# A GENERALIZATION OF THE GORESKY-KLAPPER CONJECTURE, PART II 

TODD COCHRANE, MICHAEL J. MOSSINGHOFF, CHRIS PINNER, AND C. J. RICHARDSON


#### Abstract

Suppose that $f(x)=A x^{k} \bmod p$ is a permutation of the least residues mod $p$. With the exception of the maps $f(x)=A x$ and $A x^{(p+1) / 2}$ $\bmod p$ we show that for fixed $n \geq 2$ the image of each residue class mod $n$ contains elements from every residue classe $\bmod n$, once $p$ is sufficiently large. If $f(x)=A x \bmod p$, then for each $p$ and $n$ there will be exactly $(1+o(1)) \frac{6}{\pi^{2}} n^{2}$ readily describable values of $A$ for which the image of some residue class mod $n$ misses at least one residue class $\bmod n$, even when $p$ is large relative to $n$. A similar situation holds for $f(x)=A x^{(p+1) / 2} \bmod p$.


## 1. Introduction

For an odd prime $p$ we let $I=\{1,2, \ldots, p-1\}$ denote the reduced residues mod $p$, and $f: I \rightarrow I$ a permutation of $I$ of the form

$$
\begin{equation*}
f(x)=A x^{k} \bmod p \tag{1.1}
\end{equation*}
$$

with $A, k$ integers. Generally we assume that

$$
\begin{equation*}
|A|<p / 2, \quad p \nmid A, \quad 1 \leq k<p-1, \quad \operatorname{gcd}(k, p-1)=1 \tag{1.2}
\end{equation*}
$$

although occasionally we allow $k$ to be negative with $|k|<(p-1) / 2 ; f(x)$ is determined by the value of $k \bmod (p-1)$.

Goresky \& Klapper [9] divided $I$ into the even and odd residues

$$
E=\{2,4, \ldots, p-1\}, \quad O=\{1,3, \ldots, p-2\}
$$

and asked when $f$ could also be a permutation of $E$ (equivalently of $O$ ). Apart from the identity map $(p ; A, k)=(p ; 1,1)$ they found six cases

$$
(p ; A, k)=(5 ;-2,3),(7 ; 1,5),(11 ;-2,3),(11 ; 3,7),(11 ; 5,9),(13 ; 1,5)
$$

and conjectured that there were no more for $p>13$. This was proved for sufficiently large $p$ in [3] and in full in [6], with asymptotic counts on $|f(E) \cap O|$ considered in [4]. Since $x \mapsto p-x$ switches elements of $E$ and $O$, this is the same as asking when $f(E)=O$ or $f(O)=E$, on replacing $A$ by $-A$. A related question of Lehmer [11, Problem F12, p. 381] asks how often $x \bmod p$ and its inverse, $f(x)=x^{-1} \bmod$ $p$, have opposite parity; see Zhang [21], or the generalizations by Alkan, Stan and Zaharescu [1, Lu and Yi [13, 14, Shparlinski [16, 17], Xi and Yi [19, and Yi and Zhang [20].

[^0]Regarding even and odd as a mod 2 property, we ask the same question for a general modulus $n$. Dividing $I$ up into the $n$ congruence classes mod $n$,

$$
\begin{equation*}
I_{j}:=\{x: 1 \leq x \leq p-1, x \equiv j \bmod n\}, \quad j=0, \ldots, n-1 \tag{1.3}
\end{equation*}
$$

there are now several different ways of generalizing the concept of a permutation (1.1) having $f(O)=O$ and $f(E)=E$, or $f(O)=E$ and $f(E)=O$. In [2] we identified five types of $f(x)$ :

Type (i): $f\left(I_{j}\right)=I_{j}$ for all $j=0, \ldots, n-1$.
Type (iia): $f\left(I_{0}\right), \ldots, f\left(I_{n-1}\right)$ a permutation of $I_{0}, \ldots, I_{n-1}$.
Type (iib): $f\left(I_{j}\right)=I_{j}$ for some $j$.
Type (iii): There is a pair $i, j$ with $f\left(I_{i}\right) \subseteq I_{j}$.
Type (iv): There is a pair $i, j$ with $f\left(I_{i}\right) \cap I_{j}=\emptyset$.
Notice that for $n=2$ these are all the same problem, but for general $n$ they can be quite different (indeed the $I_{j}$ may not even have the same cardinality).

In the first paper [2] our focus was primarily on the Type (i)-(iii) maps, showing that, with the exception of $f(x)= \pm x \bmod p$ when $n$ is even, and $f(x)= \pm x$ or $\pm x^{(p+1) / 2} \bmod p$ when $n$ is odd, every $f\left(I_{i}\right)$ must contain elements from at least two different $I_{j}$ once $p \geq 9 \cdot 10^{34} n^{92 / 3}$.

Here we are mainly interested in the Type (iv) maps. When

$$
\begin{equation*}
d:=\operatorname{gcd}(p-1, k-1) \tag{1.4}
\end{equation*}
$$

is suitably small we showed in [2] that the values of $f\left(I_{i}\right)$ are, from an asymptotic point of view, distributed equally in the $n$ residue classes, ruling out any Type (iv) maps. In particular we shall need the following result, Theorem 3.2, from [2].

Theorem 1.1. Let $p$ be an odd prime and $A, k, n$ integers satisfying (1.2) with $n \geq 2$, and

$$
d=\operatorname{gcd}(k-1, p-1) \leq 0.006 p^{89 / 92}
$$

For any $i, j, 0 \leq i, j<n$, we have $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ provided that

$$
p>4 \cdot 10^{29} n^{\frac{184}{3}}
$$

For small $|k|$ this bound can be improved, for example Theorem 1.1 of [2]:
Theorem 1.2. Suppose that $f(x)=A x^{k} \bmod p$, with $k \neq 1$, positive or negative. If $p \geq 16.2|k-1|^{2} n^{4}$ then $f(x)$ is not a Type (iv) map.

Note here we are thinking of $p>n^{2}$, otherwise any permutation $f(x)$ is a Type (iv) mapping. Indeed, if $p<n^{2}$ there will always be a residue class $I_{i}$, and hence its image $f\left(I_{i}\right)$, containing fewer than $n$ elements.

If we want a stronger statement avoiding cases of Type (iv) even when $d$ is large, that is, prove that the image of every residue class mod $n$ hits every residue class $\bmod n$, then we will need to exclude more examples for $n>2$. For the linear maps, $k=1$, we see in the next example that the image of each residue class $\bmod n$ will miss at least one residue class $\bmod n$ when the coefficient $A$ is sufficiently small, or more generally, of the form

$$
\begin{equation*}
A=\frac{t p-r}{s}, \quad \operatorname{gcd}(r, s)=1 \tag{1.5}
\end{equation*}
$$

for some integers $r, s, t$ with $s \neq 0$, and $r$ and $s$ sufficiently small. Note that any such representation also has $(t, s)=1$.
Example 1.1. Suppose that $f(x)=A x \bmod p$ with $A$ an integer satisfying (1.2).
(a) If $|A|<n$, or more generally,
(b) if $A$ is of the form (1.5) with $|r|+|s|+\operatorname{gcd}(n, s)-1 \leq n$,
then for each $i$ there is at least one $j$ with $f\left(I_{i}\right) \cap I_{j}=\emptyset$.
(c) If $A$ is of the form (1.5) with

$$
\begin{equation*}
|r|+|s| \leq n \tag{1.6}
\end{equation*}
$$

then at least $n / \operatorname{gcd}(n, s)$ residue classes $I_{i}$ will have $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $j$.
Indeed, letting $B:=|A|$ in case (a), $B:=|r|+|s|+\operatorname{gcd}(n, s)-2$ in case (b) and $B:=|r|+|s|-1$ in case (c), the number of missed residue classes $I_{j}$ will be at least $n-B$.

Note that (a) is a special case of (b) with $s=1, t=0, r=-A$, and (c) coincides with (b) when $\operatorname{gcd}(n, s)=1$.

A similar situation occurs for exponent $k=(p+1) / 2$, though we must halve the range of restriction, as we see in the next example.

Example 1.2. Suppose that $p \equiv 1 \bmod 4$ and $f(x)=A x^{(p+1) / 2} \bmod p$. If $A$ satisfies (1.2) and
(a) $2|A|<n$, or more generally,
(b) $A$ is of the form (1.5) with $2(|r|+|s|+\operatorname{gcd}(n, s)-2)<n$,
then for each $i$ there is at least one $j$ with $f\left(I_{i}\right) \cap I_{j}=\emptyset$. Indeed, if the restriction in parts (a) and (b) takes the form $B<n$ (as in the preceding example) then in each case the number of missed residue classes $I_{j}$ will be at least $n-B$.

The ranges in Example 1.2 can be extended to resemble Example 1.1(c) if we just want there to be at least one residue class whose image does not hit all classes.
Example 1.3. Suppose that $p \equiv 1 \bmod 4$ and $f(x)=A x^{(p+1) / 2} \bmod p$ and $2^{\beta} \| n$. If $A$ satisfies (1.2) and
(a) $2^{\beta} \mid A$ and $|A|<n$, or
(b) $2^{\beta} \nmid A$ and $|A|+\operatorname{gcd}(n, A)<n$, or
$A$ is of the form (1.5), and
(c) $n$ is odd, with $|r|+|s|+\min \{\operatorname{gcd}(n, r), \operatorname{gcd}(n, s)\}-1 \leq n$, or
(d) $n$ is even and $2^{\beta} \mid r$, with $|r|+|s|+\operatorname{gcd}(n, s)-1 \leq n$, or
(e) $n$ is even and $2^{\beta} \mid s$, with $|r|+|s|+\operatorname{gcd}(n, r)-1 \leq n$, or
(f) $n$ is even and $2^{\beta} \nmid r s$ with $|r|+|s|+\operatorname{gcd}(n, s)+\operatorname{gcd}(n, r)-1 \leq n$, then $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $i, j$.

Appropriate values for $i$ can be found in the proof of Example 1.3 and again, for those $i$ there will be at least $(n-B)$ missed residue classes $I_{j}$, when the restriction takes the form $n<B$ (although in some cases of (c) we must interchange the roles of $i$ and $j$ ).

It turns out that, as long as we avoid exponents $k=1$ or $(p+1) / 2$ with coefficients similar to those in Examples 1.1, 1.2 or 1.3 then $f\left(I_{i}\right)$ will hit all residue classes once $p$ is sufficiently large relative to $n$. To make this precise we define the set

$$
\mathscr{C}:=\left\{C \equiv A x^{k-1} \bmod p: 1 \leq x \leq p-1,|C|<p / 2\right\}
$$

Notice that for any integer $x, f(x)=A x^{k} \equiv C x \bmod p$ for some $C$ in $\mathscr{C}$. As we shall see in Section 3, when $d=\operatorname{gcd}(k-1, p-1)$ is relatively large, and so $|\mathscr{C}|$ is relatively small, it can be useful to reduce to the consideration of the linear maps
$C x \bmod p$. Note that when $k=1, \mathscr{C}=\{A\}$, while when $k=\frac{p+1}{2}, \mathscr{C}=\{A,-A\}$. In the next theorem we show that if $\mathscr{C}$ contains an element $C$ with $n \leq|C| \leq p / n$, and $p$ is sufficiently large then $f\left(I_{i}\right)$ will hit all residue classes $I_{j}$. In particular, this happens when $A$ itself satisfies $n \leq|A| \leq p / n$. This is always the case when $n=2$, other than the maps $f(x)= \pm x$ or the $\pm x^{(p+1) / 2}$.

If $\mathscr{C}$ contains only elements in the ranges $|C|<n$ or $p / n<|C|<p / 2$ then, prompted by the examples in Example 1.1, 1.2 and 1.3, we write the latter $C$ in the form

$$
\begin{equation*}
C=\frac{t p-r}{s}, \quad s>0, \quad \operatorname{gcd}(r, s)=1 \tag{1.7}
\end{equation*}
$$

If for some such $C,|r|$ is sufficiently large relative to $s$ then again we see that the image of each residue class will hit every residue class. Throughout the paper $x^{-1}$ $\bmod m$ denotes the multiplicative inverse of $x \bmod m$.

Theorem 1.3. If $\mathscr{C}$ contains an element $C$ or $C^{-1} \bmod p$ with $n \leq|C| \leq p / n$ or

$$
C=\frac{t p-r}{s}, \quad s>0, \quad \operatorname{gcd}(r, s)=1, \quad(n+3) s \leq|r| \leq \frac{p}{n}
$$

and $p \geq 4 \cdot 10^{29} n^{184 / 3}$, then $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.
We show in Section 4 that any $C$ can be written in the form (1.7) with

$$
\begin{equation*}
|r|<p / n, \quad 1 \leq s \leq n \tag{1.8}
\end{equation*}
$$

Plainly once $r, s$ are chosen there will be only one value of $t \equiv r p^{-1} \bmod s$ making $C$ an integer with $|C|<p / 2$. In particular, for fixed $n$ there will be at most $(n+2)^{3}$ values of $C$ which cannot be used in Theorem 1.3. Thus if $|\mathscr{C}|>(n+2)^{3}$ we are guaranteed a suitable $C$. It turns out that we just need $|\mathscr{C}|>2$ :

Theorem 1.4. Suppose that $f(x) \neq A x$ or $A x^{\frac{p+1}{2}} \bmod p$, and that

$$
p>4 \cdot 10^{29} n^{184 / 3}
$$

Then for any $i, j$ we have $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$.
For the exponents $k=1$ or $(p+1) / 2$, success or failure depends critically on the representation of $A$ in the manner (1.5) as we saw in Examples $1.1,1.2$ and 1.3. In the linear case we obtain a precise description of the Type (iv) maps. The restriction (1.6) in Example 1.1 is in fact sharp for $p$ sufficiently large.

Theorem 1.5. Suppose that $f(x)=A x \bmod p$.
If $p>n^{3}(n+3)$ then $f(x)$ is a Type (iv) map if and only if $A$ is of the form

$$
\begin{equation*}
A=\frac{t p-r}{s}, \quad \operatorname{gcd}(r, s)=1, \quad s>0 \quad 1 \leq|r|+s \leq n \tag{1.9}
\end{equation*}
$$

Writing

$$
S(N):=\sum_{\substack{1 \leq r, s \leq N,(r, s)=1, r+s \leq N}} 1 \sim \frac{3}{\pi^{2}} N^{2},
$$

we see that for each $n \geq 3$ and $p$ we have precisely $2 S(n) \sim\left(6 / \pi^{2}\right) n^{2}$, choices of $A \equiv r s^{-1} \bmod p$ that can give a Type (iv) map $A x \bmod p$.

We obtain the same restriction (1.6) for $k=(p+1) / 2$ when $p$ is slightly larger.

Theorem 1.6. Suppose that $p \equiv 1 \bmod 4$ and $f(x)=A x^{\frac{p+1}{2}} \bmod p$ with

$$
p>\max \left\{\left(n^{3}+1\right)^{2}, 8 \cdot 10^{4}(n \log n)^{4}\right\}
$$

If $A$ is not of the form (1.9) then for any $i, j$ we have $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$.
Example 1.3 shows that for $k=(p+1) / 2$ there are cases where the condition $|r|+s \leq n$ is sharp, for example when $\operatorname{gcd}(n, r)$ or $\operatorname{gcd}(n, s)=1$. This time not every $r, s$ satisfying (1.6) will produce a Type (iv) map, but from Example 1.3 we will get at least $2 S((n+1) / 2) \sim\left(3 / 2 \pi^{2}\right) n^{2}$ and at most $2 S(n) \sim\left(6 / \pi^{2}\right) n^{2}$ examples of Type (iv) maps $f(x)=A x^{(p+1) / 2} \bmod p$.

## 2. Computations and Conjectures

Computations looking for maps of Type (iv) were performed for the primes $p<20,000$ and moduli $n=3$ through 12 .

These computations revealed a number of families of Type (iv) maps that seemed to occur for every prime. These all had exponent $k=1$ or $k=(p+1) / 2$. Restricting to $k \neq 1$ or $(p+1) / 2$, examples of Type (iv) eventually died out. We showed in Theorem 1.4 that for a given $n$ there is indeed a $C(n)$ such that once $p>C(n)$ any $f(x)=A x^{k} \bmod p$ with $k \neq 1,(p+1) / 2$ has $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$. The value $C(n)=4 \cdot 10^{29} n^{184 / 3}$ obtained there is likely far from optimal. For each $n=3$ through 12 the five largest primes $p<20,000$ having an $f(x)=A x^{k} \bmod$ $p$ with $k \neq 1,(p+1) / 2$ and $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $(i, j)$ are recorded in Table 1 Notice that if $A x^{k}$ has this property with $2 j \equiv p \bmod n$ then so will $A x^{k^{\prime}}$ when $k^{\prime}=k \pm(p-1) / 2$ has $\left(k^{\prime}, p-1\right)=1$; a number of these pairs can be seen in the table.

In view of this data it is tempting to make the following conjecture.
Conjecture 2.1. For $n=3$ through 12 the optimal $C(n)$ is

$$
\begin{aligned}
& C(3)=127, \quad C(4)=271, \quad C(5)=601, \quad C(6)=571, \quad C(7)=1733 \\
& C(8)=1777, \quad C(9)=3433, \quad C(10)=2473, \quad C(11)=6577, \quad C(12)=3851
\end{aligned}
$$

The data suggests that one can take $C(n)=6 n^{3}$.
It is noticeable that maps of the form $f(x)=A x^{p-2}=A x^{-1} \bmod p$ appear frequently in the data; this is somewhat surprising since from Theorem 1.2 we know that there are no Type (iv) maps of this form for $p>65 n^{4}$, a much smaller bound than we have for the general $k$. But we note that this map is a self inverse, and most of the remaining examples of Type (iv) maps in our table are also self inverse maps.

The recurring Type (iv) maps $A x$ or $A x^{(p+1) / 2}$ all seemed to have $A$ small or of the form (1.7) with $r$ and $s$ small. Identifying and explaining these led to Examples 1.1, 1.2 and 1.3 . In practice these Examples went through many refinements as additional data revealed new forms. We know from Theorem 1.5 that Example 1.1(c) is sharp. The current version of Example 1.3 is able to predict all the repeat Type (iv) maps that we see in our data for $n=3$ through 12 (though higher $n$ would probably lead to new refinements). Some further fine tuning is certainly possible, for example if $r$ and $\lfloor r / \operatorname{gcd}(s, n)\rfloor$ or $s$ and $\lfloor s / \operatorname{gcd}(r, n)\rfloor$ have opposite parity then we just need $r+s \leq n$ in Example 1.3(c) (see the proof of Example 1.3 for this and other cases where the gcd term can be dropped). Computations for $n=15$, $A=(p-9) / 5$ produced no Type (iv) maps between 1489 and 2000, showing that

|  | $p$ | $A$ | $k$ | $(i, j)$ |
| :---: | :---: | :---: | :---: | :---: |
|  | 83 | 21,26 | 81 | $(1,1)$ |
|  | 89 | 17,21 | 23,67 | $(1,1)$ |
| $n=3$ | 97 | 17 | 47,95 | $(2,2)$ |
|  | 109 | 44 | 53,107 | $(2,2)$ |
|  | 127 | 45,53 | 71 | $(2,2)$ |
|  | 151 | 2 | 13 | $(1,4),(2,3)$ |
|  | 151 | 46 | 127 | $(3,1),(4,2)$ |
|  | 157 | 64 | 155 | $(2,2),(3,3)$ |
| $n=4$ | 167 | 83 | 165 | $(1,1),(2,2)$ |
|  | 193 | 16,48 | 95 | $(2,2),(3,3)$ |
|  | 193 | 49 | 95 | $(2,3),(3,2)$ |
|  | 271 | 107 | 269 | $(1,1),(2,2)$ |
|  | 479 | 142 | 477 | $(2,2)$ |
|  | 503 | 25 | 65 | $(4,4)$ |
| $n=5$ | 503 | 243 | 363 | $(4,4)$ |
|  | 521 | 215 | 259,519 | $(3,3)$ |
|  | 541 | 176 | 269,539 | $(3,3)$ |
|  | 601 | 59 | 251,551 | $(3,3)$ |
|  | 449 | 158 | 447 | $(5,5),(6,6)$ |
|  | 457 | 137 | 151 | $(3,3),(4,4)$ |
|  | 457 | 162 | 227 | $(1,1),(6,6)$ |
| $n=6$ | 457 | 80,137 | 455 | $(3,3),(4,4)$ |
|  | 479 | 214 | 477 | $(5,5),(6,6)$ |
|  | 547 | 30 | 155 | $(3,3),(4,4)$ |
|  | 571 | 118 | 341 | $(3,3),(4,4)$ |
|  | 1303 | 347 | 1301 | $(4,4)$ |
| $n=7$ | 1321 | 232 | 329,989 | $(6,6)$ |
|  | 1409 | 416 | 703,1407 | $(1,1)$ |
|  | 1489 | 653 | 371,1115 | $(6,6)$ |
|  | 1733 | 670 | 865,1731 | $(2,2)$ |
|  |  |  |  |  |


|  | $p$ | $A$ | $k$ | $(i, j)$ |
| :---: | :---: | :---: | :---: | :---: |
|  | 1249 | 36 | 623 | $(1,1),(8,8)$ |
|  | 1301 | 432 | 599 | $(5,5),(8,8)$ |
| $n=8$ | 1381 | 648 | 1379 | $(5,8),(8,5)$ |
|  | 1637 | 437 | 1635 | $(6,7),(7,6)$ |
|  | 1777 | 176 | 1775 | $(3,6),(6,3)$ |
|  | 2857 | 1383 | 713,2141 | $(2,2)$ |
|  | 3037 | 105 | 505,2023 | $(2,2)$ |
| $n=9$ | 3067 | 356 | 1871 | $(8,8)$ |
|  | 3067 | 1313 | 2363 | $(8,8)$ |
|  | 3089 | 482 | 1543,3087 | $(1,1)$ |
|  | 3433 | 1590 | 571,2287 | $(2,2)$ |
| $n=10$ | 2137 | 830 | 1067 | $(8,9),(9,8)$ |
|  | 2287 | 109 | 2285 | $(1,1),(6,6)$ |
|  | 2377 | 623 | 2375 | $(0,7),(7,0)$ |
|  | 2441 | 1169 | 1829 | $(0,0),(1,1)$ |
|  | 2473 | 803 | 1235 | $(0,3),(3,0)$ |
| $n=11$ | 4787 | 624 | 4785 | $(1,1)$ |
|  | 4987 | 2070 | 2215 | $(2,2)$ |
|  | 5281 | 964 | 2111,4751 | $(6,6)$ |
|  | 5683 | 2390 | 5681 | $(9,9)$ |
|  | 6577 | 731,3284 | 1645,4933 | $(5,5)$ |
| $n=12$ | 3457 | 1135 | 1727 | $(0,1),(1,0)$ |
|  | 3529 | 1485 | 1763 | $(0,1),(1,0)$ |
|  | 3637 | 993 | 3635 | $(0,1),(1,0)$ |
|  | 3659 | 934 | 3657 | $(0,0),(11,11)$ |
|  | 3851 | 9 | 351 | $(5,6),(6,5)$ |

TABLE 1. Type (iv): Five largest $p<20,000$ with an $f(x)=A x^{k}$ $\bmod p, k \neq 1,(p+1) / 2$ having $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $(i, j)$.
(c) can not always be weakened to $r+s \leq n$. Our existing data already showed that the gcds can not be dropped in (b),(d),(e) and (f); for example $n=12, A=9$, $(p \pm 8) / 3,(p \pm 3) / 8,(p \pm 1) / 6$ or $(p \pm 9) / 2$. In order to see that both gcds were needed in (f) computations were carried out on $n=24, A=(p-4) / 15$ and Type (iv) did not always occur.

Example 1.1(c) gives Type (iv) maps of the form $f(x)=A x \bmod p$ that will occur for every $p$ (whenever $p$ is in the correct congruence class to make that $A$ an integer). These $A$ for $n=3$ to 12 are shown in Table 2,

Similarly when $p \equiv 1 \bmod 4$ and $k=(p+1) / 2$, Example 1.3 gives us cases of Type (iv) maps $f(x)=A x^{(p+1) / 2} \bmod p$ that will occur for all $p$. These $A$ for $n=3$ to 12 are shown in Table 3.

After excluding the values of $A$ in Tables 2 and 3, few additional Type (iv) exceptions were found in a search of $p<20,000$ and $k=1$ or $(p+1) / 2$; the largest prime for each $n$ is shown in Tables 4 and 5 .

Conjecture 2.2. Suppose that $f(x)=A x$ or $A x^{(p+1) / 2} \bmod p$ where $A$ satisfies (1.2) but is not of the form

$$
|A|<n \quad \text { or } \quad A=(t p-r) / s \quad \text { with } \quad|r|+|s| \leq n, \quad \operatorname{gcd}(r, s)=1
$$

then $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$ once $p>c(n)$, with the data suggesting that one can take $c(n)=3 n^{3}$. For small $n$ the optimal values are

$$
\begin{aligned}
& c(3)=17, \quad c(4)=61, \quad c(5)=137, \quad c(6)=197, \quad c(7)=277 \\
& c(8)=937, \quad c(9)=653, \quad c(10)=2297, \quad c(11)=1061, \quad c(12)=2857
\end{aligned}
$$

By Theorems 1.5 and 1.6 this holds with $c(n)=O\left(n^{4}\right)$ for $k=1$, and $c(n)=$ $O\left(n^{6}\right)$ for $k=(p+1) / 2$.

## 3. Type (iv) intersections for Large $d$

Theorem 1.3 is an immediate consequence of the next two theorems. Recall that $\mathscr{C}$ is the set of absolute least residues

$$
\mathscr{C}:=\left\{A x^{k-1} \bmod p: 1 \leq x \leq p-1\right\}
$$

If we have a suitable $C \in \mathscr{C}$ then we can show that each residue class gets mapped to all the residue classes.

Theorem 3.1. Suppose that $\mathscr{C}$ contains an integer $C$ with $n \leq|C| \leq p / n$.
If $p>10^{6}$ and $d \geq 0.88 n^{2} p^{1 / 2} \log ^{2} p$, or if $k=1$, then $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.
Proof. We proceed as in the proof of Theorem 4.1 of [2], but with $I_{j}$ in place of $I \backslash I_{j}$. Since $C$ is in $\mathscr{C}$ we can write $C \equiv A B^{k-1} \bmod p$ for some $B$. We let

$$
L:=(p-1) / d,
$$

and

$$
\mathscr{U}=\left\{x \in I_{i}: \quad C x \bmod p \in I_{j}, \quad x \equiv B z^{L} \bmod p \text { for some } z\right\}
$$

Notice that if $x$ is in $\mathscr{U}$ we have

$$
A x^{k} \equiv C x\left(B^{-1} x\right)^{k-1} \equiv C x z^{L(k-1)}=C x\left(z^{p-1}\right)^{(k-1) / d} \equiv C x \bmod p
$$

So we can show that $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ by showing $|\mathscr{U}|>0$. Writing $\mathscr{I}_{j}(x)$ for the characteristic function of $I_{j}$, and using the Dirichlet characters of order $L$ to pick out when $B^{-1} x$ is an $L$ th power $\bmod p$, we have

$$
L|\mathscr{U}|=\sum_{\chi^{L}=\chi_{0}} \chi\left(B^{-1}\right) S(\chi), \quad S(\chi):=\sum_{x \in \mathbb{Z}_{p}} \chi(x) \mathscr{I}_{i}(x) \mathscr{I}_{j}(C x),
$$

where $\chi_{0}$ denotes the principal character. Hence

$$
L|\mathscr{U}|=M_{i j}+E,
$$

where

$$
M_{i j}:=\sum_{x \in \mathbb{Z}_{p}^{*}} \mathscr{I}_{i}(x) \mathscr{I}_{j}(C x), \quad E:=\sum_{\substack{\chi^{L}=\chi_{0} \\ \chi \neq \chi_{0}}} \chi\left(B^{-1}\right) S(\chi) .
$$

| $n$ | A |
| :---: | :---: |
| 3 | $1,2,(p-1) / 2$. |
| 4 | $1,2,3,(p-1) / 2,(p \pm 1) / 3$. |
| 5 | $1,2,3,4,(p-1) / 2,(p-3) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 1) / 4$. |
| 6 | $1,2,3,4,5,(p-1) / 2,(p-3) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 1) / 4,(p \pm 1) / 5,(2 p \pm 1) / 5$. |
| 7 | $\begin{aligned} & 1,2,3,4,5,6,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 4) / 3,(p \pm 1) / 4 \text {, } \\ & (p \pm 3) / 4,(p \pm 1) / 5,(p \pm 2) / 5,(2 p \pm 1) / 5,2(p \pm 1) / 5,(p \pm 1) / 6 . \end{aligned}$ |
| 8 | $\begin{aligned} & 1,2,3,4,5,6,7,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 4) / 3,(p \pm 5) / 3 \\ & (p \pm 1) / 4,(p \pm 3) / 4,(p \pm 1) / 5,(p \pm 2) / 5,(p \pm 3) / 5,(2 p \pm 1) / 5,2(p \pm 1) / 5,(2 p \pm 3) / 5,(p \pm 1) / 6 \\ & (p \pm 1) / 7,(2 p \pm 1) / 7,(3 p \pm 1) / 7 . \end{aligned}$ |
| 9 | $\begin{aligned} & 1,2,3,4,5,6,7,8,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p-7) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 4) / 3,(p \pm 5) / 3 \\ & (p \pm 1) / 4,(p \pm 3) / 4,(p \pm 5) / 4,(p \pm 1) / 5,(2 p \pm 1) / 5,(p \pm 2) / 5,(2 p \pm 2) / 5,(p \pm 3) / 5,(2 p \pm 3) / 5 \\ & (2 p \pm 4) / 5,(2 p \pm 4) / 5,(p \pm 1) / 6,(p \pm 1) / 7,(2 p \pm 1) / 7,(3 p \pm 1) / 7,(p \pm 2) / 7, \\ & (2 p \pm 2) / 7,(3 p \pm 2) / 7,(p \pm 1) / 8,(3 p \pm 1) / 8 \text {. } \end{aligned}$ |
| 10 | $\begin{aligned} & 1,2,3,4,5,6,7,8,9,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p-7) / 2,(p \pm 1) / 3,(p \pm 2) / 3,(p \pm 4) / 3, \\ & (p \pm 5) / 3,(p \pm 7) / 3,(p \pm 1) / 4,(p \pm 3) / 4,(p \pm 5) / 4,(p \pm 1) / 5,(2 p \pm 1) / 5,(p \pm 2) / 5,(2 p \pm 2) / 5, \\ & (p \pm 3) / 5,(2 p \pm 3) / 5,(p \pm 4) / 5,(2 p \pm 4) / 5,(p \pm 1) / 6,(p \pm 1) / 7,(2 p \pm 1) / 7,(3 p \pm 1) / 7, \\ & (p \pm 2) / 7,(2 p \pm 2) / 7,(3 p \pm 2) / 7,(p \pm 3) / 7,(2 p \pm 3) / 7,(3 p \pm 3) / 7,(p \pm 1) / 8,(3 p \pm 1) / 8, \\ & (p \pm 1) / 9,(2 p \pm 1) / 9,(4 p \pm 1) / 9 \text {. } \end{aligned}$ |
| 11 | $\begin{aligned} & 1,2,3,4,5,6,7,8,9,10,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p-7) / 2,(p-9) / 2,(p \pm 1) / 3,(p \pm 2) / 3, \\ & ((p \pm 4) / 3,(p \pm 5) / 3,(p \pm 7) / 3,(p \pm 8) / 3,(p \pm 1) / 4,(p \pm 3) / 4,(p \pm 5) / 4,(p \pm 7) / 4,(p \pm 1) / 5 \\ & (2 p \pm 1) / 5,(p \pm 3) / 5,(2 p \pm 2) / 5,(p \pm 3) / 5,(2 p \pm 3) / 5,(p \pm 4) / 5,(2 p \pm 4) / 5,(p \pm 6) / 5,(2 p \pm 6) / 5 \\ & (p \pm 1) / 6,(p \pm 5) / 6,(p \pm 1) / 7,(2 p \pm 1) / 7,(3 p \pm 1) / 7,(p \pm 2) / 7,(2 p \pm 2) / 7,(3 p \pm 2) / 7,(p \pm 3) / 7, \\ & (2 p \pm 3) / 7,(3 p \pm 3) / 7,(p \pm 4) / 7,(2 p \pm 4) / 7,(3 p \pm 4) / 7,(p \pm 1) / 8,(3 p \pm 1) / 8,(p \pm 3) / 8,(3 p \pm 3) / 8, \\ & (p \pm 1) / 9,(2 p \pm 1) / 9,(4 p \pm 1) / 9,(p \pm 2) / 9,(2 p \pm 2) / 9,(4 p \pm 2) / 9,(p \pm 1) / 10,(3 p \pm 1) / 10 . \end{aligned}$ |
| 12 | $\begin{aligned} & 1,2,3,4,5,6,7,8,9,10,11,(p-1) / 2,(p-3) / 2,(p-5) / 2,(p-7) / 2,(p-9) / 2,(p \pm 1) / 3,(p \pm 2) / 3 \\ & (p \pm 4) / 3,(p \pm 5) / 3,(p \pm 7) / 3,(p \pm 8) / 3,(p \pm 1) / 4,(p \pm 3) / 4,(p \pm 5) / 4,(p \pm 7) / 4,(p \pm 1) / 5, \\ & (2 p \pm 1) / 5,(p \pm 2) / 5,(2 p \pm 2) / 5,(p \pm 3) / 5,(2 p \pm 3) / 5,(p \pm 4) / 5,(2 p \pm 4) / 5, \\ & (p \pm 6) / 5,(2 p \pm 6) / 5,(p \pm 7) / 5,(2 p \pm 7) / 5,(p \pm 1) / 6,(p \pm 5) / 6,(p \pm 1) / 7,(2 p \pm 1) / 7,(3 p \pm 1) / 7, \\ & (p \pm 2) / 7,(2 p \pm 2) / 7,(3 p \pm 2) / 7,(p \pm 3) / 7,(2 p \pm 3) / 7,(3 p \pm 3) / 7,(p \pm 4) / 7,(2 p \pm 4) / 7,(3 p \pm 4) / 7, \\ & (p \pm 5) / 7,(2 p \pm 5) / 7,(3 p \pm 5) / 7,(p \pm 1) / 8,(3 p \pm 1) / 8,(p \pm 3) / 8,(3 p \pm 3) / 8,(p \pm 1) / 9,(2 p \pm 1) / 9, \\ & (4 p \pm 1) / 9,(p \pm 2) / 9,(2 p \pm 2) / 9,(4 p \pm 2) / 9,(p \pm 1) / 10,(3 p \pm 1) / 10,(p \pm 1) / 11,(2 p \pm 1) / 11, \\ & (3 p \pm 1) / 11,(4 p \pm 1) / 11,(5 p \pm 1) / 11 . \end{aligned}$ |

Table 2. Type (iv) examples $A x \bmod p$ from Example 1.1

Using the finite Fourier expansion $\mathscr{I}_{i}(x)=\sum_{y \in \mathbb{Z}_{p}} a_{i}(y) e_{p}(y x)$ we have

$$
S(\chi)=\sum_{\substack{u=0 \\(u, v) \neq(0,0)}}^{p-1} \sum_{i=0}^{p-1} a_{i}(u) a_{j}(v) \sum_{x \in \mathbb{Z}_{p}} \chi(x) e_{p}(u x+v C x)
$$

```
n A
3 1, 2,(p-1)/2.
4.
5 1, 2, 3,4,(p-1)/2,(p-3)/2,(p\pm1)/3,(p\pm2)/3,(p-1)/4.
6 1, 2,4,(p-1)/2,(p-1)/4.
7 1, 2, 3, 4, 5, 6, (p-1)/2, (p-3)/2,(p-5)/2,(p\pm1)/3,(p\pm2)/3,(p\pm4)/3,
    (p-1)/4,(p+3)/4,(p\pm1)/5,(2p\pm1)/5,(p\pm2)/5,(2p\pm2)/5,(p\pm1)/6.
8 1, 2, 3, 5,(p-1)/2,(p-3)/2,(p\pm1)/3,(p\pm2)/3,(p\pm1)/5,(2p\pm1)/5.
9 1, 2, 3, 4, 5, 6, 7, 8,(p-1)/2,(p-3)/2,(p-5)/2,(p-7)/2,(p\pm1)/3,(p\pm2)/3,
    (p\pm4)/3,(p\pm5)/3,(p-1)/4,(p+3)/4,(p-5)/4,(p\pm1)/5,(2p\pm1)/5,(p\pm2)/5,
    (2p\pm2)/5,(p\pm3)/5,(2p\pm3)/5,(p\pm4)/5,(2p\pm4)/5,(p\pm1)/6, (p\pm1)/7,
    (2p\pm1)/7,(3p\pm1)/7,(p\pm2)/7,(2p\pm2)/7,(3p\pm2)/7,(p-1)/8,(3p+1)/8.
10 1, 2, 3, 4, 6, 7, 8, (p-1)/2, (p-3)/2,(p-7)/2,(p\pm1)/3,(p\pm2)/3,(p\pm4)/3,(p\pm1)/4,(p\pm3)/4,
    (p\pm1)/6,(p\pm1)/7,(2p\pm1)/7,(3p\pm1)/7, (p\pm2)/7,(2p\pm2)/7, (3p\pm2)/7, (p-1)/8,(3p+1)/8.
11 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, (p-1)/2, (p-3)/2,(p-5)/2,(p-7)/2,(p-9)/2,(p\pm 1)/3,(p\pm2)/3,
    ((p\pm4)/3,(p\pm5)/3,(p\pm7)/3,(p\pm8)/3,(p-1)/4,(p+3)/4,(p-5)/4,(p+7)/4,(p\pm1)/5,
    (2p\pm1)/5,(p\pm3)/5,(2p\pm2)/5,(p\pm3)/5,(2p\pm3)/5,(p\pm4)/5,(2p\pm4)/5,(p\pm6)/5,(2p\pm6)/5,
    (p\pm1)/6, (p\pm5)/6, (p\pm1)/7, (2p\pm1)/7, (3p\pm1)/7, (p\pm2)/7, (2p\pm2)/7, (3p\pm2)/7, (p\pm3)/7,
    (2p\pm3)/7, (3p\pm3)/7,(p\pm4)/7, (2p\pm4)/7, (3p\pm4)/7, (p-1)/8, (3p+1)/8, (p+3)/8, (3p-3)/8,
    (p\pm1)/9, (2p\pm1)/9,(4p\pm1)/9,(p\pm2)/9,(2p\pm2)/9,(4p\pm2)/9, (p\pm1)/10, (3p\pm1)/10.
12 1, 2, 3, 4, 5, 7, 8, (p-1)/2, (p-3)/2, (p-5)/2,(p-7)/2,(p\pm1)/3,(p\pm2)/3,(p\pm4)/3,(p\pm5)/3,
    (p-1)/4,(p+3)/4,(p-5)/4,(p+7)/4,(p\pm1)/5,(2p\pm1)/5,(p\pm2)/5,(2p\pm2)/5,(p\pm3)/5,
    (2p\pm3)/5, (p\pm4)/5,(2p\pm4)/5,(p\pm1)/7,(2p\pm1)/7,(3p\pm1)/7,(p\pm2)/7,(2p\pm2)/7,(3p\pm2)/7,
    (p\pm4)/7,(2p\pm4)/7,(3p\pm4)/7,(p-1)/8,(3p+1)/8.
```

Table 3. Type (iv) examples $A x^{(p+1) / 2} \bmod p$ from Example 1.3

The classic Gauss sum bound, and the [8, Theorem 1] bound on $\sum_{u=0}^{p-1}\left|a_{i}(u)\right|$, give as in the proof of [2, Theorem 4.1],

$$
\begin{equation*}
|E|<0.22(L-1) p^{1 / 2} \log ^{2} p \tag{3.1}
\end{equation*}
$$

|  | $p$ | $A$ | $k$ | $(i, j)$ |
| :---: | :---: | :---: | :---: | :---: |
| $n=3$ | 13 | 5 | 1 | $(2,2)$ |
| $n=4$ | 19 | 7 | 1 | $(3,4),(4,3))$ |
|  | 19 | 8 | 1 | $(3,3),(4,4)$ |
| $n=5$ | 53 | 14,19 | 1 | $(4,4)$ |
| $n=6$ | 61 | 16,22 | 1 | $(2,4),(5,3)$ |
|  | 61 | 19 | 1 | $(3,2),(4,5)$ |
|  | 61 | 25 | 1 | $(3,5),(4,2))$ |
| $n=7$ | 131 | 27,34 | 1 | $(6,6)$ |
| $n=8$ | 151 | 31,39 | 1 | $(7,8),(8,7)$ |
| $n=9$ | 241 | 35,62 | 1 | $(8,8)$ |
| $n=10$ | 283 | 48,58 | 1 | $(4,8),(9,5)$ |
|  | 283 | 112 | 1 | $(5,4),(8,9)$ |
|  | 283 | 122 | 1 | $(5,9),(8,4)$ |
| $n=11$ | 449 | 65,76 | 1 | $(10,10)$ |
| $n=12$ | 491 | 71,83 | 1 | $(0,11),(11,0)$ |

Table 4. Type (iv): Largest $p<20,000$ having an $f(x)=A x$ $\bmod p$ with $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $(i, j)$ and $A$ not in Table 2.

|  | $p$ | $A$ | $k$ | $(i, j)$ |
| :---: | :---: | :---: | :---: | :---: |
| $n=3$ | 17 | 5 | 9 | $(2,2),(3,3)$ |
|  | 17 | 7 | 9 | $(2,3),(3,2)$ |
| $n=4$ | 61 | 6 | 31 | $(1,3),(4,2)$ |
|  | 61 | 10 | 31 | $(2,4),(3,1)$ |
| $n=5$ | 137 | 7 | 69 | $(3,2),(4,5)$ |
|  | 137 | 39 | 69 | $(2,4),(5,3)$ |
| $n=6$ | 197 | 16 | 99 | $(1,3),(4,2)$ |
|  | 197 | 37 | 99 | $(2,4),(3,1)$ |
| $n=7$ | 277 | 9,56 | 139 | $(5,4),(6,7)$ |
|  | 277 | 62 | 139 | $(4,5),(7,6)$ |
|  | 277 | 67 | 139 | $(5,7),(6,4)$ |
|  | 277 | 94,123 | 139 | $(4,6),(7,5)$ |
| $n=8$ | 937 | 188 | 469 | $(2,7),(7,2)$ |
|  | 937 | 314 | 469 | $(2,7),(7,7)$ |
| $n=9$ | 653 | 149 | 327 | $(1,1),(4,4)$ |
| $n=10$ | 2297 | 768,984 | 1149 | $(3,4),(4,3)$ |
| $n=11$ | 1061 | 337 | 531 | $(5,7),(11,9)$ |
|  | 1061 | 488 | 531 | $(7,5),(9,11)$ |
| $n=12$ | 2857 | 570,817 | 1429 | $(4,4),(9,9)$ |

TABLE 5. Type (iv): Largest $p<20,000$ having an $f(x)=$ $A x^{(p+1) / 2} \bmod p$ with $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for some $(i, j)$ and $A$ not in Table 3 .

We need a lower bound on $M_{i j}$. Suppose that we have $n \leq C \leq p / n$. If $C<0$ we replace $C$ by $-C$ and $j$ by $\bar{j}=p-j \bmod n$. Since $0<C x<C p$ we have

$$
\begin{aligned}
M_{i j} & =\sum_{u=0}^{C-1}\left|\left\{x \in I_{i}: u p \leq C x<(u+1) p, \quad C x-u p \in I_{j}\right\}\right| \\
& =\sum_{\substack{u=0 \\
u \equiv K \bmod n}}^{C-1}\left|\left\{x \in I_{i}: \frac{u p}{C} \leq x<\frac{u p}{C}+\frac{p}{C}\right\}\right|,
\end{aligned}
$$

where $K:=(C i-j) p^{-1} \bmod n$.
Note that for $p / 2 n<C \leq p / n$ we have

$$
\left\lfloor\frac{p}{n C}\right\rfloor=1>\frac{p}{2 n C}
$$

and for $C \leq p / 2 n$

$$
\left\lfloor\frac{p}{n C}\right\rfloor>\frac{p}{n C}-1 \geq \frac{p}{2 n C} .
$$

Similarly, for $n \leq C<2 n$ we have

$$
\left\lfloor\frac{C}{n}\right\rfloor=1>\frac{C}{2 n}
$$

and for $C \geq 2 n$

$$
\left\lfloor\frac{C}{n}\right\rfloor>\frac{C}{n}-1 \geq \frac{C}{2 n}
$$

Hence, observing that a general interval of length $\ell$ or an interval of the form $[0, \ell-1]$, will contain at least $\lfloor\ell / n\rfloor$ complete sets of residues $\bmod n$, we have

$$
\left|\left\{x \in I_{i}: \frac{u p}{C} \leq x<\frac{u p}{C}+\frac{p}{C}\right\}\right| \geq\left\lfloor\frac{p}{n C}\right\rfloor>\frac{p}{2 n C}
$$

and

$$
|\{0 \leq u \leq C-1: u \equiv K \bmod n\}| \geq\left\lfloor\frac{C}{n}\right\rfloor>\frac{C}{2 n}
$$

giving

$$
M_{i j}>\frac{C}{2 n} \cdot \frac{p}{2 n C}=\frac{p}{4 n^{2}}
$$

Hence, as long as we have

$$
\frac{p}{4 n^{2}} \geq 0.22 \frac{p^{3 / 2} \log ^{2} p}{d}
$$

we have $|\mathscr{U}|>0$ and $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$.
If $k=1$ then $C=A$ and as shown above, $\left|f\left(I_{i}\right) \cap I_{j}\right|=M_{i j}>p / 4 n^{2}$ whenever $n \leq|A| \leq p / n$. Note that $n \leq|C| \leq p / n$ implies $p>n^{2}$.

Theorem 3.2. Suppose that $\mathscr{C}$ contains a $C$ of the form

$$
\begin{equation*}
C \text { or } C^{-1}=\frac{(t p-r)}{s}, \quad s>0, \quad \operatorname{gcd}(r, s)=1, \quad(n+3) s \leq|r| \leq \frac{p}{n} \tag{3.2}
\end{equation*}
$$

If $p>10^{6}$ and $d \geq 1.32 n^{2} p^{1 / 2} \log ^{2} p$, or if $k=1$, then $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.
Proof. We proceed as in Theorem 3.1 If $C^{-1}$ is of the stated form we observe that counting $x$ in $I_{i}$ with $C x$ in $I_{j}$ is the same as counting $x$ in $I_{j}$ with $C^{-1} x$ in $I_{i}$ and reverse the roles of $i$ and $j$. So we suppose that $C$ is of the stated form and that $r, s>0$. If $r<0$ we can replace $C$ by $-C$ and $j$ by $\bar{j}=p-j \bmod n$.

To estimate $M_{i j}$ we split the $x$ into the different residue classes $a \bmod s$ and observe that for $x=a+s y$ we have

$$
C x=x\left(\frac{t p-r}{s}\right) \equiv \frac{(t p-r) a}{s}-r y \bmod p .
$$

Hence, writing $\frac{(t p-r) a}{s} \equiv \alpha(a) \bmod p$ with $0 \leq \alpha(a)<p$, we have

$$
M_{i j}=\sum_{a=0}^{s-1}\left|\left\{0 \leq y \leq \frac{(p-1-a)}{s}: y s+a \in I_{i}, \alpha(a)-r y \bmod p \in I_{j}\right\}\right|
$$

If $b:=\operatorname{gcd}(n, s)=1$ then the condition $y s+a \in I_{i}$ reduces to the $\bmod n$ congruence $y \equiv \lambda(a):=(i-a) s^{-1} \bmod n$. If $b>1$ then we are reduced to the $s / b$ values

$$
\mathscr{A}=\{a: 0 \leq a<s, a \equiv i \bmod b\}
$$

and the condition $y s+a \in I_{i}$ becomes $y \equiv \lambda(a):=(s / b)^{-1}(i-a) / b \bmod n / b$, that is $y \equiv \lambda_{v}(a) \bmod n, v=1, \ldots, b$ with $\lambda_{v}(a)=\lambda(a)+v n / b$.

Now any $y$ with

$$
-\left(\left\lfloor\frac{r(p-1-a)}{s p}\right\rfloor-1\right) p \leq \alpha(a)-r y<0
$$

will have $0<y \leq(p-1-a) / s$ and hence

$$
M_{i j} \geq \sum_{a \in \mathscr{A}} \sum_{v=1}^{b\left\lfloor\left\lfloor\frac{r(p-1-a)}{s p}\right\rfloor-1\right.} \sum_{u=1}^{s p} M_{i j}(a, v, u)
$$

where
$M_{i j}(a, v, u)=\left|\left\{y \equiv \lambda_{v}(a) \bmod n / b, \quad-u p \leq \alpha(a)-r y<-(u-1) p, \alpha(a)-r y \bmod p \in I_{j}\right\}\right|$.
The condition $\alpha(a)-r y \bmod p \in I_{j}$ becomes $\alpha(a)-r y+u p \equiv j \bmod n$ and $u \equiv \mu(a, v):=\left(j+r \lambda_{v}(a)-\alpha(a)\right) p^{-1} \bmod n$.

Hence

$$
M_{i j} \geq \sum_{a \in \mathscr{A}} \sum_{v=1}^{b} \sum_{\substack{u=1 \\ u \equiv \mu(a, v) \bmod n}}^{\left\lfloor\frac{r(p-1-a)}{s p}\right\rfloor-1}\left|\left\{y \equiv \lambda_{v}(a) \bmod n, \frac{(\alpha(a)+u p)}{r}-\frac{p}{r}<y \leq \frac{(\alpha(a)+u p)}{r}\right\}\right|
$$

When $n<p / r<2 n$ we observe that we are guaranteed at least one element $y \equiv \lambda_{v}(a) \bmod n$ in the interval of length $p / r>n$. When $p / r \geq 2 n$ we use that we have at least $\lfloor p / r n\rfloor>p / r n-1$ elements satisfying the congruence. Hence

$$
\left|\left\{y \equiv \lambda_{v}(a) \bmod n, \frac{(\alpha(a)+u p)}{r}-\frac{p}{r}<y \leq \frac{(\alpha(a)+u p)}{r}\right\}\right| \geq \frac{p}{2 r n}
$$

Similarly, with $(n+3) s \leq r<p / n$,

$$
\left\lfloor\frac{r(p-1-a)}{s p}\right\rfloor-1 \geq \frac{r(p-s)}{s p}-2 \geq \frac{r}{s}-3 \geq n
$$

So we get at least one $u$ in the sum satisfying $u \equiv \mu(a, v) \bmod n$ for $(n+3) \leq$ $r / s<(2 n+3)$ and $\lfloor(r / s-3) / n\rfloor>r / n s-3 / n-1$ for $(2 n+3) \leq r / s$ and

$$
\left|\left\{1 \leq u \leq\left\lfloor\frac{r(p-1-a)}{s p}\right\rfloor: u \equiv \mu(a, v) \bmod n\right\}\right| \geq \frac{r}{s(2 n+3)}
$$

Hence

$$
\begin{equation*}
M_{i j} \geq \frac{s}{b} \cdot b \cdot \frac{r}{s(2 n+3)} \cdot \frac{p}{2 r n}=\frac{p}{2 n(2 n+3)} \tag{3.3}
\end{equation*}
$$

and making this greater than $|E|<0.22(p / d) \sqrt{p} \log ^{2} p$ ensures that $\mathscr{U} \neq \emptyset$. The $k=1$ case follows as before.

## 4. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Suppose that $p>4 \cdot 10^{29} n^{184 / 3}$. Certainly $p>10^{6}$. If $d \leq$ $0.006 p^{89 / 92}$ then Theorem 1.3 follows from Theorem1.1, while if $d \geq 1.32 n^{2} p^{1 / 2} \log ^{\overline{2}} p$ it follows from Theorems 3.1 and 3.2. Otherwise we have

$$
1.32 n^{2} p^{1 / 2} \log ^{2} p>d>0.006 p^{89 / 92}
$$

and so $p^{43 / 92} / \log ^{2} p<220 n^{2}$. But this does not occur for $p>4 \cdot 10^{29} n^{184 / 3}$.
Proof of Theorem 1.4. Suppose that $k \neq 1$ or $(p+1) / 2$ and that

$$
p>\max \left\{4 \cdot 10^{29} n^{184 / 3}, 2(n+3)^{2} n^{4}\right\}=4 \cdot 10^{29} n^{184 / 3}
$$

Let $C$ be an integer with $|C|<p / 2$. By the box principle the congruence $y \equiv C x$ $\bmod p$ has a nonzero solution $x=s, y=r$ with $1 \leq s \leq n,|r|<p / n$, and $\operatorname{gcd}(r, s)=1$. Let $\mathscr{G}$ be the group of $d$-th powers $\bmod p$, and recall that

$$
\mathscr{C}:=\left\{A x^{k-1} \bmod p: 1 \leq x \leq p-1\right\}=\{A x \bmod p: x \in \mathscr{G}\}
$$

reduced to values between $-p / 2$ and $p / 2$. Each element $C \in \mathscr{C}$ has a representation as above,

$$
C \equiv r s^{-1} \bmod p, \quad 1 \leq s \leq n, \quad|r|<p / n, \quad \operatorname{gcd}(r, s)=1
$$

If for some $C \in \mathscr{C}$ we have $(n+3) s \leq|r| \leq p / n$, then Theorem 1.3 applies.
Otherwise, every $C \in \mathscr{C}$ is in

$$
\mathscr{B}:=\left\{C \in \mathscr{C}: C \equiv r s^{-1} \bmod p, 1 \leq s \leq n,|r|<(n+3) s\right\} .
$$

In this case, let $A x$ be an element in $\mathscr{C}$ having a representation $A x \equiv r s^{-1} \bmod$ $p$ from $\mathscr{B}$ with $|r / s|$ minimal. Let $y \neq \pm 1 \in \mathscr{G}$; such a $y$ exists since $|\mathscr{G}| \geq 3$ by assumption. Then $A x, A y x, A y^{-1} x$ are distinct elements of $\mathscr{C}$, and so we have representations

$$
A x \equiv r_{1} s_{1}^{-1}, \quad A y x \equiv r_{2} s_{2}^{-1}, \quad A y^{-1} x \equiv r_{3} s_{3}^{-1} \quad \bmod p
$$

with $1 \leq s_{i} \leq n,\left|r_{i}\right|<(n+3) s_{i}$. Thus,

$$
y \equiv r_{2} s_{2}^{-1} s_{1} r_{1}^{-1} \equiv r_{1} s_{1}^{-1} s_{3} r_{3}^{-1} \bmod p
$$

and so

$$
s_{1}^{2} r_{2} r_{3} \equiv r_{1}^{2} s_{2} s_{3} \bmod p .
$$

Thus if $p>2(n+3)^{2} n^{4}$ then the two sides must be equal, that is,

$$
\left(\frac{r_{1}}{s_{1}}\right)^{2}=\frac{r_{2}}{s_{2}} \frac{r_{3}}{s_{3}}
$$

which cannot happen by the minimality of $r_{1} / s_{1}$.

## 5. Proofs of Theorems 1.5 and 1.6

In order to deal with the exponents $k=1$ and $k=(p+1) / 2$, we need the following addition to Theorems 3.1 and 3.2 which deals with the case when $r, s$ are both small but (1.6) does not hold.

Theorem 5.1. Suppose that

$$
A=\frac{t p-r}{s}, \quad s>0, \quad \operatorname{gcd}(r, s)=1, \quad|r|+s>n
$$

(a) If $p>|r|$ sn, then $f(x)=A x \bmod p$ has $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.
(b) If $p>(|r| s n+1)^{2}$ then $f(x)=A x^{(p+1) / 2} \bmod p$ has $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.

For part (a) we actually prove that $\left|f\left(I_{i}\right) \cap I_{j}\right| \geq\lfloor p / r n s\rfloor$ under the given hypotheses. Using [5] we can replace the hypothesis in part (b) with the condition $p \gg(|r| n s \log (|r| n s))^{4 / 3}$.

Proof. (a) We first deal with the linear case $f(x)=A x \bmod p$. We assume that $r>0$ else we can replace $A$ by $-A$ and $j$ by $\bar{j}=p-j \bmod n$. We also assume that $r<s$ else we replace $A$ by $A^{-1}=\left(t^{\prime} p-s\right) / r$ where $t^{\prime} \equiv s p^{-1} \bmod r$ and switch the roles of $i$ and $j$.

Take $a$ with $1 \leq a \leq n$ with

$$
a \equiv(j s+r i) p^{-1} \bmod n
$$

For convenience here $p^{-1}$ will denote the inverse of $p \bmod n s$.
We define $u$ such that

$$
u \equiv i \bmod n, \quad u \equiv a t^{-1} \bmod s
$$

Writing $b:=\operatorname{gcd}(n, s)$ we see that if $b>1$ then $a t^{-1} \equiv i r p^{-1} t^{-1} \equiv i \bmod b$ so there is a solution $($ defined $\bmod n s / b)$. Note that $a p-j s-r u \equiv 0 \bmod s$ and $\bmod n$, and so when $b>1$ we can define $\lambda$ by

$$
r \lambda \equiv \frac{(a p-j s-r u)}{(n s / b)} \bmod b, \quad 0 \leq \lambda<b
$$

with $\lambda=0$ if $b=1$. Set $v=u+\lambda n s / b$. We split into two cases:
Case 1: $1 \leq a \leq s$.
We solve

$$
\begin{equation*}
x \equiv v \bmod n s, \quad 1 \leq x \leq \min \{a p / r, p-1\} . \tag{5.1}
\end{equation*}
$$

The condition $p>n r s$ ensures that $a p / r \geq p / r>n s$ so we are guaranteed a solution, and $x \equiv i \bmod n$ so $x$ is in $I_{i}$.

Since $x t \equiv a \bmod s$ we have

$$
A x \equiv \frac{a p-x r}{s}, \bmod p
$$

Notice that $0<(a p-x r) / s<a p / s \leq p$ so that this is the least residue with

$$
\frac{a p-r x}{s} \equiv \frac{a p-r v}{s}=j+n \frac{(a p-j s-r u) /(n s / b)-r \lambda}{b} \equiv j \bmod n
$$

Case 2: $s+1 \leq a \leq n$.
Notice $1 \leq a-s \leq n-s<r$. We solve

$$
\begin{equation*}
x \equiv v \bmod n s, \quad(a-s) p / r<x \leq p-1 . \tag{5.2}
\end{equation*}
$$

Since $0<(a-s) p / r \leq p-p / r<p-n s$ we are again guaranteed a solution $x$ in $I_{i}$,

$$
A x \equiv \frac{(a-s) p-x r}{s}+p, \bmod p
$$

Since $0>((a-s) p-x r) / s>-p r / s>-p$ this is the least residue and again $(a p-r x) / s \equiv j \bmod n$.

We note that the set of $x$ satisfying (5.1) or (5.2), is an arithmetic progression of length at least $\lfloor p / r n s\rfloor$. In particular, we have shown that $M_{i j}=\left\{x \in I_{i}\right.$ : $\left.A x \bmod p \in I_{j}\right\}$ satisfies $\left|M_{i j}\right| \geq\lfloor p / r s n\rfloor$.
(b) Suppose now that $k=(p+1) / 2$, and that $p>(|r| s n+1)^{2}$. Then $f(x) \equiv \pm A x$ $\bmod p$ depending on whether $x$ is a quadratic residue or not. In part (a) we saw that there was an arithmetic progression of $\lfloor p /|r| s n\rfloor \geq p /|r| s n-1>\sqrt{p}$ values of $x \in I_{i}$, with $A x \bmod p \in I_{j}$. By [12] these cannot all be quadratic nonresidues. Thus we must have a quadratic residue $x \in I_{i}$ with $f(x)=A x \bmod p$ in $I_{j}$.

Proof of Theorem 1.5. Suppose that $p>(n+3) n^{3}$ and that $f(x)=A x \bmod p$. By the box principle we can write $A \equiv r s^{-1} \bmod p$ with $(r, s)=1,1 \leq s \leq n$ and $|r|<p / n$. If $1 \leq|r|+s \leq n$, then Example 1.1(c) shows that $f(x)=A x \bmod p$ is a Type (iv) mapping. Suppose now that $|r|+s>n$. If $|r|>(n+3) s$ then the result follows from Theorem 3.2, so we can assume that $|r| \leq(n+3) s \leq(n+3) n$. Since $p>(n+3) n^{3}$ we have $p>|r| s n$, and so Theorem 5.1] gives $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ for all $i, j$.

Proof of Theorem 1.6. Suppose that $k=(p+1) / 2$ and $p>\max \left\{\left(n^{3}+1\right)^{2}, 8\right.$. $\left.10^{4}(n \log n)^{4}\right\}$. Observe that in the proof of Theorem 3.2 we have $L=2$ and hence by (3.1) will get $f\left(I_{i}\right) \cap I_{j} \neq \emptyset$ as long as $M_{i j}>0.22 \sqrt{p} \log ^{2} p$. Since $p>8 \cdot 10^{4}(n \log n)^{4}$ we have $p>8 \cdot 10^{6}$ and so by (3.3),

$$
\begin{equation*}
M_{i j} \geq p / 2 n(2 n+3) \geq p / 6 n^{2}>0.22 \sqrt{p} \log ^{2} p \tag{5.3}
\end{equation*}
$$

provided $\mathscr{C}$ contains a value $C$ satisfying (3.2).
By the box principle we can write $A \equiv r_{1} s_{1}^{-1} \bmod p$ and $A^{-1} \equiv r_{2} s_{2}^{-1} \bmod p$ with $\left(r_{i}, s_{i}\right)=1,1 \leq s_{i} \leq n$ and $\left|r_{i}\right|<p / n$. If one of these has $\left|r_{i}\right| \geq(n+3) s_{i}$ then the result follows from (5.3). If both have $\left|r_{i}\right|<(n+3) s_{i}$ then, since $r_{1} r_{2} \equiv s_{1} s_{2}$ $\bmod p$ and $\left|r_{1} r_{2}-s_{1} s_{2}\right|<(n+3)^{2} n^{2}+n^{2}<p$, we must have $r_{1} r_{2}=s_{1} s_{2}$ and $\left|r_{1}\right|=s_{2} \leq n$. Hence $A$ has a representation $A=(t p-r) / s, \operatorname{gcd}(r, s)=1$, with both $s,|r| \leq n$, and since $A$ is not of the form (1.9) by assumption, we have $|r|+s>n$. Since $p>\left(n^{3}+1\right)^{2} \geq(|r| s n+1)^{2}$ the result follows from Theorem 5.1.

## 6. Proof of examples

Proof of Example 1.1, (a) Suppose that $0<A<n$. Then each $A x, x=1, \ldots, p-1$ will lie in $[1, A(p-1)]$ with $A(p-1)<A p$. So reducing $\bmod p$ to lie in $[1, p)$ we have

$$
A x \bmod p=A x-\ell p, \quad 0 \leq \ell \leq A-1
$$

For $x$ in $I_{i}$ we have $A x-\ell p \equiv A i-\ell p \bmod n$ with at most $A$ different values of $\ell$, and so $A x \bmod p$ can take at most $A$ different values $\bmod n$. Similarly the $-A x$ $\bmod p$ take the form $p-(A x-\ell p)=(\ell+1) p-A x, 0 \leq \ell<A$, giving at most $A$ classes $\bmod n$. Therefore $f(x)=A x \bmod p$ or $-A x \bmod p$ with $A<n$ must omit at least $n-A$ classes.
(b) Suppose that $A=(t p-r) / s$ with $s>0$ and $1 \leq x<p, \operatorname{gcd}(s, r t)=1$. We divide $x$ into the various residue classes $\bmod s$. Since $\operatorname{gcd}(s, t)=1$, letting $t^{-1}$ denote the $\bmod s$ inverse of $t$, we can write

$$
x \equiv t^{-1} a \bmod s, \quad 1 \leq a \leq s
$$

Then $s \mid(a p-r x)$ and

$$
A x \equiv \frac{a p-r x}{s} \bmod p
$$

Suppose that $r>0$, otherwise replace $A$ by $-A$ and count the $p-\ell \bmod n$, and set

$$
r=h s+r_{0}, 1 \leq r_{0}<s
$$

We have

$$
\frac{a p-r x}{s}<\frac{a p}{s} \leq p
$$

and

$$
\frac{a p-r x}{s}>\frac{a p-r p}{s}=\left(-h+\frac{a-r_{0}}{s}\right) p
$$

Hence the least residue of $A x \bmod p$ is

$$
\frac{a p-r x}{s}+\ell p
$$

where $\ell$ is one of the $h+1$ possibilities $0,1, \ldots, h$ if $a \geq r_{0}$, or the $h+2$ possibilities $0,1, \ldots, h, h+1$ for $1 \leq a \leq r_{0}-1$.

Therefore, writing $m=\ell s+a$, we have $1 \leq m \leq(h+1) s+\left(r_{0}-1\right)=r+s-1$ and the least residues take the form

$$
\frac{m p-r x}{s}, \quad 1 \leq m \leq r+s-1, \quad m \equiv t x \bmod s
$$

Let $b:=\operatorname{gcd}(n, s)$ and suppose that $x$ is in $I_{i}$. If $b=1$ then, for each $m$, we have

$$
\frac{m p-r x}{s} \equiv(m p-r i) s^{-1} \bmod n
$$

and hence obtain at most $r+s-1$ residue classes $\bmod n$. If $b>1$ then $m \equiv t i$ $\bmod b$ and, for a given $m$, plainly $(m p-r x) / b \equiv(m p-r i) / b \bmod n / b$ giving

$$
\frac{m p-r x}{s} \equiv(s / b)^{-1}(m p-r i) / b \bmod n / b
$$

So we will have $b$ possible residue classes $\bmod n$ for each of the $m$ in $1 \leq m \leq r+s-1$ lying in a particular residue class $m \equiv t i \bmod b$; that is, at most

$$
\begin{equation*}
b\left\lceil\frac{r+s-1}{b}\right\rceil \leq b\left(\frac{r+s-2}{b}+1\right)=r+s+b-2 \tag{6.1}
\end{equation*}
$$

residue classes $\bmod n$. At least one residue class is missed when this is less than $n$.
(c) We proceed as in (b). For $b=1$ there is nothing to show. So suppose that $b>1$ with $(r+s-1)=b q+w, 0 \leq w<b$. We take our $i$ to satisfy $t i \equiv v \bmod b$ for any $v$ with $w<v \leq b$. This gives us $(n / b)(b-w)=n\left(1-\left\{\frac{r+s-1}{b}\right\}\right) \geq n / b$ residue classes $\bmod n$. For these $i$ the number of residue classes hit in (6.1) becomes

$$
b\left\lfloor\frac{r+s-1}{b}\right\rfloor \leq r+s-1<n .
$$

Proof of Example 1.2. Recall that $A x^{(p+1) / 2} \equiv \pm A x \bmod p$. Counting the residue classes for $A x$ or $-A x \bmod p$ gives at worst twice the total obtained in the proof of Example 1.1 for each of these, and therefore a missed residue class when this is less than $n$.

Proof of Example 1.3. (a) Suppose that $A>0$. Notice that when $n$ is odd or $n$ is even and $2^{\beta} \mid A$ and $x \equiv 2^{-1} p \bmod n / \operatorname{gcd}(A, n)$ we have

$$
A x-\ell p \equiv(A-\ell) p-A x \bmod n, \quad \ell=0, \ldots, A-1
$$

Thus, matching up the opposite ends $A x$ and $A p-A x$, we can perfectly pair the residue classes $A x, A x-p, \ldots, A x-(A-1) p$ for $A x \bmod p$ and the classes
$p-A x, 2 p-A x, \ldots, A p-A x$ for $-A x \bmod p$ in reverse order. Hence $A x^{(p+1) / 2}$ or $-A x^{(p+1) / 2} \equiv \pm A x \bmod p$ can take at most $A$ different values $\bmod n$ when $x$ is in $I_{i}$ for any of the $\operatorname{gcd}(n, A)$ values of $i$ with $i \equiv 2^{-1} p \bmod n / \operatorname{gcd}(A, n)$.
(b) If $2^{\beta} \nmid A$ then we can no longer match the end values and the best we can hope for is to match up $\operatorname{gcd}(A, n)$ steps in. That is

$$
A x-\operatorname{gcd}(A, n) p \equiv A p-A x \bmod n
$$

so that the remaining $A x-(\operatorname{gcd}(A, n)+\ell) p$ match up with the $(A-\ell) p-A x \bmod$ $n$. Thus we will just have the $A x-\ell p$ with $0 \leq \ell<\operatorname{gcd}(A, n)$ unmatched, and hence a total of $B:=A+\operatorname{gcd}(A, n)$ residue classes. This requires $2 A x \equiv(A+\operatorname{gcd}(A, n)) p$ $\bmod n$, that is $2 A / \operatorname{gcd}(A, n) x \equiv(A / \operatorname{gcd}(A, n)+1) p \bmod n / \operatorname{gcd}(A, n)$, equivalently $x \equiv \frac{1}{2}(A / \operatorname{gcd}(A, n)+1) p(A / \operatorname{gcd}(A, n))^{-1} \bmod n / 2 \operatorname{gcd}(A, n)$. Similarly we could match at the other end $p-A x \equiv A x-(A-1-\operatorname{gcd}(A, n)) p \bmod n$ for the same count. Hence if

$$
i: \equiv \frac{1}{2}\left(\frac{A}{\operatorname{gcd}(A, n)} \pm 1\right)\left(\frac{A}{\operatorname{gcd}(A, n)}\right)^{-1} p \bmod \frac{n}{\operatorname{gcd}(n, 2 A)}
$$

we have $f\left(I_{i}\right) \cap I_{j}=\emptyset$ for at least $n-B$ values of $j$.
(c), (d) and (e). Let $b:=\operatorname{gcd}(n, s)$ and $c:=\operatorname{gcd}(n, r)$.

Suppose first that $n$ is odd or $n$ even with $2^{\beta} \mid r$ and

$$
B:=r+s+b-2<n .
$$

Suppose that $i$ satisfies $i \equiv 2^{-1} p \bmod n / c$.
As in the proof of Example 1.1 for $A=(t p-r) / s, r, s>0$ the classes for $A x$ $\bmod p$ and $-A x \bmod p$ with $x$ in $I_{i}$ will take the form

$$
\left(\frac{m p-r x}{s}\right) \text { and } p-\left(\frac{m p-r x}{s}\right)
$$

respectively, with $1 \leq m \leq r+s-1$, and $m \equiv t x \bmod s$. Writing $m^{\prime}=r+s-m$ we have

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right)=\frac{(m p-r x)}{s}+\frac{r\left(x+x^{\prime}-p\right)}{s}
$$

where plainly $1 \leq m \leq r+s-1$ iff $1 \leq m^{\prime} \leq r+s-1$ and, since $r \equiv p t \bmod s$,

$$
m^{\prime} \equiv t x^{\prime} \bmod s \quad \text { iff } \quad x^{\prime} \equiv p-m t^{-1} \bmod s
$$

Note that when $b>1$, the conditions $x \equiv m t^{-1} \bmod s$ with $x$ in $I_{i}$ and $m^{\prime} \equiv$ $t x^{\prime} \bmod s, x^{\prime}$ in $I_{i}$ both imply that $m \equiv t i \bmod b$, since $i \equiv p-i \bmod b$.

If $b=1$ then the $x, x^{\prime}$ in $I_{i}$ have $x+x^{\prime}-p \equiv 2 i-p \equiv 0 \bmod n / c$ and

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right) \equiv \frac{(m p-r x)}{s} \equiv(m p-r i) s^{-1} \bmod n
$$

with the different $m$ only giving us $r+s-1$ different residue classes $\bmod n$.
Now suppose that $b>1$ and $x, x^{\prime}$ are in $I_{i}$, and that we have an $m$ with $1 \leq$ $m \leq r+s-1$ and $m \equiv t i \bmod b$. Consider the $x$ with

$$
x \equiv i \bmod n / c, \quad x \equiv m t^{-1} \bmod s
$$

If $x_{0}$ is one solution then the other $x$ will satisfy $x \equiv x_{0} \bmod n s / b c$. That is, we will have $b$ solutions $\bmod n s / c$ :

$$
x=x_{0}+\lambda n s / b c \bmod n s / c, \quad 0 \leq \lambda<b
$$

Similarly, the

$$
x^{\prime} \equiv i \bmod n / c, \quad x^{\prime} \equiv p-m t^{-1} \bmod s
$$

will have $b$ solutions $\bmod n s / c$, namely, since $p-i \equiv i \bmod n / c$,

$$
x^{\prime}=p-x_{0}-\lambda n s / b c \bmod n s / c, \quad 0 \leq \lambda<b
$$

Thus pairing up the $x$ and $x^{\prime}$ with the same $\lambda$ we get $r\left(x+x^{\prime}-p\right) \equiv 0 \bmod n s$ and

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right) \equiv \frac{(m p-r x)}{s} \bmod n
$$

perfectly pairing up the classes for $-A x^{\prime}$ and $A x$. Counting the $b$ values of $\lambda$ for each $m$ with $1 \leq m \leq r+s-1$ and $m \equiv t i \bmod b$ gives the count $B$ as before and we miss $n-B$ classes. This gives us (d), and (c) when $b=\min \{b, c\}$.

Notice that in some cases we can relax our inequality; for example if $b>1$ but $b \mid(r+s-1)$, or if $r$ and $\lfloor r / b\rfloor$ have opposite parity (so that if $r \equiv w \bmod b$ then $\left.m \equiv 2^{-1} r \equiv \frac{1}{2}(w+b) \bmod b\right)$, we never have to round up in (6.1) and so only need $r+s \leq n$.

Observe that $f\left(I_{i}\right) \cap I_{j}=\emptyset$ if and only if $f^{-1}\left(I_{j}\right) \cap I_{i}=\emptyset$ where

$$
f(x)=A x^{(p+1) / 2} \bmod p \Rightarrow f^{-1}(x)=\left(\frac{A}{p}\right) A^{-1} x^{(p+1) / 2} \bmod p
$$

with

$$
A=(t p-r) / s \Rightarrow A^{-1}=\left(t^{\prime} p-s\right) / r, \quad t^{\prime} \equiv s r^{-1} \bmod p
$$

Switching the roles of $r$ and $s$ gives (c) when $n$ is odd and (e) when $n$ is even.
(f) Suppose that $n$ is even $2^{\beta} \nmid r$ and that $i$ satisfies

$$
i \equiv \frac{1}{2}((r / c) \pm 1) p(r / c)^{-1} \bmod n / c
$$

(we just consider the plus sign, the case with the minus sign is similar). Take $m^{\prime}=r+s+c-m$ and write

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right)=\frac{m p-r x}{s}+\frac{r\left(x+x^{\prime}-p\right)-c p}{s}
$$

with $1 \leq m^{\prime} \leq r+s-1$, and hence $1+c \leq m \leq r+s+c-1$, and

$$
x^{\prime} \equiv m^{\prime} t^{-1} \equiv(r+c) t^{-1}-m t^{-1} \bmod s .
$$

Notice that if $x^{\prime}$ is in $I_{i}$ then $m=s+r+c-m^{\prime} \equiv r+\operatorname{gcd}(r, n)-t i \equiv t i \bmod b$, since 2 it $\equiv p t(r / c)^{-1}(1+(r / c)) \equiv(c+r) \bmod b$.

Suppose that $x, x^{\prime}$ are in $I_{i}$. If $b=1$ then

$$
r\left(x+x^{\prime}-p\right)-c p \equiv c(2 i(r / c)-p((r / c)+1)) \equiv 0 \bmod n
$$

and

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right) \equiv \frac{m p-r x}{s} \equiv(m p-r i) s^{-1} \bmod n
$$

For the $-A x^{\prime} \bmod p$ we need the $1+c \leq m \leq r+s-1+c$ and for $A x \bmod p$ the $1 \leq m \leq r+s-1$. Hence we have $1 \leq m \leq r+s+c-1$ and at most $r+s+c-1$ residue classes mod $n$.

Suppose that $b>1$ and $m \equiv t i \bmod b$, then taking $x_{0}$ to be a solution to

$$
x \equiv i \bmod n / c, \quad x \equiv m t^{-1} \bmod s
$$

the solutions take the form

$$
x \equiv x_{0}+\lambda n s / b c \bmod n s / c, \quad 0 \leq \lambda<b
$$

Likewise, since $(r / c)^{-1}(1+(r / c)) p-i \equiv i \bmod n / c$, the solutions to

$$
x^{\prime} \equiv i \bmod n / c, \quad x^{\prime} \equiv(r+c) t^{-1}-m t^{-1} \bmod s
$$

can be written

$$
x^{\prime} \equiv(r / c)^{-1}(1+(r / c)) p-x_{0}-\lambda n s / b c \bmod n s / c, \quad 0 \leq \lambda<b
$$

where here we take $(r / c)^{-1}$ to be an inverse of $r / c \bmod n s / c$.
Pairing up the $x$ and $x^{\prime}$ with the same $\lambda$ we have

$$
p-\left(\frac{m^{\prime} p-r x^{\prime}}{s}\right) \equiv \frac{m p-r x}{s} \equiv \frac{m p-r x_{0}}{s}-\lambda(r / c)(n / b) \bmod n .
$$

With $b$ choices of $\lambda$ for each $m \equiv t i \bmod b$ with $1 \leq m \leq r+s+c-1$ we have at most

$$
\begin{equation*}
b\left\lceil\frac{r+s+c-1}{b}\right\rceil \leq b\left(\frac{r+s+c-2}{b}+1\right)=r+s+c+b-2 \tag{6.2}
\end{equation*}
$$

residue classes $\bmod n$.
Notice that $t i \equiv(r+c) / 2 \bmod b$ and if $b \mid(r+c)$ when $s$ is odd, or $2 b \mid(r+c)$ when $s$ is even, or $b \nmid(r+c)$ and $\lfloor(r+c) / b\rfloor$ is odd, then in (f) we only need $r+s+c \leq n$. Similarly when $2^{\beta} \mid s$ the value of $i$ is only fixed $\bmod b / 2$, hence if $r+c \equiv w \bmod b$ we can pick an $i$ so that $t i \equiv(w+b) / 2 \bmod b$, and again we only need $r+s+c \leq n$, giving us (e) directly without flipping $r$ and $s$.

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Department of Mathematics, Kansas State University, Manhattan, KS 66506 USA
E-mail address: cochrane@ksu.edu, pinner@ksu.edu, crichardson@ksu.edu
Department of Mathematics \& Computer Science, Davidson College, Davidson, NC 28035, USA. E-mail address: mimossinghoff@davidson.edu


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