DEFORMATION SPACES OF DISCRETE GROUPS OF SU(2,1) IN QUATERNIONIC HYPERBOLIC PLANE: A CASE STUDY

ANTONIN GUILLOUX AND INKANG KIM

ABSTRACT. In this note, we study deformations of discrete and Zariski dense subgroups of SU(2,1) in the isometry group Sp(2,1) of quaternionic hyperbolic space. Specifically we consider two examples coming from representations of 3-manifold groups (the figure eight knot and Whitehead links complement) and show opposite behaviors: one is not deformable outside U(2,1), while the other has a big space of deformations in Sp(2,1).

1. Introduction

In 1960's, A. Weil [24] proved a local rigidity of a uniform lattice $\Gamma \subset G$ inside G: he showed that $H^1(\Gamma, \mathfrak{g}) = 0$ for any semisimple Lie group G not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. This result implies that the canonical inclusion map $i:\Gamma \hookrightarrow G$ is locally rigid up to conjugacy. In other words, for any local deformation $\rho_t:\Gamma \to G$ such that $\rho_0=i$, there exists a continuous family $g_t \in G$ such that $\rho_t=g_t\rho_0g_t^{-1}$. Weil's idea is further explored by many others but notably by Raghunathan [21] and Matsushima-Murakami [19]. Much later Goldman and Millson [10] considered the embedding of a uniform lattice Γ of $\mathrm{SU}(n,1)$

$$\Gamma \hookrightarrow \mathrm{SU}(n,1) \hookrightarrow \mathrm{SU}(n+1,1)$$

and proved that there is still a local rigidity inside SU(n+1,1) if one ignores a deformation coming from the center. More recently further examples [16, 17, 15, 18] of local rigidity of a complex hyperbolic lattice in quaternionic Kähler manifolds are described in the following situations:

$$\Gamma \hookrightarrow \mathrm{SU}(n,1) \subset \mathrm{Sp}(n,1) \subset \mathrm{SU}(2n,2) \subset \mathrm{SO}(4n,4),$$

$$\Gamma \hookrightarrow \mathrm{SU}(n,1) \subset \mathrm{SU}(p,q),$$

$$\Gamma \hookrightarrow \mathrm{SU}(n,1) \subset \mathrm{Sp}(n+1,\mathbb{R}),$$

$$\Gamma \hookrightarrow \mathrm{SU}(n,1) \subset \mathrm{SO}(2n,2).$$

¹2000 Mathematics Subject Classification. 51M10, 57S25.

²Key words and phrases. Quaternionic hyperbolic space, complex hyperbolic space, local rigidity, representation variety.

³The second author gratefully acknowledges the partial support of the grant (NRF-2017R1A2A2A05001002) and a warm support of IHES during his stay.

But all these examples deal with the standard inclusion map $\Gamma \hookrightarrow G'$ to use the Weil's original idea about L^2 -group cohomology. We look in this paper at the more general setting of a representation $\rho: \Gamma \to G \subset G'$. We focus our attention to the case where the representation is discrete and has Zariski-dense image in G. We seek the possibility of deforming ρ in G' without being conjugate to a representation landing in G.

In general, very little is known on this general problem. We study here deformations of two representations of non-uniform lattices of $SL(2,\mathbb{C})$ inside Sp(2,1). Indeed, let M_8 be the figure eight knot complement and denote by Γ_8 its fundamental group, and let M_W be the Whitehead link complement and Γ_W its fundamental group.

The character variety $\chi(\Gamma_8, SU(2, 1))$ is fully understood [7] (see section 3 for the definition of character variety), and it contains 2 (up to some equivalences) boundary unipotent irreducible representations ρ_0 and ρ_1 which are already obtained in [9], see also [8]. We will be mainly interested in the representation ρ_0 whose image is generated by the following matrices in SU(2,1):

$$\begin{bmatrix} 1 & 1 & \frac{-1 - i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{-1 - i\sqrt{3}}{2} & -1 & 1 \end{bmatrix}$$

In particular, we see that the image of ρ_0 is included in the Eisenstein-Picard arithmetic lattice of SU(2,1). It turns out that it is a *thin* subgroup, as it is Zariski-dense. We will show that ρ_0 , as its surrounding lattice, is not deformable outside U(2,1). Recently some thin subgroups of finite index in Γ_8 were constructed inside lattices in $SL(4,\mathbb{R})$, that are indeed deformable (inside $SL(4,\mathbb{R})$) [2].

Our knowledge of the character variety $\chi(\Gamma_W, \mathrm{SU}(2,1))$ is far less thorough. Boundary unipotent representations are described in [8], whereas a component of this character variety has been described in [13]. We will consider a representation ρ_W inside this component. Note that the image of ρ_W is a free product of two copies of $\mathbb{Z}/3\mathbb{Z}$ and is not contained in an arithmetic lattice. We will prove that ρ_W has a big space of deformations in $\mathrm{Sp}(2,1)$ and is therefore deformable outside $\mathrm{U}(2,1)$.

We will first describe what is known about ρ_0 and ρ_W , exhibiting structural differences. We then prove that the first one is rigid whereas the second one is deformable. It would be very interesting to understand which properties of these representations lead to the rigidity or deformability.

2. Two opposite behaviors

2.1. Rigidity of ρ_0 . The fundamental group Γ_8 has a presentation [7]:

$$\Gamma_8 = \langle a, b | b^{-1} a b a^{-1} b a b^{-1} a^{-1} b a^{-1} \rangle.$$

We consider the representation ρ_0 defined by the images of the generators:

$$\rho_0(a) = \begin{bmatrix} 1 & 1 & \frac{-1 - i\sqrt{3}}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_0(b) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{-1 - i\sqrt{3}}{2} & -1 & 1 \end{bmatrix}$$

We prove in this paper that the representation ρ_0 cannot be deformed locally outside U(2, 1). The proof is fairly straightforward, though involved computations are tedious. Here are the steps:

- (1) As we will see in section 3.2, at $[\rho_0]$, the character variety $\chi(\Gamma_8, U(2, 1))$ is 3-dimensional.
- (2) We are able to compute the tangent space to $\chi(\Gamma_8, \operatorname{Sp}(2,1))$ at $[\rho_0]$: it amounts to compute $H^1(\Gamma_8, \mathfrak{sp}(2,1)_{\operatorname{ad}(\rho_0)})$. This homological computation will be explained in section 4. The computed dimension is 3.
- (3) As we will recall in section 3.1, the natural map $\chi(\Gamma_8, U(2,1)) \rightarrow \chi(\Gamma_8, \operatorname{Sp}(2,1))$ is a local diffeomorphism onto its image.

Knowing these three facts, we see:

Proposition 2.1. Every small deformation of $\rho_0 : \Gamma_8 \to \operatorname{Sp}(2,1)$ results in a representation conjugate to a representation $\Gamma_8 \to \operatorname{U}(2,1)$.

2.2. **Deformability of** ρ_W . Following [13], the fundamental group Γ_W has a presentation:

$$\Gamma_W = \langle a, b | aba^{-3}b^2a^{-1}b^{-1}a^3b^{-2} \rangle.$$

We consider the representation ρ_W defined by the images of the generators:

$$\rho_W(a) = \begin{bmatrix} 1 & \frac{\sqrt{3} - i\sqrt{5}}{2} & -1 \\ \frac{-\sqrt{3} - i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ and } \rho_W(b) = \begin{bmatrix} 1 & -\frac{\sqrt{3} + i\sqrt{5}}{2} & -1 \\ \frac{\sqrt{3} - i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Unlike the previous example, we prove in this paper that the representation ρ_W can be deformed locally outside U(2,1). The proof is once again fairly straightforward. Here are the steps:

- (1) ρ_W factors through a quotient $\mathbb{Z}_3 * \mathbb{Z}_3$, and the whole component of the SU(2, 1)-character variety of Γ_W does (see section 3.3).
- (2) The U(2,1)-character variety has dimension 6 at ρ_W (see section 3.3).
- (3) The Sp(2, 1)-character of $\mathbb{Z}_3 * \mathbb{Z}_3$ at ρ_W has dimension at least 7 (see section 5).

We hence see that the $\mathrm{Sp}(2,1)$ -character variety of Γ_W has dimension, at ρ_W , at least 1 more than the dimension of the U(2,1)-character variety. It yields:

Proposition 2.2. There are small deformations of $\rho_W : \Gamma_W \to \operatorname{Sp}(2,1)$ which are not conjugate to any representation $\Gamma_W \to \operatorname{U}(2,1)$.

3. Character varieties

The G-character variety of $\pi_1(M)$, denoted $\chi(\pi_1(M), G)$, is the geometric invariant theory quotient of $\operatorname{Hom}(\pi_1(M), G)$ by inner automorphisms of G. Often, some components of the character variety are realized as the space of (G, X)-structures on a given manifold M. Thurston studied the Dehn surgery space of a hyperbolic knot complement in the early 70s using the idea of gluing tetrahedra in hyperbolic 3-space. In his case, the variety appears as defined by his gluing equations [23]. Thurston's approach is generalized to several different directions corresponding to different geometric structures such as spherical CR structure and real projective structure associated with Lie groups $\operatorname{SU}(2,1)$ and $\operatorname{SL}(3,\mathbb{R})$ respectively. The latter one is known as a Hitchin component consisting of convex real projective structures on a closed surface [14].

3.1. General facts and definitions. For a given reductive algebraic group $G \subset \operatorname{GL}(m,k)$ defined over k, and a finitely generated group Γ with n-generators, the **representation variety** is $R(\Gamma,G) = \operatorname{Hom}(\Gamma,G) \subset G^n$, defined by the zero set of polynomials in $k[x_1,\cdots,x_{nm^2}]$. In this paper, $k = \mathbb{R}$ or \mathbb{C} . A representation $\rho: \Gamma \to G$ is **Zariski dense** if the Zariski closure of the image is G. The group G acts on $R(\Gamma,G)$ by conjugation, and it is well-known that the orbit of ρ under conjugation is closed if ρ is Zariski dense.

Since the orbit under the conjugation is not closed in general, the quotient space of $R(\Gamma, G)$ under conjugation is not in general a Hausdorff space. To avoid this phenomenon, one takes the GIT quotient $\chi(\Gamma, G) = R(\Gamma, G)//G$ to get again an algebraic set, called the **character variety**.

In this paper, all the representations we are considering are not contained in $P \times Z(G)$ where P is a parabolic subgroup and Z(G) is the center of G. In this case, the quotient of $R(\Gamma, G)$ by the conjugation action of G is nice around ρ [11, Section 1.3], and we can assure that the Zariski tangent space of the character variety at $[\rho]$ can be computed by the first group cohomology of Γ with coefficient in $\mathfrak{g}_{Ad\rho}$.

We will need the following later.

Lemma 3.1. Let $\nu_1: \Gamma \to U(2,1)$ be a Zariski dense representation which is conjugate to $\nu_2: \Gamma \to U(2,1)$ in $\mathrm{Sp}(2,1)$. Then ν_1 is conjugate in $\mathrm{SU}(2,1)$ to either ν_2 or $\overline{\nu_2}$.

Proof: Suppose $Q\nu_1Q^{-1} = \nu_2$ for $Q \in \operatorname{Sp}(2,1)$. Since ν_1 is Zariski dense, Q stabilizes $H^2_{\mathbb{C}}$ inside $H^2_{\mathbb{H}}$. If it is holomorphic, $Q \in \operatorname{SU}(2,1)$. Suppose it is anti-holomorphic. Any anti-holomorphic element in $H^2_{\mathbb{C}}$ can be written as ι followed by an element in U(2,1) where ι is a reflection along $H^2_{\mathbb{R}}$. By absorving the element in U(2,1) we may assume that Q restricted to $H^2_{\mathbb{C}}$ is ι . Now, ι can be realized as a complex conjugate $(z,w) \to (\bar{z},\bar{w})$ in unit ball model.

Then Q is realized by a diagonal matrix with entries (j, j, j). Hence we get $Q\nu_1Q^{-1}=\overline{\nu_1}$ and $\nu_1=\overline{\nu_2}$.

3.2. Description of the U(2,1)-character variety for Γ_8 . Let Γ_8 denote the fundamental group of the figure eight knot complement in \mathbb{S}^3 . Falbel-Guilloux-Koseleff-Rouillier-Thistlethwaite [7] studied the character variety of Γ_8 in PGL(3, \mathbb{C}) and PU(2,1). They describe a Zariski open set, through a variant of the character variety: the deformation variety. They show that there exist three irreducible components of the deformation variety. Each one of these components is smooth of complex dimension two and contains a real-dimension 2 subvariety of representations landing in PU(2,1) [7, Section 5.3].

Proposition 3.2. The component of the U(2,1)-character variety $\chi(\Gamma_8, U(2,1))$ through ρ_0 has dimension 3.

Proof. First of all, ρ_0 belongs to one of the components described in [7]: it corresponds to the point $(u,v)=(-\sqrt{3}i,2)$ from [7, Section 5.3]. Hence, we know that the component of the $\mathrm{SU}(2,1)$ -character variety $\chi(\Gamma_8,\mathrm{SU}(2,1))$ through ρ_0 has real dimension 2.

Then Γ_8 is the fundamental group of a knot complement, so its abelianization is \mathbb{Z} . Hence the character variety from Γ_8 to the center U(1) of U(2,1) is of real dimension 1.

Now any representation $\Gamma_8 \to U(2,1)$ can be locally decomposed as product of a representation in its center and a representation in SU(2,1). We get that the component of the U(2,1)-character variety $\chi(\Gamma_8, U(2,1))$ through ρ_0 has dimension 3.

3.3. A known component of the U(2,1)-character variety for Γ_W . Guilloux-Will studied the character variety $\chi(\Gamma_W, \operatorname{SL}(3,\mathbb{C}))$. They showed that the representations studied by Schwartz, Deraux, Falbel, Acosta, Parker, Will [22, 5, 6, 1, 20] all belong to a common algebraic component X_0 consisting of representations that factor through the group $\pi' = \mathbb{Z}_3 * \mathbb{Z}_3$. Here X_0 is the character variety of π' consisting of representations whose images are generated by two regular order 3 elements in $\operatorname{SL}(3,\mathbb{C})$. X_0 is of complex dimension 4, and the subset of representations in $\operatorname{SU}(2,1)$ is of real dimension 4

Moreover, the representation ρ_W belongs to this component X_0 [13, Section 3.4]. Using that the abelianization of Γ_W is \mathbb{Z}^2 , we get as before:

Proposition 3.3. The component of the U(2,1)-character variety $\chi(\Gamma_W, U(2,1))$ through ρ_0 has dimension 6.

4. Fox calculus and homological computations

4.1. **General presentation.** In this section, we briefly introduce a Fox calculus which is necessary for the calculation of the first group cohomology

and the Zariski tangent space of $\operatorname{Hom}(\pi, G)$. For a detailed exposition, refer to [11] Section 3. Such computations have already been used, e.g. in [3]. Let F_n be a free group on n-generators x_1, \dots, x_n and $\mathbb{Z}F_n$ the integral group ring. The augmentation homomorphism is a ring homomorphism

$$\epsilon: \mathbb{Z}F_n \to \mathbb{Z}$$

which maps an element $\sum_{\sigma \in F_n} m_{\sigma} \sigma$ to the coefficient sum $\sum_{\sigma \in F_n} m_{\sigma}$. A derivation is a \mathbb{Z} -linear map $D : \mathbb{Z}F_n \to \mathbb{Z}F_n$ satisfying

$$D(m_1 m_2) = D(m_1)\epsilon(m_2) + m_1 D(m_2).$$

Then the set of derivations $Der(F_n)$ is freely generated as a right $\mathbb{Z}F_n$ -module by n elements $\partial_i = \frac{\partial}{\partial x_i}$ which satisfy $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$. This derivation satisfies a useful rule of differential calculus, a mean value theorem,

$$u - \epsilon(u) = \sum (\partial_i u)(x_i - 1)$$

for any $u \in \mathbb{Z}F_n$.

Let $\phi: F_n \to \operatorname{GL}(V)$ be a linear representation, which extends to a ring homomorphism $\mathbb{Z}F_n \to \operatorname{End}(V)$. Then a cocyle $u: \mathbb{Z}F_n \to V$ which satisfies the cocycle identity $u(ab) = u(a)\epsilon(b) + \phi(a)u(b)$, can be written using the mean value theorem as

$$u(w) = \sum_{i=1}^{n} \phi(\partial_i w) u(x_i).$$

Using this Fox calculus, we can describe the Zariski tangent space to $\operatorname{Hom}(\pi,G)\subset G^n$ for a group $\pi=F_n/\mathbb R$ where $\mathbb R$ is a normal subgroup of F_n consisting of relations and G is a Lie group whose Lie algebra is denoted by $\mathfrak g$. Since an element in $\operatorname{Hom}(\pi,G)$ corresponds to an element $\phi\in\operatorname{Hom}(F_n,G)$ satisfying $\phi(R)=1$ for all $R\in\mathbb R$, the Zariski tangent space to $\operatorname{Hom}(\pi,G)$ at $\phi\in\operatorname{Hom}(\pi,G)$ is the space of cocycles

$$Z^{1}(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) = \{(u_{1}, \cdots, u_{n}) \in \mathfrak{g}^{n} | \sum_{i=1}^{n} \mathrm{Ad}\,\phi(\partial_{i}R)u_{i} = 0, \text{ for all } R \in \mathcal{R}\}$$

by associating $(\mu(x_1), \dots, \mu(x_n))$ to each 1-cocycle μ .

Moreover, in order to have the Zariski tangent space to the character variety, you have to mod out by the coboundaries $B^1(\pi, \mathfrak{g}_{Ad\phi})$. In this setting, a coboundary is an element $(u_1, \ldots, u_n) \in \mathfrak{g}^n$ such that there exist some $u \in \mathfrak{g}$ with:

$$\forall 1 \leq i \leq n, \quad u_i = \operatorname{Ad} \phi(x_i)u - u.$$

4.2. Effective computations for Γ_8 . The material presented above can be tackled in a very concrete and effective manner. Let us describe the involved computations for the representation $\rho_0: \Gamma_8 \to \operatorname{Sp}(2,1)$. The actual computations are basic linear algebra, but with matrices a bit too big to be fully displayed here. A Sage Notebook [12] is available showing the computations done by a computer algebra system.

First of all, we use Fox calculus on our presentation of Γ_8 :

$$\Gamma_8 = \langle a, b | b^{-1} a b a^{-1} b a b^{-1} a^{-1} b a^{-1} \rangle.$$

Let us denote by R the relation $b^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1}$. A straightforward computation gives:

- $\begin{array}{l} \bullet \ \, \partial_a R = b^{-1} b^{-1} aba^{-1} + b^{-1} aba^{-1} b b^{-1} aba^{-1} bab^{-1} a^{-1} b^{-1} aba^{-1} bab^{-1} a^{-1} ba^{-1}. \\ \bullet \ \, \partial_b R = -b^{-1} + b^{-1} a + b^{-1} aba^{-1} b^{-1} aba^{-1} bab^{-1} + b^{-1} aba^{-1} bab^{-1} a^{-1}. \end{array}$

Let us note that in Sagemath, the Fox calculus is implemented and the result of this computation is given by the so-called Alexander matrix.

From this, the whole computation of the Zariski tangent space follows. This computation can be seen in the notebook, and the steps are:

- Choose a basis for $\mathfrak{sp}(2,1)$: its cardinality is 21. Pairs of vectors in this basis give a basis of the cochains C^1 : as presented above, a cochain is seen as an element of $\mathfrak{sp}(2,1)^2$.
- Compute both 21×21 matrices representing in this basis the adjoint action Ad(x) and Ad(y) of the generators, $x = \rho_0(a), y = \rho_0(b)$.
- Compute B^1 as the image of the 42×21 matrix $\begin{pmatrix} \operatorname{Ad}(x) \operatorname{id} \\ \operatorname{Ad}(y) \operatorname{id} \end{pmatrix}$ in the chosen basis.
- Using the different terms $\partial_a R$, $\partial_b R$ appearing in the definition of Z^1 as in the previous lemma, applying $Ad(\rho_0)$ to these expressions, we get the 21×42 matrix whose kernel is Z^1 :

$$\left(\operatorname{Ad}\rho_0(\partial_a R), \operatorname{Ad}\rho_0(\partial_b R)\right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

• Compute the dimension of Z^1/B^1 . Note that ρ_0 has entries in a number field: the computation can be done exactly and the computed dimension has a true meaning.

As a result of this computation, we get:

Proposition 4.1. The component of the character variety $\chi(\Gamma_8, \operatorname{Sp}(2,1))$ through ρ_0 has dimension 3.

Proof of Proposition 2.1: The Lie algebra $\mathfrak{sp}(2,1)$ decomposes as $\mathfrak{u}(2,1) \oplus$ $S^2\mathbb{C}^3$ under ρ_0 as a real representation, see [16]. Hence

$$H^1(\Gamma_8, \mathfrak{sp}(2,1)) = H^1(\Gamma_8, \mathfrak{u}(2,1)) + H^1(\Gamma_8, S^2\mathbb{C}^3).$$

By above Propositions 3.2 and 4.1, $H^1(\Gamma_8, S^2\mathbb{C}^3) = 0$, which implies that all the small deformations of ρ_0 in Sp(2, 1) are conjugate to the ones in U(2, 1).

5. Order 3 elements and the deformation of ρ_W

We compute in this section a lower bound on the dimension of a component of the character variety $\chi(\Gamma_W, \operatorname{Sp}(2,1))$:

Proposition 5.1. The dimension around $[\rho_W]$ of the character variety $\chi(\Gamma_W, \operatorname{Sp}(2,1))$ is at least 7.

Proof. As we saw in Section 2.2, the image of ρ_W is isomorphic to $\mathbb{Z}_3 \star \mathbb{Z}_3$, with $\rho_W(a)$ and $\rho_W(b)$ being two order 3 generators.

Moreover, as recalled in section 3.3, the whole component of the SU(2, 1)-character variety containing $[\rho_W]$ is made from representations $[\rho]$ with $\rho(a)$ and $\rho(b)$ being two order 3 elements of SU(2, 1).

Let $\mathcal{E} = \{(A, B) \in \operatorname{Sp}(2, 1)^2 \mid A^3 = B^3 = 1\}$. Then we have an inclusion $\mathcal{E}/\operatorname{Sp}(2, 1) \to \chi(\Gamma_W, \operatorname{Sp}(2, 1))$.

As a matrix of SU(2,1), the eigenvalues of $\rho_W(\alpha)$ are 1, ω , ω^2 , where $\omega^3 = 1$ in \mathbb{C} . So, inside Sp(2,1), $\rho_W(\alpha)$ is conjugate [4] to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

By deforming the pair $(\rho_W(\alpha), \rho_W(\beta))$ to a pair of order 3 matrices in $\operatorname{Sp}(2,1)$ and up to conjugation, we may assume that the first one always equals A. Its centralizer [4, Section 5.1] in $\operatorname{Sp}(2,1)$ is the subgroup of block-diagonal matrices:

$$Z = \left\{ \begin{pmatrix} x & \\ & X \end{pmatrix} \in \operatorname{Sp}(2,1) & \text{where } x \in \mathbb{H}, X \in \operatorname{U}(2) \right\}.$$

Note that the dimension of Z is 7.

The second matrix B of the pair is another order 3 matrix, conjugate to A. So we are indeed looking at the set of pairs (A, gAg^{-1}) up to conjugation. In other terms, let

$$\mathcal{E}' = \{ (A, gAg^{-1}) \text{ for } g \in \text{Sp}(2, 1) \}.$$

Then locally around $[\rho_W]$ we have $\mathcal{E}/\operatorname{Sp}(2,1) = \mathcal{E}'/Z$.

Eventually, we see that for any $g \in \text{Sp}(2,1)$ and $h \in \text{Sp}(2,1)$, the two pairs (A, gAg^{-1}) and (A, hAh^{-1}) are conjugate if and only if there exist z_1 and z_2 in Z such that $h = z_1 g z_2$.

Hence the dimension of \mathcal{E}'/Z is at least $\dim(\operatorname{Sp}(2,1)) - 2\dim(Z) = 7$. This implies that the dimension around $[\rho_W]$ of $\chi(\Gamma_W, \operatorname{Sp}(2,1))$ is at least 7. \square

Indeed, the dimension of $\chi(\Gamma_W, \operatorname{Sp}(2,1))$ around any point in the component \mathcal{C} containing $[\rho_W]$ of $\chi(\Gamma_W, \operatorname{U}(2,1))$ is at least 7 as we can see as follows. Note that any point in \mathcal{C} can be written as a pair $(\alpha C, \beta B)$ with $\alpha, \beta \in U(1)$ and $C^3 = B^3 = I$ in $\operatorname{SU}(2,1)$. This point is conjugate to $(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}h_0Ah_0^{-1})$ for some $g_0, h_0 \in \operatorname{Sp}(2,1)$. Now by varing $h \in \operatorname{Sp}(2,1)$, as in the proof above, there are at least 7-dimensional space of $\{(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}hAh^{-1})|h \in \operatorname{Sp}(2,1)\}$ near $(g_0\alpha g_0^{-1}A, g_0\beta g_0^{-1}h_0Ah_0^{-1})$ in $\chi(\Gamma_W, \operatorname{Sp}(2,1))$.

Note that the proposition 2.2 is now proven: the space of deformations of ρ_W in Sp(2,1) has bigger dimension than the space of deformations in U(2,1) showing that some deformations are not conjugate to U(2,1).

REFERENCES

[1] M. Acosta. Spherical CR Dehn Surgery. ArXiv e-prints, September 2015.

- [2] Samuel Ballas and Darren D. Long. Constructing thin subgroups commensurable with the figure-eight knot group. *Algebr. Geom. Topol.*, 15(5):3011–3024, 2015.
- [3] Leila Ben Abdelghani and Michael Heusener. Irreducible representations of knot groups into SL(n, C). Publ. Mat., 61(2):363–394, 2017.
- [4] Wensheng Cao and Krishnendu Gongopadhyay. Algebraic characterization of isometries of the complex and the quaternionic hyperbolic planes. *Geom. Dedicata*, 157:23–39, 2012.
- [5] Martin Deraux. On spherical CR uniformization of 3-manifolds. Exp. Math., 24(3):355–370, 2015.
- [6] Martin Deraux. A 1-parameter family of spherical CR uniformizations of the figure eight knot complement. Geom. Topol., 20(6):3571–3621, 2016.
- [7] E. Falbel, A. Guilloux, P.-V. Koseleff, F. Rouillier, and M. Thistlethwaite. Character varieties for SL(3, C): the figure eight knot. *Exp. Math.*, 25(2):219–235, 2016.
- [8] E. Falbel, P.-V. Koseleff, and F. Rouillier. Representations of fundamental groups of 3-manifolds into PGL(3, ℂ): exact computations in low complexity. *Geom. Dedicata*, 177:229–255, 2015.
- [9] Elisha Falbel. A spherical CR structure on the complement of the figure eight knot with discrete holonomy. J. Differential Geom., 79(1):69–110, 2008.
- [10] W. M. Goldman and J. J. Millson. Local rigidity of discrete groups acting on complex hyperbolic space. *Invent. Math.*, 88(3):495–520, 1987.
- [11] William M. Goldman. The symplectic nature of fundamental groups of surfaces. Adv. in Math., 54(2):200–225, 1984.
- [12] A. Guilloux and I. Kim. Companion sage notebook to this article. Link to the Sage Notebook, 2017.
- [13] Antonin Guilloux and Pierre Will. On SL(3,C)-representations of the Whitehead link group. preprint, 2016.
- [14] N. J. Hitchin. Lie groups and Teichmüller space. Topology, 31(3):449–473, 1992.
- [15] I. Kim and G. Zhang. Local rigidity of complex hyperbolic lattices in semisimple Lie groups. ArXiv e-prints, August 2015. To appear in Math. Proc. Cambridge Philosophical Society.
- [16] Inkang Kim, Bruno Klingler, and Pierre Pansu. Local quaternionic rigidity for complex hyperbolic lattices. J. Inst. Math. Jussieu, 11(1):133–159, 2012.
- [17] Inkang Kim and Pierre Pansu. Local rigidity in quaternionic hyperbolic space. J. Eur. Math. Soc. (JEMS), 11(6):1141–1164, 2009.
- [18] B. Klingler. Local rigidity for complex hyperbolic lattices and Hodge theory. *Invent. Math.*, 184(3):455–498, 2011.
- [19] Yozô Matsushima and Shingo Murakami. On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds. *Ann. of Math.* (2), 78:365–416, 1963.
- [20] John R. Parker and Pierre Will. A complex hyperbolic Riley slice. Geom. Topol., 21(6):3391–3451, 2017.
- [21] M. S. Raghunathan. On the first cohomology of discrete subgroups of semisimple Lie groups. Amer. J. Math., 87:103–139, 1965.
- [22] Richard Evan Schwartz. Spherical CR geometry and Dehn surgery, volume 165 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2007.
- [23] W. Thurston. The geometry and topology of 3-manifolds. http://library.msri.org/books/gt3m/, 1983.
- [24] André Weil. On discrete subgroups of Lie groups. Ann. of Math. (2), 72:369–384, 1960.

Sorbonne Université, CNRS, IMJ-PRG and INRIA, OURAGAN, 4, Place Jussieu, 75005 Paris, France

 $E\text{-}mail\ address:\ \mathtt{antonin.guilloux@imj-prg.fr}$

School of Mathematics, KIAS, Hoegiro 85, Dongdaemun-gu, Seoul, 130-722, Korea

 $E\text{-}mail\ address{:}\ \mathtt{inkang@kias.re.kr}$