# The Reidemeister spectra of low dimensional almost-crystallographic groups 

Sam Tertooy*

1st January 2021

This is an Accepted Manuscript of an article published by Taylor \& Francis in Experimental Mathematics on 11 Jul 2019, available online:<br>https://www.tandfonline.com/10.1080/10586458.2019.1636426


#### Abstract

We determine which non-crystallographic, almost-crystallographic groups of dimension 4 have the $R_{\infty}$-property. We then calculate the Reidemeister spectra of the 3 dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups.


## 1 Introduction

Let $G$ be any group and $\varphi: G \rightarrow G$ an endomorphism of this group. Define an equivalence relation $\sim_{\varphi}$ on $G$ given by

$$
\forall g, g^{\prime} \in G: g \sim_{\varphi} g^{\prime} \Longleftrightarrow \exists h \in G: g=h g^{\prime} \varphi(h)^{-1}
$$

An equivalence class $[g]_{\varphi}$ is called a Reidemeister class of $\varphi$ or $\varphi$-twisted conjugacy class. The Reidemeister number $R(\varphi)$ is the number of Reidemeister classes of $\varphi$ and is therefore always a positive integer or infinity. The Reidemeister spectrum of a group $G$ is the set of all Reidemeister numbers when considering all possible automorphisms of that group:

$$
\operatorname{Spec}_{R}(G):=\{R(\varphi) \mid \varphi \in \operatorname{Aut}(G)\}
$$

If $\operatorname{Spec}_{R}(G)=\{\infty\}$ we say that $G$ has the $R_{\infty}$-property.
Reidemeister numbers originate in Nielsen fixed point theory, where they are defined as the number of fixed point classes of a self-map of a topological space 11, although they also yield applications in algebraic geometry and representation theory [8].

It turns out that many (infinite) groups admit the $R_{\infty}$-property. This is also the case for most almost-crystallographic groups, e.g. in [6] it was shown that 207 of the 219 3-dimensional crystallographic groups and 15 of the 17 families of 3-dimensional (noncrystallographic) almost-crystallographic groups all have the $R_{\infty}$-property. Furthermore, in $[5]$ it was shown that 4692 of the 47834 -dimensional crystallographic groups admit the $R_{\infty}$ property. Moreover, the Reidemeister spectra of all crystallographic groups of dimensions 1,2 and 3 were calculated, as well as the spectra of the 4 -dimensional Bieberbach groups. In this paper we extend these results by studying the 4-dimensional almost-crystallographic groups.

[^0]This paper is structured as follows. In the next two sections, we provide the necessary preliminaries on Reidemeister numbers and almost-crystallographic groups. In section 4 we determine which almost-crystallographic groups of dimension 4 possess the $R_{\infty}$-property. Sections 5 and 6 are devoted to calculating the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups respectively. The final section summarises the obtained results.

## 2 Reidemeister numbers and spectra

In this section we introduce basic notions concerning the Reidemeister number. For a general reference on Reidemeister numbers and their connection to fixed point theory, we refer the reader to [11.

The definitions of the Reidemeister number and Reidemeister spectrum were given in the introduction. However, nothing was said on how we actually determine whether a group has the $R_{\infty}$-property, and if not, how we calculate its Reidemeister spectrum. The following lemma is an essential tool for the former.

Lemma 2.1 (see [8, Section 2.2], 9, Lemma 1.1]). Let $N$ be a normal subgroup of a group $G$ and $\varphi \in \operatorname{Aut}(G)$ with $\varphi(N)=N$. We denote the restriction of $\varphi$ to $N$ by $\left.\varphi\right|_{N}$, and the induced automorphism on the quotient $G / N$ by $\varphi^{\prime}$. We then get the following commutative diagram with exact rows:


We obtain the following properties:
(1) $R(\varphi) \geq R\left(\varphi^{\prime}\right)$,
(2) if $R\left(\varphi^{\prime}\right)<\infty, R\left(\left.\varphi\right|_{N}\right)=\infty$ and $\left|\operatorname{Fix}\left(\varphi^{\prime}\right)\right|<\infty$, then $R(\varphi)=\infty$.

A direct consequence for characteristic subgroups is the following:
Corollary 2.2. Let $N$ be a characteristic subgroup of $G$. If either
(1) the quotient $G / N$ has the $R_{\infty}$-property, or
(2) $N$ has finite index in $G$ and has the $R_{\infty}$-property,
then $G$ has the $R_{\infty}$-property as well.

## 3 Almost-crystallographic groups

Let $G$ be a connected, simply connected, nilpotent Lie group with automorphism group $\operatorname{Aut}(G)$. The affine group $\operatorname{Aff}(G)$ is the semi-direct product $\operatorname{Aff}(G)=G \rtimes \operatorname{Aut}(G)$, where multiplication is defined by $\left(d_{1}, D_{1}\right)\left(d_{2}, D_{2}\right)=\left(d_{1} D_{1}\left(d_{2}\right), D_{1} D_{2}\right)$. If $C$ is a maximal compact subgroup of $\operatorname{Aut}(G)$, then $G \rtimes C$ is a subgroup of $\operatorname{Aff}(G)$. A cocompact discrete subgroup $\Gamma$ of $G \rtimes C$ is called an almost-crystallographic group modelled on the Lie group $G$. The dimension of $\Gamma$ is defined as the dimension of $G$.

If $\Gamma$ is torsion-free, then it is called an almost-Bieberbach group. If $G=\mathbb{R}^{n}$, then it is called a crystallographic group, or a Bieberbach group if it also torsion-free.

Crystallographic groups were historically studied first, and are well understood by the three Bieberbach theorems. These theorems have since been generalised to almost-crystallographic groups, which we will briefly discuss below. We refer to [14] and [3] for more information on the original and generalised theorems respectively.

The generalised first Bieberbach theorem says that if $\Gamma \subseteq \operatorname{Aff}(G)$ is an $n$-dimensional almost-crystallographic group, then its translation subgroup $N:=\Gamma \cap G$ is a uniform lattice of $G$ and is of finite index in $\Gamma$. Moreover, $N$ is the unique maximal nilpotent normal subgroup of $\Gamma$, and is therefore characteristic in $\Gamma$. The quotient group $F:=\Gamma / N$ is a finite group called the holonomy group of $\Gamma$. In fact $F=\{A \in \operatorname{Aut}(G) \mid \exists a \in G:(a, A) \in \Gamma\}$. If $\Gamma$ is crystallographic $\left(G=\mathbb{R}^{n}\right)$, we may assume that $N=\mathbb{Z}^{n}$ and $F$ is a subgroup of $\mathrm{GL}_{n}(\mathbb{Z})$.

The generalised second Bieberbach theorem tells us more about automorphisms of al-most-crystallographic groups.

Theorem 3.1 (generalised second Bieberbach theorem). Let $\varphi: \Gamma \rightarrow \Gamma$ be an automorphism of an almost-crystallographic group $\Gamma \subseteq \operatorname{Aff}(G)$ with holonomy group $F$. Then there exists $a(d, D) \in \operatorname{Aff}(G)$ such that $\varphi(\gamma)=(d, D) \circ \gamma \circ(d, D)^{-1}$ for all $\gamma \in \Gamma$. To shorten notation, we will write $\varphi=\xi_{(d, D)}$.

An automorphism $\Phi: G \rightarrow G$ of a Lie group $G$ induces an automorphism $\Phi_{*}: \mathfrak{g} \rightarrow \mathfrak{g}$ of the associated Lie algebra $\mathfrak{g}$. We will henceforth always denote an induced automorphisms on a Lie algebra with a star $(*)$ subscript, for example $A_{*}$ is the Lie algebra automorphism induced by some $A \in F$ where $F \subseteq \operatorname{Aut}(G)$ is the holonomy group of an almost-crystallographic group. In particular, an automorphism $\varphi=\xi_{(d, D)}$ of an almostcrystallographic group has an associated matrix $D_{*}$.

The generalised third Bieberbach theorem is less straightforward to generalise. Unlike for crystallographic groups, it is not true that there are only finitely many $n$-dimensional almost-crystallographic groups for a given dimension $n$. However, we can state that for a given finitely generated torsion-free nilpotent group $N$, there are (up to isomorphism) only finitely many almost-crystallographic groups $\Gamma$ such that the translation subgroup of $\Gamma$ is isomorphic to $N$.

In 3, Section 2.5], this generalisation is proved using the concept of an isolator, which shall prove useful to us as well.

Definition 3.2. Let $G$ be a group with subgroup $H$. The isolator of $H$ in $G$ is defined as

$$
\sqrt[G]{H}:=\left\{g \in G \mid g^{k} \in H \text { for some } k \geq 1\right\}
$$

Although much can be said about isolators, for the purposes of this paper we only care about a very specific result.

Lemma 3.3 (see [3, Lemma 2.4.2]). Let $\Gamma$ be an almost-crystallographic group with translation subgroup $N$ of nilpotency class $c$. Then the isolator $\sqrt[N]{\gamma_{c}(N)} \leq Z(N)$ is a characteristic subgroup of $\Gamma$. Moreover, the quotient group $\Gamma / \sqrt[N]{\gamma_{c}(N)}$ is an almost-crystallographic group whose translation subgroup $N / \sqrt[N]{\gamma_{c}(N)}$ has nilpotency class $c-1$. If $c=2$, then this quotient is a crystallographic group.

We will now give the most important results for Reidemeister theory applied to almostcrystallographic groups. A first result allows us to easily determine whether an almostcrystallographic group admits the $R_{\infty}$-property or not.

Theorem 3.4 (see [6, Corollary 3.10]). Let $\Gamma$ be an $n$-dimensional almost-crystallographic group with holonomy group $F \subseteq \operatorname{Aut}(G)$ and $\varphi=\xi_{(d, D)} \in \operatorname{Aut}(\Gamma)$ (where we use the notation
of theorem 3.1). Then

$$
\begin{aligned}
& R(\varphi)=\infty \\
& \Longleftrightarrow \exists A \in F \text { such that } \operatorname{det}\left(\mathbb{1}_{n}-A_{*} D_{*}\right)=0 \\
& \Longleftrightarrow \exists A \in F \text { such that } A_{*} D_{*} \text { has eigenvalue } 1 .
\end{aligned}
$$

The second result only holds for almost-Bieberbach groups, and allows for an easy computation of the Reidemeister number of an automorphism.

Theorem 3.5 (averaging formula, see [10, Theorem 6.11] and [12, Theorem 4.3]). Let $\Gamma$ be an n-dimensional almost-Bieberbach group with holonomy group $F \subseteq \operatorname{Aut}(G)$, and $\varphi=\xi_{(d, D)} \in \operatorname{Aut}(\Gamma)$ with $R(\varphi)<\infty$. Then

$$
R(\varphi)=\frac{1}{\# F} \sum_{A \in F}\left|\operatorname{det}\left(\mathbb{1}_{n}-A_{*} D_{*}\right)\right| .
$$

In general, this formula does not hold for automorphisms of almost-crystallographic groups, examples can be found in [5 and later in this paper. Therefore, the calculation of the Reidemeister spectra usually requires a deeper understanding of how the Reidemeister classes are formed in a specific group.

## 4 The $R_{\infty}$-property for 4 -dimensional almost-crystallographic groups

Every almost-crystallographic group of dimension 1 or 2 is crystallographic. In 6 it was determined which 3-dimensional almost-crystallographic groups admit the $R_{\infty}$-property. We extend these results to dimension 4 . In this case the translation subgroup $N$ is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class at most 3. Nilpotency class 1 is of course the crystallographic case, which was done in [5].

### 4.1 Nilpotency class 2

Let $\Gamma$ be an almost-crystallographic group whose translation subgroup $N$ is a nilpotent group of rank 4 and nilpotency class 2 . In [3] it was shown that $N$ can be given the following presentation:

$$
\left\langle e_{1}, e_{2}, e_{3}, e_{4} \left\lvert\, \begin{array}{ll}
{\left[e_{2}, e_{1}\right]=1} & {\left[e_{3}, e_{2}\right]=e_{1}^{l_{1}}} \\
{\left[e_{3}, e_{1}\right]=1} & {\left[e_{4}, e_{2}\right]=e_{1}^{l_{2}}} \\
{\left[e_{4}, e_{1}\right]=1} & {\left[e_{4}, e_{3}\right]=e_{1}^{l_{3}}}
\end{array}\right.\right\rangle .
$$

Moreover, let $G$ be the Lie group that $\Gamma$ is modelled on. By [2, Theorem 4.1], there exists a faithful affine representation $\lambda: G \rtimes \operatorname{Aut}(G) \rightarrow \operatorname{Aff}\left(\mathbb{R}^{4}\right)$ such that its restriction to $\Gamma$ is
again a faithful affine representation. In particular,

$$
\begin{array}{ll}
\lambda\left(e_{1}\right)=\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right), & \lambda\left(e_{2}\right)=\left(\begin{array}{ccccc}
1 & 0 & -\frac{l_{1}}{2} & -\frac{l_{2}}{2} & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right), \\
\lambda\left(e_{3}\right)=\left(\begin{array}{ccccc}
1 & \frac{l_{1}}{2} & 0 & -\frac{l_{3}}{2} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right), & \lambda\left(e_{4}\right)=\left(\begin{array}{ccccc}
1 & \frac{l_{2}}{2} & \frac{l_{3}}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{array}\right),
\end{array}
$$

where the values of $l_{1}, l_{2}$ and $l_{3}$ are determined by the relations $\left[e_{3}, e_{2}\right]=e_{1}^{l_{1}},\left[e_{4}, e_{2}\right]=e_{1}^{l_{2}}$ and $\left[e_{4}, e_{3}\right]=e_{1}^{l_{3}}$.

Lemma 3.3 tells us that the subgroup $\left\langle e_{1}\right\rangle=\sqrt[N]{\gamma_{2}(N)}$ is characteristic and the quotient $\Gamma^{\prime}:=\Gamma /\left\langle e_{1}\right\rangle$ is a 3 -dimensional crystallographic group. Using corollary 2.2, we know that if $\Gamma^{\prime}$ has the $R_{\infty}$-property, then so does $\Gamma$. In [3, 4] the almost-crystallographic groups were classified into families based on which crystallographic group $\Gamma^{\prime}$ is. Since only twelve 3-dimensional crystallographic groups do not have the $R_{\infty}$-property, we need only consider the corresponding twelve families of 4 -dimensional almost-crystallographic groups.

Each of these families can be split in smaller subfamilies, determined by the action of $F$ on $\sqrt[N]{\gamma_{2}(N)}$ : every $A \in F$ acts on $e_{1}$ by ${ }^{A} e_{1}=e_{1}^{\epsilon_{A}}$ with $\epsilon_{A} \in\{-1,1\}$. The following proposition quickly deals with the subfamilies where $F$ does not act trivially on $\sqrt[N]{\gamma_{2}(N)}$.
Proposition 4.1. Let $\Gamma$ be an almost-crystallographic group with translation subgroup $N$ of rank 4 and nilpotency class 2 , and holonomy group $F$. If $F$ acts non-trivially on $\sqrt[N]{\gamma_{2}(N)}$, then $\Gamma$ has the $R_{\infty}$-property.
Proof. Let $A \in F$ arbitrary and $\varphi=\xi_{(d, D)} \in \operatorname{Aut}(\Gamma)$. Since $A$ acts on $\left\langle e_{1}\right\rangle=\sqrt[N]{\gamma_{2}(N)}$ by ${ }^{A} e_{1}=e_{1}^{\epsilon_{A}}$ with $\epsilon_{A} \in\{-1,1\}$ and $\varphi\left(e_{1}\right)=e_{1}^{\nu}$ with $\nu \in\{-1,1\}, A_{*}$ and $D_{*}$ must have the following forms:

$$
A_{*}=\left(\begin{array}{cccc}
\epsilon_{A} & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{array}\right), \quad D_{*}=\left(\begin{array}{cccc}
\nu & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{array}\right) .
$$

Thus, $\mathbb{1}_{4}-A_{*} D_{*}$ is of the form

$$
\mathbb{1}_{4}-A_{*} D_{*}=\left(\begin{array}{cccc}
1-\nu \epsilon_{A} & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{array}\right) .
$$

Now let us look at specific $A \in F$. First, let $A$ be the neutral element of $F$, which necessarily acts trivially on $e_{1}$. The above matrix then has upper left entry $1-\nu$, hence $\operatorname{det}\left(\mathbb{1}_{4}-D_{*}\right) \neq 0$ if and only if $\nu=-1$.

Second, let $A$ be an element of $F$ for which $\epsilon_{A}=-1$. Such element exists since we assumed $F$ acts non-trivially on $\sqrt[N]{\gamma_{2}(N)}$. Then the matrix $\mathbb{1}_{4}-A_{*} D_{*}$ has upper left entry $1+\nu$, and $\operatorname{det}\left(\mathbb{1}_{4}-A_{*} D_{*}\right) \neq 0$ if and only if $\nu=1$.

As $\nu$ cannot be -1 and 1 at the same time, we always have some $A \in F$ for which $\operatorname{det}\left(\mathbb{1}_{4}-A_{*} D_{*}\right)=0$, and by theorem 3.4 this means that $R(\varphi)=\infty$. Since this holds for any automorphism, $\Gamma$ has the $R_{\infty}$-property.

| Family | $\delta$ |
| :---: | :---: |
| 1,2 | $\left(\begin{array}{lllll}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0\end{array}\right)$ |
| 3,4 | $\left(\begin{array}{llllll}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0\end{array}\right)$ |
| 5 | $\left(\begin{array}{llllll}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0\end{array}\right)$ |
| 143 | $\left(\begin{array}{llllll}1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0\end{array}\right)$ |
| 146 | $\left(\begin{array}{ccccc}1 \\ 0 & -\frac{k_{1}}{2}+k_{2}+2 k_{3} & -k_{2}+k_{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0\end{array}\right)$ |

Table 1: Conjugacy matrices between representations

From the proof of the theorem above, we can also conclude the following:
Proposition 4.2. Let $\Gamma$ be an almost-crystallographic group with translation subgroup $N$ of rank 4 and nilpotency class 2, and let $e_{1}$ be a generator of $\sqrt[N]{\gamma_{2}(N)}$. If $\varphi \in \operatorname{Aut}(\Gamma)$ has finite Reidemeister number, then $\varphi\left(e_{1}\right)=e_{1}^{-1}$.

We will number the twelve families under consideration according to the crystallographic group $\Gamma / \sqrt[N]{\gamma_{2}(N)}$, using the classification in the International Tables in Crystallography [1]: they are families $1-5,16,19,22-24,143$ and 146 . When we write $\Gamma_{n / m}$, we mean the $n$-dimensional crystallographic group with IT-number $m$.

Using the techniques in [3, Section 5.4], we find that for an almost-crystallographic group belonging to one of the families 16,19 or $22-24, F$ acting trivially on $\sqrt[N]{\gamma_{2}(N)}$ implies that the group is actually crystallographic. Therefore we may omit these families and we are left with only 7 families to study.

Note that the presentations given in this paper may vary from those in [3, 4]. Let $\Gamma$ and $\lambda$ denote a group and its faithful representation as given in this paper, and let $\Gamma^{\prime}$ and $\mu$ be the corresponding group and representation as given by [3] or 4]. Table 1] contains a matrix $\delta$ such that

$$
\lambda(\Gamma)=\delta \mu\left(\Gamma^{\prime}\right) \delta^{-1}
$$

hence $\lambda(\Gamma)$ and $\mu\left(\Gamma^{\prime}\right)$ are conjugate subgroups of $\operatorname{Aff}\left(\mathbb{R}^{4}\right)$ and therefore $\Gamma$ and $\Gamma^{\prime}$ are isomorphic.

Family 1. This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 4. It was shown in [7, Section 3.2] that these groups do not have the $R_{\infty}$-property.

Family 2. Every group in this family has a presentation of the form

$$
\begin{aligned}
& \left\langle\begin{array}{ll} 
& {\left[e_{2}, e_{1}\right]=1} \\
& \alpha e_{1}=e_{1} \alpha \\
{\left[e_{3}, e_{1}\right]=1} & \alpha e_{2}=e_{1}^{k_{4}} e_{2}^{-1} \alpha \\
e_{1}, e_{2}, e_{3}, e_{4}, \alpha \mid & {\left[e_{4}, e_{1}\right]=1} \\
{\left[e_{3}, e_{2}\right]=e_{1}^{k_{1}}} & \alpha e_{3}=e_{1}^{k_{5}} e_{3}^{-1} \alpha \\
{\left[e_{4}, e_{2}\right]=e_{1}^{k_{2}}} & \alpha e_{1}^{k_{6}} e_{4}^{-1} \alpha
\end{array}\right\rangle, e_{1}^{k_{7}}, \\
& {\left[e_{4}, e_{3}\right]=e_{1}^{k_{3}}}
\end{aligned}
$$

and the faithful representation $\lambda$ is given by

$$
\lambda(\alpha)=\left(\begin{array}{ccccc}
1 & k_{4} & k_{5} & k_{6} & \frac{k_{7}}{2} \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right) .
$$

Set $k:=\operatorname{gcd}\left(k_{1}, k_{2}, k_{3}\right)$ and $g:=e_{2}^{k_{3} / k} e_{3}^{-k_{2} / k} e_{4}^{k_{1} / k}$, then the centre $Z(N)$ of the translation subgroup is generated by $e_{1}$ and $g$. Let $\varphi: \Gamma \rightarrow \Gamma$ be any automorphism. Since $\left\langle e_{1}\right\rangle$ and $Z(N)$ are both characteristic in $\Gamma$, we have that $\varphi(g)=g^{\epsilon} e_{1}^{m}$ for some $\epsilon \in\{-1,1\}$ and $m \in \mathbb{Z}$. Consider the induced automorphism $\varphi^{\prime}=\xi_{\left(d^{\prime}, D^{\prime}\right)}$ on $\Gamma /\left\langle e_{1}\right\rangle \cong \Gamma_{3 / 2}$. Then

$$
\varphi^{\prime}\left(g\left\langle e_{1}\right\rangle\right)=D^{\prime}\left(g\left\langle e_{1}\right\rangle\right)=\varphi(g)\left\langle e_{1}\right\rangle=g^{\epsilon}\left\langle e_{1}\right\rangle .
$$

Depending on the value of $\epsilon, D_{*}^{\prime}$ has either eigenvalue 1 , in which case $\operatorname{det}\left(\mathbb{1}_{3}-D_{*}^{\prime}\right)=0$, or eigenvalue -1 , in which case $\operatorname{det}\left(\mathbb{1}_{3}+D_{*}^{\prime}\right)=0$. Since the holonomy group of $\Gamma_{3 / 2}$ is $\left\{\mathbb{1}_{3},-\mathbb{1}_{3}\right\}$, we obtain by theorem 3.4 that $R\left(\varphi^{\prime}\right)=\infty$ and by lemma 2.1 that therefore $R(\varphi)=\infty$. Since this holds for an arbitrary automorphism, $\Gamma$ has the $R_{\infty}$-property.

Families 3, 4 and 5. Every group in one of these families has a presentation of the form

$$
\left\langle\begin{array}{ll}
{\left[e_{2}, e_{1}\right]=1} & \alpha e_{1}=e_{1} \alpha \\
{\left[e_{1}, e_{2}, e_{3}, e_{4}, \alpha\right.} & {\left[\begin{array}{ll}
{\left[e_{3}, e_{1}\right]=1} & \alpha e_{2}=e_{2} \alpha \\
{\left[e_{4}, e_{1}\right]=1} & \alpha e_{3}=e_{1}^{k_{2}} e_{2}^{-\nu} e_{3}^{-1} \alpha \\
{\left[e_{3}, e_{2}\right]=1} & \alpha e_{4}=e_{1}^{k_{3}} e_{4}^{-1} \alpha \\
{\left[e_{4}, e_{2}\right]=1} & \alpha^{2}=e_{1}^{k_{4}} e_{2}^{\mu} \\
& {\left[e_{4}, e_{3}\right]=e_{1}^{k_{1}}}
\end{array}\right.}
\end{array}\right.
$$

and the faithful representation $\lambda$ is given by

$$
\lambda(\alpha)=\left(\begin{array}{ccccc}
1 & 0 & k_{2} & k_{3} & \frac{k_{4}}{2} \\
0 & 1 & -\nu & 0 & \frac{\mu}{2} \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right) .
$$

Family 3 is given by $\mu, \nu=0$, family 4 by $\mu=1, \nu=0$ and family 5 by $\mu=0, \nu=1$. Define an automorphism $\varphi=\xi_{(d, D)}$ by

$$
\begin{aligned}
& \varphi\left(e_{1}\right)=e_{1}^{-1}, \\
& \varphi\left(e_{2}\right)=e_{2}^{-1} \\
& \varphi\left(e_{3}\right)=e_{1}^{k_{1}-k_{2}-k_{3}} e_{2}^{\nu} e_{3} e_{4}^{2}, \\
& \varphi\left(e_{4}\right)=e_{1}^{3 k_{1}-k_{2}-2 k_{3}} e_{2}^{\nu} e_{3}^{2} e_{4}^{3}, \\
& \varphi(\alpha)=e_{1}^{-k_{4}} e_{2}^{-\mu} \alpha,
\end{aligned}
$$

then $D_{*}$ is of the form

$$
D_{*}=\left(\begin{array}{cccc}
-1 & * & * & * \\
0 & -1 & * & * \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 3
\end{array}\right)
$$

We can apply theorem 3.4 to show that $R(\varphi)<\infty$ and hence $\Gamma$ does not have the $R_{\infty^{-}}$ property.

Families 143 and 146. Every group in one of these families has a presentation of the form

$$
\begin{aligned}
& \begin{array}{ll}
{\left[e_{2}, e_{1}\right]=1} & \alpha e_{1}=e_{1} \alpha \\
{\left[e_{3}, e_{1}\right]=1} & \alpha e_{2}=e_{2} \alpha
\end{array} \\
& \left\langle e_{1}, e_{2}, e_{3}, e_{4}, \alpha \left\lvert\, \begin{array}{ll}
{\left[e_{4}, e_{1}\right]=1} & \alpha e_{3}=e_{1}^{k_{2}} e_{4} \alpha \\
{\left[e_{3}, e_{2}\right]=1} & \alpha e_{4}=e_{1}^{k_{3}} e_{2}^{\mu} e_{3}^{-1} e_{4}^{-1} \alpha
\end{array}\right.\right\rangle, \\
& {\left[e_{4}, e_{2}\right]=1 \quad \alpha^{3}=e_{1}^{k_{4}}} \\
& {\left[e_{4}, e_{3}\right]=e_{1}^{k_{1}}}
\end{aligned}
$$

and the faithful representation $\lambda$ is given by

$$
\lambda(\alpha)=\left(\begin{array}{ccccc}
1 & 0 & k_{2} & -\frac{k_{1}}{2}+k_{3} & \frac{k_{4}}{3} \\
0 & 1 & 0 & \mu & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Family 143 is given by $\mu=0$ and family 146 by $\mu=1$. Using an argument identical to the proof of [6, Theorem 4.4, family 13], we may conclude that all groups in these families have the $R_{\infty}$-property.

### 4.2 Nilpotency class 3

By an argument analogous to [9, Example 5.2], a finitely-generated, torsion-free, nilpotent group of nilpotency class 3 and rank 4 has the $R_{\infty}$-property. Applying corollary 2.2 then proves that every 4-dimensional almost-crystallographic group with translation subgroup of nilpotency class 3 has the $R_{\infty}$-property.

## 5 The Reidemeister spectra of the 3-dimensional al-most-crystallographic groups

Let $\Gamma$ be an almost-crystallographic group whose translation subgroup $N$ is a nilpotent group of rank 3 and nilpotency class 2 . Such $N$ can be given the following presentation:

$$
\left\langle e_{1}, e_{2}, e_{3} \left\lvert\, \begin{array}{l}
{\left[e_{2}, e_{1}\right]=1} \\
{\left[e_{3}, e_{1}\right]=1}
\end{array} \quad\left[e_{3}, e_{2}\right]=e_{1}^{l_{1}}\right.\right\rangle
$$

with $l_{1}>0$. Moreover, let $G$ be the Lie group that $\Gamma$ is modelled on. By [2, Theorem 4.1], there exists a faithful affine representation $\lambda: G \rtimes \operatorname{Aut}(G) \rightarrow \operatorname{Aff}\left(\mathbb{R}^{3}\right)$ such that its restriction to $\Gamma$ is again a faithful affine representation. In particular,

$$
\lambda\left(e_{1}\right)=\left(\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad \lambda\left(e_{2}\right)=\left(\begin{array}{cccc}
1 & 0 & -\frac{l_{1}}{2} & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad \lambda\left(e_{3}\right)=\left(\begin{array}{cccc}
1 & \frac{l_{1}}{2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where the value of $l_{1}$ is determined by the relation $\left[e_{3}, e_{2}\right]=e_{1}^{l_{1}}$. Like in section 4.1, we have that the subgroup $\left\langle e_{1}\right\rangle=\sqrt[N]{\gamma_{2}(N)}$ is characteristic in $\Gamma$, and an automorphism $\varphi$ must satisfy $\varphi\left(e_{1}\right)=e_{1}^{-1}$ to have finite Reidemeister number.

As mentioned before, in [6, Theorem 4.4] it was shown that there are only 2 families of almost-crystallographic groups that do not admit the $R_{\infty}$-property. We again number these families according to the IT-number of the quotient $\Gamma / \sqrt[N]{\gamma_{2}(N)}$.

Family 1. The groups in this family are exactly the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 3. In [13, Section 3] it was shown that these groups have Reidemeister spectrum $2 \mathbb{N} \cup\{\infty\}$. This was shown specifically for the case $k_{1}=1$, but the argument holds for any $k_{1}>0$.

Family 2. Every group in this family has a presentation of the form

$$
\left\langle e_{1}, e_{2}, e_{3}, \alpha \left\lvert\, \begin{array}{ll}
{\left[e_{2}, e_{1}\right]=1} & \alpha e_{1}=e_{1} \alpha \\
{\left[e_{3}, e_{1}\right]=1} & \alpha e_{2}=e_{1}^{k_{2}} e_{2}^{-1} \alpha \\
{\left[e_{3}, e_{2}\right]=e_{1}^{k_{1}}} & \alpha e_{3}=e_{1}^{k_{1}} e_{3}^{-1} \alpha \\
\alpha^{2}=e_{1}^{k_{4}}
\end{array}\right.\right\rangle
$$

and the faithful representation $\lambda$ is given by

$$
\lambda(\alpha)=\left(\begin{array}{cccc}
1 & k_{2} & k_{3} & \frac{k_{4}}{2} \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

Let $\varphi$ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation $\lambda$, this automorphism will correspond to a matrix $\delta \in \operatorname{Aff}\left(\mathbb{R}^{4}\right)$ such that

$$
\lambda(\varphi(\gamma))=\delta \lambda(\gamma) \delta^{-1}
$$

for all $\gamma \in \Gamma$. Since we assumed that $R(\varphi)<\infty$, we have that $\varphi\left(e_{1}\right)=e_{1}^{-1}$. Moreover, $\varphi$ induces an automorphism $\varphi^{\prime}$ on $\Gamma^{\prime}:=\Gamma /\left\langle e_{1}\right\rangle$. Thus, $\delta$ must be of the form

$$
\delta=\left(\begin{array}{cccc}
-1 & n_{1} & n_{2} & 0 \\
0 & m_{1} & m_{3} & d_{1} / 2 \\
0 & m_{2} & m_{4} & d_{2} / 2 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

where the constants $m_{i}, d_{j}$ are integers, $m_{1} m_{4}-m_{2} m_{3}=-1$ and $n_{1}, n_{2} \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of $n_{1}, n_{2}$ and $l_{1}, l_{2}, l_{3}$ such that

$$
\begin{aligned}
\delta \lambda\left(e_{2}\right) \delta^{-1} & =\lambda\left(e_{1}\right)^{l_{1}} \lambda\left(e_{2}\right)^{m_{1}} \lambda\left(e_{3}\right)^{m_{2}}, \\
\delta \lambda\left(e_{3}\right) \delta^{-1} & =\lambda\left(e_{1}\right)^{l_{2}} \lambda\left(e_{2}\right)^{m_{3}} \lambda\left(e_{3}\right)^{m_{4}}, \\
\delta \lambda(\alpha) \delta^{-1} & =\lambda\left(e_{1}\right)^{l_{3}} \lambda\left(e_{2}\right)^{d_{1}} \lambda\left(e_{3}\right)^{d_{2}} \lambda(\alpha) .
\end{aligned}
$$

From the obtained values of $l_{1}, l_{2}$ and $l_{3}$, we get

$$
\begin{aligned}
& \varphi\left(e_{1}\right)=e_{1}^{-1}, \\
& \varphi\left(e_{2}\right)=e_{1}^{\frac{k_{1}}{2}\left(m_{1} m_{2}+m_{1} d_{2}-m_{2} d_{1}\right)-\frac{k_{2}}{2}\left(m_{1}+1\right)-\frac{k_{3}}{2} m_{2}} e_{2}^{m_{1}} e_{3}^{m_{2}}, \\
& \varphi\left(e_{3}\right)=e_{1}^{\frac{k_{1}}{2}\left(m_{3} m_{4}+m_{3} d_{2}-m_{4} d_{1}\right)-\frac{k_{2}}{2} m_{3}-\frac{k_{3}}{2}\left(m_{4}+1\right)} e_{2}^{m_{3}} e_{3}^{m_{4}}, \\
& \varphi(\alpha)=e_{1}^{\frac{k_{1}}{2} d_{1} d_{2}-\frac{k_{2}}{2} d_{1}-\frac{k_{3}}{2} d_{2}-k_{4}} e_{2}^{d_{1}} e_{3}^{d_{2}} \alpha,
\end{aligned}
$$

where all exponents must be integers. This places four conditions on the $m_{i}$ and $d_{j}$ :
(a) $k_{1}\left(m_{1} m_{2}+m_{1} d_{2}-m_{2} d_{1}\right)-k_{2}\left(m_{1}+1\right)-k_{3} m_{2} \equiv 0 \bmod 2$,
(b) $k_{1}\left(m_{3} m_{4}+m_{3} d_{2}-m_{4} d_{1}\right)-k_{2} m_{3}-k_{3}\left(m_{4}+1\right) \equiv 0 \bmod 2$,
(c) $k_{1} d_{1} d_{2}-k_{2} d_{1}-k_{3} d_{2} \equiv 0 \bmod 2$,
(d) $m_{1} m_{4}-m_{2} m_{3}=-1$.

For ease of notation, let us set

$$
M:=\left(\begin{array}{ll}
m_{1} & m_{3} \\
m_{2} & m_{4}
\end{array}\right) \in \mathrm{GL}_{2}(\mathbb{Z}), \quad d:=\binom{d_{1}}{d_{2}} \in \mathbb{Z}^{2}
$$

We will determine $R(\varphi)$ in a very similar way to the proof of [5, Proposition 5.11]. Let $[x]_{\varphi}$ be a Reidemeister class of $\Gamma$, then for any $k \in \mathbb{Z}$,

$$
x=\left(e_{1}^{-k}\right) x e_{1}^{2 k} \varphi\left(e_{1}^{-k}\right)^{-1},
$$

therefore $x \sim_{\varphi} x e_{1}^{2 k}$ for all $k \in \mathbb{Z}$. Consider the quotient group $\Gamma^{\prime}=\Gamma /\left\langle e_{1}\right\rangle$ and let $\varphi^{\prime}=\xi_{(d / 2, M)}$ be the induced automorphism on this quotient. Since we assumed that $R(\varphi)<\infty$, we have that $R\left(\varphi^{\prime}\right)<\infty$ as well. [5. Proposition 5.10] tells us that $R\left(\varphi^{\prime}\right)=$ $|\operatorname{tr}(M)|+O\left(\mathbb{1}_{2}-M, d\right)$ with

$$
O(A, a):=\#\left\{\bar{x} \in \mathbb{Z}_{2}^{2} \mid \bar{A} \bar{x}=\bar{a}\right\}
$$

where the bar-notation denotes the element-wise projection to $\mathbb{Z}_{2}$. A Reidemeister class $\left[x\left\langle e_{1}\right\rangle\right]_{\varphi^{\prime}}$ of $\Gamma^{\prime}$ will lift to at most 2 Reidemeister classes of $\Gamma:[x]_{\varphi}$ and $\left[x e_{1}\right]_{\varphi}$; so the number of lifts is either 2 (when $x \not \chi_{\varphi} x e_{1}$ ) or 1 (when $x \sim_{\varphi} x e_{1}$ ). The latter happens if and only if

$$
\begin{equation*}
\exists z \in \Gamma: x e_{1}=z x \varphi(z)^{-1} \tag{1}
\end{equation*}
$$

Projecting this to the quotient $\Gamma^{\prime}$, we have

$$
\begin{equation*}
\exists z \in \Gamma: x\left\langle e_{1}\right\rangle=z x \varphi(z)^{-1}\left\langle e_{1}\right\rangle \tag{2}
\end{equation*}
$$

Since $e_{1}$ is central in $\Gamma$ and $x$ appears exactly once on each side of the equality sign in (1), the $e_{1}$-component of $x$ does not matter. Set $x=e_{2}^{x_{2}} e_{3}^{x_{3}} \alpha^{\epsilon_{x}}$ and $z=e_{1}^{z_{1}} e_{2}^{z_{2}} e_{3}^{z_{3}} \alpha^{\epsilon_{z}}$. Let us first assume that $\epsilon_{z}=0$, then (2) is equivalent to

$$
\exists z_{2}, z_{3} \in \mathbb{Z}:\left(\mathbb{1}_{2}-A M\right)\binom{z_{2}}{z_{3}}=0
$$

with $A$ the holonomy part of $x\left\langle e_{1}\right\rangle$. As $R\left(\varphi^{\prime}\right)<\infty$, we must have $z_{2}=z_{3}=0$. But then $z=e_{1}^{z_{1}}$, and (11) then becomes $x e_{1}=x e_{1}^{2 z_{1}}$. As $z_{1}$ is an integer, this is impossible. So, let us assume that $\epsilon_{z}=1$. Writing out (1) component-wise, we find that this condition is equivalent to the following:

There exist $z_{1}, z_{2}, z_{3} \in \mathbb{Z}$ such that:
(i) $2\binom{x_{2}}{x_{3}}=\left(\mathbb{1}_{2}-(-1)^{\epsilon_{x}} M\right)\binom{z_{2}}{z_{3}}-(-1)^{\epsilon_{x}} d$,
(ii) $k_{1} z_{2} z_{3}-k_{2} z_{2}-k_{3} z_{3}-k_{4}+1=2 z_{1}$.

Condition (i) is independent of the $e_{1}$-components, and hence can be interpreted in terms of the quotient group $\Gamma^{\prime}$. In the proof of [5, Proposition 5.11] it was shown that, for a fixed value of $\epsilon_{x}$, the number of Reidemeister classes $\left[x\left\langle e_{1}\right\rangle\right]_{\varphi^{\prime}}$ for which a pair $\left(z_{2}, z_{3}\right)$ satisfying (i) exists is exactly $O\left(\mathbb{1}_{2}-M, d\right)$, i.e. the number of solutions $\left(\bar{z}_{2}, \bar{z}_{3}\right) \in \mathbb{Z}_{2}^{2}$ of the linear system of equations

$$
\text { (i') } \quad\left(\overline{\mathbb{1}_{2}-M}\right)\binom{\bar{z}_{2}}{\bar{z}_{3}}=\bar{d}
$$

Note that the above equation is exactly condition (i) taken modulo 2.
Since $\epsilon_{x}$ can take two values $(1$ and -1$)$, there are in total $2 O\left(\mathbb{1}_{2}-M, d\right)$ Reidemeister classes $\left[x\left\langle e_{1}\right\rangle\right]_{\varphi^{\prime}}$ satisfying condition (i). On the other hand, there are $|\operatorname{tr}(M)|-O\left(\mathbb{1}_{2}-M, d\right)$ Reidemeister classes of $\Gamma^{\prime}$ for which condition (i) does not hold (see [5, Section 5]).

Recall that the variable $z_{1}$ appears only in condition (ii). If we have a Reidemeister class $\left[x\left\langle e_{1}\right\rangle\right]_{\varphi^{\prime}}$ and a pair $\left(z_{2}, z_{3}\right)$ for which (i) holds, then we can find a $z_{1} \in \mathbb{Z}$ to make condition (ii) hold if and only if

$$
\text { (ii') } \quad \bar{k}_{1} \bar{z}_{2} \bar{z}_{3}-\bar{k}_{2} \bar{z}_{2}-\bar{k}_{3} \bar{z}_{3}-\bar{k}_{4}+\overline{1}=\overline{0}
$$

which is exactly condition (ii) taken modulo 2 .
We partition the solutions of (i') into those that do not satisfy condition (ii') and those that do. Let $S$ be the number of the former and $T$ the number of the latter, then $S+T=$ $O\left(\mathbb{1}_{2}-M, d\right)$. Of the $2 O\left(\mathbb{1}_{2}-M, d\right)$ Reidemeister classes $\left[x\left\langle e_{1}\right\rangle\right]_{\varphi^{\prime}}$ satisfying condition (i), $2 S$ lift to two distinct Reidemeister classes $[x]_{\varphi}$ and $\left[x e_{1}\right]_{\varphi}$, and $2 T$ lift to a single Reidemeister class $[x]_{\varphi}$. All together, we have

$$
\begin{aligned}
R(\varphi) & =2(|\operatorname{tr}(M)|-S-T)+2(2 S)+2 T \\
& =2(|\operatorname{tr}(M)|+S)
\end{aligned}
$$

In particular, we get that $R(\varphi) \in 2 \mathbb{N}$. Taking the parity of $\operatorname{tr}(M)$ into account, we can further determine the possible Reidemeister numbers:

$$
R(\varphi) \in\left\{\begin{array}{lll}
4 \mathbb{N}+2 S & \text { if } \operatorname{tr}(M) \equiv 0 & (\bmod 2) \\
4 \mathbb{N}+2 S-2 & \text { if } \operatorname{tr}(M) \equiv 1 & (\bmod 2)
\end{array}\right.
$$

where

$$
S \leq O\left(\mathbb{1}_{2}-M, d\right) \leq \begin{cases}4 & \text { if } \operatorname{tr}(M) \equiv 0 \quad(\bmod 2) \\ 1 & \text { if } \operatorname{tr}(M) \equiv 1 \quad(\bmod 2)\end{cases}
$$

There is one special case, however. If $M \equiv \mathbb{1}_{2} \bmod 2$ all entries of $\mathbb{1}_{2}-M$ will be multiples of 2 ; so $\left|\operatorname{det}\left(\mathbb{1}_{2}-M\right)\right|=|\operatorname{tr}(M)| \in 4 \mathbb{N}$ and therefore $R(\varphi) \in 8 \mathbb{N}+2 S$.

For a fixed group $\Gamma$ in this family (i.e. a fixed 4 -tuple of parameters $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ ), an automorphism $\varphi \in \operatorname{Aut}(\Gamma)$ is uniquely determined by the matrix $M \in \mathrm{GL}_{2}(\mathbb{Z})$ and the vector $d \in \mathbb{Z}^{2}$. Our goal is to find out, for each group in the family (or equivalently, for each tuple ( $\left.k_{1}, k_{2}, k_{3}, k_{4}\right)$ ), which $M$ and $d$ satisfy conditions (a) - (d) and thus produce an automorphism.

Conditions (a) - (c) are actually conditions over $\mathbb{Z}_{2}$, and none of the parameters $k_{i}$ appear in condition (d). Therefore, only the parity of the $k_{i}$ will play a role, so we need to check 16 cases, each corresponding to an element of $\mathbb{Z}_{2}^{4}$. Furthermore, a group with parameters $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ is isomorphic to the group with parameters $\left(-k_{1}, k_{3}, k_{2}, k_{4}\right)$, which allows us to omit the cases $(0,1,0,0),(0,1,0,1),(1,1,0,0)$ and $(1,1,0,1)$, leaving only 12 cases. Rather than trying to find all couples $(M, d)$ (of which there are likely to be infinitely many), we can start by finding all couples $(\bar{M}, \bar{d}) \in \mathrm{GL}_{2}\left(\mathbb{Z}_{2}\right) \times \mathbb{Z}_{2}^{2}$ satisfying conditions (a)-(c).

The function MakeList defined in algorithm 1 does exactly this. Moreover, it assigns to every couple a set $R$, which is the set of possible Reidemeister numbers the corresponding automorphisms can have. The results can be found in tables 2 to 13 . The Reidemeister spectrum of a group is a subset of (or the entirety of) the union of all these sets $R$.

Next, for each quadruplet of parameters, we tried to find a family of automorphisms whose Reidemeister numbers produce the union of these sets $R$. We succeeded in this for every choice of parameters, hence the Reidemeister spectrum always equals the union of the $R$. These automorphisms and their Reidemeister spectra, for all $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$, can be found in table 14. For the sake of brevity, we omitted $\infty$ from the spectra in this table.

We may thus conclude that, depending on the parity of the parameters $k_{1}, k_{2}, k_{3}$ and $k_{4}$, the Reidemeister spectrum is $2 \mathbb{N} \cup\{\infty\}, 4 \mathbb{N} \cup\{\infty\},(4 \mathbb{N}-2) \cup\{\infty\}$ or $(2 \mathbb{N}+2) \cup\{\infty\}$. Note that all almost-Bieberbach groups have parameters with parities $(0,0,0,1)$ and therefore have spectrum $2 \mathbb{N} \cup\{\infty\}$.

```
Algorithm 1 MakeList function
    function MAKELIST( \(k_{1}, k_{2}, k_{3}, k_{4}\) )
        AutList \(:=\varnothing\)
        for \(\bar{M} \in \mathrm{GL}_{2}\left(\mathbb{Z}_{2}\right), \bar{d} \in \mathbb{Z}_{2}^{2}\) do
            if conditions (1), (2), (3) are met then
                \(S:=0\)
                for \(\bar{z} \in \mathbb{Z}_{2}^{2}\) do
                        if \(\bar{z}\) satisfies (i') but not (ii') then
                    \(S:=S+1\)
                        end if
                end for
                if \(\operatorname{tr}(M) \equiv 0 \bmod 2\) then
                    if \(M \equiv \mathbb{1}_{2} \bmod 2\) then
                    \(R:=8 \mathbb{N}+2 S\)
                    else
                    \(R:=4 \mathbb{N}+2 S\)
                    end if
                else
                    \(R:=4 \mathbb{N}+2 S-2\)
                end if
                AutList \(:=\) AutList \(\cup\{(\bar{M}, \bar{d}, R)\}\)
            end if
        end for
        return AutList
    end function
```


## 6 Spectra of 4D almost-Bieberbach groups

We already determined in section 4 which families of four-dimensional almost-crystallographic groups do not have the $R_{\infty}$-property. In [3] it is determined which groups among these families are almost-Bieberbach groups. We use the presentations from section 4.

Family 1. Every group in this family is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class 2. In [7] Section 3.2] it was shown that the Reidemeister spectrum of such group is always $4 \mathbb{N} \cup\{\infty\}$.

Family 3. The almost-Bieberbach groups in this family are those with $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)=$ $(2 k, 0,0,1)$ for some $k \in \mathbb{N}$. An automorphism $\varphi=\xi_{(d, D)}$ with $R(\varphi)<\infty$ must be of the form

$$
\begin{aligned}
& \varphi\left(e_{1}\right)=e_{1}^{-1} \\
& \varphi\left(e_{2}\right)=e_{1}^{l} e_{2}^{-1} \\
& \varphi\left(e_{3}\right)=e_{1}^{k\left(m_{1} m_{2}+m_{1} d_{2}-m_{2} d_{1}\right)} e_{3}^{m_{1}} e_{4}^{m_{2}} \\
& \varphi\left(e_{4}\right)=e_{1}^{k\left(m_{3} m_{4}+m_{3} d_{2}-m_{4} d_{1}\right)} e_{3}^{m_{3}} e_{4}^{m_{4}} \\
& \varphi(\alpha)=e_{1}^{k d_{1} d_{2}-1} e_{3}^{d_{1}} e_{4}^{d_{2} \alpha}
\end{aligned}
$$

with $m_{1}, m_{2}, m_{3}, m_{4}, d_{1}, d_{2}, l \in \mathbb{Z}$ and $m_{1} m_{4}-m_{2} m_{3}=-1$. Then $D_{*}$ is of the form

$$
D_{*}=\left(\begin{array}{cccc}
-1 & * & * & * \\
0 & -1 & * & * \\
0 & 0 & m_{1} & m_{3} \\
0 & 0 & m_{2} & m_{4}
\end{array}\right)
$$

Using theorem 3.5, we find that $R(\varphi)=4\left|m_{1}+m_{4}\right| \in 4 \mathbb{N}$. Now, take the automorphism $\varphi_{m}$ given by

$$
\begin{array}{ll}
\varphi_{m}\left(e_{1}\right)=e_{1}^{-1}, & \varphi_{m}\left(e_{4}\right)=e_{1}^{k m} e_{3} e_{4}^{m}, \\
\varphi_{m}\left(e_{2}\right)=e_{2}^{-1}, & \varphi_{m}(\alpha)=e_{1}^{-1} \alpha, \\
\varphi_{m}\left(e_{3}\right)=e_{4}, &
\end{array}
$$

with $m \in \mathbb{N}$. Then $R\left(\varphi_{m}\right)=4 m$ and hence $\operatorname{Spec}_{R}(\Gamma)=4 \mathbb{N} \cup\{\infty\}$.
Family 4. The almost-Bieberbach groups in this family are those where either $\left(k_{1}, k_{2}, k_{3}\right.$, $\left.k_{4}\right)=(k, 0,0,0)$ with $k \in \mathbb{N}$ or $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)=(2 k, 1,0,0)$ with $k \in \mathbb{N}$. In the former case, such almost-Bieberbach group can be seen as an internal semidirect product $H_{k} \rtimes \mathbb{Z}$, where $H_{k}=\left\langle e_{1}, e_{3}, e_{4}\right\rangle$ and $\mathbb{Z}=\langle\alpha\rangle$. Similarly, in the latter case, a group is an internal semidirect product $H_{2 k} \rtimes \mathbb{Z}$.

Both of these semidirect products were studied in [7, Proposition 5.23], their Reidemeister spectra are respectively $4 \mathbb{N} \cup\{\infty\}$ and $8 \mathbb{N} \cup\{\infty\}$.

Family 5. The almost-Bieberbach groups in this family are those where $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)=$ $(k, 0,0,1)$ with $k \in \mathbb{N}$. An automorphism $\varphi=\xi_{(d, D)}$ with $R(\varphi)<\infty$ must be of the form

$$
\begin{aligned}
& \varphi\left(e_{1}\right)=e_{1}^{-1} \\
& \varphi\left(e_{2}\right)=e_{2}^{-1} e_{1}^{k\left(2 m_{1} m_{2}+2 m_{1} d_{2}-2 m_{2} d_{1}-m_{2}-d_{2}\right)-2 l} \\
& \varphi\left(e_{3}\right)=e_{2}^{m_{1}} e_{3}^{-1+2 m_{1}} e_{4}^{m_{2}} e_{1}^{l} \\
& \varphi\left(e_{4}\right)=e_{2}^{m_{3}} e_{3}^{2 m_{3}} e_{4}^{1+2 m_{4}} e_{1}^{k\left(2 m_{3} m_{4}+m_{3} d_{2}+m_{3}-2 m_{4} d_{1}-d_{1}\right)} \\
& \varphi(\alpha)=e_{2}^{d_{1}} e_{3}^{2 d_{1}} e_{4}^{d_{2}} e_{1}^{k d_{1} d_{2}-1} \alpha
\end{aligned}
$$

with $m_{1}, m_{2}, m_{3}, m_{4}, d_{1}, d_{2}, l \in \mathbb{Z}$ and $m_{1}-m_{4}+2 m_{1} m_{4}-m_{2} m_{3}=0$. Then $D_{*}$ is of the form

$$
D_{*}=\left(\begin{array}{cccc}
-1 & * & * & * \\
0 & -1 & * & * \\
0 & 0 & -1+2 m_{1} & 2 m_{3} \\
0 & 0 & m_{2} & 1+2 m_{4}
\end{array}\right)
$$

Using theorem 3.5, we find that $R(\varphi)=8\left|m_{1}+m_{4}\right| \in 8 \mathbb{N} \cup\{\infty\}$. Now, take the automorphism $\varphi_{m}$ given by

$$
\begin{array}{ll}
\varphi_{m}\left(e_{1}\right)=e_{1}^{-1}, & \varphi_{m}\left(e_{4}\right)=e_{1}^{k m} e_{2}^{m} e_{3}^{2 m} e_{4}, \\
\varphi_{m}\left(e_{2}\right)=e_{1}^{k(2 m-1)} e_{2}^{-1}, & \varphi_{m}(\alpha)=e_{1}^{-1} \alpha, \\
\varphi_{m}\left(e_{3}\right)=e_{2}^{m} e_{3}^{2 m-1} e_{4}, &
\end{array}
$$

with $m \in \mathbb{N}$. Then $R\left(\varphi_{m}\right)=8 m$ and hence $\operatorname{Spec}_{R}(\Gamma)=8 \mathbb{N} \cup\{\infty\}$.

## 7 Conclusion

We have determined which (non-crystallographic) almost-crystallographic groups of dimension 4 admit the $R_{\infty}$ property, and calculated the Reidemeister spectra of the noncrystallographic 3-dimensional almost-crystallographic groups, as well as the spectra of the non-crystallographic 4-dimensional almost-Bieberbach groups. Together with the results of [5], this completes the calculation of the Reidemeister spectra of the 3-dimensional almostcrystallographic groups and of the 4-dimensional almost-Bieberbach groups.

Acknowledgement The author would like to thank the referee for their careful reading and useful suggestions for the paper.

## References

[1] Mois I. Aroyo, editor. International tables for crystallography. Vol. A. Wiley, 2016. Space-group symmetry.
[2] Karel Dekimpe. The construction of affine structures on virtually nilpotent groups. Manuscripta Math., 87(1):71-88, 1995.
[3] Karel Dekimpe. Almost-Bieberbach Groups: Affine and Polynomial Structures, volume 1639 of Lect. Notes in Math. Springer-Verlag, 1996.
[4] Karel Dekimpe and Bettina Eick. Computational aspects of group extensions and their applications to topology. Experimental Math., 11:183-200, 2002.
[5] Karel Dekimpe, Tom Kaiser, and Sam Tertooy. The reidemeister spectra of low dimensional crystallographic groups. Journal of Algebra, 533:353-375, 2019.
[6] Karel Dekimpe and Pieter Penninckx. The finiteness of the Reidemeister number of morphisms between almost-crystallographic groups. J. Fixed Point Theory Appl., 9(2):257-283, 2011.
[7] Karel Dekimpe, Sam Tertooy, and Iris Van den Bussche. Reidemeister spectra for solvmanifolds in low dimensions. Topol. Methods Nonlinear Anal., 53(2):575-601, 2019.
[8] Alexander Fel'shtyn and Evgenij Troitsky. Aspects of the property $R_{\infty}$. J. Group Theory, 18(6):1021-1034, 2015.
[9] Daciberg L. Gonçalves and Peter Wong. Twisted conjugacy classes in nilpotent groups. J. Reine Angew. Math., 633:11-27, 2009.

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{0}{1}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 0 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$ | $8 \mathbb{N}+8$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | 8 N |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | 8 N |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\binom{$ 0 }{0} | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{0}{1}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{1}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}$ |

Table 2: $\operatorname{MakeList}(0,0,0,0)$
[10] Ku Yong Ha, Jong Bum Lee, and Pieter Penninckx. Formulas for the Reidemeister, Lefschetz and Nielsen coincidence number of maps between infra-nilmanifolds. Fixed Point Theory Appl., pages 2012:39, 23, 2012.
[11] Boju Jiang. Nielsen Fixed Point Theory, volume 14 of Contemp. Math. American Mathematical Society, 1983.
[12] Jong Bum Lee and Kyung Bai Lee. Averaging formula for Nielsen numbers of maps on infra-solvmanifolds of type (R). Nagoya Math. J., 196:117-134, 2009.
[13] Vitaly Roman'kov. Twisted conjugacy classes in nilpotent groups. J. Pure Appl. Algebra, 215(4):664-671, 2011.
[14] Andrzej Szczepański. Geometry of crystallographic groups, volume 4 of Algebra and Discrete Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

Sam Tertooy
KU Leuven Campus Kulak Kortrijk
Etienne Sabbelaan 53
8500 Kortrijk
Belgium
Sam.Tertooy@kuleuven.be

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{lll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{lll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1 \\ 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{0}{1}$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}-2$ |

Table 3: $\operatorname{MakeList}(0,0,0,1)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $8 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |

Table 4: $\operatorname{MakeList}(0,0,1,0)$
$\left.\begin{array}{c|c|l}\bar{M} & \bar{d} & R \\ \hline\left(\begin{array}{l}1 \\ 0\end{array} 0\right. \\ 1\end{array}\right)$

Table 5: MakeList $(0,0,1,1)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1 \\ 1 & 0\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{1}{1}$ | 8 N |

Table 6: MakeList( $0,1,1,0$ )

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}0 & 1 \\ 0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+4$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{1}{1}$ | 8 N |

Table 7: $\operatorname{MakeList}(0,1,1,1)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+6$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}-2$ |

Table 8: MakeList $(1,0,0,0)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$ | $8 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |

Table 9: $\operatorname{MakeList}(1,0,0,1)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | ( $\left.\begin{array}{l}1 \\ 1\end{array}\right)$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\left(\begin{array}{l}1 \\ 1 \\ 0\end{array}\right)$ | $4 \mathbb{N}-2$ |
| $\left(\begin{array}{l}1 \\ 1 \\ 0\end{array} 0\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+6$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}-2$ |

Table 10: $\operatorname{MakeList}(1,0,1,0)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{1}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $4 \mathbb{N}$ |

Table 11: $\operatorname{MakeList}(1,0,1,1)$

| $\bar{M}$ | $\bar{d}$ | $R$ |
| :---: | :---: | :---: |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |
| $\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $8 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 0 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}+2$ |
| $\left(\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right)$ | $\binom{0}{0}$ | $4 \mathbb{N}$ |

Table 12: MakeList( $1,1,1,0$ )
$\left.\begin{array}{c|c|l}\bar{M} & \bar{d} & R \\ \hline\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right) & \binom{0}{0} & 4 \mathbb{N}+2 \\ \left(\begin{array}{l}0 \\ 1 \\ 1\end{array} 1\right. \\ (1) & \binom{0}{1} & 4 \mathbb{N}-2 \\ 0 & 0 \\ 0\end{array}\right)$

Table 13: $\operatorname{MakeList}(1,1,1,1)$

| $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ | M | $d$ | $R(\varphi)$ | $\operatorname{Spec}_{R}(\Gamma)$ |
| :---: | :---: | :---: | :---: | :---: |
| (0, $0,0,0)$ | $\left(\begin{array}{ll}0 \\ 1 \\ 1 & 1 \\ 1\end{array}\right)$ | $\binom{0}{1}$ | $4 m$ | $4 \mathbb{N}$ |
| (0, 0, 0, 1) | $\left(\begin{array}{lll}0 & 1 \\ 0 & 1 \\ 1 & m\end{array}\right)$ | $\binom{0}{0}$ | $2 m$ | $2 \mathbb{N}$ |
| (0, $0,1,0)$ | $\left(\begin{array}{cc}1 & 1 \\ 2 m & 1 \\ 1 & 2 m-1\end{array}\right)$ | $\binom{1}{0}$ | $4 m$ | $4 \mathbb{N}$ |
| (0, $0,1,1$ ) | $\left(\begin{array}{ccc}1 & 1 \\ 2 m & 2 m-1\end{array}\right)$ | $\binom{0}{0}$ | $4 m$ | $4 \mathbb{N}$ |
| (0, 1, 1, 0) | $\left(\begin{array}{lll}0 & 1 \\ 1 & 1 & m\end{array}\right)$ | $\binom{1}{1}$ | $4 m$ | $4 \mathbb{N}$ |
| (0, 1, 1, 1) | $\left(\begin{array}{ll}0 \\ 0 & 1 \\ 1 & 2\end{array}\right)$ | $\binom{0}{0}$ | $4 m$ | $4 \mathbb{N}$ |
| (1, 0, 0, 0) | $\left(\begin{array}{ll}0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 m\end{array}\right)$ | $\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right)$ | $4 m-2$ | $4 \mathbb{N}-2$ |
| (1, 0, 0, 1) | $\left(\begin{array}{ll}1 & 1 \\ m & m\end{array}\right)$ | $\binom{1}{0}$ | $2 m+2$ | $2 \mathbb{N}+2$ |
| (1, $0,1,0)$ | $\left(\begin{array}{lll}0 & 1 \\ 1 & 2 m-1\end{array}\right)$ | $\binom{1}{0}$ | $4 m-2$ | $4 \mathbb{N}-2$ |
| $(1,0,1,1)$ | $\left(\begin{array}{cc}m & 1 \\ 1 & 0\end{array}\right)$ | $\binom{1}{1}$ | $2 m+2$ | $2 \mathbb{N}+2$ |
| (1, 1, 1, 0) | $\left(\begin{array}{lll}0 & 1 \\ 1 & m\end{array}\right)$ | $\binom{0}{0}$ | $2 m+2$ | $2 \mathbb{N}+2$ |
| ( $1,1,1,1$ ) | $\left(\begin{array}{cc}0 & 1 \\ 1 & 1 \\ 1 & 1\end{array}\right)$ | $\binom{0}{0}$ | $4 m-2$ | $4 \mathbb{N}-2$ |

Table 14: Automorphisms and Reidemeister spectra and for all $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$


[^0]:    *Supported by long term structural funding - Methusalem grant of the Flemish Government.

