Generalized Samuel Multiplicities of Monomial Ideals and Volumes

Published Version:
Generalized Samuel Multiplicities of Monomial Ideals and Volumes / Achilles R.; Manaresi M.. - In: EXPERIMENTAL MATHEMATICS. - ISSN 1058-6458. - STAMPA. - 31:2(2022), pp. 611-620. [10.1080/10586458.2019.1671919]

This version is available at: https://hdl.handle.net/11585/729776 since: 2022-11-07

Published:

DOI: http://doi.org/10.1080/10586458.2019.1671919

UEXM#1671919, VOL 0, ISS 0

Generalized Samuel Multiplicities of Monomial Ideals and Volumes

Rüdiger Achilles and Mirella Manaresi

QUERY SHEET

This page lists questions we have about your paper. The numbers displayed at left are hyperlinked to the location of the query in your paper.

The title and author names are listed on this sheet as they will be published, both on your paper and on the Table of Contents. Please review and ensure the information is correct and advise us if any changes need to be made. In addition, please review your paper as a whole for typographical and essential corrections.

Your PDF proof has been enabled so that you can comment on the proof directly using Adobe Acrobat. For further information on marking corrections using Acrobat, please visit http://journalauthors.tandf.co.uk/production/acrobat.asp; https://authorservices.taylorandfrancis.com/how-to-correct-proofs-with-adobe/

The CrossRef database (www.crossref.org/) has been used to validate the references.

AUTHOR QUERIES

Q1 Please confirm that the numbering of equations has been set correctly.

Q2 The Ref. [10] is listed in the references list but is not cited in the text. Please either cite the reference or remove it from the references list.

Q3 Please provide complete author information and year of publication for Ref. [11]

Q4 Please provide the missing editors name for Ref. [17].

Q5 The ORCID details of the authors have been validated against ORCID registry. please check the ORCID ID details of the authors.

EXPERIMENTAL MATHEMATICS
https://doi.org/10.1080/10586458.2019.1671919

Generalized Samuel Multiplicities of Monomial Ideals and Volumes

Rüdiger Achilles and Mirella Manaresi

Dipartimento di Matematica, Università di Bologna, Bologna, Italy

ABSTRACT

We describe conjecturally the generalized Samuel multiplicities $c_0, \ldots, c_{d-1}$ of a monomial ideal $I \subset K[x_1, \ldots, x_d]$ in terms of its Newton polyhedron NP(I). More precisely, we conjecture that c_i equals the sum of the normalized $(d-i)$-volumes of pyramids over the projections of the $(d-i-1)$-dimensional compact faces of NP(I) along the infinite-directions of i-unbounded facets in which they are contained. For c_0 proofs are known (Guibert, Jeffries and Montaño) and for c_{d-1} a proof is given.

KEYWORDS

monomial ideal; generalized Samuel multiplicity; Newton polyhedron

2010 MATHEMATICS SUBJECT CLASSIFICATION

Primary 13H15; Secondary 13F20; 52B20

1. Introduction

In this paper, based on computations with the free softwares Germenes [14] and REDUCE [11], we give a conjecture that in the case of monomial ideals links the generalized multiplicities defined algebraically in [3] with volumes derived from the Newton polyhedra of the ideals, thus extending a result of Teissier [17].

In 1988, Teissier [17, p. 131] proved that for an $\mathfrak{m}$-primary monomial ideal I of a local ring A the Samuel multiplicity is equal to the normalized volume of the complement of the Newton polyhedron of the ideal I. In 1999, Guibert [9] generalized Teissier's result. Precisely, Guibert defines the local Segre class of an ideal generated by a set of germs of holomorphic functions and, under a non-degeneracy condition, he describes such a class by Minkowski mixed volumes of polytopes. As a special case he obtains that for a certain class of monomial ideals the local Segre class is a normalized volume of the simplex generated by the origin and the vertices of the Newton polyhedron, see [9, 4.2]. By [4], the local Segre class is the so called j-multiplicity of the ideal. In 2013, Jeffries and Montaño [13] gave a different proof that the j-multiplicity of a monomial ideal is the normalized volume of the pyramid of the ideal.

The j-multiplicity of an ideal is different from zero if and only if its analytic spread is maximal, that is, equal to the Krull-dimension d of A. A result of Bivià-Ausina [6] states that the analytic spread diminished by one is the maximum of the dimensions of compact faces of the Newton polyhedron of I.

According to [3] the j-multiplicity is the first coordinate of the generalized Samuel multiplicity vector $c(I) = (c_0(I), \ldots, c_d(I))$. Here we present and illustrate a conjecture which expresses the other components of $c(I)$ in terms of the Newton polyhedron of I. Our conjecture holds for $c_0(I)$ by the known result of Guibert and of Jeffries and Montaño, and we shall prove it here for $c_{d-1}(I)$.

2. Generalized Samuel multiplicities

This section is a quick review of a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we introduced in [3].

Let A be a d-dimensional Noetherian local ring $(A, \mathfrak{m})$ with unique maximal ideal $\mathfrak{m}$ or a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is a field and $\mathfrak{m} = (A_1)A$ is the unique homogeneous maximal ideal of A. Let $I \subset A$ be an arbitrary ideal (not necessarily $\mathfrak{m}$-primary).

In order to define the generalized Samuel multiplicity $c(I)$, consider $G_I(A) := \bigoplus_{j \geq 0} I^j/I^{j+1}$, the associated graded ring of A with respect to I and the bigraded ring

$$T = \bigoplus_{i,j \geq 0} T_{ij} = G_{\mathfrak{m}}(G_I(A)) = \bigoplus_{i,j \geq 0} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}},$$

where $T_{00} = A/\mathfrak{m} = K$ is a field.

Let $H^{(0,0)}(i,j) := \dim_K T_{ij}$ be the Hilbert function of the bigraded ring T and let

$$H^{(1,1)}(i,j) := \sum_{q=0}^{j} \sum_{p=0}^{i} H^{(0,0)}(p,q)$$

be its twofold sum transform. For both $i, j \gg 1$ this function becomes a polynomial in (i, j), which can be written in the form

$$\sum_{k+l \leq d} a_{k,l}^{(1,1)} \binom{i+k}{k} \binom{j+l}{l}.$$

Following [3] define the *generalized Samuel multiplicity* to be the vector

CONTACT Mirella Manaresi mirella.manaresi@unibo.it Dipartimento di Matematica, Università di Bologna, Bologna, 40126 Italy.
Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uexm.

$$\left(a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \dots, a_{d,0}^{(1,1)}\right) =: \left(c_0(T), c_1(T), \dots, c_d(T)\right) =: c(T)$$
$$=: (c_0(I), c_1(I), \dots, c_d(I)) =: c(I) \ .$$

The first coefficient $c_0(I)$ plays an important role as an intersection number and was introduced in [2]. It is called the *j-multiplicity* $j(I) := c_0(I)$. The last coefficient $c_d(I)$ is zero if $\dim(A/I) < d = \dim(A)$, see [3, Proposition 2.3]. In particular, if $I \neq 0$ is a monomial ideal in a polynomial ring, then $c_d(I) = 0$.

The generalized Samuel multiplicities depend only on the highest dimensional components of T, see [18] or [3, Proposition 1.2]:

Proposition 1. *Let Q_i, $i = 1, \dots, q$, be the highest dimensional primary ideals in a minimal primary decomposition of the zero ideal of the bigraded ring $T = G_{\mathfrak{m}}(G_I(R))$ and P_i their associated prime ideals. Then there is an equality of vectors*

$$c(I) = c(T) = \sum_{i=1}^{q} \operatorname{length}(T_{P_i}) \cdot c(T/P_i) = \sum_{i=1}^{q} c(T/Q_i) \ .$$

By analogy with the application of $c(I)$ to intersection theory, we shall call $c_k(T/Q_i) \neq 0$ a *movable contribution* to $c_k(I)$ if there is an integer $\ell > k$ such that $c_\ell(T/Q_i) \neq 0$.

3. A conjecture and some results

Let I be an ideal in $R = K[x_1, \dots, x_d] = K[\mathbf{x}]$ (K a field) minimally generated by the monomials

$$\mathbf{x}^{v_1} := {x_1}^{v_1(1)} \cdots {x_d}^{v_1(d)}, \dots, \mathbf{x}^{v_r} := {x_1}^{v_r(1)} \cdots {x_d}^{v_r(d)},$$

that is, $v_1 = (v_1(1), \dots, v_1(d)), \dots, v_r = (v_r(1), \dots, v_r(d))$ are the points of $\mathbb{Z}^d_{\geq 0}$ corresponding to the exponents of the generators of I.

The *Newton polyhedron* $\mathrm{NP}(I)$ of I is defined as the convex hull of $\{v \in \mathbb{Z}^d_{\geq 0} | \mathbf{x}^v\} \in I\}$ in $\mathbb{R}^d$, that is,

$$\mathrm{NP}(I) := \operatorname{conv}(\{v \in \mathbb{Z}^d_{\geq 0} | {x_1}^{v(1)} \cdots {x_d}^{v(d)} \in I\})$$
$$= \operatorname{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}^d_{\geq 0},$$

where + denotes the Minkowski sum (for the equality see [15, Lemma 4.3]). It is well-known (see, for example, [12, Proposition 1.4.6] that the set of integer lattice points of $\mathrm{NP}(I)$ equals the exponent set of the integral closure of I, which is again a monomial ideal. Our conjectures involve both $\mathrm{NP}(I)$ and the generalized Samuel multiplicities $c(I)$, which are also known to be invariant up to the integral closure of I, see [7, Proposition 2.7].

A hyperplane

$$H = \{v \in \mathbb{R}^d \mid \langle v, a \rangle = b\} \quad (\text{with } a \in \mathbb{R}^d_{\geq 0}, b \in \mathbb{R})$$

is called a *supporting hyperplane* of the Newton polyhedron $\mathrm{NP}(I)$ if

$$\mathrm{NP}(I) \subset H^+ = \{v \in \mathbb{R}^d \mid \langle v, a \rangle \geq b\} \text{ and } \mathrm{NP}(I) \cap H \neq \emptyset.$$

A subset $F \subset \mathrm{NP}(I)$ is called a *proper face* of $\mathrm{NP}(I)$ if there exists a supporting hyperplane H of $\mathrm{NP}(I)$ such that $F = \mathrm{NP}(I) \cap H$. The boundary of $\mathrm{NP}(I)$ is a set of faces of dimension $d - 1$, called *facets* of $\mathrm{NP}(I)$, some of them may be compact.

The zero-dimensional faces are called *vertices* or *extreme points* of $\mathrm{NP}(I)$. We shall denote the set of vertices by $\mathit{vert}(I)$. Note that the monomials corresponding to the points in $\mathit{vert}(I)$ are part of the set of minimal generators of I, so by renumbering we will assume that

$$\mathit{vert}(I) = \{v_1, \dots, v_s\} \text{ with some } s \leq r,$$

hence

$$\mathrm{NP}(I) = \operatorname{conv}(\{v_1, \dots, v_r\}) + \mathbb{R}^d_{\geq 0} = \operatorname{conv}(\{v_1, \dots, v_s\}) + \mathbb{R}^d_{\geq 0}.$$

The monomials corresponding to the points in $\mathit{vert}(I)$ generate the unique minimal monomial reduction ideal of I, see [16, Proposition 2.1].

Any face F can be described using its vertices and infinite-directions. Let e_j denote the unit vector with non-zero jth component, let H be a supporting hyperplane such that $F = \mathrm{NP}(I) \cap H$ and let a be a normal vector to H. We call the coordinate direction e_j an *infinite-direction* of F if the jth component $a(j)$ of a is zero. If $v_{i_1}, \dots, v_{i_s}$ are the vertices of F, then

$$F = \operatorname{conv}(\{v_{i_1}, \dots, v_{i_s}\}) + \sum_{j:a(j)=0} \mathbb{R}_{\geq 0} \ e_j.$$

Often we shall write simply $v_{i_1} \dots v_{i_s}$ instead of $\operatorname{conv}(\{v_{i_1}, \dots, v_{i_s}\})$. Of course, the compact or bounded faces are precisely those that do not have infinite-directions e_j.

By the Minkowski-Weyl Theorem for convex polyhedra, there are uniquely determined finitely many closed half spaces

$$H_i^+ = \{v \in \mathbb{R}^d \mid \langle v, a_i \rangle \geq b_i\} (\text{with } a_i \in \mathbb{Z}^d_{\geq 0}, b_i \in \mathbb{Z}_{\geq 0}),$$
$$i = 1, \dots, t,$$

such that

$$\mathrm{NP}(I) = H_1^+ \cap \dots \cap H_t^+.$$

Then $F_i := H_i \cap \mathrm{NP}(I)$, $i = 1, \dots, t$, are the facets of $\mathrm{NP}(I)$. We will assume that $H_1, \dots, H_u$ are the hyperplanes corresponding to the unbounded facets and that $H_{u+1}, \dots, H_t$ are those corresponding to the compact facets.

To each bounded facet $F = \operatorname{conv}(\{v_{i1}, \dots, v_{is}\})$ of $\mathrm{NP}(I)$ we associate the polytope (or pyramid)

$$\hat{F} := \operatorname{conv}(0, F) = \operatorname{conv}(\{0, v_{i1}, \dots, v_{is}\})$$

and denote by $\mathrm{vol}_d(\hat{F})$ its d-dimensional volume and by

$$\mathrm{Vol}_d(\hat{F}) := d!\mathrm{vol}_d(\hat{F})$$

its *normalized volume*.

The normalized volumes of pyramids over projections of bounded faces of $\mathrm{NP}(I)$ play a crucial role in our conjectures. Since it was known that $c_0(I)$ equals the sum of the normalizes d-dimensional volumes of the pyramids over the bounded facets of $N(I)$, our guess was that $c_{d-(k+1)}(I)$ should be a sum of $(k+1)$-dimensional volumes coming from the bounded faces F^k of dimension k. We succeeded in proving this for $k = 0$ by projecting the vertices of $\mathrm{NP}(I)$ on the coordinate axes, see Theorem 2. Thus, in order to obtain $c_{d-(k+1)}(I)$, we tried to sum up the normalized $(k+1)$-dimensional volumes of the pyramids over the projections of the bounded faces F^k of $\mathrm{NP}(I)$ on coordinate $(k+1)$-planes. This did not work well, and by [5, Section 2.2] we realized that one should consider only projections along the infinite-directions of facets F^{d-1} that contain F^k. Then we refined our guess by computing many examples. To formulate our conjectures, we proceed as follows.

We call a facet $F \subset \mathrm{NP}(I)$ an *h-unbounded facet* if the normal vector a to its supporting hyperplane has at least $h>0$ coordinates $a(j)$ which are zero, that is, if the facet has at least h infinite-directions e_j.

Let $\mathcal{F}(k)$ be the set of all $(d-(k+1))$-unbounded facets containing at least one k-dimensional compact face F^k, $0 \le k \le d-2$. We define $\mathcal{F}(d-1)$ to be the set of all compact or bounded facets of $\mathrm{NP}(I)$.

If $F^{d-1} \in \mathcal{F}(k)$ and F^k is a k-dimensional compact face contained in F^{d-1}, then we associate to the pair (F^k, F^{d-1}) a $(k+1)$-dimensional normalized volume $\mathrm{Vol}(F^k, F^{d-1})$ as follows. Let $L \subset \{1,\dots,d\}$ be such that $\{e_\ell : \ell \in L\}$ is the set of all infinite-directions of F^{d-1} and let $M_1,\dots,M_q$ be all subsets of L consisting of exactly $d-(k+1)$ infinite-directions. For $i=1,\dots,q$, let

$$\pi_i : \mathbb{R}^d \to \sum_{1\le j\le d,\, j\notin M_i} \mathbb{R}e_j \cong \mathbb{R}^{k+1}$$

be the orthogonal projection which removes from $v \in \mathbb{R}^d$ the coordinates $v(m)$ with $m \in M_i$. Then $\pi_i(F^k)$ is a polytope of dimension at most k. By renumbering, assume that $\dim(\pi_i(F^k)) = k$ for $1 \le i \le p$ and $\dim(\pi_i(F^k)) < k$ for $p+1 \le i \le q$.

The volume associated to the pair (F^k, F^{d-1}) is

$$\mathrm{Vol}(F^k, F^{d-1}) := \min_{1\le i\le p} \mathrm{Vol}_{k+1}(\mathrm{conv}(\{0, \pi_i(F^k)\})). \tag{3–1}$$

Conjecture 1. *For each $k=0,\dots,d-1$, the generalized Samuel multiplicity of a monomial ideal I is*

$$c_{d-(k+1)}(I) = \sum_{F^{d-1}\in\mathcal{F}(k)} \min_{F^k}\{\mathrm{Vol}(F^k, F^{d-1})\}, \tag{3–2}$$

where the minimum is taken over all compact faces F^k of $\mathrm{NP}(I)$ that are contained in the facet F^{d-1}.

In the extremal cases $k=d-1$ and $k=0$ the formula (3.2) can be simplified.

The case $\mathbf{k=d-1}$. Since $\mathcal{F}(d-1)$ is the set of compact facets of $\mathrm{NP}(I), F^{d-1} \in \mathcal{F}(d-1)$ does not have infinite-directions. Hence L above is empty, $q=1$, $M_1=\emptyset$ and π_1 is the identity map on $\mathbb{R}^d$. The only $(d-1)$-dimensional compact face contained in F^{d-1} is F^{d-1} itself and

$$\begin{aligned}\mathrm{Vol}(F^{d-1}, F^{d-1}) &:= \mathrm{Vol}_d(\mathrm{conv}(\{0, \pi_1(F^{d-1})\})) \\ &= \mathrm{Vol}_d(\mathrm{conv}(\{0, F^{d-1}\})).\end{aligned}$$

Then the formula (3–2) reads

$$c_0(I) = \sum_{F^{d-1}\in\mathcal{F}(d-1)} \mathrm{Vol}_d(\mathrm{conv}(\{0, F^{d-1}\})). \tag{3–3}$$

The case $\mathbf{k=0}$. At first we describe $\mathcal{F}(0)$.

Proposition 2. *Let $I \subset K[x_1,\dots,x_d] = K[\mathbf{x}]$ be an ideal generated by the monomials $\mathbf{x}^{v_1},\dots,\mathbf{x}^{v_r}$. For $j=1,\dots,d$, set $m_j := \min\{v_1(j),\dots,v_r(j)\}$ and*

$$F_j := \mathrm{conv}(\{v \in \mathrm{vert}(I) | v(j) = m_j\}) + \sum_{1\le i\le d,\, i\ne j} \mathbb{R}_{\ge 0}\, e_i.$$

Then $\mathcal{F}(0) = \{F_1,\dots,F_d\}$.

Proof. Since each $v \in \mathrm{NP}(I)$ is the sum of a convex combination of the vertices $v_1,\dots,v_s$ of $\mathrm{NP}(I)$ and of some $w \in \mathbb{R}^d_{\ge 0}$, we have

$$v(j) \ge \min\{v_1(j),\dots,v_s(j)\} + w(j) \ge \min\{v_1(j),\dots,v_s(j)\},$$

hence

$$m_j := \min\{v_1(j),\dots,v_r(j)\} = \min\{v_1(j),\dots,v_s(j)\} = \min_{v\in\mathrm{NP}(I)}\{v(j)\}.$$

It follows that $F_1,\dots,F_d$ are precisely the $(d-1)$-unbounded facets of $\mathrm{NP}(I)$, that is, $\mathcal{F}(0) = \{F_1,\dots,F_d\}$. □

By the preceding proposition, for $k=0$ the formula (3–2) reads

$$c_{d-1}(I) = \sum_{j=1}^{d} \min_v\{\mathrm{Vol}(v, F_j)\}, \tag{3–4}$$

where the minimum is taken over all vertices v of $\mathrm{NP}(I)$ that are contained in the facet F_j. In order to compute $\mathrm{Vol}(v, F_j)$, note that each F_j has $d-1$ infinite-directions, more precisely, $L = \{1,\dots,\hat{j},\dots,d\}$, hence $q=1$, $L = M_1$, and $\pi_1 : \mathbb{R}^d \to \mathbb{R}e_j$ sends each $v \in F_j$ to its jth coordinate $v(j) = m_j$. It follows that $\mathrm{Vol}(v, F_j) = \mathrm{Vol}_1(\mathrm{conv}(\{0, \pi_1(v)\})) = v(j) = m_j$ for each vertex v of $\mathrm{NP}(I)$ that is contained in F_j, and no minimum has to be taken in (3–4). Thus the formula of Conjecture 1 becomes

$$c_{d-1}(I) = m_1 + \cdots + m_d. \tag{3–5}$$

Conjecture 2. *With the notation of Proposition 1, for each $k=0,\dots,d-1$, there is a one-to-one correspondence between the non-zero $c_{d-(k+1)}(T/Q_i)$ and the non-zero summands $\min_{F^k}\{\mathrm{Vol}(F^k, F^{d-1})\}$ in the formula of Conjecture 1 such that the corresponding numbers are equal.*

In particular, the number of compact facets of $\mathrm{NP}(I)$ is equal to the number of d-dimensional associated prime ideals of T that contain $\mathfrak{m} = (x_1,\dots,x_d)R$.

Moreover, if $\mathrm{Vol}(F^k, F^{d-1}) \ne 0$ contributes to $c_{d-(k+1)}(I)$ and can be obtained by more than one projection p_i then it is a movable contribution.

Note that in general, if K is algebraically closed and

$$T = G_{\mathfrak{m}}(G_I(R)) \cong K[x_1,\dots,x_d, y_1,\dots,y_r]/\mathfrak{n},$$

then the bigraded ideal $\mathfrak{n}$ is a binomial but not a monomial ideal, see [8, Corollary 1.9].

Our conjectures are confirmed by many examples, but so far we do not have a proof except for Conjecture 1 in the extremal cases $k=0$ (formula (3–3)) and $k=d-1$ (formula (3–5)), as it is stated in the following two theorems.

Theorem 1. (Jeffries and Montaño [13, Theorem 3.2]). *If $I \subset K[x_1,\dots,x_d]$ is a monomial ideal and $F_{r+1},\dots,F_t$*

are the compact facets of the Newton polyhedron NP(I), *then*

$$c_0(I) = \sum_{i=r+1}^{t} d!\mathrm{vol}(\hat{F}_i) = \sum_{i=r+1}^{t} \mathrm{Vol}(\hat{F}_i).$$

Theorem 2. *Let I be a monomial ideal in $R = K[x_1, \ldots, x_d] = K[\mathbf{x}]$ generated by $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_r}$ and $m_j = \min\{v_1(j), \ldots, v_r(j)\}, j = 1, \ldots, d$. Then*

$$c_{d-1}(I) = m_1 + \cdots + m_d.$$

Proof. By [3, Proposition 2.3], $c_{d-1}(I) \neq 0$ if and only if $\dim R/I = d-1$. If $\dim R/I < d-1$, then none of the variables x_j appears in all monomials generating I, hence $m_j = 0$ for all j, and the result is true. If $\dim R/I = d-1$, then again by [3, loc. cit.],

$$c_{d-1}(I) = \sum_P e(IR_P) \cdot e(R/P),$$

where P runs through all $(d-1)$-dimensional associated prime ideals of R/I, that is, prime ideals of the form (x_j) for some j. Therefore $IR_P = (x_j^{m_j})R_P$ and $e(IR_P) = m_j$. The $(d-1)$-dimensional part of the primary decomposition of I is $(x_1^{m_1}) \cap (x_2^{m_2}) \cap \cdots \cap (x_d^{m_d})$, which is of degree $m_1 + \cdots + m_d$. □

Corollary 3. *Let $I \subset K[x_1, x_2]$ be a monomial ideal generated by $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_r}$. We assume that $v_1, \ldots, v_s$ $(1 \leq s \leq r)$ are the vertices of the Newton polyhedron* NP(I) *numbered such that $v_1(1) > \cdots > v_s(1)$, hence $v_1(2) < \cdots < v_s(2)$. Then the set of unbounded facets of* NP(I) *is*

$$\mathcal{F}(0) = \{F_1 = v_s + \mathbb{R}_{\geq 0}\ e_2, F_2 = v_1 + \mathbb{R}_{\geq 0}\ e_1\},$$

the set of bounded facets of NP(I) *is*

$$\mathcal{F}(1) = \{v_1v_2, v_2v_3, \ldots, v_{s-1}v_s\} (= \emptyset \text{ if } s = 1),$$

and the generalized Samuel multiplicities are

$$c_0(I) = \det\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \cdots + \det\begin{pmatrix} v_{s-1} \\ v_s \end{pmatrix},$$
$$c_1(I) = v_1(2) + v_s(1), c_2 = 0.$$

Proof. The corollary follows immediately from Proposition 2, Theorem 1, Theorem 2 and [3, Proposition 2.3 (i)]. □

4. Examples

We illustrate the theorems and the conjecture by examples of monomial ideals $I \subset R = K[x_1, \ldots, x_d]$, K an arbitrary field. We set $\mathfrak{m} := (x_1, \ldots, x_d)R$ and $T := G_{\mathfrak{m}}(G_I(R))$. All the examples will show a close relation between the summands in the formula of Conjecture 1 and the highest dimensional primary components of T and confirm Conjecture 2.

In our first two examples we consider monomial ideals in polynomial rings of dimension two. Note that Conjecture 1 holds in this case (formulas (3–3) and (3–5)), see the Theorems 1 and 2 and Corollary 3.

Example 1 (Figure 1). We begin with the simplest case of a monomial ideal generated by one monomial in two variables in order to illustrate Corollary 3 if the Newton polyhedron has only one vertex.

Let $I = (x^3y^2) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (0, 5, 0) = 2 \cdot (0, 1, 0) + 3 \cdot (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron NP(I) has only one vertex $v = (3, 2)$ and two (unbounded) facets $F_1 = v + \mathbb{R}_{\geq 0}e_2$ and $F_2 = v + \mathbb{R}_{\geq 0}e_1$ (see Figure 1), hence $\mathcal{F}(1) = \emptyset$ and $c_0(I) = 0$. We have $\mathcal{F}(0) = \{F_1, F_2\}$ and $\mathrm{Vol}(v, F_1) = 3$ and $\mathrm{Vol}(v, F_2) = 2$, hence $c_1(I) = 5$.

Example 2 (Figure 2). This example is to illustrate Corollary 3 in the case of two and more vertices. The vertices are numbered as in the corollary.

Let $I = (x^6y, x^4y^2, x^2y^5, x^3y^4) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I))$$
$$= (24, 3, 0) = 16 \cdot (1, 0, 0) + 8 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron NP(I) has three vertices $v_1 = (6, 1), v_2 = (4, 2), v_3 = (2, 5)$, two unbounded facets $F_1 = v_3 + \mathbb{R}_{\geq 0}e_2$, $F_2 = v_1 + \mathbb{R}_{\geq 0}e_1$ and two bounded facets: the line segments $F_3 = \mathrm{conv}(v_1, v_2)$, $F_4 = \mathrm{conv}(v_2, v_3)$ (see Figure 2), hence $\mathcal{F}(1) = \{F_3, F_4\}$ and

$$c_0(I) = \mathrm{Vol}(\mathrm{conv}(0, F_3)) + \mathrm{Vol}(\mathrm{conv}(0, F_4))$$
$$= \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 8 + 16.$$

We have $\mathcal{F}(0) = \{F_1, F_2\}$ and

$$c_1(I) = \mathrm{Vol}(v_3, F_1) + \mathrm{Vol}(v_1, F_2) = 1 + 2 = 3.$$

Example 3 (Figures 3–5). The purpose of this example is twofold. It shows that there can be compact faces of NP(I) that do not contribute to the generalized Samuel multiplicity $c(I)$. Furthermore it aims to discuss a movable contribution (to $c_1(I)$). In this example, because of the movable contribution, the number of the highest dimensional components of T is one less than the number of summands in the conjectured formula (3–2).

Let $I = (x^2y, x^2z, xy^2, xz^2) \subset K[x, y, z]$. By a computer computation (using [1]) we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I))$$
$$= (9, 3, 1, 0) = 3 \cdot (3, 0, 0, 0) + (0, 1, 0, 0) + (0, 2, 1, 0),$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 2 in the last vector is a

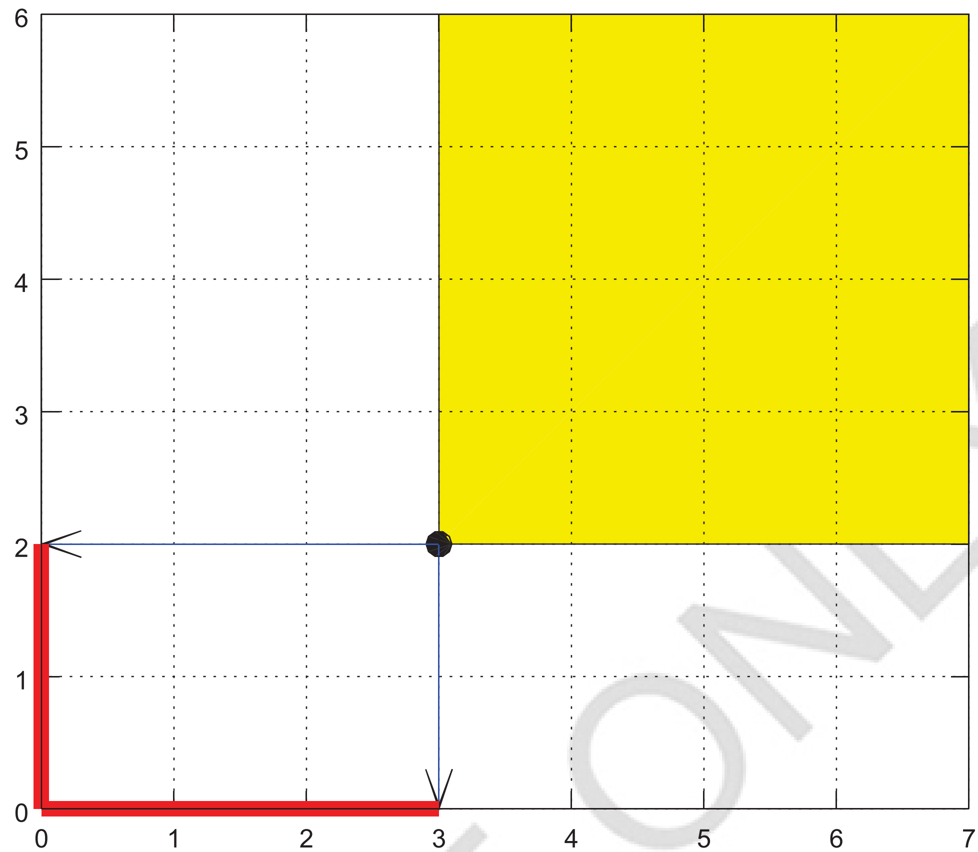

Figure 1. Projection along the infinite-directions of the facets gives $c_1(I)$, which is the red distance.

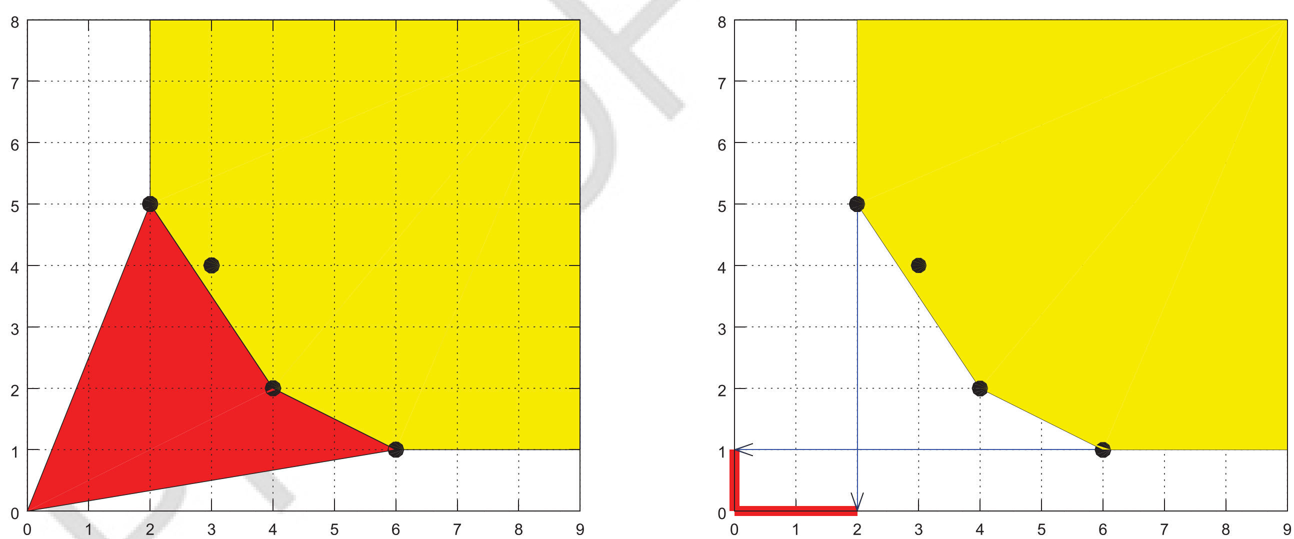

Figure 2. The red area is $c_0(I)/2$, the red distance $c_1(I)$.

movable contribution to $c_1(I)$. This can be read off also from the Newton polyhedron $\mathrm{NP}(I)$, see Figure 5.

According to the program Germenes [14], the compact faces of $\mathrm{NP}(I)$ are the vertices $v_1 = (2,1,0), v_2 = (2,0,1), v_3 = (1,2,0), v_4 = (1,0,2)$, the line segments $v_1v_2, v_1v_3, v_2v_4, v_3v_4$ and the quadrilateral facet $v_1v_2v_4v_3$. The unbounded facets are

$$F_1 = v_3v_4 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3, \quad F_2 = v_2v_4 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_3,$$
$$F_3 = v_1v_3 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_2, \quad F_4 = v_1v_2 + \mathbb{R}_{\geq 0}\ e_1.$$

We observe that the set of bounded facets is $\mathcal{F}(2) = \{v_1v_2v_4v_3\}$ and

$$c_0(I) = \mathrm{Vol}(\mathrm{conv}(0, v_1, v_2, v_4, v_3)) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$
$$= 3 + 6 = 9,$$

see Figure 4.

The set of 1-unbounded facets that contain a compact one-dimensional face is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\mathrm{Vol}(v_1v_2, F_4) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

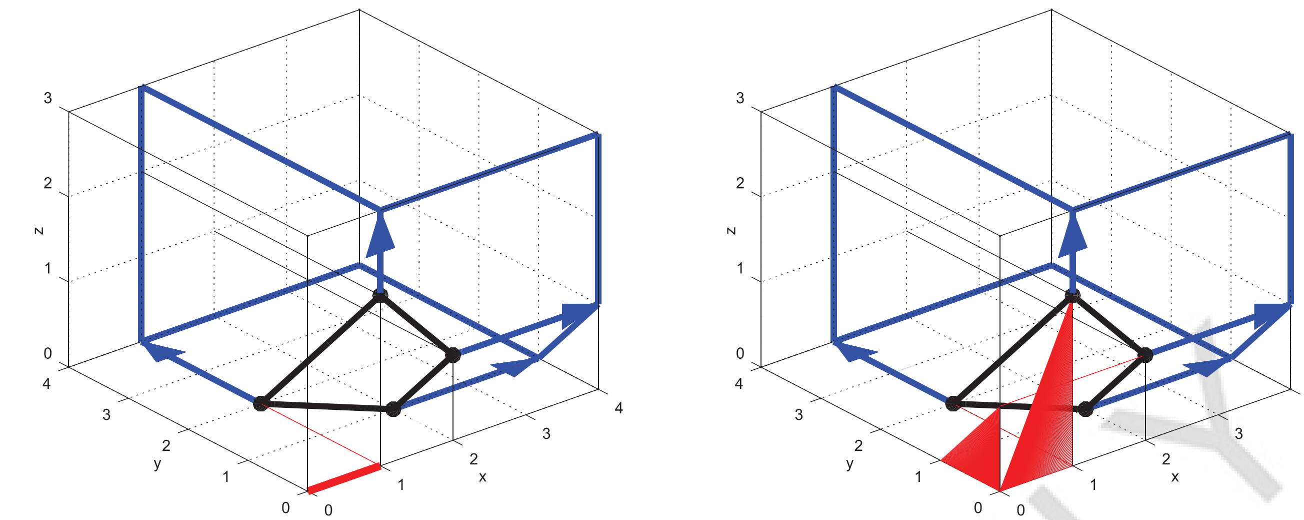

Figure 3. Infinite-directions (blue arrows) of the unbounded facets, $c_2(I)$ (red distance) and $c_1(I)/2$ (red area).

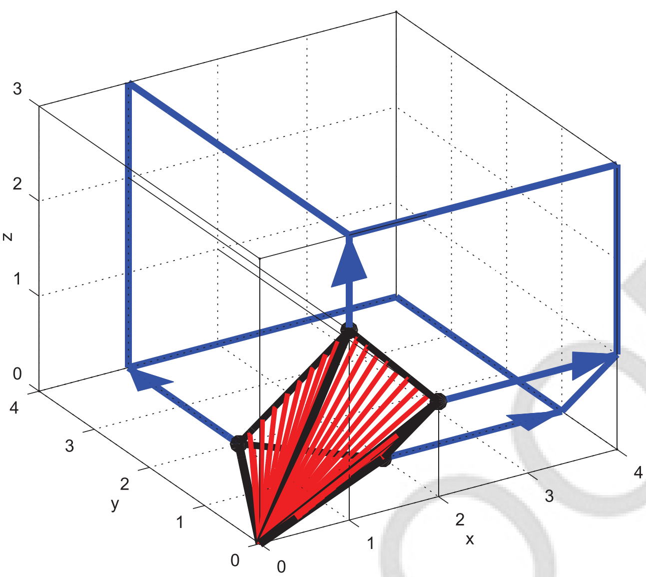

Figure 4. The volume of the red pyramid is $c_0/6$.

$$\mathrm{Vol}(v_1v_3, F_3) = \min\left\{\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}\right\} = 0,$$

$$\mathrm{Vol}(v_2v_4, F_2) = \min\left\{\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}\right\} = 0,$$

$$\mathrm{Vol}(v_3v_4, F_1) = \min\left\{\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\right\} = 2$$

(the last minimum is given by two different projections and is a movable contribution, see Figure 5), hence

$$\begin{aligned} c_1(I) &= \mathrm{Vol}(v_1v_2, F_4) + \mathrm{Vol}(v_1v_3, F_3) + \mathrm{Vol}(v_2v_4, F_2) \\ &\quad + \mathrm{Vol}(v_3v_4, F_1) \\ &= 1 + 0 + 0 + 2 = 3. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$. We have $\mathrm{Vol}(v_3, F_1) = 1, \mathrm{Vol}(v_4, F_1) = 1$, $\mathrm{Vol}(v_4, F_2) = 0, \mathrm{Vol}(v_1, F_3) = 0, \mathrm{Vol}(v_3, F_3) = 0$, hence

$$\begin{aligned} c_2(I) = \min\{\mathrm{Vol}(v_3, F_1), \mathrm{Vol}(v_4, F_1)\} + \mathrm{Vol}(v_4, F_2) + \\ + \min\{\mathrm{Vol}(v_1, F_3), \mathrm{Vol}(v_3, F_3)\} = 1 + 0 + 0 = 1. \end{aligned}$$

Example 4 (Figure 6). Here we give a monomial ideal I such that its Newton polyhedron has compact edges that do not lie on any 1-unbounded facet and therefore, according to Conjecture 1, have not to be taken into account in order to compute $c_1(I)$.

Let $I = (xy^4z^5, x^2y^5z^2, xy^5z^3, x^5yz^2, x^2yz^5, x^5y^2z) \subset K[x, y, z]$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) = (168, 26, 3, 0) = \\ &= 19 \cdot (1,0,0,0) + 103 \cdot (1,0,0,0) + 22 \cdot (1,0,0,0) + \\ &\quad + 24 \cdot (1,0,0,0) + 7 \cdot (0,1,0,0) + (0,3,1,0) + \\ &\quad + 8 \cdot (0,1,0,0) + (0,1,0,0) + 4 \cdot (0,1,0,0) + \\ &\quad + 3 \cdot (0,1,0,0) + (0,0,1,0) + (0,0,1,0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 3 in the sixth vector is a movable contribution to $c_1(I)$.

The program Germenes [14] gives the following description of the Newton polyhedron $\mathrm{NP}(I)$. The compact faces of $\mathrm{NP}(I)$ are the 6 vertices $v_1 = (1,4,5)$, $v_2 = (2,5,2), v_3 = (1,5,3), v_4 = (5,1,2), v_5 = (2,1,5), v_6 = (5,2,1)$, the 9 line segments $v_4v_6, v_2v_6, v_5v_6, v_4v_5, v_3v_6, v_3v_2, v_3v_5, v_1v_5, v_1v_3$ and the 4 triangles (bounded facets) $v_4v_5v_6, v_2v_3v_6, v_3v_5v_6, v_1v_3v_5$. There are 7 unbounded facets:

$$\begin{aligned} &F_1 = v_1v_3 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3, \quad F_2 = v_4v_5 + \mathbb{R}_{\geq 0}\ e_1 \\ &\qquad + \mathbb{R}_{\geq 0}\ e_3, \\ &F_3 = v_6 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_2, \quad F_4 = v_1v_5 + \mathbb{R}_{\geq 0}\ e_3, \\ &F_5 = v_2v_3 + \mathbb{R}_{\geq 0}\ e_2, \quad F_6 = v_2v_6 + \mathbb{R}_{\geq 0}\ e_2, \\ &F_7 = v_4v_6 + \mathbb{R}_{\geq 0}\ e_1. \end{aligned}$$

From the set of bounded facets $\mathcal{F}(2) = \{v_4v_5v_6, v_2v_3v_6, v_3v_5v_6, v_1v_3v_5\}$ we get

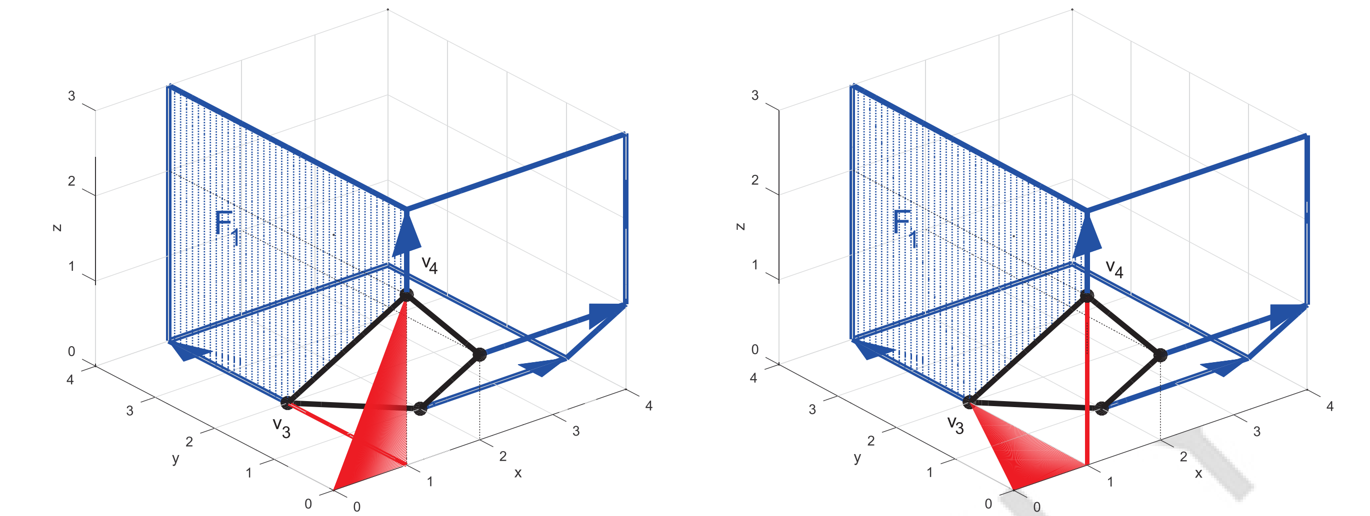

Figure 5. A movable contribution (to $c_1(I)/2$, red area) can be realized by at least two projections. Here the volume associated to the pair (v_3v_4, F_1) is obtained both by the projection of v_3v_4 along the y-axis and the z-axis, that is, along the two inifinite-directions of F_1.

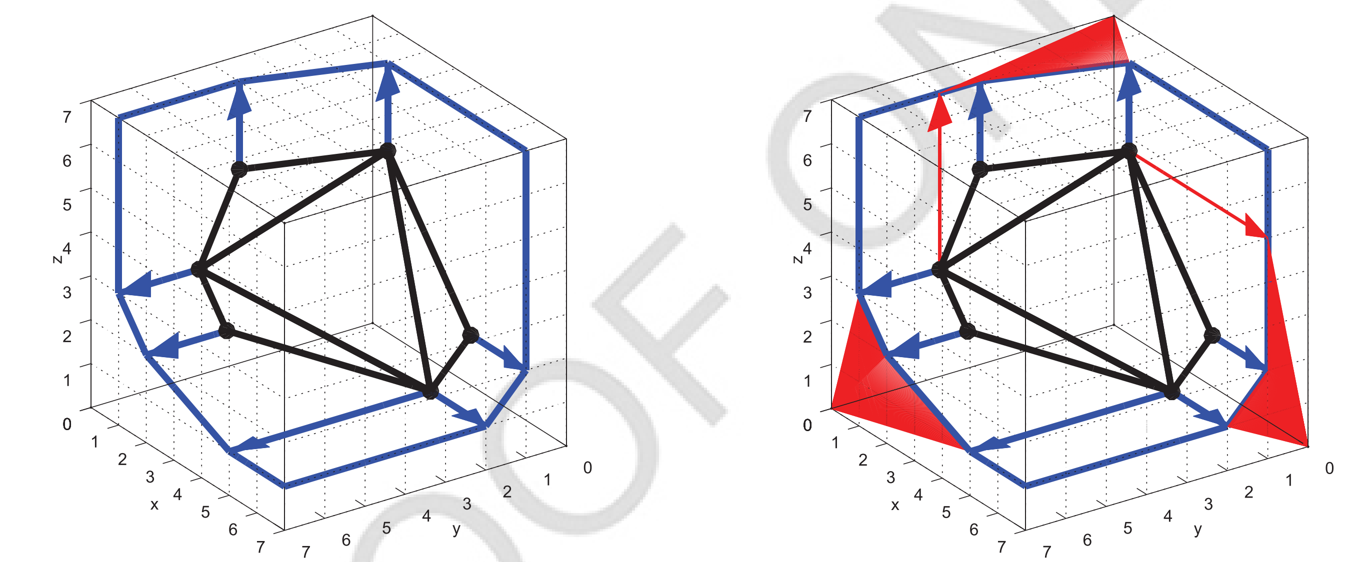

Figure 6. NP(I) with compact (black) and unbounded (blue) facets; projections of compact edges along infinite-directions (blue arrows) give $c_1(I)/2$ (red area).

$$
\begin{aligned}
c_0(I) &= \mathrm{Vol}(\mathrm{conv}(0, v_4, v_5, v_6)) + \mathrm{Vol}(\mathrm{conv}(0, v_2, v_3, v_6)) + \\
&\quad + \mathrm{Vol}(\mathrm{conv}(0, v_3, v_5, v_6)) + \mathrm{Vol}(\mathrm{conv}(0, v_1, v_3, v_5)) = \\
&= \begin{vmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 2 \\ 1 & 5 & 3 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = \\
&= 24 + 22 + 103 + 19 = 168.
\end{aligned}
$$

We have $\mathcal{F}(1) = \{F_1, F_2, F_4, F_5, F_6, F_7\}$ and

$$
\mathrm{Vol}(v_1v_3, F_1) = \min\left\{ \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 2\} = 1,
$$

$$
\mathrm{Vol}(v_4v_5, F_2) = \min\left\{ \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \right\} = \min\{3, 3\} = 3
$$

(here the minimum is attained twice, that is, by two different projections which indicates a movable contribution),

$$
\mathrm{Vol}(v_1v_5, F_4) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \mathrm{Vol}(v_2v_3, F_5) = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4,
$$

$$
\mathrm{Vol}(v_2v_6, F_6) = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 8, \quad \mathrm{Vol}(v_4v_6, F_7) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,
$$

hence

$$
\begin{aligned}
c_1(I) &= \mathrm{Vol}(v_1v_3, F_1) + \mathrm{Vol}(v_4v_5, F_2) + \mathrm{Vol}(v_1v_5, F_4) + \\
&\quad + \mathrm{Vol}(v_2v_3, F_5) + \mathrm{Vol}(v_2v_6, F_6) + \mathrm{Vol}(v_4v_6, F_7) = \\
&= 1 + 3 + 7 + 4 + 8 + 3 = 26.
\end{aligned}
$$

We observe that the compact 1-dimensional faces v_5v_6, v_3v_6, v_3v_5, that is, the edges of the big triangle $v_3v_5v_6$, do not contribute to $c_1(I)$ since they lie on no 1-unbounded facet. Moreover, as in the previous example, there is a movable contribution, namely $\mathrm{Vol}(v_4v_5, F_2) = 3$.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$, and we have $\mathrm{Vol}(v_1, F_1) = 1, \mathrm{Vol}(v_3, F_1) = 1, \mathrm{Vol}(v_4, F_2) = 1, \mathrm{Vol}(v_5, F_2) = 1, \mathrm{Vol}(v_6, F_3) = 1$, hence

$$
\begin{aligned}
c_2(I) &= \min\{\mathrm{Vol}(v_1, F_1), \mathrm{Vol}(v_3, F_1)\} \\
&\quad + \min\{\mathrm{Vol}(v_4, F_2), \mathrm{Vol}(v_5, F_2)\} + \\
&\quad + \mathrm{Vol}(v_6, F_3) = 1 + 1 + 1 = 3.
\end{aligned}
$$

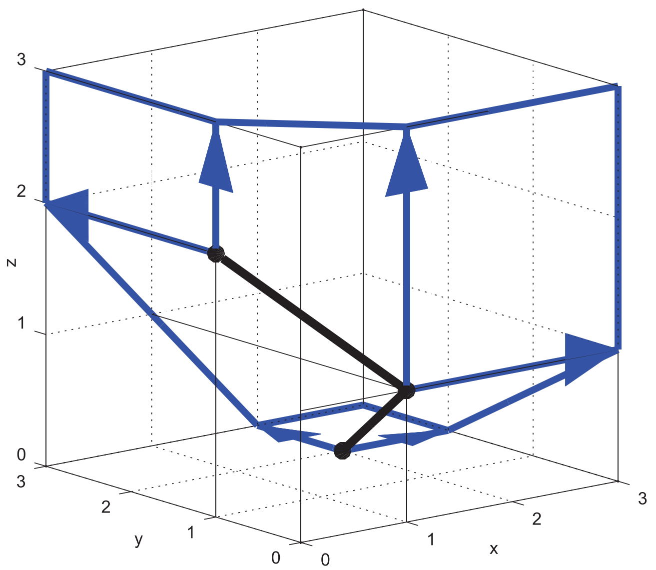

Figure 7. The triangle defined by 3 affinely independent vertices is not a compact facet.

Example 5 (Figure 7). The scope of this example is to give a monomial ideal in dimension three such that its Newton polyhedron has three affinely independent vertices v_1, v_2, v_3, but its local Segre class is zero and thus not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as erroneously stated in [9, 4.2].

Let $I = (xz, x^2y^2, yz^2) \subset K[x, y, z]$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) = (0, 7, 0, 0) = \\ &= 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + (0, 1, 0, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

A computation with the program Germenes [14] shows that the compact faces of $NP(I)$ are the vertices $v_1 = (1, 0, 1), v_2 = (2, 2, 0), v_3 = (0, 1, 2)$ and the line segments v_1v_2, v_1v_3. There are no compact facets, but 6 unbounded facets:

$$\begin{aligned} &F_1 = v_3 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3, \quad F_2 = v_1 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_3, \\ &F_3 = v_2 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_2, \quad F_4 = v_1v_2 + \mathbb{R}_{\geq 0}\ e_1, \\ &F_5 = v_1v_3 + \mathbb{R}_{\geq 0}\ e_3, \quad F_6 = v_1v_2v_3 + \mathbb{R}_{\geq 0}\ e_2. \end{aligned}$$

We observe that $\mathcal{F}(2) = \emptyset$, hence

$$c_0(I) = 0 \neq \mathrm{Vol}_3(\mathrm{conv}(\{0, v_1, v_2, v_3\})) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 6.$$

This means that v_1, v_2, v_3 are affinely independent, but the local Segre class is zero and not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as claimed in [9, 4.2]. The reason is that the triangle $v_1v_2v_3$ is not a compact facet of $NP(I)$.

The set of 1-unbounded facets which contain a compact 1-dimensional face is $\mathcal{F}(1) = \{F_4, F_5, F_6\}$, and we have

$$\begin{aligned} \mathrm{Vol}(v_1v_2, F_4) &= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2, \quad \mathrm{Vol}(v_1v_2, F_6) = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2, \\ \mathrm{Vol}(v_1v_3, F_5) &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \mathrm{Vol}(v_1v_3, F_6) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \end{aligned}$$

hence

$$\begin{aligned} c_1(I) &= \mathrm{Vol}(v_1v_2, F_4) + \mathrm{Vol}(v_1v_2, F_6) + \mathrm{Vol}(v_1v_3, F_5) \\ &\quad + \mathrm{Vol}(v_1v_3, F_6) \\ &= 2 + 2 + 1 + 2 = 7. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$ and we have

$$\mathrm{Vol}(v_3, F_1) = 0, \quad \mathrm{Vol}(v_1, F_2) = 0, \quad \mathrm{Vol}(v_2, F_3) = 0,$$

hence $c_2(I) = 0$.

Example 6. With the notation of the paragraph before Conjecture 1, this example is to have pairs (F^k, F^{d-1}) such that $\dim(\pi_i(F^k)) < k$ for some of the projections π_i. In our example we have $k = 1$, $d = 4$, and the pairs (v_1v_2, F_2) and (v_1v_2, F_3) have the desired property.

Let $d = 4$, $I = (x_1^3x_2x_3x_4, x_1x_2x_3x_4^2)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I)) = (0, 0, 7, 4, 0) = \\ &= 5 \cdot (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + \\ &\quad + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The compact faces of the Newton polyhedron $NP(I)$ are the vertices $v_1 = (3, 1, 1, 1), v_2 = (1, 1, 1, 2)$ and the line segment v_1v_2. There are no compact facets, but 5 unbounded facets:

$$\begin{aligned} F_1 &= v_2 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3 + \mathbb{R}_{\geq 0}\ e_4, \\ F_2 &= v_1v_2 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_3 + \mathbb{R}_{\geq 0}\ e_4, \\ F_3 &= v_1v_2 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_4, \\ F_4 &= v_1 + \mathbb{R}_{\geq 0}\ e_1 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3, \\ F_5 &= v_1v_2 + \mathbb{R}_{\geq 0}\ e_2 + \mathbb{R}_{\geq 0}\ e_3. \end{aligned}$$

Obviously $\mathcal{F}(3) = \mathcal{F}(2) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(1) = \{F_2, F_3, F_5\}$ and

$$\begin{aligned} \mathrm{Vol}(v_1v_2, F_2) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \\ \mathrm{Vol}(v_1v_2, F_3) &= \min\left\{ \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right\} = \min\{2, 1\} = 1, \\ \mathrm{Vol}(v_1v_2, F_5) &= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5. \end{aligned}$$

We observe that in the computations of $\mathrm{Vol}(v_1v_2, F_2)$ and $\mathrm{Vol}(v_1v_2, F_3)$ the projection of the line segment v_1v_2 on the $\{x_2, x_3\}$-plane gives the point (1, 1) and must not be considered. We obtain

$$\begin{aligned} c_2(I) &= \mathrm{Vol}(v_1v_2, F_2) + \mathrm{Vol}(v_1v_2, F_3) + \mathrm{Vol}(v_1v_2, F_5) \\ &= 1 + 1 + 5 = 7. \end{aligned}$$

The set of 3-unbounded facets is $\mathcal{F}(0)=\{F_1,F_2,F_3,F_4\}$, and we have

$$\begin{array}{lll} \mathrm{Vol}(v_2,F_1)=1, & \mathrm{Vol}(v_1,F_2)=1, & \mathrm{Vol}(v_2,F_2)=1,\\ \mathrm{Vol}(v_1,F_3)=1, & \mathrm{Vol}(v_2,F_3)=1, & \mathrm{Vol}(v_1,F_4)=1, \end{array}$$

hence

$$\begin{aligned} c_3(I) &= \mathrm{Vol}(v_2,F_1)+\min\{\mathrm{Vol}(v_1,F_2),\mathrm{Vol}(v_2,F_2)\}+\\ &\quad +\min\{\mathrm{Vol}(v_1,F_3),\mathrm{Vol}(v_2,F_3)\}+\mathrm{Vol}(v_1,F_4)=\\ &= 1+1+1+1=4. \end{aligned}$$

Example 7. This example is to show that in the formula of Conjecture 1 with $k>0$ one needs to take the minimum over all compact faces F^k of $\mathrm{NP}(I)$ that are contained in the facet F^{d-1}. In our example we have $d=5$, $k=1$ and three compact 1-dimensional faces lie on the same 3-unbounded facet F_3 or F_4.

Let $d=5$, $I=(x_1^3x_2x_3x_4x_5, x_1x_2^2x_3x_4x_5, x_1x_2x_3x_4x_5^5)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I),c_1(I),c_2(I),c_3(I),c_4(I),c_5(I))=(0,0,26,6,5,0)=\\ &= 22\cdot(0,0,1,0,0,0)+2\cdot(0,0,1,0,0,0)+2\cdot(0,0,1,0,0,0)+\\ &\quad +2\cdot(0,0,0,1,0,0)+(0,0,0,1,0,0)+(0,0,0,1,0,0)+\\ &\quad +(0,0,0,1,0,0)+(0,0,0,1,0,0)+(0,0,0,0,1,0)+\\ &\quad +(0,0,0,0,1,0)+(0,0,0,0,1,0)+(0,0,0,0,1,0)+\\ &\quad +(0,0,0,0,1,0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The program Germenes [14] shows that the compact faces of $\mathrm{NP}(I)$ are the vertices $v_1=(3,1,1,1,1)$, $v_2=(1,2,1,1,1)$, $v_3=(1,1,1,1,5)$, the line segments v_1v_2, v_1v_3, v_2v_3 and the triangle $v_1v_2v_3$. The facets, all of them unbounded, are:

$$\begin{aligned} F_1 &= v_2v_3+\mathbb{R}_{\geq 0}\ e_2+\mathbb{R}_{\geq 0}\ e_3+\mathbb{R}_{\geq 0}\ e_4+\mathbb{R}_{\geq 0}\ e_5,\\ F_2 &= v_1v_3+\mathbb{R}_{\geq 0}\ e_1+\mathbb{R}_{\geq 0}\ e_3+\mathbb{R}_{\geq 0}\ e_4+\mathbb{R}_{\geq 0}\ e_5,\\ F_3 &= v_1v_2v_3+\mathbb{R}_{\geq 0}\ e_1+\mathbb{R}_{\geq 0}\ e_2+\mathbb{R}_{\geq 0}\ e_4+\mathbb{R}_{\geq 0}\ e_5,\\ F_4 &= v_1v_2v_3+\mathbb{R}_{\geq 0}\ e_1+\mathbb{R}_{\geq 0}\ e_2+\mathbb{R}_{\geq 0}\ e_3+\mathbb{R}_{\geq 0}\ e_5,\\ F_5 &= v_1v_2+\mathbb{R}_{\geq 0}\ e_1+\mathbb{R}_{\geq 0}\ e_2+\mathbb{R}_{\geq 0}\ e_3+\mathbb{R}_{\geq 0}\ e_4,\\ F_6 &= v_1v_2v_3+\mathbb{R}_{\geq 0}\ e_3+\mathbb{R}_{\geq 0}\ e_4. \end{aligned}$$

Obviously $\mathcal{F}(4)=\mathcal{F}(3)=\emptyset$, hence $c_0(I)=c_1(I)=0$. We have $\mathcal{F}(2)=\{F_3,F_4,F_6\}$ and

$$\begin{aligned} \mathrm{Vol}(v_1v_2v_3,F_3) &= \mathrm{Vol}(v_1v_2v_3,F_4)\\ &= \min\left\{\begin{vmatrix}3&1&1\\1&2&1\\1&1&1\end{vmatrix},\begin{vmatrix}1&1&1\\2&1&1\\1&1&5\end{vmatrix}\right\}=\\ &= \min\{2,4\}=2,\\ \mathrm{Vol}(v_1v_2v_3,F_6) &= \begin{vmatrix}3&1&1\\1&2&1\\1&1&5\end{vmatrix}=22. \end{aligned}$$

Observe that in the computations of $\mathrm{Vol}(v_1v_2v_3,F_3)$ and $\mathrm{Vol}(v_1v_2v_3,F_4)$ four projections of the compact triangle $v_1v_2v_3$ give a line segment and must not be considered. Summing up we get

$$\begin{aligned} c_2(I) &= \mathrm{Vol}(v_1v_2v_3,F_3)+\mathrm{Vol}(v_1v_2v_3,F_4)+\mathrm{Vol}(v_1v_2v_3,F_6)\\ &= 2+2+22=26. \end{aligned}$$

The set of 3-unbounded facets containing 1-dimensional compact faces is $\mathcal{F}(1)=\{F_1,F_2,F_3,F_4,F_5\}$, and we have

$$\begin{aligned} &\mathrm{Vol}(v_2v_3,F_1)=\min\left\{\begin{vmatrix}1&1\\1&2\end{vmatrix},\begin{vmatrix}1&1\\1&5\end{vmatrix}\right\}=\min\{1,4\}=1,\\ &\mathrm{Vol}(v_1v_3,F_2)=\min\left\{\begin{vmatrix}3&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&5\end{vmatrix}\right\}=\min\{2,4\}=2,\\ &\mathrm{Vol}(v_1v_2,F_3)=\mathrm{Vol}(v_1v_2,F_4)=\min\left\{\begin{vmatrix}3&1\\1&1\end{vmatrix},\begin{vmatrix}2&1\\1&1\end{vmatrix}\right\}\\ &\quad=\min\{2,1\}=1,\\ &\mathrm{Vol}(v_1v_3,F_3)=\mathrm{Vol}(v_1v_3,F_4)=\min\left\{\begin{vmatrix}3&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&5\end{vmatrix}\right\}\\ &\quad=\min\{2,4\}=2,\\ &\mathrm{Vol}(v_2v_3,F_3)=\mathrm{Vol}(v_2v_3,F_4)=\min\left\{\begin{vmatrix}2&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&5\end{vmatrix}\right\}\\ &\quad=\min\{1,4\}=1,\\ &\mathrm{Vol}(v_1v_2,F_5)=\min\left\{\begin{vmatrix}3&1\\1&1\end{vmatrix},\begin{vmatrix}2&1\\1&1\end{vmatrix}\right\}=\min\{2,1\}=1, \end{aligned}$$

hence

$$\begin{aligned} c_3(I) &= \mathrm{Vol}(v_2v_3,F_1)+\mathrm{Vol}(v_1v_3,F_2)+\\ &\quad +\min\{\mathrm{Vol}(v_1v_2,F_3),\mathrm{Vol}(v_1v_3,F_3),\mathrm{Vol}(v_2v_3,F_3)\}+\\ &\quad +\min\{\mathrm{Vol}(v_1v_2,F_4),\mathrm{Vol}(v_1v_3,F_4),\mathrm{Vol}(v_2v_3,F_4)\}+\\ &\quad +\mathrm{Vol}(v_1v_2,F_5)=1+2+1+1+1=6. \end{aligned}$$

From the list of the facets we see that there are five 4-unbounded facets, precisely $\mathcal{F}(0)=\{F_1,F_2,F_3,F_4,F_5\}$ and we have

$$\begin{array}{lll} \mathrm{Vol}(v_2,F_1)=1, & \mathrm{Vol}(v_3,F_1)=1, & \mathrm{Vol}(v_1,F_2)=1,\\ \mathrm{Vol}(v_3,F_2)=1, & & \\ \mathrm{Vol}(v_1,F_3)=1, & \mathrm{Vol}(v_2,F_3)=1, & \mathrm{Vol}(v_3,F_3)=1,\\ \mathrm{Vol}(v_1,F_4)=1, & & \\ \mathrm{Vol}(v_2,F_4)=1, & \mathrm{Vol}(v_3,F_4)=1, & \mathrm{Vol}(v_1,F_5)=1,\\ \mathrm{Vol}(v_2,F_5)=1, & & \end{array}$$

$$\begin{aligned} c_4(I) &= \min\{\mathrm{Vol}(v_2,F_1),\mathrm{Vol}(v_3,F_1)\}+\min\{\mathrm{Vol}(v_1,F_2),\\ &\quad \mathrm{Vol}(v_3,F_2)\}+\\ &\quad +\min\{\mathrm{Vol}(v_1,F_3),\mathrm{Vol}(v_2,F_3),\mathrm{Vol}(v_3,F_3)\}+\\ &\quad +\min\{\mathrm{Vol}(v_1,F_4),\mathrm{Vol}(v_2,F_4),\mathrm{Vol}(v_3,F_4)\}+\\ &\quad +\min\{\mathrm{Vol}(v_1,F_5),\mathrm{Vol}(v_2,F_5)\}=1+1+1+1+1=5. \end{aligned}$$

Acknowledgments

We are grateful to the referees, whose comments improved the presentation.

ORCID

Rüdiger Achilles http://orcid.org/0000-0003-0935-2697
Mirella Manaresi http://orcid.org/0000-0003-2693-9641

References

[1] Achilles, R, Aliffi, D. Segre, a script for the Reduce package Cali, Bologna, 1999–2017. Available at: http://www.dm.unibo.it/achilles/segre/

[2] Achilles, R, Manaresi, M. (1993). Multiplicity for ideals of maximal analytic spread and intersection theory. *J. Math. Kyoto Univ*. 33(4): 1029–1578.

[3] Achilles, R, Manaresi, M. (1997). Multiplicities of a bigraded ring and intersection theory. *Math. Ann*. 309(4): 573–591.

[4] Achilles, R, Rams, S. (2001). Intersection numbers, Segre numbers and generalized Samuel multiplicities. *Arch. Math*. 77(5): 391–398.

[5] Aluffi, P. (2016). Segre classes as integrals over polytopes. *J. Eur. Math. Soc*. 18(12): 2849–2863.

[6] Bivià-Ausina, C. (2003). The analytic spread of monomial ideals. *Commun. Algebra*. 31(7): 3487–3496.

[7] Ciupercă, C. (2003). A numerical characterization of the S_2-ification of a Rees algebra. *J. Pure Appl. Algebra*. 178: 25–48.

[8] Eisenbud, D, Sturmfels, B. (1996). Binomial ideals. *Duke Math. J*. 84(1): 1–45.

[9] Guibert, G. (1999). Classe de Segre locale d'un idéal et frontières de Newton. *C. R. Acad. Sci. Paris Sér. I Math*. 329(4): 315–320.

[10] Hearn, A. C. (1968). Reduce: A user-oriented interactive system for algebraic simplification. In: Klerer, M., Reinfelds, J., eds. *Interactive Systems for Experimental Applied Mathematics*. New York, NY: Academic Press, pp. 79–90.

[11] Hearn et al. REDUCE: A portable general-purpose computer algebras system. Available at: https://reduce-algebra.sourceforge.io/.

[12] Huneke, C, Swanson, I. (2006). *Integral Closure of Ideals, Rings, and Modules*. London Mathematical Society Lecture Note Series, 336. Cambridge: Cambridge University Press.

[13] Jeffries, J, Montaño, J. (2013). The j-multiplicity of monomial ideals. *Math. Res. Lett*. 20(4): 729–744.

[14] Montesinos Amilibia, A. Germenes, a program for computing Newton polyhedra. Available at: http://www.uv.es/montesin/

[15] Reid, L, Vitulli, M. A. (1999). The weak subintegral closure of a monomial ideal. *Commun. Algebra*. 27(11): 5649–5667.

[16] Singla, P. (2007). Minimal monomial reductions and the reduced fiber ring of an extremal ideal. *Illinois J. Math*. 51(4): 1085–1102.

[17] Teissier, B. (1988). Monômes, volumes et multiplicités. In *Introduction à la Théorie Des Singularités*, Vol. II. Travaux en Cours, 37. Paris: Hermann, pp. 127–141.

[18] van der Waerden, B. L. (1929). On Hilbert's function, series of composition of ideals and a generalisation of the theorem of Bezout. *Proc. Roy. Acad. Amsterdam*. 31: 749–770.