SOME SINGULAR CURVES AND SURFACES ARISING FROM INVARIANTS OF COMPLEX REFLECTION GROUPS

by

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Abstract. — We construct highly singular projective curves and surfaces defined by invariants of primitive complex reflection groups.

It is a classical problem to determine the maximal number of singularities of a given type that a curve or a surface might have. Several kinds of upper bounds have been given [**Sak**], [**Bru**], [**Miy**], [**Var**], [**Wah**]..., and these bounds have been approached for small degrees [**Iv**], [**Bar**], [**Esc1**], [**Esc2**], [**End**], [**EnPeSt**], [**Lab**], [**Sar3**], [**Sta**]... or general degrees [**Chm**].

In [**Bar**], [**Sar1**], [**Sar2**], [**Sar3**], Barth and Sarti used pencils of surfaces constructed from invariants of some finite Coxeter subgroups of $\mathbf{GL}_4(\mathbb{R})$ to obtain surfaces of degree 6, 10, 12 with the biggest number of nodes known up to now. We have decided to explore more systematically pencils of curves and surfaces constructed from invariants of finite complex reflection subgroups of $\mathbf{GL}_3(\mathbb{C})$ or $\mathbf{GL}_4(\mathbb{C})$. In this paper, we gather the results of these computations (made with MAGMA [**Magma**]) obtained from the *primitive* complex reflection. As the reader will see, not all the primitive complex reflection groups lead to interesting examples but these investigations have lead to the discovery of the following curves or surfaces, which improve some known lower bounds and are quite close to upper bounds found by Sakai [**Sak**] for curves or Miyaoka [**Miy**] for surfaces (we refer to Shephard-Todd notation [**ShTo**] for complex reflection groups; for Coxeter groups, we also use the notation $W(\Gamma)$, where Γ is a Coxeter graph):

(a) Using the complex reflection group $G_{24} \subset \mathbf{GL}_3(\mathbb{C})$, we construct a curve of degree 14 with 42 cusps (i.e. singularities of type A_2): this improves known lower bounds (see Example 3.2). Note that the known upper bound for the number of cusps of a curve of degree 14 in $\mathbf{P}^2(\mathbb{C})$ is 55.

(b) Using the complex reflection group G_{26}, we construct a curve of degree 18 with 36 singular points of type E_6 (see Example 3.4). We do not know if such a bound was already reached.

(c) Let $\mu_{D_4}(d)$ denote the maximal number of quotient singularities of type D_4 that an irreducible surface in $\mathbf{P}^3(\mathbb{C})$ might have. Miyaoka [**Miy**] proved that

$$\mu_{D_4}(d) \leqslant \frac{16}{117} d(d-1)^2.$$

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For $d = 8$, 12, or 24, this reads

$$\mu_{D_4}(8) \leqslant 53, \qquad \mu_{D_4}(12) \leqslant 198 \qquad \text{and} \qquad \mu_{D_4}(24) \leqslant 1\,736.$$

Using respectively the complex reflection groups $G_{28} = \mathrm{W}(F_4)$, G_{29} and G_{32}, we prove that

$$\mu_{D_4}(8) \geqslant 48, \qquad \mu_{D_4}(12) \geqslant 160 \qquad \text{and} \qquad \mu_{D_4}(24) \geqslant 1\,440$$

(see Examples 4.3 and 4.5(3) and Table IV). This improves considerably the last known lower bounds [**Esc2**]. Recall that, by standard arguments, this implies that $\mu_{D_4}(8k) \geqslant 48k^3$, $\mu_{D_4}(12k) \geqslant 160k^3$ and $\mu_{D_4}(24k) \geqslant 1\,440k^3$ for all $k \geqslant 1$. Note that the fact that $\mu_{D_4}(24) \geqslant 1\,440$ was first announced in [**Bon1**] (and a previous lower bound $\mu_{D_4}(8) \geqslant 44$ was also obtained): see Section 6 for details.

We also found examples which do not improve known lower bounds but might possibly be interesting for the number and the type of singularities they contain (with "big" multiplicities or "big" Milnor numbers): see Examples 3.4, 4.5, 4.7. The examples might also be interesting for their big group of automorphisms.

These computations also show that Miyaoka bounds are quite sharp, even for singularities that are not of type A. Contrary to previous constructions, the singular points of our curves or surfaces are in general not all real[(1)] (even though most of these varieties are defined over $\mathbb{Q}$). By contrast, note also that, using a theorem of Marin-Michel on automorphisms of reflection groups [**MaMi**], we can show that the Sarti dodecic can be defined over $\mathbb{Q}$ (this was still an open question).

For the smoothness of the exposition, we have decided to include most of the MAGMA codes in separate texts [**Bon1**] (for varieties associated with G_{32}) and [**Bon2**] (for the other examples), as well as some explicit polynomials: these two texts are not intended to be published, but are made for the reader interested in checking the computations by himself.

1. Notation, preliminaries

We fix an n-dimensional $\mathbb{C}$-vector space V and a finite subgroup W of $\mathbf{GL}_{\mathbb{C}}(V)$. We set

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}}(V^s) = n-1\}.$$

Hypothesis. *We assume throughout this paper that*

$$W = \langle \mathrm{Ref}(W) \rangle.$$

*In other words, W is a **complex reflection group**. We also assume that W acts **irreducibly** on V. The number n is called the **rank** of W.*

(1) There is an important exception to this remark: all the singular points of the surface of degree 8 with 48 singularities of type D_4 constructed in Example 4.3 have rational coordinates.

1.A. Invariants. — We denote by $\mathbb{C}[V]$ the ring of polynomial functions on V (identified with the symmetric algebra $\mathrm{S}(V^*)$ of the dual V^* of V) and by $\mathbb{C}[V]^W$ the ring of W-invariant elements of $\mathbb{C}[V]$. By Shephard-Todd/Chevalley-Serre Theorem [**Bro**, Theorem 4.1], there exist n algebraically independent homogeneous elements f_1, f_2,..., f_n of $\mathbb{C}[V]^W$ such that

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, f_2, \dots, f_n].$$

Let $d_i = \deg(f_i)$. We will assume that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. A family $(f_1, f_2, \dots, f_n)$ satisfying the above property is called a *family of fundamental invariants* of W. Whereas such a family is not uniquely defined, the list $(d_1, d_2, \dots, d_n)$ is well-defined and is called the list of *degrees* of W. If $f \in \mathbb{C}[V]$ is homogeneous, we will denote by $\mathscr{Z}(f)$ the projective (possibly reduced) hypersurface in $\mathbf{P}(V) \simeq \mathbf{P}^{n-1}(\mathbb{C})$ defined by f. Its singular locus will be denoted by $\mathscr{Z}_{\mathrm{sing}}(f)$. A homogeneous element $f \in \mathbb{C}[V]$ is called a *fundamental invariant* if it belongs to a family of fundamental invariants.

Recall that a subgroup G of $\mathbf{GL}_{\mathbb{C}}(V)$ is called *primitive* if there does not exist a decomposition $V = V_1 \oplus \cdots \oplus V_r$ with $V_i \neq 0$ and $r \geqslant 2$ such that G permutes the V_i's. We will be mainly interested in *primitive* (often called *exceptional*) complex reflection groups, and we will refer to Shephard-Todd numbering [**ShTo**] for such groups (there are 34 isomorphism classes, named G_i for $4 \leqslant i \leqslant 37$). Almost all the computations(2) have been done using the software MAGMA [**Magma**].

1.B. Marin-Michel Theorem. — Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and we set $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Using the classification of finite reflection groups, Marin-Michel [**MaMi**] proved that there exists a $\mathbb{Q}$-structure $V_{\mathbb{Q}}$ of V such that:

(1) $V_{\overline{\mathbb{Q}}} = \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$ is stable under the action of W (so that W might be viewed as a subgroup of $\mathbf{GL}_{\overline{\mathbb{Q}}}(V_{\overline{\mathbb{Q}}})$).
(2) The action of Γ on $\mathbf{GL}_{\overline{\mathbb{Q}}}(V_{\overline{\mathbb{Q}}})$ induced by the $\mathbb{Q}$-form $V_{\mathbb{Q}}$ stabilizes W.

This implies that $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$ is a $\overline{\mathbb{Q}}$-form of $\mathbb{C}[V]$ stable under the action of W and that the action of Γ on $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$ induced by the $\mathbb{Q}$-form $\mathbb{Q}[V_{\mathbb{Q}}]$ stabilizes the invariant ring $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$.

Proposition 1.1. — *The Sarti dodecic can be defined over* $\mathbb{Q}$.

Remark 1.2. — An explicit polynomial with rational coefficients defining the Sarti dodecic is given in [**Bon2**]. ■

Proof. — Assume here that W is a Coxeter group of type H_4 acting on a vector space V of dimension 4. We fix a $\mathbb{Q}$-form $V_{\mathbb{Q}}$ as above. Let f be a homogeneous invariant of W of degree 12 defining the Sarti dodecic: it belongs to $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$. We fix a $\mathbb{Q}$-basis $(h_1, h_2, \dots, h_{455})$ of the homogeneous component of degree 12 of $\mathbb{Q}[V_{\mathbb{Q}}]$. It is also a $\overline{\mathbb{Q}}$-basis of the homogeneous component of degree 12 of $\overline{\mathbb{Q}}[V_{\overline{\mathbb{Q}}}]$. By multiplying f by a scalar if necessary, we may assume that there exists $i \in \{1, 2, \dots, 455\}$ such that the coefficient of f on h_i is 1.

Now, if $\gamma \in \Gamma$, then ${}^{\gamma}f$ is also an invariant of W of degree 12 defining an irreducible projective surface with 600 nodes. By the unicity of such an invariant [**Sar3**], this forces ${}^{\gamma}f = \xi f$ for some $\xi \in \overline{\mathbb{Q}}^{\times}$. But $\xi = 1$ because the coefficient of f on h_i is 1. So $f \in \mathbb{Q}[V_{\mathbb{Q}}]$. □

(2)Some Milnor and Tjurina numbers were computed with SINGULAR [**DGPS**].

Remark 1.3. — In our computations made with MAGMA, reflection groups W are represented as subgroups of $\mathbf{GL}_n(K)$ where K is a number field depending on W. There are of course infinitely many possibilities for representing W in this way, but it turns out that the choice of this model have a considerable impact on the time used for computations, and on the form of the defining polynomials for the singular varieties we obtain. Let us explain which choices we have made and for which reasons:

- We do not use the MAGMA command
  `ShephardTodd(k)`
  for defining the complex reflection group G_k. Indeed, the MAGMA model for G_k is generally not stable under the Galois action, and leads to very lengthy computations (and sometimes to computations that do not conclude after hours) and to very ugly defining polynomials for the singular varieties found by our methods.

- In his CHAMP package for MAGMA intended to study the representation theory of Cherednik algebras [**Thi**], Thiel used the model implemented in the CHEVIE package of GAP3 by Michel [**Mic**]. These models are almost all stable under the action of the Galois group (except for the Coxeter groups $G_{23} = \mathrm{W}(H_3)$ and $G_{30} = \mathrm{W}(H_4)$) and leads to much shorter computations and much nicer defining polynomials for singular varieties (for instance, they almost all have rational coefficients).

- We have decided to create our own models for the Coxeter groups $G_{23} = \mathrm{W}(H_3)$ and $G_{30} = \mathrm{W}(H_4)$: they are stable under the Galois action (so fit with Marin-Michel Theorem). This again shortens the computations and lead to polynomials with rational coefficients for defining singular varieties: this is how we found en explicit polynomial with rational coefficients defining the Sarti dodecic [**Bon2**]. These models are implemented in a file `primitive-complex-reflection-groups.m` available in [**Bon2**] and are accessible through the command
  `PrimitiveComplexReflectionGroup(k)`
  once this file is downloaded. Note that:
  - This file copies almost entirely Thiel's file except for the Coxeter groups $G_{23} = \mathrm{W}(H_3)$, $G_{28} = \mathrm{W}(F_4)$ and $G_{30} = \mathrm{W}(H_4)$.
  - For $G_{23} = \mathrm{W}(H_3)$ and $G_{30} = \mathrm{W}(H_4)$, we have given our own models defined over the field $\mathbb{Q}(\rho)$, where $\rho^4 = 5\rho^2 - 5$ (i.e. $\rho = \sqrt{(5+\sqrt{5})/2}$). We do not pretend it is the best possible model but, for our purposes, it is the best model available as of today.
  - For $G_{28} = \mathrm{W}(F_4)$, we have used a version which contains the Coxeter group $\mathrm{W}(B_4)$ in its standard form (that is, as the group of monomial matrices whose non-zero coefficients belong to $\boldsymbol{\mu}_2 = \{1, -1\}$) as a subgroup of index 3. This implies in particular that invariant polynomials can be expressed in terms on elementary symmetric functions.

Of course, as explained in the introduction, the fact that most of the singular varieties we construct are defined over $\mathbb{Q}$ do not imply that the coordinates of all the singular points are rational, or even real. Some of the varieties have in fact no real points. The only example where singular points have rational coordinates is given in Example 4.3 (see Figure I). ∎

n	W	$\|W\|$	$\|W/\mathrm{Z}(W)\|$	$(d_1,d_2,\dots,d_n)$	d_r
3	$G_{23}=\mathrm{W}(H_3)$	120	60	$2,6,10$	6
	G_{24}	336	168	$4,6,14$	14
	G_{25}	648	108	$6,9,12$	12
	G_{26}	1 296	216	$6,12,18$	12
	G_{27}	2 160	360	$6,12,30$	12
4	$G_{28}=\mathrm{W}(F_4)$	1 152	576	$2,6,8,12$	6
	G_{29}	7 680	1 920	$4,8,12,20$	8
	$G_{30}=\mathrm{W}(H_4)$	14 400	7 200	$2,12,20,30$	12
	G_{31}	46 080	11 520	$8,12,20,24$	20
	G_{32}	155 520	25 920	$12,18,24,30$	24
5	G_{33}	51 840	25 920	$4,6,10,12,18$	10
6	G_{34}	39 191 040	6 531 840	$6,12,18,24,30,42$	12
	$G_{35}=\mathrm{W}(E_6)$	51 840	25 920	$2,5,6,10,12,14,18$	6
7	$G_{36}=\mathrm{W}(E_7)$	2 903 040	1 451 520	$2,6,8,10,12,14,18$	6
8	$G_{37}=\mathrm{W}(E_8)$	696 729 600	348 364 800	$2,8,12,14,18,20,24,30$	8

TABLE I. Degrees of primitive complex reflection groups in rank $\geqslant 3$

2. Strategy for finding some "singular" invariants in rank $n \geqslant 3$

If $n = 2$, then the varieties $\mathscr{Z}(f)$ are just collections of points, and so are uninteresting for our purpose.

Hypothesis and notation. *From now on, and until the end of this paper, we assume moreover that $n \geqslant 3$ and that W is* ***primitive****. We denote by r the minimal natural number such that the space of homogeneous invariants of W of degree d_r has dimension $\geqslant 2$.*

Note that this implies that W is one of the groups G_i, with $23 \leqslant i \leqslant 37$, in Shephard-Todd classification. We recall in Table I the degrees $(d_1, d_2, \dots, d_n)$ of these groups. We also give the following informations: the order of W, the order of $W/\mathrm{Z}(W)$ (which is the group which acts faithfully on $\mathbf{P}(V)$), the degree d_r and, whenever W is a Coxeter group, we recall its type ($\mathrm{W}(X_i)$ denotes the Coxeter group of type X_i). Recall from general theory that $|W| = d_1 d_2 \cdots d_n$ and $|\mathrm{Z}(W)| = \mathrm{Gcd}(d_1, d_2, \dots, d_n)$.

Using MAGMA, we first determine by computer calculations some fundamental invariants $f_1, \dots, f_r$. By the definition of r, the fundamental invariants $f_1, \dots, f_{r-1}$ are uniquely determined up to scalar. By inspection of Table I, we see that $d_1 < d_2 < \cdots < d_n$ and that there is a unique f of the form $f_1^{m_1} \cdots f_{r-1}^{m_{r-1}}$ which has degree d_r. So the space of homogeneous invariants of degree d_r has dimension 2, and is spanned by f_r and f. Moreover, all fundamental invariants of degree d_r are, up to a scalar, of the form $f_r + uf$, for some $u \in \mathbb{C}$.

This means that we need to determine the values of u such that $\mathscr{Z}(f_r + uf)$ is singular. For this, we use the basis $(x_1,\dots,x_n)$ of V^* chosen by MAGMA and we set

$$F_u = f_r + uf \qquad \text{and} \qquad F_u^{\mathrm{aff}}(x_1,\dots,x_{n-1}) = F_u(x_1,\dots,x_{n-1},1).$$

This basis allows to identify $\mathbf{P}(V)$ with $\mathbf{P}^{n-1}(\mathbb{C})$ and we denote by $\mathbf{A}^{n-1}(\mathbb{C})$ the affine open subset of $\mathbf{P}^{n-1}(\mathbb{C})$ defined by $x_n \neq 0$. Then $\mathscr{Z}^{\mathrm{aff}}(F_t^{\mathrm{aff}})$ denotes the affine open subset of $\mathscr{Z}(F_u)$ defined by $x_n \neq 0$. Note the following easy fact:

(2.2) *Any W-orbit of points in $\mathbf{P}^{n-1}(\mathbb{C})$ meets $\mathbf{A}^{n-1}(\mathbb{C})$.*

Proof. — Indeed, the linear span of a W-orbit of a non-zero vector in V must be equal to V, because W acts irreducibly. So it cannot be fully contained in the orthogonal of x_n. $\square$

One deduces immediately the following fact, which will be useful for saving much time during computations:

(2.3) *$\mathscr{Z}(F_u)$ is singular if and only if $\mathscr{Z}^{\mathrm{aff}}(F_u^{\mathrm{aff}})$ is singular.*

Now, let

$$\mathscr{X} = \{(\xi,u) \in \mathbf{A}^{n-1}(\mathbb{C}) \times \mathbf{A}^1(\mathbb{C}) \mid F_u^{\mathrm{aff}}(\xi) = 0\}.$$

We denote by $\phi : \mathscr{X} \to \mathbf{A}^1(\mathbb{C})$ the second projection. Then the fiber $\phi^{-1}(u)$ is the variety $\mathscr{Z}^{\mathrm{aff}}(F_u^{\mathrm{aff}})$. We can then define

$$\mathscr{X}_{\mathrm{sfib}} = \{(\xi,u) \in \mathscr{X} \mid \frac{\partial F_u^{\mathrm{aff}}}{\partial x_1}(\xi) = \cdots = \frac{\partial F_u^{\mathrm{aff}}}{\partial x_{n-1}}(\xi) = 0\}.$$

Then $\mathscr{X}_{\mathrm{sfib}}$ is not necessarily the singular locus of $\mathscr{X}$, but the points in $\phi(\mathscr{X}_{\mathrm{sfib}})$ are the values of u for which the fiber $\phi^{-1}(u) = \mathscr{Z}^{\mathrm{aff}}(F_u^{\mathrm{aff}})$ (or, equivalently, $\mathscr{Z}(F_u)$) is singular. We set $U_{\mathrm{sing}} = \phi(\mathscr{X}_{\mathrm{sfib}})$ and we denote by $U_{\mathrm{sing}}^{\mathrm{irr}}$ the set of $u \in U_{\mathrm{sing}}$ such that $\mathscr{Z}(F_u)$ is irreducible. This provides an algorithm for finding these values of u: it turns out that ϕ is not dominant in our examples, so that there are only finitely many such values of u. We then study more precisely these finite number of cases (number of singular points, nature of singularities, Milnor number,...). Let us see on a simple example how it works:

***Example 2.4* (Coxeter group of type H_3).** — Assume here, and only here, that $W = G_{23} = \mathrm{W}(H_3)$. Then $(d_1,d_2,d_3) = (2,6,10)$ so that $r = 2$ and $d_r = 6$. Then $F_u = f_2 + uf_1^3$. We first define W (see Remark 1.3 for the choice of a model) and the fundamental invariants f_1 and f_2:

```
> load 'primitive-complex-reflection-groups.m';
> W:=PrimitiveComplexReflectionGroup(23);
> K<a>:=CoefficientRing(W);
> R:=InvariantRing(W);
> P<x1,x2,x3>:=PolynomialRing(R);
> f1:=InvariantsOfDegree(W,2)[1];
> f2:=InvariantsOfDegree(W,6)[1];
> Gcd(f1,f2);
1
```

Note that the last command shows that the invariant f_2 of degree 6 we have chosen is indeed a fundamental invariant. We now define F_u^{aff} and $\mathscr{X}_{\mathrm{sfib}}$ and then determine the set U_{sing} of values of u such that $\mathscr{Z}(F_u)$ is singular:

```
> P2:=Proj(P);
> A2xA1<xx1,xx2,u>:=AffineSpace(K,3);
> A1<U>:=AffineSpace(K,1);
> phi:=map<A2xA1->A1 | [u]>;
> f1aff:=Evaluate(f1,[xx1,xx2,1]);
> f2aff:=Evaluate(f2,[xx1,xx2,1]);
> Fuaff:=f2aff + u * f1aff^3;
> X:=Scheme(A2xA1,Fuaff);
> Xsfib:=Scheme(X,[Derivative(Fuaff,i) : i in [1,2]]);
> Psing:=MinimalBasis(phi(Xsfib));
> # Psing;
1
> Factorization(Psing[1]);
[
    <T + 1, 1>,
    <T + 9/10, 1>,
    <T + 63/64, 1>
]
> Using:=[-1, -9/10, -63/64];
```

We next determine for which values $u \in U_{\mathrm{sing}} = \{u_1, u_2, u_3\}$ the curve $\mathscr{Z}(F_u)$ is irreducible:

```
> F:=[f2+ui*f1^3 : ui in Using]; // the polynomials F_{t_i}
> Z:=[Curve(P2,f) : f in F];
> [IsAbsolutelyIrreducible(i) : i in Z];
[ true, true, false ]
```

We then study the singular locus of the irreducible curves $\mathscr{Z}(F_u)$ for $u = u_1$ or u_2. Let us see how to do it for $u = u_1$:

```
> Z1sing:=SingularSubscheme(Z[1]);
> Z1sing:=ReducedSubscheme(Z1sing);
> Degree(Z1sing);
10
> points:=SingularPoints(Z[1]);
> # points;
10
> pt:=points[1];
> IsNode(Z[1],pt);
true
> # ProjectiveOrbit(W,pt);
10
```

The command `Degree(Z1sing)` shows that $\mathscr{Z}(F_{u_1})$ contains exactly 10 singular points. The command `# points` shows that they are all defined over the field `K` $(= \mathbb{Q}(\sqrt{5}))$. The command `# ProjectiveOrbit(W,p1)` shows that they are all in the same W-orbit (the function `ProjectiveOrbit` has been defined by the author for computing orbits in projective spaces (see **[Bon1]** or **[Bon2]** for the code). So all these singularities are equivalent and the command `IsNode(Z[1],pt)` shows that they are all nodes.

One can check similarly that $\mathscr{Z}(F_{u_2})$ has 6 nodes, all belonging to the same W-orbit. ■

In the next sections, we will give tables of singular curves and surfaces obtained in this way. Inspection of these tables (and Examples 5.1 and 5.2) leads to the following result:

Proposition 2.5. — *Apart from the two singular surfaces $\mathscr{S}$ and $\mathscr{S}'$ of degree 8 with 80 nodes defined by invariants of G_{29}, all the singular curves and surfaces described in Tables II, III and IV can be defined over $\mathbb{Q}$. The singular surfaces $\mathscr{S}$ and $\mathscr{S}'$ are Galois conjugate over $\mathbb{Q}$.*

Proof. — One could just check that the polynomials given thanks to the MAGMA codes contained in [**Bon2**] have coefficients in $\mathbb{Q}$. But one could also follow the same argument as in Proposition 1.1, based on Marin-Michel Theorem, by using the fact that all these singular curves and surfaces are characterized by their number of singular points or their type. □

Proposition 2.6. — *If $W = G_k$, with $23 \leqslant k \leqslant 35$ and $k \neq 34$, and if $u \in U_{\mathrm{sing}}^{\mathrm{irr}}$, then W acts transitively on $\mathscr{Z}_{\mathrm{sing}}(F_u)$.*

3. Singular curves from groups of rank 3

Hypothesis. *We still assume that W is primitive but, in this section, we assume moreover that $n = 3$.*

This means that W is one of the groups G_i, for $23 \leqslant i \leqslant 27$. We denote by (f_1, f_2, f_3) a set of fundamental invariants provided by MAGMA. Table II gives the list of curves obtained through the methods detailed in Section 2. This table contains the degree d_r, the cardinality of $U_{\mathrm{sing}}^{\mathrm{irr}}$, the number of singular points and the type of the singularity (since all singular points belong to the same W-orbit by Proposition 2.6, they are all equivalent singularities). Details of MAGMA computations are given in [**Bon2**] (they follow the lines of Example 2.4). We use standard notation for the types of the singularities of curves [**AGV**]. For instance (here, we denote by m the multiplicity, μ the Milnor number and τ the Tjurina number):

- A_1 is a *node*, i.e. a singularity equivalent to xy: in this case, $m = 2$ and $\mu = \tau = 1$.
- A_2 is a *cusp*, i.e. a singularity equivalent to $y^2 - x^3$: in this case, $m = 2 = \mu = \tau = 2$.
- D_4 is a singularity equivalent to $x(y^2 - x^2)$: in this case, $m = 3$, $\mu = \tau = 4$.
- X_9 is a singularity equivalent to $xy(y-x)(y+x)$: in this case, $m = 4$, $\mu = \tau = 9$.
- E_6 is a singularity equivalent to $y^3 - x^4$: in this case, $m = 3$, $\mu = \tau = 6$.

Example 3.2. — A plane curve is called *cuspidal* if all its singular points are of type A_2. By [**Sak**, (0.4)], a cuspidal plane curve of degree 14 has at most 55 singular points of type A_2. But it is not known if this is the sharpest bound: to the best of our knowledge, no cuspidal plane curve of degree 14 with 42 or more singular points of type A_2 was known before the above example of $\mathscr{Z}(F_{u_3})$ for $W = G_{24}$.

Also, a cuspidal plane curve of degree 12 can have at most 40 singular points [**Sak**, (0.4)], but it is not known if this bound can be achieved. However, there exists at least one cuspidal curve of degree 12 with 39 cusps [**C-ALi**, Example 6.3]. Our example obtained from invariants of G_{25}, with 36 cusps, approaches these bounds and has an automorphism group of order $\geqslant 108$. ■

W	d_r	$\lvert U^{\mathrm{irr}}_{\mathrm{sing}}\rvert$	u_i	$\lvert\mathscr{Z}_{\mathrm{sing}}(F_{u_i})\rvert$	Singularity
$G_{23} = \mathrm{W}(H_3)$	6	2	u_1	6	A_1
			u_2	10	A_1
G_{24}	14	3	u_1	21	A_1
			u_2	28	A_1
			u_3	42	A_2
G_{25}	12	2	u_1	12	D_4
			u_2	36	A_2
G_{27}	12	2	u_1	45	A_1
			u_2	36	A_1

TABLE II. Singularities of the curves $\mathscr{Z}(F_u)$ for $t \in U^{\mathrm{irr}}_{\mathrm{sing}}$

(u,v)	$\lvert\mathscr{Z}_{\mathrm{sing}}(F_{u,v})\rvert$	W-orbits	Singularity
(u_1,v_1)	63	9	X_9
		54	A_2
(u_2,v_2)	21	9	X_9
		12	D_4
(u_3,v_3)	45	9	X_9
		36	A_2
(u_4,v_4)	36	36	E_6
(u_5,v_5)	84	12	D_4
		72	A_2

TABLE III. Some singular curves of degree 18 defined by invariants of G_{26}

Remark 3.3. — Note that G_{26} does not appear in Table II. The reason is the following: if $W = G_{26}$, then $d_r = 12$ but G_{26} contains $W' = G_{25}$ as a normal subgroup of index 2 and it turns out that invariants of degree 12 of G_{25} and G_{26} coincide. This makes the computation for G_{26} unnecessary in this case. Note, however, the next Example 3.4, where we construct singular curves of degree 18 using invariants of G_{26}. ■

***Example 3.4* (The group G_{26})**. — We assume in this example that $W = G_{26}$. Recall that $(d_1,d_2,d_3) = (6,12,18)$. Up to a scalar, any fundamental invariant of degree 18 of W is of the form $F_{u,v} = f_3 + u f_1 f_2 + v f_1^3$ for some $(u,v) \in \mathbf{A}^2(\mathbb{C})$. Using MAGMA, one can check the following facts. First, the set $\mathscr{C}$ of $(u,v) \in \mathbf{A}^2(\mathbb{C})$ such that $\mathscr{Z}(F_{u,v})$ is singular is a union of three affine lines $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ and a smooth curve $\mathscr{E}$ isomorphic to $\mathbf{A}^1(\mathbb{C})$. The singular locus $\mathscr{C}_{\mathrm{sing}}$ of $\mathscr{C}$ consists of 7 points and it turns out that there are only 5 points $(u_i, v_i)_{1 \leqslant i \leqslant 5}$ in $\mathscr{C}_{\mathrm{sing}}$ such that $\mathscr{Z}(F_{u_i,v_i})$ is irreducible. Table III gives the information about singularities of these varieties $\mathscr{Z}(F_{u_i,v_i})$ (with the numbering used in our MAGMA programs [**Bon2**]).

Note that a cuspidal curve of degree 18 has at most 94 singularities of type A_2 [**Sak**, (0.3)]. Note also that there exists a cuspidal curve of degree 18 with 81 cusps [**Iv**]. ■

W	d_r	$\lvert U^{\mathrm{irr}}_{\mathrm{sing}}\rvert$	u_i	$\lvert \mathscr{Z}_{\mathrm{sing}}(F_{u_i})\rvert$	Singularity
$G_{28} = \mathrm{W}(F_4)$	6	4	u_1	12	A_1
			u_2	12	A_1
			u_3	48	A_1
			u_4	48	A_1
G_{29}	8	5	u_1	40	A_1
			u_2	20	Ordinary, $m=3, \mu=11, \tau=10$
			u_3	160	A_1
			u_4	80	A_1
			u_5	80	A_1
$G_{30} = \mathrm{W}(H_4)$	12	4	u_1	300	A_1
			u_2	60	A_1
			u_3	360	A_1
			u_4	600	A_1
G_{31}	20	5	u_1	480	A_1
			u_2	960	A_1
			u_3	1 920	A_1
			u_4	640	A_1
			u_5	1 440	A_1
G_{32}	24	4	u_1	40	Ordinary, $m=6, \mu=125, \tau=125$
			u_2	360	Non-ordinary, $m=3, \mu=18, \tau=18$
			u_3	1 440	D_4
			u_4	540	Non-simple, non-ordinary, $m=2, \mu=9, \tau=9$

TABLE IV. Singularities of the surfaces $\mathscr{Z}(F_u)$ for $u \in U^{\mathrm{irr}}_{\mathrm{sing}}$

4. Singular surfaces from groups of rank 4

Hypothesis. *We still assume that W is primitive but, in this section, we assume moreover that $n = 4$.*

This means that W is one of the groups G_i, for $28 \leqslant i \leqslant 32$. We denote by (f_1, f_2, f_3, f_4) a set of fundamental invariants provided by MAGMA and we denote by $U^{\mathrm{irr}}_{\mathrm{sing}}$ the set of elements $u \in \mathbb{C}$ such that $\mathscr{Z}(F_u)$ is irreducible and singular. Table IV gives the list of surfaces obtained through the methods detailed in Section 2. This table contains the degree d_r, the number of values of t such that $\mathscr{Z}(F_u)$ is irreducible and singular, the number of singular points and informations about the singularity (since all singular points belong to the same W-orbit by Proposition 2.6, they are all equivalent singularities). The number m (resp. μ, resp. τ) denotes the multiplicity (resp. the Milnor number, resp. the Tjurina number).

The example with 1 440 singularities of type D_4 obtained from G_{32} is detailed in section 6: one can derive from the construction a surface of degree 8 with 44 singularities of type D_4 (see also **[Bon1]**).

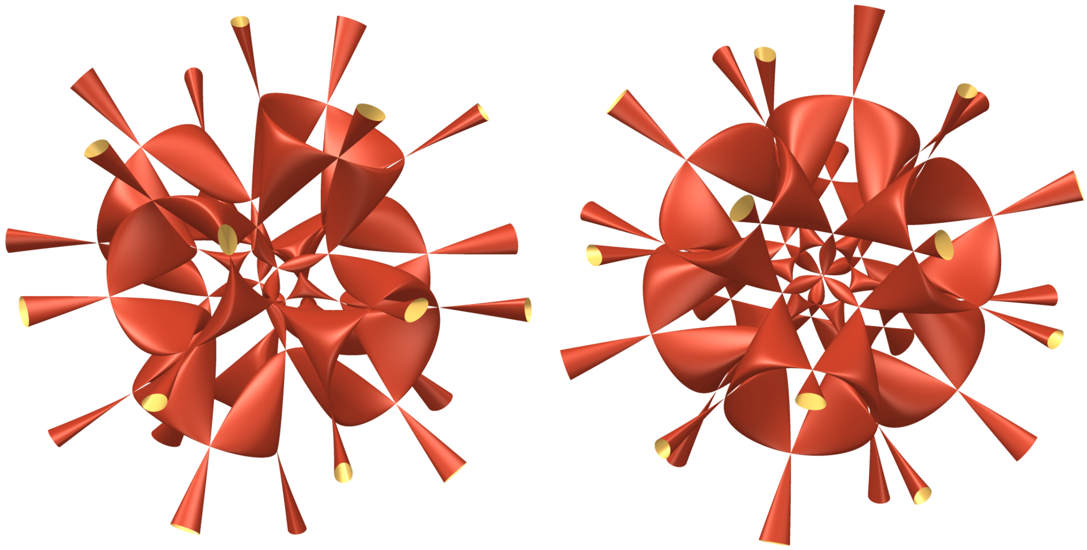

FIGURE I. Part of the real locus of $\mathscr{Z}(\varphi_2)$ for $W = G_{28} = \mathrm{W}(F_4)$.

***Remark 4.2* (Coxeter groups of rank 4).** — In Table IV, the cases of Coxeter groups of type F_4 and H_4 (i.e. the primitive reflection groups G_{28} and G_{30}) was dealt with by Sarti [**Sar1**]. ■

***Examples 4.3* (Coxeter group of type F_4).** — Assume in this example, and only in this example, that $W = G_{28} = \mathrm{W}(F_4)$ is the Coxeter group of type F_4, in the form explained in Remark 1.3. We denote by σ_1, σ_2, σ_3, σ_4 the elementary symmetric polynomials in x_1, x_2, x_3, x_4 and if $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ and $k \geqslant 1$, we set $f[k] = f(x_1^k, x_2^k, x_3^k, x_4^k)$.

Let φ_1 and φ_2 be the following two polynomials:

$$\varphi_1 = 7\sigma_1[2]^4 - 72\sigma_1[2]^2\sigma_2[2] + 4\,320\sigma_4[2] + 432\sigma_2[4]$$

and
$$\varphi_2 = \sigma_1[2]^4 - 9\sigma_1[2]^2\sigma_2[2] + 27\sigma_2[2]^2 - 27\sigma_1[2]\sigma_3[2] + 324\sigma_4[2].$$

Then it is easily checked that $\varphi_i \in \mathbb{C}[V]^W$ and that the two varieties $\mathscr{Z}(\varphi_i)$ are isomorphic (because there is an element g of $\mathrm{N}_{\mathbf{GL}_4(\mathbb{C})}(W)$ such that $\varphi_2 = {}^g\varphi_1$) and have the following properties:

- The reduced singular locus $\mathscr{Z}_{\mathrm{sing}}(\varphi_i)$ has dimension 0 and consists of 48 points which are all quotient singularities of type D_4.
- The group G_{28} acts transitively on $\mathscr{Z}_{\mathrm{sing}}(\varphi_i)$ and all elements of $\mathscr{Z}_{\mathrm{sing}}(\varphi_i)$ have coordinates in $\mathbb{Q}$.

This shows in particular that

$$\mu_{D_4}(8) \geqslant 48, \tag{4.4}$$

as announced in the introduction. Figure I shows part of the real locus of $\mathscr{Z}(\varphi_2)$. ■

***Examples 4.5* (The group G_{29}).** — Assume in this example, and only in this example, that $W = G_{29}$, in the version implemented by Jean Michel in the `Chevie` package of `GAP3` [**Mic**]. Then it contains the symmetric group $\mathfrak{S}_4$ (viewed as the subgroup of $\mathbf{GL}_4(\mathbb{C})$ consisting of permutation matrices). We use the notation of Example 4.3 for elementary symmetric functions and evaluation at powers of the indeterminates.

(1) Recall that the *Endraß octic* [**End**] has degree 8 and 168 nodes and its automorphism group has order 16. As shown in Table IV, $\mathscr{Z}(F_{u_3})$ is an irreducible surface in $\mathbf{P}^3(\mathbb{C})$ with 160 nodes and a group of automorphisms of order at least 1 920, thus approaching Endraß' record but with more symmetries. However, this surface has no real point. Up to a scalar, we have

$$F_{u_3} = \sigma_1[8] + 3\sigma_1[2]^2\sigma_2[2] + 2\sigma_2[4] - 30\sigma_1[2]\sigma_3[2] + 240\sigma_4[2].$$

It is still an open question to determine whether one can find a surface of degree 8 in $\mathbf{P}^3(\mathbb{C})$ with more than 168 nodes (being aware that the maximal number of nodes cannot exceed 174, see [**Miy**]).

(2) For the surface $\mathscr{Z}(F_{u_2})$, it can be shown with the software SINGULAR that the singularities are all of type $T_{4,4,4}$ that is, are equivalent to the singularity $xyz + x^4 + y^4 + z^4$. Up to a scalar, we have

$$F_{u_2} = \sigma_1[2]^4 - 32\sigma_1[2]\sigma_3[2] + 256\sigma_4[2].$$

Figure II shows part of the real locus of $\mathscr{Z}(F_{u_2})$.

(3) On the other hand, if we set

$$\varphi_1 = \sigma_1[2]^6 - \frac{3}{2}\sigma_1[2]^4\sigma_2[2] - 78\sigma_1[2]^2\sigma_2[2]^2 + \frac{585}{2}\sigma_1[2]^3\sigma_3[2] + 208\sigma_2[2]^3$$
$$-990\sigma_1[2]\sigma_2[2]\sigma_3[2] + 1710\sigma_1[2]^2\sigma_4 + 1350\sigma_3[2]^2 - 2880\sigma_2[2]\sigma_4[2],$$

we can check that $\varphi_1 \in \mathbb{C}[V]^W$ and that:

- $\mathscr{Z}(\varphi_1)$ has exactly 160 singular points, which are all singularities of type D_4.
- $\mathscr{Z}_{\mathrm{sing}}(\varphi_1)$ is a single G_{29}-orbit.

This shows that

$$\mu_{D_4}(12) \geqslant 160, \tag{4.6}$$

as announced in the introduction. This improves considerably known lower bounds (to the best of our knowledge, it was only known that $\mu_{D_4}(12) \geqslant 96$, see [**Esc2**]). Recall also that Miyaoka's bound says that $\mu_{D_4}(12) \leqslant 198$. Figure III shows part of the real locus of $\mathscr{Z}(\varphi_1)$.

(4) Let us keep going on with fundamental invariants of degree 12. Let

$$\varphi_2 = \sigma_3[2]\sigma_1[2]^3 - 4\sigma_1[2]\sigma_2[2]\sigma_3[2] + 4\sigma_1[2]^2\sigma_4[2] + 4\sigma_3[2]^2$$

(up to a scalar). Then $\varphi_2 \in \mathbb{C}[V]^W$ is irreducible over $\mathbb{C}$ (this has been checked with SINGULAR) and computations with MAGMA show that:

- $\mathscr{Z}_{\mathrm{sing}}(\varphi_2)$ has pure dimension 1 and is the union of 30 lines.
- G_{29} acts transitively on these 30 lines.
- The set of points belonging to at least two of these 30 lines has cardinality 60, and splits into two G_{29}-orbits (one of cardinality 40, the other of cardinality 20).

Figure IV shows part of the real locus of $\mathscr{Z}(\varphi_2)$. ■

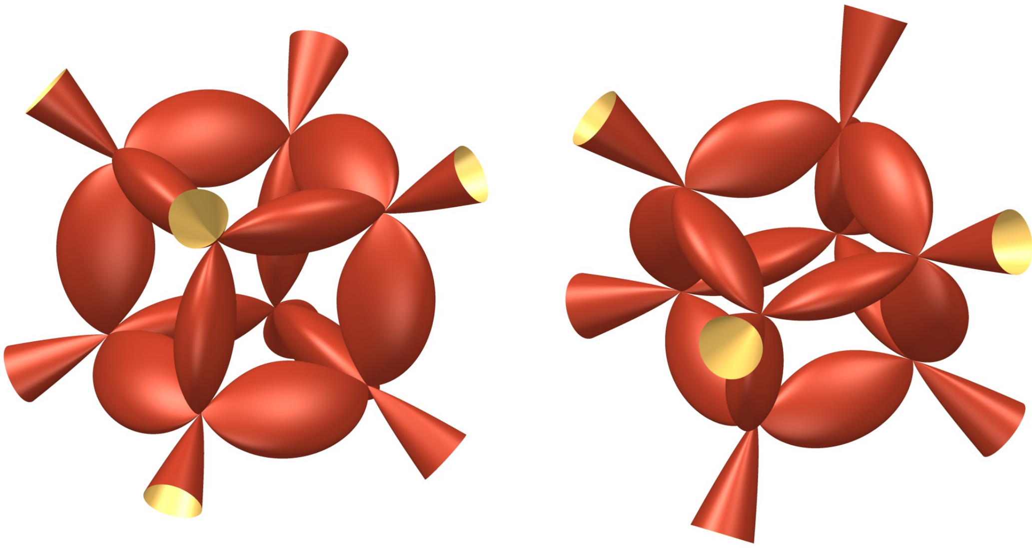

FIGURE II. Part of the real locus of $\mathscr{Z}(F_{u_3})$ for $W = G_{29}$.

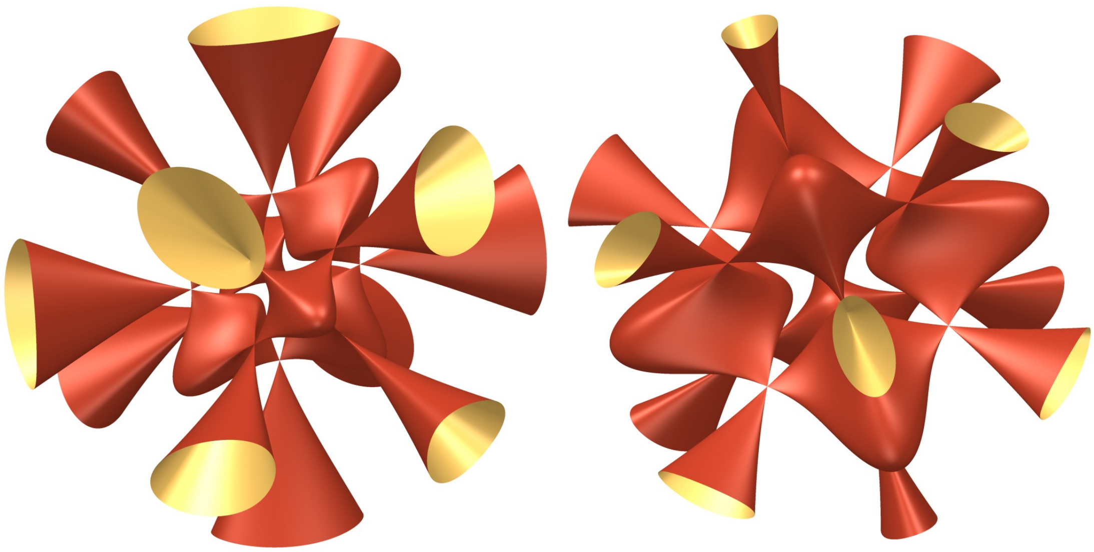

FIGURE III. Part of the real locus of $\mathscr{Z}(\varphi_1)$ for $W = G_{29}$.

***Example 4.7* (The group G_{31}).** — Recall that the *Chmutov surface* [**Chm**] of degree 20 has 2 926 nodes and that an irreducible surface in $\mathbf{P}^3(\mathbb{C})$ of degree 20 cannot have more than 3 208 nodes [**Miy**]. The third surface associated with G_{31} in Table IV has "only" 1 920 nodes and most of them are not real (contrary to the Chmutov surface). However, it has a big group of automorphisms (of order a least 11 520). ■

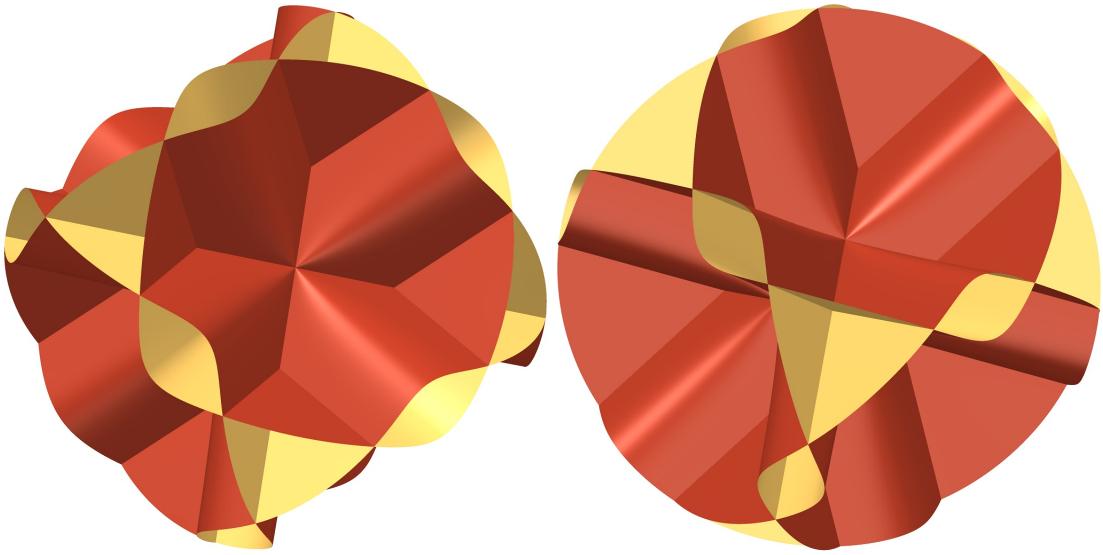

FIGURE IV. Part of the real locus of $\mathscr{Z}(\varphi_2)$ for $W = G_{29}$.

5. Examples in higher dimension

Example 5.1 **(The group G_{33}).** — Computations with MAGMA show that there are no fundamental invariant f_3 of degree 10 of G_{33} such that $\mathscr{Z}(f_3)$ is singular [**Bon2**]. ■

Example 5.2 **(Coxeter group of type E_6).** — Assume in this Example, and only in this Example, that $W = G_{35}$ is a Coxeter group of type E_6. Then $r = 3$ and $(d_1, d_2, d_3) = (2, 5, 6)$, so that any fundamental invariant of degree 6 of W is of the form $F_u = f_3 + u f_1^3$ for some $u \in \mathbb{C}$. Computations with MAGMA show that [**Bon2**]:

(a) $U_{\mathrm{sing}} = U_{\mathrm{sing}}^{\mathrm{irr}}$ has cardinality 8.
(b) For each $u \in U_{\mathrm{sing}}$, $\mathscr{Z}_{\mathrm{sing}}(F_u)$ has dimension 0, W acts transitively on $\mathscr{Z}_{\mathrm{sing}}(F_u)$, and all these singular points are nodes.
(c) The hypersurfaces $\mathscr{Z}(F_u)$, $u \in U_{\mathrm{sing}}^{\mathrm{irr}}$, have respectively 27, 36, 135, 216, 360, 432, 1080 and 1080 singular points. ■

The other exceptional groups have been investigated but the computations are somewhat too long (note that $n \geqslant 5$).

6. The case of G_{32}

Hypothesis. *We assume in this section, and only in this section, that W is the primitive complex reflection group G_{32}.*

In Table IV, it is said that the surface $\mathscr{Z}(F_{u_3})$ attached to G_{32} has 1 440 singularities of type D_4. We give here a detailed account of this example, and show that it also produces surfaces of degree 8 and 16 with many singularities of type D_4. The MAGMA codes are contained in the ARXIV version of this section [**Bon1**].

We need some more notation. If $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ is homogeneous, we denote by $f[k]$ the homogeneous polynomial $f(x_1^k, x_2^k, x_3^k, x_4^k)$. Let W_1 be the subgroup of $\mathbf{GL}_4(\mathbb{C})$ generated by

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad s_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let ζ_3 (resp. ζ_4) be a primitive third (resp. fourth) root of unity. Let W_2 be the subgroup of $\mathbf{GL}_4(\mathbb{C})$ generated by

$$s_1' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta_4 \end{pmatrix}, \quad s_2' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta_4 \end{pmatrix} \quad \text{and} \quad s_3' = \begin{pmatrix} -\zeta_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally, let $W = W_3$ denote the subgroup of $\mathbf{GL}_4(\mathbb{C})$ generated by

$$s_1'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2'' = \begin{pmatrix} \frac{\zeta_3+2}{3} & \frac{\zeta_3-1}{3} & \frac{\zeta_3-1}{3} & 0 \\ \frac{\zeta_3-1}{3} & \frac{\zeta_3+2}{3} & \frac{\zeta_3-1}{3} & 0 \\ \frac{\zeta_3-1}{3} & \frac{\zeta_3-1}{3} & \frac{\zeta_3+2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_3'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_4'' = \begin{pmatrix} \frac{\zeta_3+2}{3} & \frac{1-\zeta_3}{3} & 0 & \frac{1-\zeta_3}{3} \\ \frac{1-\zeta_3}{3} & \frac{\zeta_3+2}{3} & 0 & \frac{\zeta_3-1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1-\zeta_3}{3} & \frac{\zeta_3-1}{3} & 0 & \frac{\zeta_3+2}{3} \end{pmatrix}.$$

Commentaries. The following facts are checked using MAGMA, as explained in [**Bon1**]. Let $\mathrm{Z}(W_i)$ denote the center of W_i. In all cases, it is isomorphic to a group of roots of unity acting by scalar multiplication. Then:

(a) The group W_1 has order 48 and is isomorphic to the non-trivial double cover $\mathbf{GL}_2(\mathbb{F}_3)$ of the symmetric group $\mathfrak{S}_4 \simeq W_1/\mathrm{Z}(W_1)$.

(b) The group W_2 has order 768, contains a normal abelian subgroup H of order 32 and $W_2/H \simeq \mathfrak{S}_4$. The group $W_2/\mathrm{Z}(W_2)$ has order 192, but is not isomorphic to a Coxeter group of type D_4.

(c) The group W_3 is the complex reflection group denoted by G_{32} in the Shephard–Todd classification [**ShTo**] (it has order 155 920). Recall that the group $W_3/\mathrm{Z}(W_3)$ is a simple group of order 25 920 and is isomorphic to the derived subgroup of the Weyl group of type E_6 (i.e. to the derived subgroup of the special orthogonal group $\mathbf{SO}_5(\mathbb{F}_3)$). It contains the group W_1 as a subgroup, as well as a subgroup of diagonal matrices isomorphic to $(\boldsymbol{\mu}_3)^4$, where $\boldsymbol{\mu}_d$ is the group of d-th roots of unity.

Note that we have used the version of G_{32} implemented by Michel in the `Chevie` package of `GAP3` [**Mic**]. ■

If $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4)$ is a partition of 8 of length at most 4, we denote by Ω_λ^- (resp. Ω_λ^+) be the orbit of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} x_4^{\lambda_4}$ under the action of W_1 (resp. the symmetric group $\mathfrak{S}_4$) and we set

$$m_\lambda^\varepsilon = \sum_{m \in \Omega_\lambda^\varepsilon} m$$

for $\varepsilon \in \{+,-\}$. Then m_λ^+ is the symmetric function traditionnally denoted by m_λ. If all the λ_i's are even, then $m_\lambda^- = m_\lambda^+$ but note for instance that

$$\begin{aligned} m_{611}^+ \neq m_{611}^- &= x_1^6 x_2 x_3 + x_1^6 x_2 x_4 - x_1^6 x_3 x_4 + x_1 x_2^6 x_3 - x_1 x_2^6 x_4 + x_2^6 x_3 x_4 \\ &\quad + x_1 x_2 x_3^6 + x_1 x_3^6 x_4 - x_2 x_3^6 x_4 - x_1 x_2 x_4^6 - x_1 x_3 x_4^6 - x_2 x_3 x_4^6. \end{aligned}$$

Now, let

$$\begin{aligned} g &= m_8^- - 6m_{62}^- - 60m_{611}^- + 2\,240m_{521}^- - 14m_{44}^- + 10\,180m_{431}^- + 40\,412m_{422}^- \\ &\quad -23\,440m_{4211}^- + 111\,980m_{332}^- + 154\,704m_{2222}^-. \end{aligned}$$

By construction, m_λ^- is invariant under the action of W_1 and so g is invariant under the action of $W_1 \simeq \tilde{\mathfrak{S}}_4$. One can check with MAGMA the following facts [**Bon1**, Proposition 1]:

Proposition 6.1. — *If $1 \leqslant k \leqslant 3$, then the polynomial $g[k]$ is invariant under the action of W_k.*

One can also check that $g[3]$ is the polynomial denoted by F_{u_3} (suitably normalized) in Table IV (in the G_{32} example).

Theorem 6.2. — *The homogeneous polynomial g satisfies the following statements:*

(a) *$\mathscr{Z}(g)$ is an irreducible surface of degree 8 in $\mathbf{P}^3(\mathbb{C})$ with exactly 44 singular points which are all quotient singularities of type D_4.*

(b) *If $k \geqslant 1$, then $\mathscr{Z}(g[k])$ is an irreducible surface of degree $8k$, whose singular locus has dimension 0 and contains at least $44k^3$ quotient singularities of type D_4.*

(c) *$\mathscr{Z}(g[2])$ is an irreducible surface of degree 16 with exactly 472 singular points: 24 quotient singularities of type A_1, 96 quotient singularities of type A_2 and 352 quotient singularities of type D_4.*

(d) *$\mathscr{Z}(g[3])$ is an irreducible surface of degree 24 in $\mathbf{P}^3(\mathbb{C})$ with exactly 1 440 singular points which are all quotient singularities of type D_4.*

Remark 6.3. — Note that g has coefficients in $\mathbb{Q}$ but the singular points of $\mathscr{Z}(g)$, $\mathscr{Z}(g[2])$ and $\mathscr{Z}(g[3])$ have coordinates in various field extensions of $\mathbb{Q}$, and most of the singular points are not real (at least in this model). ■

We now turn to the study of the singularities of the varieties $\mathscr{Z}(g[i])$ for $i \in \{1,2,3\}$. Note the following fact, checked using MAGMA [**Bon1**, Lemma 3], that will be used further:

Lemma 6.4. — *If $1 \leqslant i < j \leqslant 4$, then the closed subscheme of $\mathbf{P}^3(\mathbb{C})$ defined by the homogeneous ideal $\langle g, \frac{\partial g}{\partial x_i}, \frac{\partial g}{\partial x_j} \rangle$ has dimension 0.*

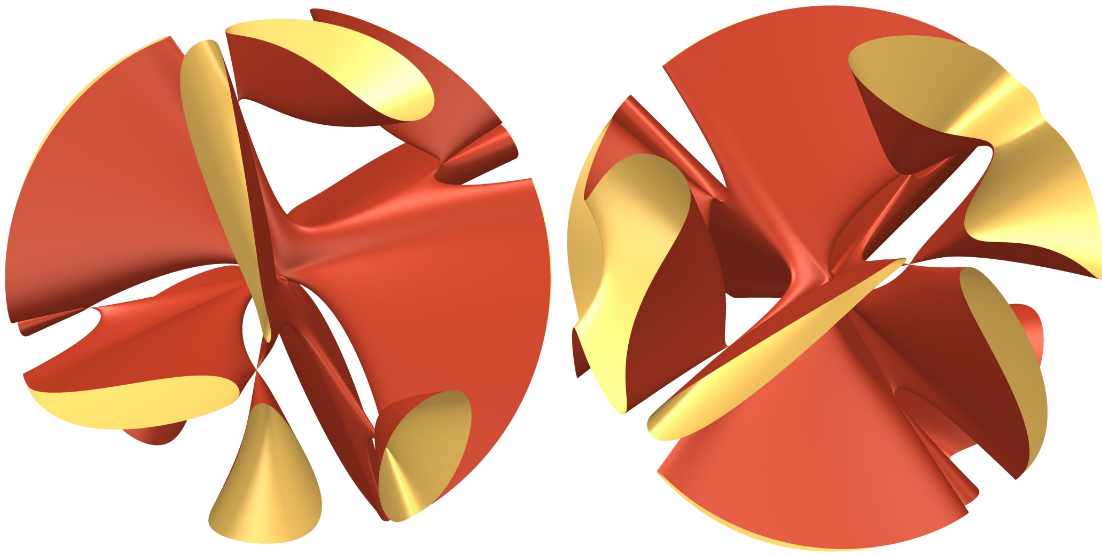

FIGURE V. Part of the real locus of $\mathscr{Z}(g)$

6.A. Degree 8. — The MAGMA computations leading to the proof of the statement (a) of Theorem 6.2 are detailed in [**Bon1**, §1]. Along these computations, the following facts are obtained (here, $\mathscr{U}$ denotes the open subset of $\mathbf{P}^3(\mathbb{C})$ defined by $x_1 x_2 x_3 x_4 \neq 0$):

Proposition 6.5. — *We have:*

(a) $\dim \mathscr{Z}_{\mathrm{sing}}(g) = 0$*, so* $\mathscr{Z}(g)$ *is irreducible.*
(b) $\mathscr{Z}_{\mathrm{sing}}(g)$ *is contained in* $\mathscr{U}$*.*
(c) *The group* W_1 *has* 3 *orbits in* $\mathscr{Z}_{\mathrm{sing}}(g)$*, of respective length* 8, 12 *and* 24.

Note that the points in the W_1-orbit of cardinality 8 are the only real singular points of $\mathscr{Z}(g)$. Figure V shows part of the real locus of $\mathscr{Z}(g)$.

6.B. Degree 8*k*. — Let $\mathscr{U}$ denote the open subset of $\mathbf{P}^3(\mathbb{C})$ defined by $x_1 x_2 x_3 x_4 \neq 0$ and let $\sigma_k : \mathbf{P}^3(\mathbb{C}) \to \mathbf{P}^3(\mathbb{C})$, $[x_1; x_2; x_3; x_4] \mapsto [x_1^k; x_2^k; x_3^k; x_4^k]$. The restriction of σ_k to a morphism $\mathscr{U} \to \mathscr{U}$ is an étale Galois covering, with group $(\boldsymbol{\mu}_k)^4/\Delta\boldsymbol{\mu}_k$ (here, $\Delta : \boldsymbol{\mu}_k \hookrightarrow (\boldsymbol{\mu}_k)^4$ is the diagonal embedding). We have $\mathscr{Z}(g[k]) = \sigma_k^{-1}(\mathscr{Z}(g))$.

Let us first prove that $\mathscr{Z}(g[k])$ is irreducible. We may assume that $k \geqslant 2$, as the result has been proved for $k = 1$ in the previous section. Recall that

$$\frac{\partial g[k]}{\partial x_i} = k x_i^{k-1} \left(\frac{\partial g}{\partial x_i} \circ \sigma_k\right),$$

so the singular locus of $\mathscr{Z}(g[k])$ is contained in

$$\{p_1, p_2, p_3, p_4\} \cup \Bigl(\bigcup_{i \neq j} \sigma_k^{-1}(\mathscr{Z}_{i,j})\Bigr),$$

where $p_i = [\delta_{i1}; \delta_{i2}; \delta_{i3}; \delta_{i4}]$ (and δ_{ij} is the Kronecker symbol) and $\mathscr{Z}_{i,j}$ is the subscheme of $\mathbf{P}^3(\mathbb{C})$ defined by the ideal $\langle g, \frac{\partial g}{\partial x_i}, \frac{\partial g}{\partial x_j} \rangle$ (and which has dimension 0 by Lemma 6.4). Since σ_k is finite, this implies that $\mathscr{Z}_{\mathrm{sing}}(g[k])$ has dimension 0, so $\mathscr{Z}(g[k])$ is irreducible.

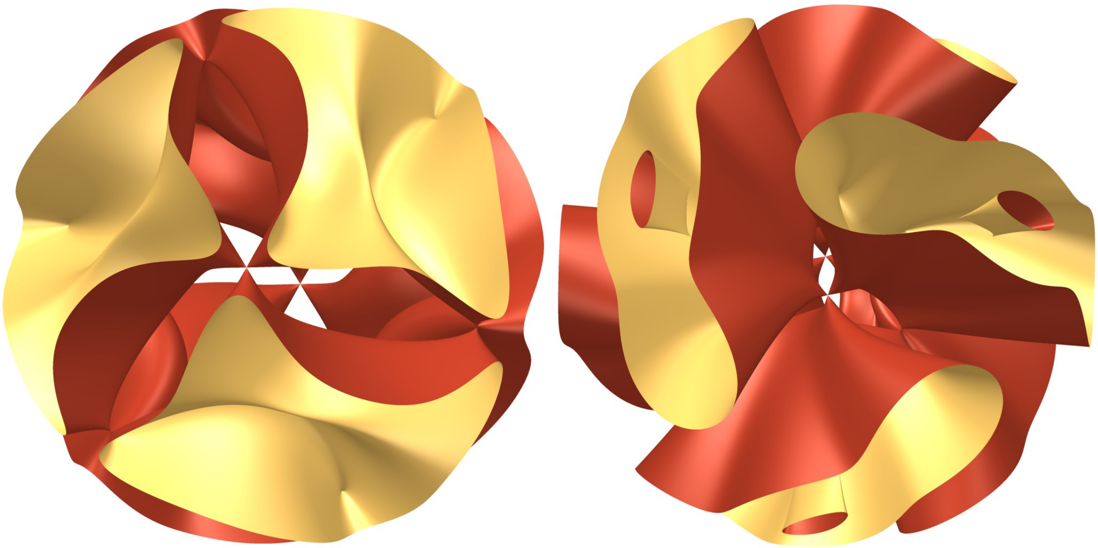

FIGURE VI. Part of the real locus of $\mathscr{Z}(g[2])$

Now, $\sigma_k : \mathscr{U} \to \mathscr{U}$ is étale and the singular locus of $\mathscr{Z}(g)$ is contained in $\mathscr{U}$ (see Proposition 4(b)). Therefore, the 44 singularities of $\mathscr{Z}(g)$ lift to $44k^3$ singularities in $\mathscr{Z}(g[k]) \cap \mathscr{U}$ of the same type, i.e. quotient singularities of type D_4. This proves the statement (b) of Theorem 6.2.

Note that, for $k = 2$, 3 and 4 (and maybe for bigger k) we will prove in the next sections that $\mathscr{Z}(g[k])$ contains singular points outside of $\mathscr{U}$.

6.C. Degree 16. — Using the morphism σ_2 defined in the previous section, we get that $\mathscr{Z}(g[2]) \cap \mathscr{U}$ has exactly 352 singular points, which are all quotient singularities of type D_4. The other singularities are determined thanks to MAGMA computations that are detailed in [**Bon1**, §3], and which confirm the statement (c) of Theorem 6.2. Note that we also need the software SINGULAR [**DGPS**] for computing some Milnor numbers and identifying the singularity A_2. Note also that W_2 acts transitively on the 24 quotient singularities of type A_1 and also acts transitively on the 96 quotient singularities of type A_2. Figure VI shows part of the real locus of $\mathscr{Z}(g)$.

6.D. Degree 24. — Using the morphism σ_3 defined in Section 6.B, we get that $\mathscr{Z}(g[3]) \cap \mathscr{U}$ has exactly $44 \times 3^3 = 1\,188$ singular points, which are all quotient singularities of type D_4. The other singularities are determined thanks to MAGMA computations that are detailed in [**Bon2**] or [**Bon1**, §4], and which confirm the statement (d) of Theorem 6.2. Note also that, in the given model, the surface $\mathscr{Z}(g[3])$ has only 32 real singular points: Figure VII gives partial views of its real locus.

6.E. Complements. — From Section 6.B, we deduce that $\mathscr{Z}_{\mathrm{sing}}(g[4])$ has $2\,816$ quotient singularities of type D_4 lying in the open subset $\mathscr{U}$ and it can be checked that it has 432 other singular points not lying in $\mathscr{U}$, for which we did not determine the type.

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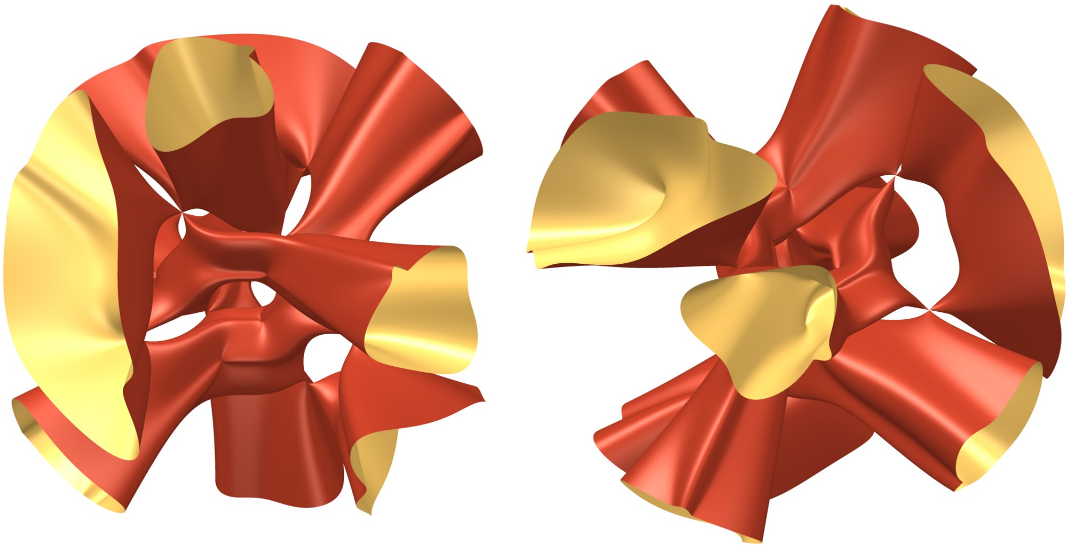

FIGURE VII. Part of the real locus of $\mathscr{X}(g[3])$

Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester. The hidden computations which led to the discovery of the polynomial g were done using the High Performance Computing facilities of the MSRI.

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References

[AGV] V. I. ARNOLD, S. M. GUSEIN-ZADE & A. N. VARCHENKO, *Singularities of differentiable maps*, Vol. I, The classification of critical points, caustics and wave fronts, volume **82** of Monographs in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1985.

[Bar] W. BARTH, *Two projective surfaces with many nodes, admitting the symmetries of the icosahedron*, J. Algebraic Geom. **5** (1996), 173–186.

[Bon1] C. BONNAFÉ, *A surface of degree* 24 *with* 1440 *singularities of type D_4*, preprint (2018), `arXiv:1804.08388`.

[Bon2] C. BONNAFÉ, *Magma codes for "Some singular curves and surfaces arising from invariants of complex reflection groups"*, preprint (2018), available at `hal.archives-ouvertes.fr/hal-01897587`; also available at `http://imag.umontpellier.fr/~bonnafe/`.

[Bro] M. BROUÉ, *Introduction to complex reflection groups and their braid groups*, Lecture Notes in Mathematics **1988**, Springer-Verlag, Berlin, 2010, xii+138 pp.

[Bru] J.W. BRUCE, *An upper bound for the number of singularities on a projective hypersurface*, Bull. Lond. Math. Soc. **13** (1981), 47-51.

[Chm] S. V. CHMUTOV, *Examples of projective surfaces with many singularities*, J. Algebraic Geom. **1** (1992), 191-196.

[C-ALi] J.-I. COGOLLUDO-AGUSTÍN & A. LIBGOBER, *Mordell-Weil groups of elliptic threefolds and the Alexander module of plane curves*, J. Reine Angew. Math. **697** (2014), 15-55.

[DGPS] W. DECKER, G.-M. GREUEL, G. PFISTER & H. SCHÖNEMANN, *Singular 4-1-1 — A computer algebra system for polynomial computations*, `http://www.singular.uni-kl.de` (2018).

[End] S. ENDRAß, *A projective surface of degree eight with 168 nodes*, J. Algebraic Geom. **6** (1997), 325-334.

[EnPeSt] S. ENDRAß, U. PERSSON & J. STEVENS, *Surfaces with triple points*, J. Algebraic Geom. **12** (2003), 367-404.

[Esc1] J. G. ESCUDERO, *Hypersurfaces with many A_j singularities: explicit constructions*, J. Comput. Appl. Math. **259** (2014), 87-94.

[Esc2] J. G. ESCUDERO *Arrangements of real lines and surfaces with A and D singularities*, Experimental Mathematics **23** (2014), 482-491.

[Iv] K. IVINSKIS, *Normale Flächen und die Miyaoka-Kobayashi Ungleichung*, Diplomarbeit (Bonn), 1985.

[Lab] O. LABS, *A septic with 99 real nodes*, Rend. Sem. Mat. Univ. Pad. **116** (2006), 299-313.

[Magma] W. BOSMA, J. CANNON & C. PLAYOUST, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235-265.

[MaMi] I. MARIN & J. MICHEL, *Automorphisms of complex reflection groups*, Represent. Theory **14** (2010), 747-788.

[Mic] J. MICHEL, *The development version of the CHEVIE package of GAP3*, J. of Algebra **435** (2015), 308–336.

[Miy] Y. MIYAOKA, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159-171.

[Sak] F. SAKAI, *Singularities of plane curves*, Geometry of complex projective varieties (Cetraro, 1990), 257-273, Sem. Conf., **9**, Mediterranean, Rende, 1993.

[Sar1] A. SARTI, *Pencils of Symmetric Surfaces in* $\mathbf{P}^3$, J. Algebra **246** (2001), 429-452.

[Sar2] A. SARTI, *Symmetric surfaces with many singularities*, Comm. Algebra **32** (2004), 3745-3770.

[Sar3] A. SARTI, *Symmetrische Flächen mit gewöhnlichen Doppelpunkten*, Math. Semesterber. **55** (2008), 1-5.

[ShTo] G. C. SHEPHARD & J. A. TODD, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274-304.

[Sta] E. STAGNARO, *A degree nine surface with 39 triple points*, Ann. Univ. Ferrara Sez. VII **50** (2004), 111-121.

[Sur] `www.imaginary.org/program/surfer`.

[Var] A.N. VARCHENKO, *On the semicontinuity of the spectrum and an upper bound for the number of singular points of a projective hypersurface*, J. Soviet Math. **270** (1983), 735-739.

[Wah] J. WAHL, *Miyaoka-Yau inequality for normal surfaces and local analogues*, in Classification of algebraic varieties (L'Aquila, 1992), 381-402, Contemp. Math. **162**, Amer. Math. Soc., Providence, RI, 1994.

[Thi] U. THIEL, *CHAMP: A Cherednik Algebra Magma Package*, LMS J. Comput. Math. **18** (2015), 266-307.

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