. By [10, Eq. (2.6)], $m_{2n}(S) = \tau(((pq)^*(pq))^n) = \tau((pqp)^n) = \tau((pqp)^n)$. Thus the image measure $\phi(\mu_S)$ of μ_S by the map $\phi : t \mapsto t^2$ is $\phi(\mu_S) = \mu_{pqp} = \mu \boxtimes \nu$. Substituting $t^2 = x$ in Eq. (18), we get

$$\mu_S(t) = \frac{1}{\pi} \frac{\sqrt{(\lambda_1^2 - t^2)(t^2 - \lambda_2^2)}}{t(1 - t^2)} 1_{[\lambda_2, \lambda_1]}(t) dt + c_0 \delta_0.$$

Symmetrizing,

$$\mu_S(t) = \frac{1}{2\pi} \frac{\sqrt{(\lambda_1^2 - t^2)(t^2 - \lambda_2^2)}}{t(1 - t^2)} 1_{[-\lambda_1, -\lambda_2] \cup [\lambda_2, \lambda_1]} dt + c_0 \delta_0,$$

and by [10, Eq (2.7)],

$$\|pq\| = \|S\| = \max(\text{supp}(\mu_S)) = \lambda_1.$$

□

Example 4.2 (free case). Let $\Gamma := \mathbb{Z}_2 \star \mathbb{Z}_3$ be the free product with generators $s \in \mathbb{Z}_2$, $c \in \mathbb{Z}_3$. Let $p = (e + s)/2$, $q = (e + c + c^2)/3 \in \mathbb{C}\Gamma$. Then by Theorem 4.1

$$\|pq\| = \sqrt{\frac{1}{2}(1 - \frac{1}{3})} + \sqrt{\frac{1}{3}(1 - \frac{1}{2})} = \frac{1 + \sqrt{2}}{\sqrt{6}} = 0.985 \dots$$

Let $\Gamma' := \mathbb{Z}_3 \star \mathbb{Z}_4$ be the free product with generators $c \in \mathbb{Z}_3$, $d \in \mathbb{Z}_4$. and let $q = (e + c + c^2)/3$, $r = (e + d + d^2 + d^3)/4 \in \mathbb{C}\Gamma'$. Then by Theorem 4.1

$$\|qr\| = \sqrt{\frac{1}{3}(1 - \frac{1}{4})} + \sqrt{\frac{1}{4}(1 - \frac{1}{3})} = \frac{\sqrt{2} + \sqrt{3}}{\sqrt{12}} = 0.908 \dots$$

5. RESULTS

Let $P, Q, R \in L(T)$ be the projections given by

$$P := \frac{I + S}{2} \quad Q := \frac{I + C + C^2}{3} \quad R := \frac{I + D + D^2 + D^3}{4},$$

where $S := D^2$ and C , and D are the generators of T shown in Figure 1. Let τ be the trace on $\mathbb{C}T$ coming from the group von Neumann algebra $L(T)$, as described in

[10, Section 2]. Using the trace properties of τ , the even moments for the self-adjoint operator

$$(20) \quad \tilde{h} := \sqrt{12} \begin{pmatrix} 0 & (QR)^* \\ QR & 0 \end{pmatrix} \in M_2(C_r^*(T))$$

are given by

$$m_{2n}(\tilde{h}) = \tau((\tilde{h})^{2n}) = 12^n \tau((QRQ)^n) = 12^n \tau((QR)^n), \quad n \in \mathbb{N},$$

where

$$h := \sqrt{12}QR = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3).$$

Our goal is to compute as many as possible of the moments

$$(21) \quad \begin{aligned} m_n &:= m_n(h^*h) \\ &= m_{2n}(\tilde{h}) \\ &= \tau(((I + C + C^2)(I + D + D^2 + D^3))^n), \quad n \in \mathbb{N}, \end{aligned}$$

because by Section 4 in [10] we can use them to estimate the norm $\|h\|$ (which is equal to $\|\tilde{h}\|$):

$$\lim_{n \rightarrow \infty} m_n^{1/2n} = \lim_{n \rightarrow \infty} \sqrt{12} \tau((QRQ)^n)^{1/2n} = \sqrt{12} \|QRQ\|^{1/2} = \sqrt{12} \|QR\| = \|h\|.$$

Rather than computing the moments m_n directly, we compute the ‘‘cyclic reduced’’ numbers

$$\zeta_n := \tau(((C + C^2)(D + D^2 + D^3))^n), \quad n \in \mathbb{N}.$$

These numbers are related by the formula

$$(22) \quad \zeta_n - \frac{5 + (-2)^n}{12} = 2 \sum_{k=0}^n \frac{c_{n,k}}{12^k} m_k, \quad n \in \mathbb{N}_0,$$

where $m_0 := 1/4$, ($\zeta_0 := 1$), and $c_{n,k}$ is the coefficient of t^k in the substituted Chebyshev polynomial of the first kind $6^{n/2} T_n(\sqrt{6}(t - \frac{5}{12}))$. To obtain this formula (22), we use Lemma 3.1 and Theorem 3.2 with $\alpha = \tau(Q) = 1/3$, $\beta = \tau(R) = 1/4$, and note that from Lemma 3.1, $\int_{[0,1]} t^k d\nu = 2m_k/12^k$.

To compute the cyclic reduced numbers ζ_n , we use the inner product

$$\langle x, y \rangle := \tau(y^*x)$$

on $L(T)$ associated with the trace τ . Recall that the trace restricted to the group ring $\mathbb{C}T$ (considered as a subalgebra of $C_r^*(T) \subset L(T)$) satisfies

$$\tau\left(\sum_{x \in Y} c_x x\right) = \begin{cases} c_e & \text{if } e \in Y \\ 0 & \text{else} \end{cases},$$

for any finite subset $Y \subset T$ and any set of complex numbers $(c_x)_{x \in T}$ indexed by T . In particular, $\tau(x) = 1$ if $x = e$ else it is zero. For instance, expanding $(I + C + C^2)(I + D + D^2 + D^3)$ we see that the identity occurs only once and thus $m_1 = 1$. Thus

$$(23) \quad \left\langle \sum_{x \in X} c_x x, \sum_{y \in Y} c_y y \right\rangle = \sum_{x \in X} \sum_{y \in Y} c_x \overline{c_y} \langle x, y \rangle,$$

for any finite subsets $X, Y \subset T$, and any $c_x, c_y \in \mathbb{C}$.

Define

$$a := C + C^2 \quad b := D + D^2 + D^3,$$

and recall that $C^3 = I$, $D^4 = I$, and that $a = a^*$ and $b = b^*$. Then

$$\zeta_n = \tau((ab)^n), \quad n \in \mathbb{N}.$$

Using computers, we compute ζ_n as follows. We store in one file the terms of the expanded sum $(ab)^n$, (one term per line), and in another file those of $(ba)^n$. Here a term is a word of length n in the letters $C, C^{-1}, D, D^{-1}, D^2$. Composition of the letters in the word is done using the algorithm described in section 2, which yields a reduced doubletree. This doubletree is then converted into a sequence of zeroes and ones and separators and pointers by serializing the range and domain trees using the preorder traversal method. We save this sequence in base 64 in a single line. To save time and space we read the file for $(ab)^n$ and apply ab (i.e. we multiply each word with $CD, CD^2, CD^{-1}, C^{-1}D, C^{-1}D^2$, and $C^{-1}D^{-1}$) in order to obtain the file for $(ab)^{n+1}$. To save space we store not only the word but also its frequency. Taking the inverse of each word in the file for $(ab)^n$ gives the file for $(ba)^n$. The inverse of a doubletree $x \in T$ with range tree $r(x)$ and domain tree $d(x)$ is the doubletree with range tree $d(x)$ and domain tree $r(x)$ (i.e. it simply swaps the domain and range trees). The inner product

$$\zeta_{2n} = \langle (ab)^n, (ba)^n \rangle, \quad n \in \mathbb{N}$$

is the intersection of the sorted files for $(ab)^n$ and $(ba)^n$. We used the GNU sort program to sort the files. The odd “cyclic reduced” numbers ζ_{2n+1} are computed similarly

$$\zeta_{2n+1} = \langle (ab)^{n+1}, (ba)^n \rangle = \langle (ab)^n, (ba)^{n+1} \rangle, \quad n \in \mathbb{N}_0.$$

To reduce the size of the files by about one half, we compute instead the numbers ζ_n^s, ζ_n^e given in Eqs. (8),(9); the idea is the following: Since $(ab)^n = Cb(ab)^n + C^{-1}b(ab)^n$, and $b(ab)^n$ is self-adjoint, we can obtain the terms of the expanded sum for $C^{-1}b(ab)^n$ from those of $Cb(ab)^n$ by using the “reverse-inverse” map $\pi \circ J = Ad(D^2) \circ R$ given in Eq. (7). For instance the reverse-inverse of the word CDC^2D^2 is $D^2R(CDC^2D^2)D^2 = C^{-1}D^{-1}C^{-2}D^{-2}$, which corresponds to reversing the order of the letters and taking the inverse. The letter s (resp. e) in ζ_n^s (resp. ζ_n^e) stands for that we are only keeping the words which are smaller than (resp. equal to) their corresponding reverse-inverses. The comparison is done using the lexicographic order of the serialized form of the doubletrees. The relation between these numbers and the reduced cyclic number ζ_n is given by Lemma 2.2

$$\zeta_n = 2\zeta_n^s + 4\zeta_n^e, \quad n \geq 2.$$

We wrote two programs in $C\#$ and Haskell, both using parallel programming, to calculate the numbers ζ_n^s, ζ_n^e , which can be downloaded at

<https://github.com/shaagerup/ThompsonGroupT/>

<https://github.com/mariars/ThompsonGroupT/>

The size of each of the two files for computing ζ_{28}^s is about 281 GB and 285 GB. They were run in a desktop computer with 2 TB of SSD hard disk, and on the Abacus 2.0 supercomputer from the DeIC National HPC Centre. The series of numbers $\zeta_n^s, \zeta_n^e, \zeta_n, m_n$ are shown in Table 1.

In comparison, when one considers the free product $\mathbb{Z}_3 \star \mathbb{Z}_4$ on two generators $c \in \mathbb{Z}_3, d \in \mathbb{Z}_4$, the measure μ_{free} in Eq. (3) based on c, d instead of C, D , is computed

explicitly using Theorem 4.1 with $\alpha = \tau(\frac{e+c+c^2}{3}) = 1/3$, $\beta = \tau(\frac{e+d+d^2+d^3}{4}) = 1/4$, (and substituting $x = \sqrt{12}t$). Hence the moments m_n^{free} can be computed explicitly with Eq. (3) together with Lemma A.1. One gets $m_1^{\text{free}} = 1$, and

$$m_n^{\text{free}} = 12m_{n-1}^{\text{free}} - 6 \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} 6^j 5^{-2j+n-2} \frac{\binom{2j}{j}}{j+1} \binom{n-2}{2j}, \quad n \geq 2.$$

Estimations of the norm $\|h\|$ are done as follows. By [10, Section 4] and the theory of orthogonal polynomials we can define a sequence of positive numbers

$$\alpha_n := \left(\frac{D_{n-2}D_n}{D_{n-1}^2} \right)^{1/2}, \quad n \in \mathbb{N},$$

where

$$D_{-1} := 1, \quad D_n := \det([m_{i+j}(\tilde{h})]_{i,j=0}^n).$$

n	ζ_n^s	ζ_n^e	ζ_n	$m_n(h^*h)$
1	0	0	0	1
2	0	0	0	6
3	0	0	0	42
4	0	0	0	318
5	1	0	2	2528
6	0	0	0	20790
7	0	0	0	175344
8	4	0	8	1508158
9	36	0	72	13177554
10	70	3	152	116636378
11	64	1	132	1043596346
12	524	1	1052	9423929906
13	2228	4	4472	85780131568
14	8160	16	16384	786252907282
15	21617	32	43362	7251162207110
16	81644	102	163696	67241091321510
17	279531	697	561850	626619942680948
18	1006816	4990	2033592	5865627675769158
19	3429416	13057	6911060	55130780282172364
20	12284412	35247	24709812	520110723876289138
21	43215686	89274	86788468	4923701716098043110
22	154863150	246102	310710708	46759540919860581346
23	550890233	763137	1104833014	445382340814268264936
24	1982133410	2484953	3974206632	4253954798148920432622
25	7128125209	9275681	14293353142	40735421620966779279998
26	25797672490	34858087	51734777328	391022235546378412228050
27	93561508424	119608865	187601452308	3761992784005490950198026
28	341014479116	411320336	683674239576	36271465945557216051920334

TABLE 1. The series of numbers for $h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3)$.

For instance

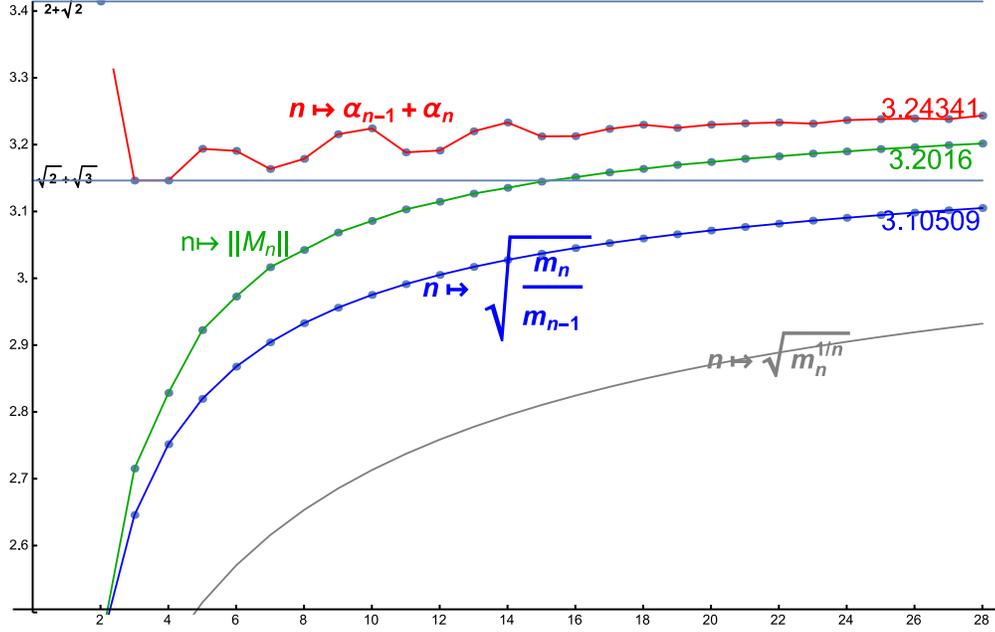
$$D_4 = \det \begin{pmatrix} m_0 & 0 & m_1 & 0 & m_2 \\ 0 & m_1 & 0 & m_2 & 0 \\ m_1 & 0 & m_2 & 0 & m_3 \\ 0 & m_2 & 0 & m_3 & 0 \\ m_2 & 0 & m_3 & 0 & m_4 \end{pmatrix},$$

where $m_0 := 1/4$ and m_n is defined in Eq. (21). Let

$$M_n = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \alpha_2 & & 0 \\ & \alpha_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \alpha_n \\ 0 & & & \alpha_n & 0 \end{pmatrix}.$$

n	$m_n^{\frac{1}{2n}}$	$\sqrt{\frac{m_n}{m_{n-1}}}$	α_n	$\lambda_{\max}(M_n)$	$\alpha_{n-1} + \alpha_n$
1	1.00000	2.00000	2.00000	2.	-----
2	1.56508	2.44949	1.41421	2.44949	3.41421
3	1.86441	2.64575	1.73205	2.71519	3.14626
4	2.05496	2.75162	1.41421	2.82843	3.14626
5	2.18916	2.81952	1.77951	2.9224	3.19373
6	2.28992	2.86773	1.41111	2.97266	3.19062
7	2.36899	2.90414	1.75248	3.01653	3.16359
8	2.43306	2.93277	1.42622	3.04235	3.17870
9	2.48626	2.95593	1.78930	3.06842	3.21552
10	2.53129	2.97509	1.43494	3.08578	3.22424
11	2.57000	2.99123	1.75349	3.10309	3.18844
12	2.60371	3.00504	1.43767	3.11454	3.19117
13	2.63339	3.01701	1.78223	3.12665	3.21990
14	2.65975	3.02753	1.45087	3.13534	3.23310
15	2.68337	3.03685	1.76147	3.14458	3.21234
16	2.70467	3.04518	1.45124	3.15121	3.21271
17	2.72399	3.05270	1.77229	3.15841	3.22353
18	2.74163	3.05953	1.45749	3.16373	3.22978
19	2.75780	3.06577	1.76751	3.16956	3.22500
20	2.77270	3.07150	1.46238	3.17394	3.22989
21	2.78647	3.07679	1.76938	3.1788	3.23176
22	2.79926	3.08169	1.46384	3.18249	3.23321
23	2.81116	3.08625	1.76757	3.1866	3.23141
24	2.82228	3.09051	1.46904	3.18976	3.23662
25	2.83269	3.09449	1.76906	3.19332	3.23810
26	2.84247	3.09824	1.47004	3.19607	3.23909
27	2.85168	3.10176	1.76808	3.19917	3.23811
28	2.86036	3.10509	1.47534	3.2016	3.24341

TABLE 2. Estimating the norm $\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\|$.


 FIGURE 12. Estimating the norm $\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\|$.

Then by [10, Proposition 4.4]

$$(24) \quad \liminf_{n \rightarrow \infty} (\alpha_{n-1} + \alpha_n) \leq \|h\| \leq \sup_{n \geq 3} (\alpha_{n-1} + \alpha_n).$$

Moreover by [10, Proposition 4.1 and Proposition 4.7], the roots $(m_n^{\frac{1}{2n}})_{n=1}^{\infty}$, the ratios $(\sqrt{\frac{m_n}{m_{n-1}}})_{n=1}^{\infty}$, and the norms $(\|M_n\|)_{n=1}^{\infty}$ are increasing sequences that converge to $\|h\|$ and satisfy

$$m_n^{\frac{1}{2n}} \leq \sqrt{\frac{m_n}{m_{n-1}}} \leq \|M_n\| \leq \frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\|, \quad n \in \mathbb{N}.$$

We list these sequences in Table 2, and plot them in Figure 12 for $n = 1, \dots, 28$. When $n \geq 16$, the norm $\|h\| \geq \sqrt{2} + \sqrt{3}$. The best lower bound for h that we can obtain from our data is

$$\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\| \geq 3.2016,$$

and a very likely lower bound for h is

$$\frac{1}{\sqrt{12}}\|(I + C + C^2)(I + D + D^2 + D^3)\| \geq \alpha_{27} + \alpha_{28} = 3.24341,$$

because of Eq (24) and because the sequence $(\alpha_{n-1} + \alpha_n)_{n=1}^{\infty}$ appear to be monotonically increasing for $n \geq 19$. By making a least squares fitting of the numbers $\lambda_{\max}^2(M_{18}), \dots, \lambda_{\max}^2(M_{28})$ to a function of the form $f(n) = a - b(n - c)^{-d}$ we get

$$f(n) = 10.79 - \frac{8.952}{(n + 0.2)^{0.841}}.$$

In particular, the extrapolation method suggests that

$$(25) \quad \frac{1}{\sqrt{12}} \|(I + C + C^2)(I + D + D^2 + D^3)\| \approx \sqrt{10.79} = 3.28.$$

This and Theorem 2.4 suggest that F might be non-amenable.

5.1. Spectral distribution measure. Recall that $h = \sqrt{12}QR$. Since \tilde{h} defined in Eq. (20) is self-adjoint, we get by the spectral theorem that there is a unique probability measure $\mu_{\tilde{h}}$ to \mathbb{R} with support (cf. [10, Section 2])

$$\text{supp}(\mu_{\tilde{h}}) \subset [-\|\tilde{h}\|, \|\tilde{h}\|] \subset [-(1+q), (1+q)],$$

where $\|\tilde{h}\| = \|h\| \leq 2 + \sqrt{2} = 1 + q$ by Theorem 2.4. In this subsection, we will estimate the spectral distribution measure $\mu_{\tilde{h}}$, as it was done in [10, Section 5]. The Hilbert space $L^2([-(q+1), (q+1)], dt)$ can be equipped with the orthonormal basis

$$\sqrt{\frac{n + \frac{1}{2}}{1 + q}} P_n\left(\frac{t}{q+1}\right), \quad n \in \mathbb{N}_0,$$

where P_n , $n \in \mathbb{N}_0$, are the Legendre polynomials. The Hilbert space $L^2([-(q+1), q+1], \frac{1}{\pi\sqrt{(1+q)^2-t^2}} dt)$ can be equipped with the orthonormal basis

$$T_0\left(\frac{t}{q+1}\right), \quad \sqrt{2}T_n\left(\frac{t}{q+1}\right), \quad n \in \mathbb{N},$$

where T_n , $n \in \mathbb{N}_0$ are the Chebyshev polynomials of the first kind.

By [10, Eq. (5.3)], the density of $\mu_{\tilde{h}}$ with respect to the Lebesgue measure can be approximated by

$$\rho'_N(t) = \sum_{n=0}^{2N} \frac{n + \frac{1}{2}}{q+1} \left(\int_{-(q+1)}^{q+1} P_n\left(\frac{s}{q+1}\right) d\mu_{\tilde{h}}(s) \right) P_n\left(\frac{t}{q+1}\right)$$

and by

$$(26) \quad \rho_N(t) = \sum_{n=0}^{2N} c_n^2 \left(\int_{-(q+1)}^{q+1} T_n\left(\frac{s}{q+1}\right) d\mu_{\tilde{h}}(s) \right) T_n\left(\frac{t}{q+1}\right) \frac{1}{\pi\sqrt{(1+q)^2-t^2}},$$

where $c_0 := 1$ and $c_n := \sqrt{2}$, $n \in \mathbb{N}$. Since $\mu_{\tilde{h}}$ is a symmetric measure, all the odd terms in Eq. (26) are zero. Using the moments m_n in Eq. (21) (with $m_0 = 1/4$) and the formula

$$\int_{-(q+1)}^{q+1} t^{2n} d\mu_{\tilde{h}} = m_{2n}(\tilde{h}) = m_n, \quad n \in \mathbb{N}_0,$$

we compute the ‘‘Chebyshev’’ density ρ_{28} for $0 \leq t \leq 2 + \sqrt{2}$ and plot it in Figure 13, together with the corresponding ‘‘free’’ density. We are more interested in the tail of the measure, however, because it gives us an estimate of the norm $\|\tilde{h}\| = \|h\|$:

$$\|h\| = \max\{|t| \mid t \in \text{supp}(\mu_{\tilde{h}})\},$$

(cf. [10, Section 2]). We plot ρ_{28} in the tail interval $[\sqrt{2} + \sqrt{3}, 2 + \sqrt{2}]$ in Figure 14. This shows that $\mu_{\tilde{h}}$ has very little mass in [3.22, 3.414], hence $\|h\|$ can be any number in [3.22, 3.414] including the extrapolated number 3.28 found in Eq. (25).

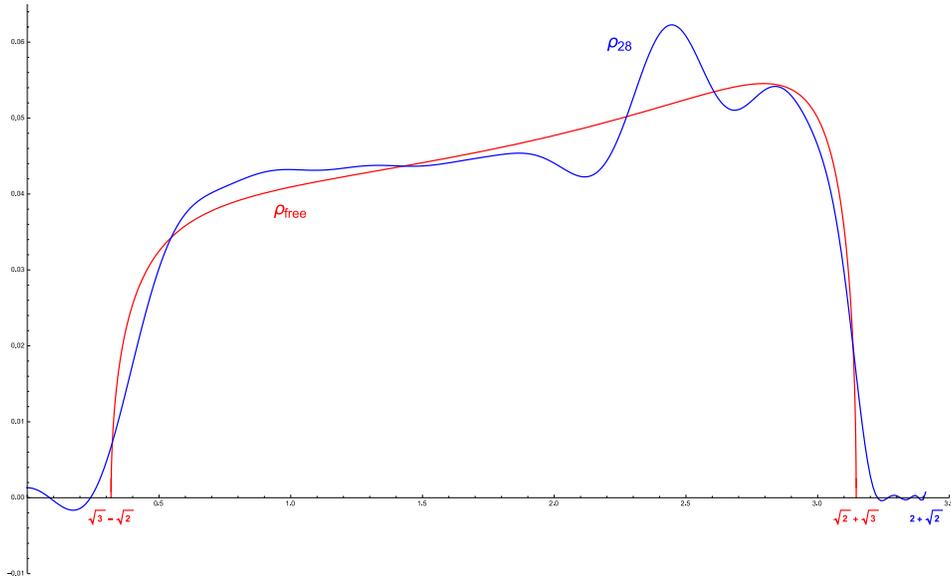


FIGURE 13. Estimating the Chebyshev density for $\mu_{\bar{h}}$ where $h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3)$.

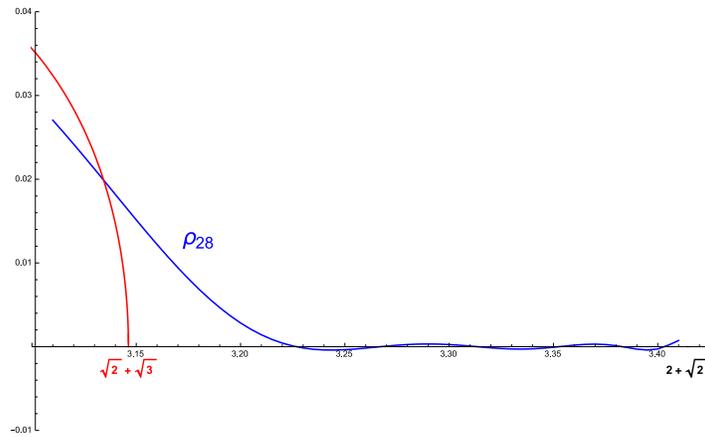


FIGURE 14. Estimating the tail of Chebyshev density for $\mu_{\bar{h}}$ where $h = \frac{1}{\sqrt{12}}(I + C + C^2)(I + D + D^2 + D^3)$.

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APPENDIX A. FREE MOMENTS

Lemma A.1. *Let*

$$(27) \quad m_n^{\text{free}} := \int_{\sqrt{3}-\sqrt{2}}^{\sqrt{3}+\sqrt{2}} \frac{t^{2n}}{\pi} \frac{\sqrt{24-(t^2-5)^2}}{t(12-t^2)} dt, \quad n \in \mathbb{N}.$$

Then $m_1^{\text{free}} = 1$ and

$$m_{n+1}^{\text{free}} = 12m_n^{\text{free}} - 6 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} C_j \binom{n-1}{2j} 6^j 5^{n-1-2j}, \quad n \in \mathbb{N}$$

where $C_j := \frac{1}{j+1} \binom{2j}{j}$, $j \in \mathbb{N}_0$ are the Catalan numbers.

Proof. Substituting $x = t^2$ in Eq.(27), we get

$$m_n^{\text{free}} = \int_{5-2\sqrt{6}}^{5+2\sqrt{6}} \frac{x^{n-1}}{2\pi} \frac{\sqrt{24-(x-5)^2}}{12-x} dx, \quad n \in \mathbb{N}.$$

Then

$$12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = \frac{1}{2\pi} \int_{5-2\sqrt{6}}^{5+2\sqrt{6}} x^n \sqrt{24-(x-5)^2} dx, \quad n \in \mathbb{N}.$$

Substituting $\frac{x-5}{\sqrt{6}} = y$, we get

$$(28) \quad 12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = \frac{1}{2\pi} \int_{-2}^2 6(\sqrt{6}y+5)^n \sqrt{4-y^2} dy, \quad n \in \mathbb{N}.$$

Using the binomial formula, we get

$$12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = 6 \sum_{k=0}^n \frac{1}{2\pi} \int_{-2}^2 y^k \sqrt{4-y^2} dy \binom{n}{k} 6^{k/2} 5^{n-k}, \quad n \in \mathbb{N}.$$

Observing that the integral is zero when k is odd,

$$12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = 6 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{1}{2\pi} \int_{-2}^2 y^{2j} \sqrt{4-y^2} dy \right) \binom{n}{2j} 6^j 5^{n-2j}, \quad n \in \mathbb{N}.$$

Hence

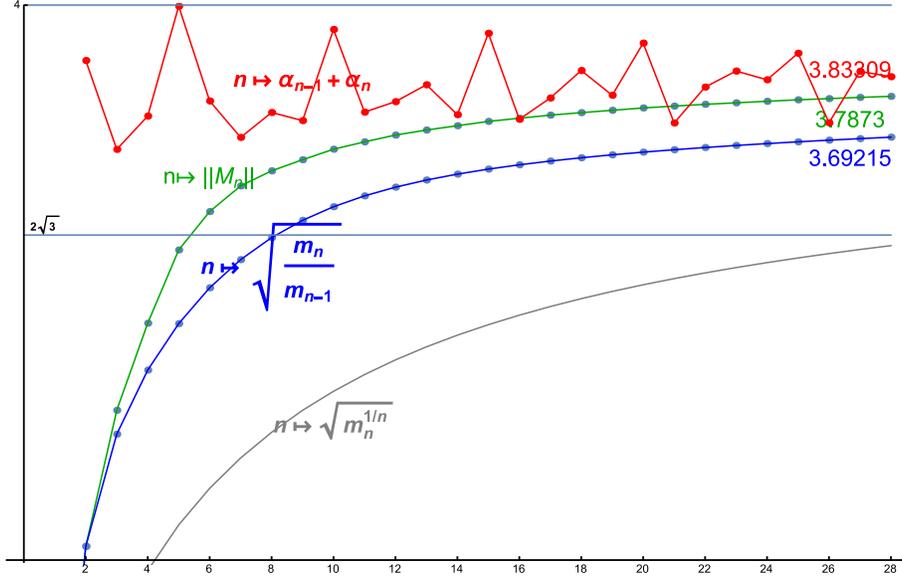
$$12m_{n+1}^{\text{free}} - m_{n+2}^{\text{free}} = 6 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_j \binom{n}{2j} 6^j 5^{n-2j}, \quad n \in \mathbb{N},$$

where $C_j := \frac{1}{j+1} \binom{2j}{j}$ is a Catalan number.

Eq. (28), holds also if we let $n = -1$:

$$12m_0^{\text{free}} - m_1^{\text{free}} = \frac{6}{2\pi} \int_{-2}^2 \frac{\sqrt{4-y^2}}{\sqrt{6}y+5} dy = 2.$$

It follows that $m_1^{\text{free}} = 1$ because by Eq. (19) (with $\alpha = 1/3$, $\beta = 1/4$), $m_0^{\text{free}} = 1/4$.


 FIGURE 15. Estimating the norm $\|C + D + C^{-1} + D^{-1}\|$.

□

 APPENDIX B. **ESTIMATING THE NORM** $\|C + D + C^{-1} + D^{-1}\|$

Here we will describe briefly our estimation of the norm of

$$h := C + D + C^{-1} + D^{-1} \in \mathbb{C}T,$$

where C, D are the generators of T whose graphs are shown in Figure 1. We use the same procedure as we did in [10]. By [10, Theorem 1.3]

$$2\sqrt{3} < \|C + D + C^{-1} + D^{-1}\| < 4.$$

The upper bound is never attained because the Thompson group T is not amenable. We compute the first 28 even moments $m_n(h^*h)$ by first computing the sequence (with $q = 3$)

$$\begin{aligned} h_0 &= e \\ h_1 &= h \\ h_2 &= hh_1 - (q+1)h_0 \\ h_{n+1} &= hh_n - qh_{n-1}, \quad n \geq 2. \end{aligned}$$

Then we compute the sequences

$$\begin{aligned} \xi_n &= \|h_n\|_2^2 - (q+1)q^{n-1}, \\ \eta_n &= \xi_n - (q-1)(\xi_{n-1} + \xi_{n-2} + \cdots + \xi_1), \\ \zeta_n &= \eta_n - (q-1)(\eta_{n-1} + \eta_{n-2} + \cdots + \eta_1) \\ m_n &= \binom{2n}{n} q^n + \sum_{k=1}^n \binom{2n}{n-k} (\zeta_k + 1 - q) q^{n-k}, \quad n \in \mathbb{N}. \end{aligned}$$

n	$\ h_n\ ^2$	η_n	ζ_n	$m_n(h^*h)$
1	4	0	0	4
2	14	2	2	30
3	46	6	2	270
4	182	50	34	2678
5	856	360	244	28418
6	3888	1680	844	317246
7	20536	10552	6356	3681822
8	111366	60310	35010	44027350
9	642550	368762	222842	538815546
10	3850086	2291198	1407754	6714321830
11	23444600	14185540	8719700	84869473770
12	145705728	89557468	55720548	1085055369622
13	915645208	568085492	355133636	14001672259722
14	5816241006	3637390874	2288268034	182071429751606
15	37273096250	23461764106	14837859518	2382930531465042
16	240608480566	152250955922	96703523122	31360608130235654
17	1563526262404	993951776628	633902431984	414711515674495370
18	10219209908952	6522582898368	4174630000468	5507403086681142854
19	67146704535028	43011657706540	27618539011904	73415226964469375622
20	443323828665766	284895372767222	183478938659506	981973882890399349286
21	2939893937656674	1894817824426598	1223610644784438	13175045740884220099018
22	19575351631144042	12650487642600618	8189644814105262	177267112861509055927594
23	130835022206113204	84759454955281696	54997636841585104	2391279755795301975623294
24	877529231836455728	569783620173397812	370502892149137828	32335124616320091148224950
25	5905019922806515884	3842215847470546512	2503367879099490904	438214977894105234044150738
26	39857811116595156626	25984967195646155486	16961687532334006854	5951190674684154918623110822
27	269809546538449104054	176221080384309789662	115227866329705330058	80978038411680591548914827558
28	1831388211478414017418	1198180652247376494918	784745277424152455990	1103891232023903090341589166522

TABLE 3. The series of numbers for $h = C + C^{-1} + D + D^{-1}$.

These series of numbers are listed in Table 3. Next, we compute the increasing sequences of “roots”, “ratios”, “norms” that converge to the norm (cf. Section 5). We list them in Table 4 and plot them in Figure 15. The best lower bound for $\|h\|$ that we can obtain from our results is

$$\|C + D + C^{-1} + D^{-1}\| \geq 3.7873.$$

By making a least squares fitting of the numbers $\lambda_{\max}^2(M_{18}), \dots, \lambda_{\max}^2(M_{28})$ to a function of the form $f(n) = a - b(n - c)^{-d}$ we get

$$f(n) = 14.8 - \frac{18.5975}{(n + 0.2)^{1.1097}}.$$

In particular, the extrapolation method predicts that

$$\|C + D + C^{-1} + D^{-1}\| \approx \sqrt{14.8} = 3.84,$$

which is closer to 4.

n	$m_n^{\frac{1}{2^n}}$	$\sqrt{\frac{m_n}{m_{n-1}}}$	α_n	$\lambda_{\max}(M_n)$	$\alpha_{n-1} + \alpha_n$
1	2.00000	2.00000	2.00000	2.	- - - - -
2	2.34035	2.73861	1.87083	2.73861	3.87083
3	2.54230	3.00000	1.79284	3.0557	3.66367
4	2.68211	3.14937	1.94854	3.25861	3.74139
5	2.78843	3.25755	2.04888	3.42926	3.99743
6	2.87375	3.34119	1.72771	3.51875	3.77659
7	2.94445	3.40670	1.96392	3.57859	3.69163
8	3.00423	3.45804	1.78580	3.61369	3.74972
9	3.05548	3.49831	1.94497	3.63956	3.73078
10	3.09992	3.53005	1.99819	3.66407	3.94316
11	3.13878	3.55529	1.75237	3.68118	3.75056
12	3.17305	3.57561	2.02259	3.69704	3.77496
13	3.20348	3.59223	1.79177	3.70881	3.81436
14	3.23068	3.60604	1.95285	3.71884	3.74462
15	3.25515	3.61772	1.98142	3.72895	3.93427
16	3.27727	3.62774	1.75239	3.73636	3.73381
17	3.29738	3.63648	2.03111	3.74354	3.78351
18	3.31575	3.64418	1.81639	3.74935	3.84750
19	3.33261	3.65107	1.97352	3.75482	3.78992
20	3.34813	3.65727	1.93786	3.7603	3.91138
21	3.36249	3.66291	1.78748	3.76455	3.72534
22	3.37581	3.66807	2.02147	3.76876	3.80895
23	3.38821	3.67283	1.82478	3.77227	3.84625
24	3.39979	3.67724	2.00115	3.77583	3.82593
25	3.41062	3.68134	1.88660	3.77924	3.88775
26	3.42079	3.68518	1.83914	3.78205	3.72575
27	3.43036	3.68877	2.00637	3.7849	3.84551
28	3.43939	3.69215	1.82673	3.7873	3.83309

TABLE 4. Estimating the norm $\|C + C^{-1} + D + D^{-1}\|$.

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