REFINED GOLDBACH CONJECTURES WITH PRIMES IN PROGRESSIONS

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ABSTRACT. We formulate some refinements of Goldbach's conjectures based on heuristic arguments and numerical data. For instance, any even number greater than 4 is conjectured to be a sum of two primes with one prime being 3 mod 4. In general, for fixed m and a, b coprime to m, any positive even $n \equiv a + b \bmod m$ outside of a finite exceptional set is expected to be a sum of two primes p and q with $p \equiv a \bmod m$, $q \equiv b \bmod m$. We make conjectures about the growth of these exceptional sets.

1. INTRODUCTION

Let p and q denote prime numbers. The binary (or strong) Goldbach conjecture asserts that any even $n > 2$ is of the form $p + q$. There is both strong heuristic evidence that this is true for sufficiently large n and enormous numerical evidence that this is true for $n > 2$. The same heuristic evidence, together with equidistibution of primes congruence classes mod m, suggests the following:

Conjecture 1.1. *Fix $a, b, m \in \mathbb{Z}$ with $\gcd(a, m) = \gcd(b, m) = 1$. For sufficiently large even $n \equiv a+b \bmod m$, we can write $n = p+q$ for some primes $p \equiv a \bmod m$, $q \equiv b \bmod m$.*

While we do not know an explicit statement of this conjecture in the literature, we do not claim any originality in its formulation.

Denote by $E_{a,b,m}$ the set of positive even $n \equiv a + b \bmod m$ which are not of the form asserted in the conjecture. This is called the exceptional set for (a, b, m), and the conjecture asserts $E_{a,b,m}$ is finite. Note for $a = b = 1$ and $m = 2$, this is a (still unknown) weak form of the binary Goldbach conjecture. Specifically, the binary Goldbach conjecture is equivalent to the statement that $E_{1,1,2} = \{2, 4\}$.

In this note, we present some heuristic and numerical investigations on the behavior of these exceptional sets. This leads to both explicit forms of Goldbach's conjecture with primes in arithmetic progressions and conjectures about the growth of $E_{a,b,m}$.

First we state a few explicit Goldbach-type conjectures:

Conjecture 1.2. *Any positive even n as below is of the form $p+q$ with p and q satisfying the following congruence conditions:*

(i) any even $n > 4$, where $p \equiv 3 \bmod 4$;
(ii) any $n \equiv 0 \bmod 4$ except $n = 4$, where $p \equiv 1 \bmod 4$, $q \equiv 3 \bmod 4$;
(iii) any $n \equiv 2 \bmod 4$ except $n = 2$, where $p \equiv q \equiv 3 \bmod 4$;
(iv) any $n \equiv 2 \bmod 4$ except $n = 2, 6, 14, 18, 62$, where $p \equiv q \equiv 1 \bmod 4$.

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We note that (ii) is already implied by Goldbach's conjecture, whereas (iii) and (iv) are not. Then (ii) and (iii) imply (i), which may be viewed as a refinement of the binary Goldbach conjecture. The notion that the exceptional set should be smaller in case (iii) rather than (iv) makes sense in light of prime number races, specifically that there are more small primes which are 3 mod 4 than 1 mod 4.

For moduli $m \neq 4$, we just list some sample conjectures about when every (or almost every) multiple of m is of the form $p+q$ with p and q each coming from single progressions mod m:

Conjecture 1.3. *Any positive even n as below is of the form $p+q$ with p and q satisfying the following congruence conditions:*

(i) *any even $n \equiv 0 \bmod 3$ except $n = 6$, where $p \equiv -q \equiv 1 \bmod 3$;*
(ii) *any even $n \equiv 0 \bmod 5$ (resp. except n=10, 20), where $p \equiv -q \equiv 2 \bmod 5$ (resp. $p \equiv -q \equiv 1 \bmod 5$);*
(iii) *any even $n \equiv 0 \bmod 7$, where $p \equiv -q \equiv 3 \bmod 7$;*
(iv) *any even $n \equiv 0 \bmod 11$, where $p \equiv -q \equiv 3 \bmod 11$;*
(v) *any $n \equiv 0 \bmod 8$, where $p \equiv -q \equiv 3 \bmod 8$;*
(vi) *any $n \equiv 0 \bmod 16$, where $p \equiv -q \equiv 3 \bmod 16$;*
(vii) *any $n \equiv 0 \bmod 60$, where $p \equiv -q \equiv a \bmod 60$ and a is any fixed integer coprime to 60 which is not $\pm 1, \pm 11 \bmod 60$.*

Of these, only the first is a direct consequence of the binary Goldbach conjecture, which we list simply for means of comparison. The others come from calculations that we describe in Section 3.

Namely, we compute the following. First, we may as well assume m is even. Then we call (a, b) or (a, b, m) admissible if $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$. We compute the exceptions n in $E_{a,b,m}$ for all admissible (a, b, m) with $m \leq 200$ up to at least $n = 10^7$. This appears sufficiently large in our cases of consideration to believe that we find the full exceptional sets for such (a, b, m).

Likely of more interest than numerous explicit such conjectures is a general understanding of the behaviour of the exceptional sets. The first question to ask is:

Question 1.4. *How fast can $E_{a,b,m}$ grow?*

There are a few ways to interpret this: we can look at the growth of the size of $|E_{a,b,m}|$ or the growth of the sizes of the individual exceptions $n \in E_{a,b,m}$, and either of these can be interpreted in an average sense or in the sense of looking for an absolute or asymptotic bound. Let $E_{\max}(m)$ be the maximum of the exceptions in $E_{a,b,m}$ (ranging over $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$) and let $L_{\text{avg}}(m)$ be the average length (size) of the exceptional sets $E_{a,b,m}$ for a fixed m. Then the heuristics we discuss in Section 2 suggest the following:

Conjecture 1.5. *As $m \to \infty$, we have*

(i) $E_{\max}(m) := \max\{n \in E_{a,b,m} : n \in \mathbb{Z},\ a, b \in (\mathbb{Z}/m\mathbb{Z})^\times\} = O(m^2(\log m)^2)$; *and*
(ii) $L_{\text{avg}}(m) := \frac{1}{\phi(m)^2} \sum_{a,b \in (\mathbb{Z}/m\mathbb{Z})^\times} |E_{a,b,m}| = O(m^\varepsilon)$ *for any $\varepsilon > 0$.*

Admittedly our heuristics are rather simplistic (they are too simplistic to suggest precise asymptotics), but since the numerical data we present in Section 3 is in strong agreement with these heuristics, it seems reasonable to believe the above conjecture. In addition, our numerical data suggests that $E_{\max}(m)$ grows roughly like a quadratic function of

m, which suggests that the growth bound in (i) is not too much of an overestimate (see Fig. 2).

Our data also suggests that as m grows, while the proportion of admissible (a, b) with $E_{a,b,m} = \emptyset$ may decrease on average, there are still many (a, b) with no exceptional sets (see Table 2), leading to:

Conjecture 1.6. *There infinitely many tuples* (a, b, m) *such that* $E_{a,b,m} = \emptyset$.

We are less confident in this conjecture as precise heuristics about this seem a bit more delicate (we do not attempt them here), but it at least seems plausible in connection with Conjecture 1.5. Namely, for fixed m the expected length of an exceptional set may be something roughly logarithmic in m, but we have $\phi(m)^2$ such exceptional sets, so there will be a good chance some of them are empty as long as the variance of $|E_{a,b,m}|$ is not too small.

We note that a number of authors (e.g., [Lav61], [BW13], [Bau17]) have studied the behaviour of $E_{a,b,m}$ analytically. However, present analytic methods seem to still be far from showing finiteness of $E_{a,b,m}$, let alone attacking the finer questions we explore here.

Lastly, we remark that one can similarly look at versions of the ternary Goldbach conjecture with primes in progressions. However, an answer to the binary case will also give results about the ternary case, in the same way that the strong Goldbach conjecture implies the weak Goldbach conjecture (the latter of which is now a theorem [Hel13]). We simply state one ternary analogue of Conjecture 1.2(i):

Conjecture 1.7. *Any odd integer* $n > 5$ *is of the form* $n = p + q + r$ *for primes* p, q, r *with* $p \equiv q \equiv 2 \bmod 3$.

To see this, our calculations suggest $E_{5,5,6} = \{4\}$ (see Section 3.2), which would imply $E_{2,2,3} = \emptyset$, so in the conjecture we can take $p+q$ to be an arbitrary even number which is $1 \bmod 3$, and then $r \in \{3, 5, 7\}$. Unlike the usual ternary Goldbach conjecture, this does not seem to be known at present. See [LP10], [Sha14], [She16] for results in this direction.

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2. Heuristics

Let $n > 2$ be even, and $g_2(n)$ be the number of ways to write $n = p + q$ for primes p and q. In 1922, Hardy and Littlewood conjectured

$$g_2(n) \sim \mathfrak{S}(n) \frac{n}{(\log n)^2},$$

where $\mathfrak{S}(n)$ is the singular series, which is 0 for odd n and on average it is is 2 for even n. For a refinement, see [Gra07]. For our heuristics, we will naively approximate $g_2(n)$ by the conjectural average $\frac{2n}{(\log n)^2}$. This simplification is justified in our heuristic upper bounds as $\mathfrak{S}(n) > 1$ for even n.

Fix an even modulus m. Let r be the integer part of expected number of admissible $(a, b) \in ((\mathbb{Z}/m\mathbb{Z})^\times)^2$ such that $a+b \equiv n$ (averaged over even $0 \le n < m$), i.e., $r = [\frac{2\phi(m)^2}{m}]$. We want to estimate the probability that $n \in E_{a,b,m}$ for some admissible pair $(a, b) \in ((\mathbb{Z}/m\mathbb{Z})^\times)^2$. We will use the following simplistic but reasonable model: we think of ordered pairs of primes (p,q) solving $p+q=n$ as a collection of $g_2(n)$ independent random events, with the reduction mod m of (p,q) landing in any of the r admissible classes $(a+m\mathbb{Z}, b+m\mathbb{Z})$ with equal probability. (Obviously (p,q) and (q,p) are not independent, but this is not so important for our heuristics.) Immediately this suggests Conjecture 1.1, but we want to speculate more precisely on the growth rate of the exceptional sets $E_{a,b,m}$ as $m \to \infty$.

We recall the coupon collector problem. Say we have r initially empty boxes, and at each time $t \in \mathbb{N}$, a coupon is placed in one box at chosen random. Assume at each stage, each box is selected with equal probability $\frac{1}{r}$. Let $W = W_r$ be the random variable representing the waiting time until all boxes have at least 1 coupon.

Let X_s denote a geometric random variable such that $P(X_s = k) = (1-s)^{k-1}s$ is the probability of initial success after exactly k trials, where each trial has independent probability of success s. The problem is to determine the expected value $E[W]$. It is easy to see that $W = X_{r/r} + X_{(r-1)/r} + \cdots X_{1/r}$ (where the X_s's are independent), and thus $E[W] = rH_r$, where $H_r = \sum_{j=1}^r \frac{1}{j}$ is the r-th harmonic number.

Thus, in our model, the probability that $n \in E_{a,b,m}$ for some (a,b) is simply $P(W_r > g_2(n))$. One has

$$P(W > k) = 1 - \sum_{j=0}^{k} \frac{r!}{r^j} \left\{ {j-1 \atop r-1} \right\} = 1 - \frac{r!}{r^k} \left\{ {k \atop r} \right\} = \sum_{j=0}^{r-1} (-1)^{r-j+1} \binom{r}{j} \left(\frac{j}{r}\right)^k,$$

where $\left\{ {k \atop r} \right\}$ denotes the Stirling number of the second kind, i.e., the number of ways to partition a set of size k into r nonempty subsets. Now we can bound each term on the right by $\binom{r}{[r/2]}(1-1/r)^k$, which will be less than $\frac{\varepsilon}{k^2r^4}$ for r large if

$$-k\log(1-\frac{1}{r}) = k(\frac{1}{r} - \frac{1}{r^2} + \cdots) \gtrsim r\log 2 + 2\log k + 4\log r - \log\varepsilon \gtrsim \log\binom{r}{[r/2]} - \log\frac{\varepsilon}{k^2r^4}.$$

We can make this asymptotic inequality hold by taking $k = Cr^2$, where C depends on ε, and then

$$P(W_r > k) < \frac{\varepsilon}{k^2r^3}.$$

Consider the set Σ_C of $\{(r,k) : r = [\frac{2\phi(m)^2}{m}],\ k = [\frac{2n}{(\log n)^2}] > Cr^2\}$. Note for r and k of this form, the condition $k > Cr^2$ is satisfied when $\frac{n}{(\log n)^2} > 2Cm^2$, and thus when $n > cm^2(\log m)^2$ for a suitable constant c. Thus, for suitably large c, we have a heuristic upper bound on the probability that some $E_{a,b,m}$ contains an element $n > cm^2(\log m)^2$ of

$$\sum_{(r,k)\in\Sigma_C} rP(W_r > k) \leq \sum_{(r,k)\in\Sigma_C} \frac{\varepsilon}{k^2r^2} < c_0\varepsilon,$$

for a uniform constant c_0. The factor of r on the left inside the sum comes from accounting for each of the (on average) r classes of pairs (a,b) given m, n.

This suggests the following bound on the growth of exceptional sets stated in Conjecture 1.5(i):

$$E_{\max}(m) = O(m^2(\log m)^2) \quad \text{as } m \to \infty. \tag{2.1}$$

Now let us consider the lengths $|E_{a,b,m}|$ of the exceptional sets. For fixed m, we will model $|E_{a,b,m}|$ as a random variable $L(m)$. For a fixed $n \equiv a+b \bmod m$, the probability (using the model described above) that $n \in E_{a,b,m}$ is simply $(1-\frac{1}{r})^{g_2(n)}$. Hence the expected size of an exceptional set is

$$E[L(m)] = \sum_{n \equiv a+b \bmod m} \alpha^{g_2(n)}, \quad \alpha = 1 - \frac{1}{r}.$$

We approximate this with the sum (over all integers $n \geq 2$):

$$E[L(m)] \approx \frac{1}{m} \sum_{n=2}^{\infty} \alpha^{\frac{2n}{(\log n)^2}}.$$

Then for $0 < \delta < 1$, we have

$$mE[L(m)] \ll \int_0^\infty \alpha^{2x^\delta}\, dx = \frac{1}{2\delta |\ln \alpha|^{1/\delta}} \int_0^\infty u^{1/\delta - 1} e^{-u}\, du = \frac{1}{2\delta |\ln \alpha|^{1/\delta}} \Gamma(\frac{1}{\delta}).$$

Note $\frac{d}{dr}|\ln \alpha| = \frac{d}{dr}(\ln(r-1) - \ln r) = \frac{1}{r^2 - r}$, so as $r \to \infty$, we have $\frac{1}{|\ln \alpha|} \sim r$. This gives

$$E[L(m)] \ll \frac{r^{1/\delta}}{2m} = O(m^\varepsilon), \quad \varepsilon = \frac{1}{\delta} - 1, \tag{2.2}$$

as stated in Conjecture 1.5(ii).

3. Numerics

Now we present numerical data on the exceptional sets $E_{a,b,m}$ for (even) $m \leq 200$.

3.1. **The method and computational issues.** Our approach, similar to many numerical verifications of Goldbach's conjecture, was roughly as follows. To find $\{n \in E_{a,b,m} : n \leq N\}$, we start with two sets of primes $P = \{p \equiv a \bmod m : p \leq M\}$ and $Q = \{q \equiv b \bmod m : q \leq N\}$ and determine which $n \leq N$ are not of the form $p + q$ for $p \in P$, $q \in Q$, with M on the order of 10^4 or 10^5 depending on m. Then any potential exceptions below M are guaranteed to actually lie in $E_{a,b,m}$, and any larger potential exceptions we checked individually by testing primality (deterministically) of $n - p$ for various p.

For each even $m \leq 200$, and $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$, we checked up to at least $N = 10^7$ using Sage. (We checked Conjecture 1.2 up to $N = 10^8$.) We note that the binary Goldbach conjecture has been numerically verified for a much, much larger range (up to $4 \cdot 10^{18}$ in [OeSHP14]). While one could certainly extend our calculations for larger N (and m) with more efficient implementation and computing resources, our goal here is not to push the limits of calculation, but rather to generate a reasonable amount of data to help formulate and support our conjectures.

That said, there are a couple of obstacles to do a similar amount of verification for various $E_{a,b,m}$. First of all, we want to test many triples (a, b, m) which increases the amount of computation involved. Second, and much more significant, when we look for representations $n = p + q$ with p and q in arithmetic progressions, the minimum value of p, say, for which such a representation is possible seems to increase much faster than without placing congruence conditions on p and q. In other words, to rule out almost all potential exceptions in the first stage of our algorithm above, for the same N we need to take M larger and larger with m. For instance, when $m = 2$ (the usual Goldbach conjecture) one can always take $M < 10^4$ (i.e., the least prime in a Goldbach partition) to rule out all exceptions for $N \leq 4 \cdot 10^{18}$ (see [OeSHP14]), however this is not a sufficiently large value of M for many of our calculations. Already when $N = 10^6$ and $M = 10^4$, there are 41 non-exceptions $< 10^6$ that we cannot rule out when $m = 50$, 24981 when $m = 100$, and 1148651 when $m = 148$.

3.2. **Data and observations.** For simplicity of exposition, we define our notation under the hypothesis: *there are no exceptions $n > 10^7$ for $m \leq 200$.* This is believable as the largest exception we find is approximately 10^5, and for a given m (with a, b varying) the gaps between one exception and the next largest exception appear to grow at most quadratically in the number of total exceptions.

Fix m. Let $E_{\max} = E_{\max}(m)$ and $L_{\text{avg}} = L_{\text{avg}}(m)$ be as in the introduction. Let $L_{\min} = L_{\min}(m)$ (resp. $L_{\max} = L_{\max}(m)$) be the minimum (resp. maximum) of the lengths $|E_{a,b,m}|$ over $a, b \in (\mathbb{Z}/m\mathbb{Z})^\times$. Let e_m (resp. $\tilde{e}_m$) denote the number of exceptions without (resp. with) multiplicity, i.e., the size of the set (resp. multiset) $\bigcup_{(a,b)} E_{a,b,m}$, where a, b run over $(\mathbb{Z}/m\mathbb{Z})^\times$.

We also consider the above quantities with the additional restriction that $b \equiv -a \bmod m$ so as to treat the special case where n is a multiple of m. In this situation, we denote the analogous quantities with a superscript 0, e.g., $E^0_{\max}$ is the maximal $n \in E_{a,b,m}$ such that n is a multiple of m.

We list the first few explicit calculations of exceptional sets (under our hypothesis):

- $E_{1,1,2} = \{2, 4\}$, which is equivalent to the binary Goldbach conjecture
- $E_{1,1,4} = \{2, 6, 14, 38, 62\}$, $E_{1,3,4} = \{4\}$, and $E_{3,3,4} = \{2\}$
- $E_{1,1,6} = \{2, 8\}$, $E_{1,5,6} = \{6\}$, and $E_{5,5,6} = \{4\}$
- $E_{1,1,8} = \{2, 10, 18, 26, 42, 50, 66, 74, 98, 122, 218, 242, 362, 458\}$, $E_{1,3,8} = \{4, 12, 68, 188\}$, $E_{1,5,8} = \{6, 14, 38, 62\}$, $E_{1,7,8} = \{8, 16, 32, 56\}$, $E_{3,3,8} = E_{3,5,8} = \emptyset$, $E_{3,7,8} = E_{5,5,8} = \{2\}$, $E_{5,7,8} = 4$ and $E_{7,7,8} = \{6, 22, 166\}$
- $E_{1,1,10} = \{2, 12, 32, 152\}$, $E_{1,3,10} = E_{7,7,10} = \{4\}$, $E_{1,7,10} = \{8\}$, $E_{1,9,10} = \{10, 20\}$, $E_{3,3,10} = E_{3,7,10} = \emptyset$, $E_{3,9,10} = \{2, 12\}$, $E_{7,9,10} = \{6, 16\}$, and $E_{9,9,10} = \{8, 18, 28, 68\}$

We summarize the data from our calculations in Table 1. Note that $E_{\max}$ and $E^0_{\max}$, as well as the total number of exceptions, tend to be relatively larger when m is a power of 2 or twice a prime. In these cases $\phi(m)$ is relatively large, i.e., we have relatively more admissible pairs (a, b) to consider, so it makes sense that we pick up more exceptions. We illustrate this coincidence in the fluctuations of $E_{\max}(m)$ and $\phi(m)$ in Fig. 1.

While our numerics are somewhat limited, they suggest that the growth of $E_{\max}(m)$ appears to be strictly slower than that of $m^2(\log m)^2$ as stated in Conjecture 1.5, and the true growth rate appears to be closer to $O(m^2)$ or $O(m^2 \log m)$—see Fig. 2 for an overlay of the graphs of $E_{\max}(m)$ (black dots, with the scale on the left) and $\phi(m)$ (gray x's, with the scale on the right).

Finally, we observe it often happens that (under our hypothesis), for fixed m, at least one admissible pair (a, b) has no exceptions, i.e., every $n \equiv a + b \bmod m$ is of the form $p + q$ with $p \equiv a \bmod m$ and $q \equiv b \bmod m$. Both the number of such pairs (a, b) and the fraction of such pairs out of the total number $\phi(m)^2$ of admissible pairs are tabulated in Table 2. It appears that $z_m = \#\{(a, b) \in ((\mathbb{Z}/m\mathbb{Z})^\times)^2 : E_{a,b,m} = \emptyset\}$ tends to grow at least on average in several of the columns in Table 2 (e.g., when m is a multiple of 5 or 6). This suggests that z_m is unbounded, and in particular suggests Conjecture 1.6.

m	$L^0_{\min}$	L^0_{avg}	$L^0_{\max}$	$E^0_{\max}$	e^0_m	$\tilde{e}^0_m$	$L_{\min}$	L_{avg}	$L_{\max}$	$E_{\max}$	e_m	$\tilde{e}_m$
2	2	2.0	2	4	2	2	2	2.0	2	4	2	2
4	1	1.0	1	4	1	2	1	2.0	5	62	6	8
6	1	1.0	1	6	1	2	1	1.25	2	8	4	5
8	0	2.0	4	56	4	8	0	2.875	14	458	28	46
10	0	1.0	2	20	2	4	0	1.563	4	152	13	25
12	0	0.5	1	12	1	2	0	1.0	4	62	9	16
14	0	1.333	3	98	3	8	0	2.056	7	512	32	74
16	0	2.5	4	368	6	20	0	3.469	17	1298	94	222
18	0	0.333	1	18	1	2	0	0.861	2	52	13	31
20	0	1.5	3	200	4	12	0	1.828	8	542	46	117
22	0	2.2	4	418	7	22	0	2.86	9	1568	102	286
24	0	1.0	4	192	4	8	0	1.344	10	458	39	86
26	1	3.0	7	754	11	36	0	3.313	13	4688	146	477
28	1	2.833	7	616	11	34	0	3.063	15	1598	145	441
30	0	0.25	1	30	1	2	0	0.719	3	152	17	46
32	0	3.875	8	1184	15	62	0	4.969	26	5014	316	1272
34	1	3.5	7	1088	14	56	0	4.355	17	5228	289	1115
36	0	0.833	2	216	4	10	0	1.319	5	478	69	190
38	2	3.889	6	1558	16	70	0	4.864	21	5032	373	1576
40	1	2.375	6	920	9	38	0	2.973	19	2282	225	761
42	0	0.333	1	42	1	4	0	0.896	5	512	39	129
44	2	4.0	6	3344	18	80	0	4.42	24	6106	415	1768
46	2	4.273	8	1564	20	94	1	5.285	20	8104	541	2558
48	0	1.0	3	288	5	16	0	1.629	12	1298	132	417
50	0	2.3	4	550	8	46	0	2.893	10	3182	273	1157
52	2	4.333	8	3380	19	104	0	5.035	23	8546	580	2900
54	0	0.889	2	216	4	16	0	1.503	6	1096	130	487
56	1	3.583	6	2072	12	86	0	4.224	24	8318	491	2433
58	3	6.071	13	3422	29	170	0	6.342	31	10366	870	4972
60	0	0.375	2	180	2	6	0	0.855	5	542	66	219
62	3	6.267	10	4712	28	188	0	6.714	28	11416	975	6043
64	2	6.438	12	4736	32	206	0	7.262	35	16126	1173	7436
66	0	0.9	2	198	3	18	0	1.417	6	1568	141	567
68	3	6.0	10	5848	26	192	0	6.387	25	13718	1044	6540
70	0	1.833	3	1540	9	44	0	2.092	9	5002	280	1205
72	0	1.333	4	864	6	32	0	1.922	13	2834	268	1107
74	2	6.611	11	6068	31	238	0	7.355	28	16046	1370	9532
76	3	6.333	11	5624	30	228	0	6.679	29	23426	1250	8656
78	0	1.0	3	624	5	24	0	1.51	7	4688	205	870
80	1	3.813	11	3680	21	122	0	3.982	19	6200	729	4078
82	3	7.2	14	8528	39	288	0	7.764	27	25616	1651	12423
84	0	0.833	4	1008	5	20	0	1.368	11	1598	202	788
86	3	7.714	15	12556	41	324	0	7.942	34	26782	1805	14009
88	2	5.8	13	5720	29	232	0	6.382	32	19274	1378	10211
90	0	0.333	2	180	2	8	0	1.017	5	976	148	586

m	$L^0_{\min}$	L^0_{avg}	$L^0_{\max}$	$E^0_{\max}$	e^0_m	$\tilde{e}^0_m$	$L_{\min}$	L_{avg}	$L_{\max}$	$E_{\max}$	e_m	$\tilde{e}_m$
92	2	7.0	13	7636	36	308	0	7.73	35	21538	1891	14966
94	3	8.0	16	12032	39	368	0	8.43	38	30916	2098	17838
96	0	1.375	3	864	6	44	0	2.194	17	5014	437	2247
98	2	5.286	10	4214	25	222	0	5.503	28	17014	1389	9708
100	0	3.8	6	4300	19	152	0	4.366	21	12134	1043	6986
102	0	1.25	2	918	8	40	0	1.848	11	5228	383	1892
104	3	6.708	13	9256	36	322	0	7.236	36	22886	1990	16671
106	4	8.577	16	11978	48	446	0	9.009	42	39842	2661	24359
108	0	2.0	4	1944	13	72	0	2.279	13	7142	548	2954
110	1	2.85	8	3410	15	114	0	3.331	14	13316	822	5329
112	3	5.667	12	6272	29	272	0	6.11	34	23038	1750	14077
114	0	1.278	3	684	6	46	0	2.064	12	5032	482	2675
116	4	8.714	15	13688	44	488	1	8.961	42	36326	2892	28102
118	4	8.793	17	17228	50	510	0	9.446	45	53614	3144	31776
120	0	0.625	3	360	3	20	0	1.3	12	2282	297	1331
122	4	9.4	18	13298	48	564	0	9.601	43	39818	3377	34563
124	4	8.667	15	12896	49	520	0	9.114	40	42778	3265	32811
126	0	1.056	4	1008	7	38	0	1.535	11	7598	389	1989
128	5	9.25	15	13568	47	592	0	10.406	43	48346	3933	42622
130	1	3.458	8	5590	18	166	0	3.955	19	19406	1242	9112
132	0	1.2	4	1584	9	48	0	1.915	14	6106	540	3064
134	2	9.455	16	14204	52	624	0	10.166	48	52894	3992	44281
136	3	8.219	17	16048	47	526	0	8.813	52	42734	3413	36100
138	0	1.682	3	1656	8	74	0	2.332	11	8104	715	4515
140	0	2.792	6	5180	17	134	0	3.18	17	14198	1077	7327
142	5	9.657	17	23146	55	676	0	10.306	45	54526	4328	50498
144	0	2.167	5	2592	15	104	0	2.771	18	19858	959	6384
146	5	10.222	22	15476	54	736	0	10.663	54	54928	4684	55278
148	2	9.528	17	21608	55	686	0	10.128	43	70222	4374	52501
150	0	0.85	2	1800	7	34	0	1.295	6	3182	387	2072
152	4	8.972	18	17176	54	646	0	9.674	49	62554	4331	50152
154	2	4.6	9	10472	25	276	0	4.954	22	29476	1983	17836
156	0	1.625	4	4212	9	78	0	2.204	16	8546	778	5078
158	6	10.718	21	21488	66	836	1	11.011	47	51248	5245	66991
160	1	5.5	15	13760	32	352	0	5.83	30	31942	2479	23879
162	0	2.148	7	1944	12	116	0	2.826	15	9908	1157	8241
164	5	10.6	20	19352	65	848	0	10.768	52	67546	5438	68912
166	6	10.268	19	22576	64	842	1	11.067	58	65776	5817	74414
168	0	1.583	8	2352	11	76	0	1.849	15	8318	675	4260
170	0	4.313	10	10880	27	276	0	4.765	31	42862	2162	19517
172	4	11.024	18	59168	65	926	0	11.098	47	104494	5771	78306
174	0	2.286	5	7482	16	128	0	2.768	15	10366	1166	8681
176	3	8.5	16	18832	53	680	0	8.926	52	62002	4683	57129
178	5	10.955	18	38092	71	964	1	11.528	50	88622	6344	89275
180	0	0.875	3	1080	5	42	0	1.481	9	7562	560	3412

m	$L^0_{\min}$	L^0_{avg}	$L^0_{\max}$	$E^0_{\max}$	e^0_m	$\tilde{e}^0_m$	$L_{\min}$	L_{avg}	$L_{\max}$	$E_{\max}$	e_m	$\tilde{e}_m$
182	2	5.139	14	14378	33	370	0	5.818	30	33536	2932	30161
184	5	10.023	18	41768	65	882	1	10.665	57	88106	6112	82588
186	0	1.967	5	2232	11	118	0	2.869	18	11784	1285	10329
188	5	10.826	17	34028	70	996	0	11.423	55	82594	6778	96686
190	2	4.639	9	15200	27	334	0	5.052	27	34652	2669	26189
192	0	2.469	4	7296	15	158	0	3.167	20	16126	1552	12972
194	6	11.417	24	33368	75	1096	1	12.054	56	96728	7361	111093
196	2	7.119	15	20776	41	598	0	7.783	39	56738	4514	54914
198	0	1.533	4	4356	9	92	0	2.373	16	13436	1152	8542
200	2	5.6	11	16400	33	448	0	6.228	32	46922	3621	39862

TABLE 1. Data on exceptional sets mod m for $p \equiv a \bmod m$, $q \equiv b \bmod m$

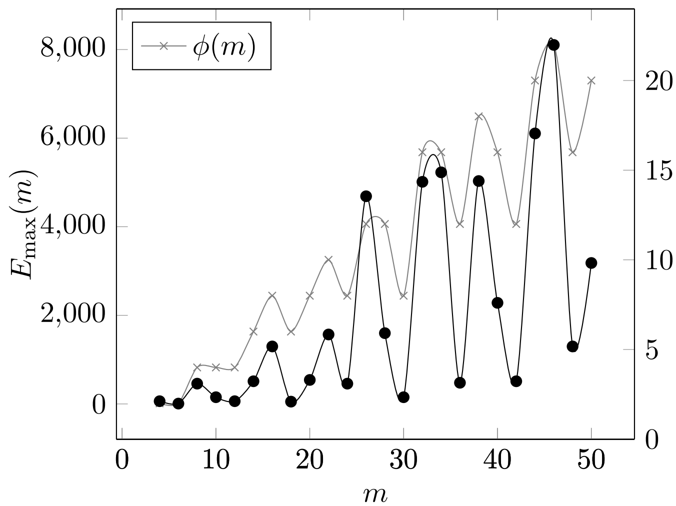

FIGURE 1. Comparing $E_{\max}(m)$ with $\phi(m)$

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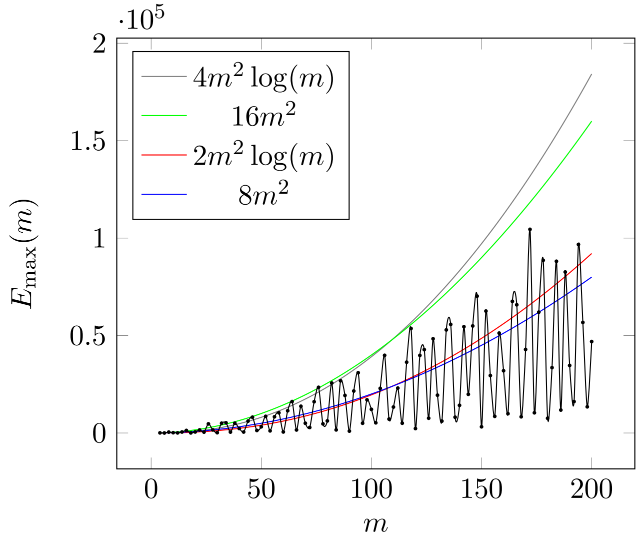

FIGURE 2. Comparing $E_{\max}(m)$ with quadratic and quadratic times logarithmic growth

m	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
#	0	0	0	3	3	3	6	6	9	11	2	20	10	16	21
%	0.0	0.0	0.0	18.8	18.8	18.8	16.7	9.4	25.0	17.2	2.0	31.3	6.9	11.1	32.8
m	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
#	12	12	33	11	26	44	8	0	68	32	10	75	18	4	97
%	4.7	4.7	22.9	3.4	10.2	30.6	2.0	0.0	26.6	8.0	1.7	23.1	3.1	0.5	37.9
m	62	64	66	68	70	72	74	76	78	80	82	84	86	88	90
#	8	4	108	24	60	105	4	14	118	42	6	160	14	2	195
%	0.9	0.4	27.0	2.3	10.4	18.2	0.3	1.1	20.5	4.1	0.4	27.8	0.8	0.1	33.9
m	92	94	96	98	100	102	104	106	108	110	112	114	116	118	120
#	6	14	163	26	18	147	20	6	171	84	12	173	0	6	326
%	0.3	0.7	15.9	1.5	1.1	14.4	0.9	0.2	13.2	5.2	0.5	13.3	0.0	0.2	31.8
m	122	124	126	128	130	132	134	136	138	140	142	144	146	148	150
#	6	20	286	4	64	297	4	8	194	107	2	241	6	2	422
%	0.2	0.6	22.1	0.1	2.8	18.6	0.1	0.2	10.0	4.6	0.0	10.5	0.1	0.0	26.4
m	152	154	156	158	160	162	164	166	168	170	172	174	176	178	180
#	4	71	272	0	95	240	6	0	473	100	4	314	8	0	609
%	0.1	2.0	11.8	0.0	2.3	8.2	0.1	0.0	20.5	2.4	0.1	10.0	0.1	0.0	26.4

TABLE 2. Counting the number (a, b) for which $|E_{a,b,m}| = 0$

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