Computer-aided study of double extensions of restricted Lie superalgebras preserving the non-degenerate closed 2-forms in characteristic 2

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Abstract. A Lie (super)algebra with a non-degenerate invariant symmetric bilinear form B is called a nis-(super)algebra. The double extension $\mathfrak{g}$ of a nis-(super)algebra $\mathfrak{a}$ is the result of simultaneous adding to $\mathfrak{a}$ a central element and a derivation so that $\mathfrak{g}$ is a nis-algebra. Loop algebras with values in simple complex Lie algebras are most known among the Lie (super)algebras suitable to be doubly extended. In characteristic 2 the notion of double extension acquires specific features.
Restricted Lie (super)algebras are among the most interesting modular Lie superalgebras. In characteristic 2, using Grozman's Mathematica-based package SuperLie, we list double extensions of restricted Lie superalgebras preserving the non-degenerate closed 2-forms with constant coefficients. The results are proved for the number of indeterminates ranging from 4 to 7 — sufficient to conjecture the pattern for larger numbers. Considering multigradings allowed us to accelerate computations up to 100 times.

Keywords: Restricted Lie superalgebra, characteristic 2, double extension

1 Introduction

1.1 Setting of the problem For a given Lie (super)algebra $\mathfrak{a}$ over the ground field $\mathbb{K}$, the notion of double extension $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$, where $\mathscr{K} := \mathbb{K}c$, and $\mathscr{K}^* := \mathbb{K}D$, was recently distinguished, see [MR]. It simultaneously involves

1) a central extension of $\mathfrak{a}$ (with center $\mathscr{K} := \mathbb{K}c$),

2) a derivation D of $\mathfrak{a}$,

3) a non-degenerate invariant symmetric bilinear form (briefly: NIS) on $\mathfrak{a}$ extendable to $\mathfrak{g}$. Hereafter any Lie (super)algebra with a NIS is called a *nis-(super)algebra*.

Most known examples of double extensions are affine Kac–Moody algebras $\mathfrak{g}$ over $\mathbb{C}$ or $\mathbb{R}$ important in physics and mathematics (Google returns 482K entries); here $\mathfrak{a}$ is a loop algebra with values in a simple finite-dimensional Lie algebra $\mathfrak{s}$. Less known examples: the Lie superalgebra $\mathfrak{gl}(a|a)$ over $\mathbb{C}$, a double extension of $\mathfrak{psl}(a|a)$, and $\mathfrak{gl}(np)$ in characteristic $p>0$, a double extension of $\mathfrak{psl}(np)$.

In what follows we show that for the double extension to be "interesting", i.e., not just the direct sum of two ideals ($\mathfrak{a}$ and $\mathscr{K}\oplus\mathscr{K}^*$), the following conditions should be satisfied:

a) the derivation D of $\mathfrak{a}$ *must be outer* for any p; if $p=2$, and $D\in(\mathfrak{out}\ \mathfrak{a})_{\bar 1}$, then condition $D^2=0$ is a must, see [BKLS];

b) the central extension has to be non-trivial, see Subsection 2.4.

The simple Lie (super)algebras with NIS are classified for various types of Lie (super)algebras over the field $\mathbb{K}$ of characteristic $p\neq 2$ in review [BKLS].

The new and most interesting results of [BeB1] are general constructions and examples of double extensions of Lie superalgebras for $p=2$. Here we recall the main definitions and general results of [BeB1] carefully pointing at the difference between the cases where $p\neq 2$ and the cases where $p=2$.

In particular, in [BeB1], the double extensions of simple NIS-Lie superalgebras $\mathfrak{h}_{\Pi}^{(1)}(0|4)$ and $\mathfrak{h}_{\Pi}^{(1)}(0|5)$, where $\mathfrak{g}^{(1)}$ is the derived of $\mathfrak{g}$, see eq. (18), were classified. Clearly, $\mathfrak{h}_{\Pi}^{(1)}(0|n)$ has the Poisson Lie superalgebra $\mathfrak{po}_{\Pi}(0|n)$ as its double extension, but there are more: several new double extensions were found; one of them gave an interpretation of a result in [BGLL1].

P. Deligne advised one of co-authors of this note to consider restricted algebras first of all, as pertaining to geometry, see [LL]. In [BBH], the notion of double extension was extended to *restricted* Lie algebras. If $p=2$, there are several notions of restrictedness, see [BLLSq]; here we consider the "classical" one. Vectorial Lie (super)algebras can only be restricted if the shearing vector is equal to $\mathbb{1}:=(1,\dots,1)$.

Hereafter $\mathbb{K}$ is an algebraically closed field of characteristic 2 and the shearing vector is equal to $\mathbb{1}$, so we do not indicate it. For details of description of simple Lie superalgebras we study in what follows, see [BGLLS].

Our results. There are two types of super analogs of the Hamiltonian Lie algebra: series $\mathfrak{h}$, and series $\mathfrak{le}$ introduced in [Le]. If $\operatorname{char}\mathbb{K}>0$, there are several analogs of these series; here we consider the "standard" ones, i.e., with constant coefficients, see [BKLS, Subsection 4.7.12]:

a) the restricted Lie (super)algebras $\mathfrak{h}_{B}^{(1)}(a|b)$ on a even and b odd indeterminates $X=(X_1,\dots,X_{a+b})$ with forms $B=\Pi\Pi$, ΠI, $I\Pi$ and II on the superspace spanned by the X_i (in particular, $B=\Pi$ and I on the space);

b) the restricted Lie superalgebras $\mathfrak{le}^{(1)}(a|a)$ on a even and a odd indeterminates.

We sharpen the conjecture from [BeB1] on the shape of double extensions $\mathfrak{h}_{\Pi}^{(1)}(0|n)$ for $n>5$ and prove it. We computed double extensions of $\mathfrak{h}_{B}^{(1)}(a|b)$ for $4\leq a+b\leq 6$ and $\mathfrak{h}_{\Pi}^{(1)}(0|7)$, and $\mathfrak{le}^{(1)}(a|a)$ for $a=2,3$: this suffices to see the pattern for any a and b. The deforms of series $\mathfrak{h}$ and $\mathfrak{le}$ preserving the 2-forms with non-constant coefficients are being considered elsewhere.

1.2 Preliminaries: gradings A. Lebedev argued that in the modular case, the space of roots, although a particular case of the space of weights, should be considered over $\mathbb{R}$, not over the ground field $\mathbb{K}$ as weights **are** considered in any representation except for the adjoint one, see [BGL, Subsect. 4.3]. The same applies to the notion of $\mathbb{Z}$-graded Lie (super)algebras: all roots of $\mathfrak{der}\ \mathfrak{g}$, in particular, $\mathbb{Z}$-gradings, should be considered over $\mathbb{R}$. (It is strange that this observation was not made ca 50 years earlier or any time later.) This interpretation of the gradings and roots is implemented in SuperLie; it allowed us to accelerate computations up to 100 times: e.g., the computation time for algebras with 6 indeterminates reduces from 8 hours to 5 minutes.

1.3 Preliminaries: Bilinear form and brackets A. Lebedev proved ([LeD]) that if $p = 2$ and $\dim V$ is **odd, there is one equivalence class** of the **even** non-degenerate symmetric bilinear forms on the linear space V, whereas if $\dim V$ is **even, there are two classes**:

<u>type *I*)</u> the Gram matrix has at least one non-zero element on the main diagonal,

<u>type *Π*)</u> if all elements on the main diagonal of the Gram matrix are zero.

For the normal shapes of these Gram matrices we take

$$\tilde{\Pi}_{2n} := \begin{pmatrix} 1_2 & 0 \\ 0 & \Pi_{2n-2} \end{pmatrix} \text{ if } B \text{ is of type } I \text{ and } \dim V = 2n, \tag{1}$$

$$\Pi_{2a} := \begin{pmatrix} 0 & 1_a \\ 1_a & 0 \end{pmatrix} \text{ if } B \text{ is of type } \Pi \text{ and } \dim V = 2a, \text{ or} \tag{2}$$

$$\Pi_{2a+1} := \begin{pmatrix} 0 & 0 & 1_a \\ 0 & 1 & 0 \\ 1_a & 0 & 0 \end{pmatrix} \text{ if } \dim V = 2a+1 \text{ and } B \text{ of any type.} \tag{3}$$

We denote the Lie algebra preserving the form B by $\mathfrak{o}_B(V)$ or $\mathfrak{o}_B(a)$ if $\dim V = a$.

If V is a superspace, the even form B on it is the direct sum of the forms on the even and odd parts of V, and hence the non-degenerate symmetric forms can be of the types $B = II$, $I\Pi$, ΠI, and $\Pi\Pi$ — short for $I \oplus I$, etc. We denote the ortho-orthogonal Lie superalgebra preserving B by $\mathfrak{oo}_B(V)$ or $\mathfrak{oo}_B(a|b)$ if $\operatorname{sdim} V = a|b$.

A. Lebedev proved ([LeD]) that for any p, there is one equivalence class of non-degenerate **odd** symmetric bilinear forms B on V. In this case, $\operatorname{sdim} V = a|a$ and for the normal shape of B one can take $\Pi_{a|a} = \Pi_{2a}$. We denote the Lie superalgebra preserving B by $\mathfrak{pe}_B(V)$ or $\mathfrak{pe}_B(a)$.

• Define the *Poisson bracket* on the space of Grassmann algebra generated by either $2n$ odd generators $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ (case Π), or n odd generators

$\theta_1, \dots, \theta_n$ (case I), in accordance with (1)–(3):

$$\{f,g\}_{\Pi} := \sum \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i}\right); \quad (4)$$

$$\{f,g\}_{I} := \sum \left(\frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i}\right); \quad (5)$$

$$\{f,g\}_{I} := \sum_{i\leq n-2} \left(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i}\right) + \begin{cases} \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta} & \text{for } n \text{ odd} \\ \sum_{i\leq 2}\left(\frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i}\right) & \text{for } n \text{ even.}\end{cases} \quad (6)$$

• If, instead of the Grassmann algebra, we consider the algebra of truncated polynomials generated by either $2n$ even generators $q_1, \dots, q_n$ and $p_1, \dots, p_n$ (case Π), or n even generators $z_1, \dots, z_n$ (case I), the *Poisson bracket* becomes

$$\{f,g\}_{\Pi} := \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right); \quad (7)$$

$$\{f,g\}_{I} := \sum \left(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_i}\right); \quad (8)$$

$$\{f,g\}_{I} := \sum_{i\leq n-2} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) + \begin{cases} \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} & \text{for } n \text{ odd} \\ \sum_{i\leq 2}\left(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_i}\right) & \text{for } n \text{ even.}\end{cases} \quad (9)$$

We call the above (super)spaces with the Poisson brackets the *Poisson algebras*. Observe that the space of truncated polynomials with the bracket $\{\cdot,\cdot\}_I$ is not a Lie algebra, but a Leibniz[1] algebra: $\{z_i, z_i\}_I = 1$, not 0. The quotient modulo center is, however, a Lie algebra. For all the above Poisson brackets and the combinations thereof on a even and b odd indeterminates, the quotients modulo center (generated by constants) is the *Lie algebra of Hamiltonian vector fields* $\mathfrak{h}_B(a|b)$, where $B = \Pi\Pi$, ΠI, $I\Pi$, or II.

• We also consider the *Buttin superalgebra* $\mathfrak{b}(a|a)$; its space is the tensor product of Grassmann algebra by the algebra of truncated polynomials generated by n even generators $q_1, \dots, q_n$ and n odd generators $\pi_1, \dots, \pi_n$ with the *Schouten bracket* a.k.a. *Buttin bracket* a.k.a. *anti-bracket*:

$$\{f,g\}_{B.b} := \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial \pi_i} + \frac{\partial f}{\partial \pi_i}\frac{\partial g}{\partial q_i}\right).$$

Set $\mathfrak{le}(a|a) = \mathfrak{b}(a|a)/\mathfrak{c}$ and $\mathfrak{h}_B(a|b) = \mathfrak{po}_B(a|b)/\mathfrak{c}$, where $\mathfrak{c}$ is the center (spanned by constants).

On each of the above Poisson (Leibniz) algebras $\mathfrak{po}_B(a|b)$ and the Buttin algebras $\mathfrak{b}(a|a)$, define NIS by means of the *Berezin integral* = the coefficient of the highest monomial: $B(f,g) = \int fg \,\mathrm{vol}$, where vol is the volume element. This NIS is of the same parity as the number of odd indeterminates and induces a

[1] The (left) *Leibniz* algebra L satisfies $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ for any $x, y, z \in L$; if in addition L it is anti-commutative, it is a Lie algebra. Superization is immediate, via the Sign Rule.

NIS of the simple subquotients, $\mathfrak{h}_B^{(1)}(a|b)$ and $\mathfrak{le}^{(1)}(a|a)$. Clearly, $\mathfrak{h}_B^{(1)}(a|b)$ (resp. $\mathfrak{le}^{(1)}(a|a)$) have the Poisson or Leibniz algebras (resp., the Buttin algebra) as their double extensions; but they also have other double extensions.

For Lie superalgebras with Cartan matrices, see [BGL]; for descriptions in terms of Cartan-Tanaka-Shchepochkina prolongations, see [BGLLS1,BGLLS]; for the classification of simple Lie superalgebras, see [BGL,BLLSq].

2 Background: Double extensions for $p \neq 2$ (after [BKLS,ABB,Be])

2.1 Lemma (On a central extension) (Lemma 3.6, page 73 in [BB]) *Let $\mathfrak{a}$ be a Lie (super)algebra over a field $\mathbb{K}$, let $B_\mathfrak{a}$ be an $\mathfrak{a}$-invariant NIS on $\mathfrak{a}$, let $D \in \mathfrak{der}\ \mathfrak{a}$ be a derivation such that $B_\mathfrak{a}$ is D-invariant, i.e.,*

$$B_\mathfrak{a}(Da, b) + (-1)^{p(a)p(D)} B_\mathfrak{a}(a, Db) = 0 \quad \textit{for any } a, b \in \mathfrak{a}. \tag{10}$$

Then the bilinear form

$$\omega(a, b) := B_\mathfrak{a}(Da, b) \tag{11}$$

is a 2-cocycle of the Lie (super)algebra $\mathfrak{a}$; clearly, $p(\omega) = p(B) + p(D)$.

Thus, under assumptions of Lemma 2.1, we can construct a central extension $\mathfrak{a}_\omega$ of $\mathfrak{a}$ with the center spanned by an element c, given by cocycle ω so that $\mathfrak{a}_\omega/\mathbb{K}c \simeq \mathfrak{a}$; moreover, we can construct a semidirect sum $\mathfrak{g} = \mathfrak{a}_\omega \ltimes \mathbb{K}D$.

On the Lie (super)algebra $\mathfrak{g}$, define a bilinear form B by setting for any $x \in \mathfrak{a}$

$$B|_\mathfrak{a} = B_\mathfrak{a}, \quad B(D, c) = 1, \quad B(c, x) = B(D, x) = B(c, c) = B(D, D) = 0. \tag{12}$$

2.2 Lemma [On NIS] (For Lie algebras: [K, Exercise 2.10]; for Lie superalgebras: [BB, Theorem 1, page 68]) *The form B defined by* (12) *is a NIS.*

Thus constructed Lie (super)algebra $\mathfrak{g}$ with NIS B on it is called the *double extension* or, for emphasis, *D-extension*, of $\mathfrak{a}$. If the derivation D is inner, then $\mathfrak{g}$ is a direct sum of its ideal $\mathfrak{a}$ and a 2-dimensional commutative ideal; this — *decomposable* — case is not interesting.

2.3 Lie (super)algebra $\mathfrak{g}$ that can be a double extension of a Lie (super)algebra $\mathfrak{a}$ Let $\mathfrak{g}$ be a Lie algebra over any field $\mathbb{K}$ or a Lie superalgebra over a field $\mathbb{K}$ of characteristic $p \neq 2$; let B be a NIS on $\mathfrak{g}$, and $c \neq 0$ a central element of $\mathfrak{g}$.

The invariance of the form B implies that $B(c, [x, z]) = 0$ for any $x, z \in \mathfrak{g}$, i.e., the space $c^\perp$ contains the commutant $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$, and hence is an ideal. Since the form B is non-degenerate, the codimension of this ideal is equal to 1.

If $B(c, c) \neq 0$, then the Lie (super)algebra $\mathfrak{g}$ is just a direct sum $\mathfrak{g} = \mathbb{K}c \oplus c^\perp$.

Theorem *Let $\mathfrak{g}$ be a Lie algebra over any field $\mathbb{K}$ or a Lie superalgebra over a field $\mathbb{K}$ of characteristic $p \neq 2$; let B be a NIS on $\mathfrak{g}$, and $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$. If $\mathfrak{g}^{(1)} \cap \mathfrak{z}(\mathfrak{g}) \neq 0$, then $\mathfrak{g}$ is a double extension of a Lie (super)algebra $\mathfrak{a}$.*

2.4 Central extension must be nontrivial for DE to be indecomposable Let $\mathfrak{g}$ be a finite-dimensional Lie (super)algebra, and B a NIS on it. Let $D \in \mathfrak{der}\,\mathfrak{g}$ preserve B, and the 2-cocycle $\omega(a,b) := B(Da,b)$ be trivial, i.e., there exists $\alpha \in \mathfrak{g}^*$ such that $\omega(a,b) = \alpha([a,b])$ for all $a,b \in \mathfrak{g}$. Let us prove that in this case D is an inner derivation, so the double extension is decomposable.

Since B is non-degenerate, there exists an $x \in \mathfrak{g}$ such that $\alpha(y) = B(x,y)$ for all $y \in \mathfrak{g}$. Then

$$B(Da,b) = \alpha([a,b]) = B(x,[a,b]) = B([x,a],b) \quad \text{for all } a,b \in \mathfrak{g}.$$

For this x, we have $Da = [x,a]$ for all $a \in \mathfrak{g}$, q.e.d.

2.5 Remark on affine Kac–Moody (super)algebras Let $\mathfrak{a}$ be a simple finite-dimensional Lie (super)algebra with a NIS b. Let $\mathfrak{a}^{\ell(1)} := \mathfrak{a} \otimes \mathbb{C}[x^{-1},x]$, where $x = \exp(i\varphi)$ for the angle parameter φ on the circle, be the $\mathfrak{a}$-valued Lie (super)algebra of loops expandable into Laurent polynomials. It is easy to see that for any $n \in \mathbb{Z}$, the bilinear form

$$\begin{array}{l} B_n(f,g) := \operatorname{Res} b(f,g)x^n dx, \quad \text{where} \\ \operatorname{Res} f(x)dx = \text{coeff. of } \frac{dx}{x} \text{ in the Laurent series expansion of } f(x)dx, \end{array} \tag{13}$$

is a NIS on $\mathfrak{a}^{\ell(1)}$. The non-trivial central extension given by the cocycle with values in $\mathbb{C}c$

$$\omega(f,g) := \operatorname{Res} b(f,dg) = B_0(f, \frac{d}{dx}g) \tag{14}$$

and $D_1 = x\frac{d}{dx}$, where $D_n = x^n\frac{d}{dx}$, make $\mathfrak{g} := (\mathbb{C}c \ltimes \mathfrak{a}^{\ell(1)}) \ltimes \mathbb{C}D_1$ a double extension of $\mathfrak{a}^{\ell(1)}$ called *affine Kac–Moody* Lie (super)algebra. If $\mathfrak{a}$ has Cartan matrix, then $\mathfrak{g}$ also has Cartan matrix; the Dynkin graph of $\mathfrak{g}$ is the extended Dynkin graph of $\mathfrak{a}$.

The above description made us wonder: (1) In the case where $\mathfrak{der}\,\mathfrak{a} = \mathfrak{a}$ for simplicity, the space of outer derivations of $\mathfrak{a}^{\ell(1)}$ is $\mathfrak{vect}(1) = \mathfrak{der}\,\mathbb{C}[x^{-1},x]$, What is so special in D_1 to be selected for the role of outer derivation of $\mathfrak{a}^{\ell(1)}$? (2) Why instead of the cocycle (11) with $D = D_1$, everybody uses eq. (14) with $D = D_0$?

Let $f = x^s$ and $g = x^t$ be $\mathfrak{a}$-valued functions with values equal to u and v, respectively. Then condition (10) for B_0, see (13), turns into

$$(s+t)\operatorname{Res} b(u,v)x^{n+s+t-1}dx = 0$$

true if either $n+s+t \neq 0$, or $s+t = 0$. Therefore, D_n preserves NIS B_{-n}, see (13), on $\mathfrak{a}^{\ell(1)}$, and the cocycle (14) can be obtained as

$$\omega(f,g) := B_n(f, D_{-n}g) \quad \text{for any } n \in \mathbb{Z}. \tag{15}$$

Since only D_1 is of degree 0, and only D_0 preserves B_0, we see that only D_1 and B_0 lead to a Cartan matrix of $\mathfrak{g}$. This answers the above questions.

3 Background: Double extensions for $p = 2$ (after [BeB1])

3.1 Quadratic and bilinear forms when $p = 2$ A given map $q : V \to \mathbb{K}$, where V is a $\mathbb{K}$-vector space, is called a *quadratic form* if

$$\begin{gathered} q(\lambda v) = \lambda^2 q(v) \text{ for any } \lambda \in \mathbb{K} \text{ and for any } v \in V, \text{ and the map} \\ (u, v) \mapsto B_q(u, v) := q(u + v) - q(u) - q(v) \text{ is bilinear.} \end{gathered}$$

The form B_q is called the *polar form* of q. Recently, Lebedev classified non-degenerate bilinear forms over a perfect field $\mathbb{K}$ (i.e., such that $\mathbb{K}^2 = \mathbb{K}$), see [LeD]. This is a non-trivial result not related with a well-known classification of quadratic forms in any characteristic, because in characteristic 2, each of the arrows

$$q \longleftrightarrow B_q$$

is not necessarily onto, has a kernel, and two quadratic forms with different Arf invariants may have identical polar forms.

3.2 Lie superalgebras for $p = 2$ For $p \neq 2$, superization of many notions of Linear Algebra is performed, as is now well-known, with the help of the Sign Rule. If $p = 2$, **additional** conditions appear. We recall basic definitions retaining the minus sign from definitions for $p \neq 2$: for clarity.

• A *Lie superalgebra* in characteristic 2 is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ over a field $\mathbb{K}$ such that the even part $\mathfrak{g}_{\bar 0}$ is a Lie algebra, the odd part $\mathfrak{g}_{\bar 1}$ is a $\mathfrak{g}_{\bar 0}$-module made two-sided by *anti*-symmetry, and on the odd part $\mathfrak{g}_{\bar 1}$ a *squaring* is defined as a map

$$\begin{gathered} s_{\mathfrak{g}} : \mathfrak{g}_{\bar 1} \to \mathfrak{g}_{\bar 0} \;\text{ given by } f \mapsto f^2 \text{ such that} \\ (\lambda f)^2 = \lambda^2 f^2 \text{ for any } f \in \mathfrak{g}_{\bar 1} \text{ and } \lambda \in \mathbb{K}, \text{ and the map} \\ [f, g] := (f + g)^2 - f^2 - g^2 \;\text{ for any } f, g \in \mathfrak{g}_{\bar 1} \\ \text{is a bilinear form on } \mathfrak{g}_{\bar 1} \text{ with values in } \mathfrak{g}_{\bar 0}. \end{gathered} \tag{16}$$

The bracket on $\mathfrak{g}_{\bar 0}$, as well as the action of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$, is denoted also by the same symbol $[\cdot, \cdot]$, or $[\cdot, \cdot]_{\mathfrak{g}}$ for clarity. The Jacobi identities for three even elements, and involving one or two odd elements, are the same as for $p \neq 2$; the following new identity replaces the one with three odd elements: it involves the squaring:

$$[f^2, g] = [f, [f, g]] \;\text{ for any } f \in \mathfrak{g}_{\bar 1} \text{ and } g \in \mathfrak{g}. \tag{17}$$

• Desuperizing (retain only the bracket) we get a $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$.

• For any Lie superalgebra $\mathfrak{g}$ in characteristic 2, its *derived algebras* are

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathbb{K}\{f^2 \mid f \in (\mathfrak{g}^{(i)})_{\bar 1}\}. \tag{18}$$

A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* of the Lie superalgebra $\mathfrak{g}$ if

$$D([f, g]) = [D(f), g] + [f, D(g)] \quad \text{for any } f, g \in \mathfrak{g} \tag{19}$$

and, **additionally** (generalization of (17), where $D = \mathrm{ad}_g$),

$$D(f^2) = [D(f), f] \quad \text{for any } f \in \mathfrak{g}_{\bar 1}. \tag{20}$$

We denote the space of all derivations of $\mathfrak{g}$ by $\mathfrak{der}(\mathfrak{g})$.

An even linear map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a *representation of the Lie superalgebra* $\mathfrak{g}$ in the superspace V, called $\mathfrak{g}$-*module*, if

$$\rho([f,g]) = [\rho(f), \rho(g)] \text{ for any } f,g \in \mathfrak{g}; \text{ and } \rho(f^2) = (\rho(f))^2 \text{ for any } f \in \mathfrak{g}_{\bar{1}}. \tag{21}$$

We say that a bilinear form B on $\mathfrak{g}$ is *symmetric* if

$$\begin{array}{l} B(f,g) = B(g,f) \text{ for any } f,g \in \mathfrak{g}; \text{ and,} \\ \textbf{additionally}, B(f,f) = 0 \text{ for any } f \in \mathfrak{g}_{\bar{1}}. \end{array}$$

We say that D *preserves* B if

$$\begin{array}{l} B(D(f),g) + B(f,D(g)) = 0 \text{ for any } f,g \in \mathfrak{g}; \text{ and,} \\ \textbf{additionally}, B(D(f),f) = 0 \text{ for any } f \in \mathfrak{g}_{\bar{1}} \text{ and any } g \in \mathfrak{g}. \end{array} \tag{22}$$

We denote the *nis-superalgebra* with a NIS $B_{\mathfrak{g}}$ by $(\mathfrak{g}, B_{\mathfrak{g}})$. A nis-superalgebra $(\mathfrak{g}, B_{\mathfrak{g}})$ is said to be *decomposable* if it can be decomposed into direct sums of ideals, namely $\mathfrak{g} = \oplus I_i$, such that all I_i are orthogonal to each other.

3.3 The case where $B_{\mathfrak{a}}$ is even Here $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$ as spaces, where $\mathscr{K} := \mathbb{K}c$ and $\mathscr{K}^* := \mathbb{K}D$.

$D_{\bar{0}}$-extensions. The quadratic form q

3.3.1. Theorem. *Let $(\mathfrak{a}, B_{\mathfrak{a}})$ be a nis-superalgebra, $p(B_{\mathfrak{a}}) = \bar{0}$, let $D \in \mathfrak{der}_{\bar{0}}(\mathfrak{a})$ preserve $B_{\mathfrak{a}}$. Let $q : \mathfrak{a}_{\bar{1}} \to \mathbb{K}$ be a quadratic form; let its polar form B_q satisfy*

$$B_{\mathfrak{a}}(a, D(b)) = B_q(a,b) \textit{ for any } a,b \in \mathfrak{a}_{\bar{1}}. \tag{23}$$

Then there exists a nis-superalgebra structure on $\mathfrak{g}$, defined as follows, cf. Subsection 2.1 *and* (12). *The squaring is given by*

$$s_{\mathfrak{g}}(a) := s_{\mathfrak{a}}(a) + q(a)c \qquad \textit{for any } a \in \mathfrak{g}_{\bar{1}} \ (= \mathfrak{a}_{\bar{1}}).$$

The bracket on $\mathfrak{g}$ is defined for any $a,b \in \mathfrak{a}$ as follows:

$$[a,b]_{\mathfrak{g}} := [a,b]_{\mathfrak{a}} + B_{\mathfrak{a}}(D(a),b)c, \quad [D,a]_{\mathfrak{g}} := D(a) \textit{ and } [c,\mathfrak{g}]_{\mathfrak{g}} = 0. \tag{24}$$

The NIS B on $\mathfrak{g}$ is given by formulas (12) *with one modification: $B(D,D)$ can be arbitrary.*

What is the difference of this Theorem from its analog for $p \neq 2$?

<u>First</u>, the extra condition in eq. (22): $B_{\mathfrak{a}}(D(a),a) = 0$. For $p \neq 2$, it follows from eq. (22), whereas for $p = 2$ it should be required as a part of invariance condition.

<u>Second</u>, the reasons for the quadratic form q to appear. Indeed, for $p \neq 2$, to determine a central extension, we only need a cocycle, whereas for $p = 2$, we have to adjust the squaring to match this cocycle and this is precisely what the form q is needed for.

Everything else is the same as for $p \neq 2$, same as with the desuperisations of $\mathfrak{a}$, when only even derivations and forms remain, i.e., no extra conditions.

We call the Lie superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.3.1 a $D_{\bar 0}$-*extension* of $(\mathfrak{a}, B_{\mathfrak{a}})$ by means of D and q.

Now, let the *"special center" of* $\mathfrak{g}$ *relative to* $B_{\mathfrak{g}}$ be

$$\mathfrak{z}_s(\mathfrak{g}) := \mathfrak{z}(\mathfrak{g}) \cap s_{\mathfrak{g}}(\mathfrak{g}_{\bar 1})^{\perp}, \quad \text{where } s_{\mathfrak{g}}(\mathfrak{g}_{\bar 1}) := \mathrm{Span}\{x^2 \mid x \in \mathfrak{g}_{\bar 1}\}. \tag{25}$$

Observe that $\mathfrak{z}_s(\mathfrak{g})_{\bar 1} = \mathfrak{z}(\mathfrak{g})_{\bar 1}$ and $\mathfrak{z}_s(\mathfrak{g})_{\bar 0} = \mathfrak{z}(\mathfrak{g})_{\bar 0} \cap s_{\mathfrak{g}}(\mathfrak{g}_{\bar 1})^{\perp}$. Moreover, $\mathfrak{z}_s(\mathfrak{g})$ is not necessarily an ideal.

3.3.2. Proposition. *Let $(\mathfrak{g}, B_{\mathfrak{g}})$ be an indecomposable nis-superalgebra. Let $\mathfrak{z}_s(\mathfrak{g})_{\bar 0} \neq \{0\}$. Then $(\mathfrak{g}, B_{\mathfrak{g}})$ is obtained as an $D_{\bar 0}$-extension from a nis-superalgebra $(\mathfrak{a}, B_{\mathfrak{a}})$.*

What is the difference of the case where $p = 2$ from that where $p \neq 2$? The "indecomposability" condition can be weakened to the existence of a nonzero even element c belonging to the intersection of the center of $\mathfrak{g} \cap [\mathfrak{g}, \mathfrak{g}] \cap s_{\mathfrak{g}}(\mathfrak{g}_{\bar 1})^{\perp}$. Now we see that the only extra condition is related to squaring and is precisely due to the fact that $c^{\perp}$ must be an ideal.

$D_{\bar 1}$-extensions. The element $A \in \mathfrak{a}_{\bar 0}$

3.3.3. Theorem. *Let $(\mathfrak{a}, B_{\mathfrak{a}})$ be a nis-superalgebra, $p(B_{\mathfrak{a}}) = \bar 0$. Let $D \in \mathfrak{der}_{\bar 1}(\mathfrak{a})$ preserve $B_{\mathfrak{a}}$; let $A \in \mathfrak{a}_{\bar 0}$ satisfy the following conditions* (see [ABB,Be])*:*

$$D^2 = \mathrm{ad}_A; \quad D(A) = 0. \tag{26}$$

Then there exists a nis-superalgebra structure on $\mathfrak{g}$, defined as follows. The squaring is given by

$$s_{\mathfrak{g}}(rc + a + tD) := s_{\mathfrak{a}}(a) + tD(a) + t^2 A \qquad \text{for any } a \in \mathfrak{a}_{\bar 1} \text{ and } r, t \in \mathbb{K}.$$

The bracket is given by eq. (24)*. The NIS B on $\mathfrak{g}$ is given by formulas* (12)*.*

We call the nis-superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.3.3 a $D_{\bar 1}$-*extension* of $(\mathfrak{a}, B_{\mathfrak{a}})$ by means of D and A.

3.3.3a. Proposition. *Let $(\mathfrak{g}, B)$ be a nis-superalgebra. Define the cone*

$$\mathscr{C}(\mathfrak{g}, B) := \{x \in \mathfrak{g}_{\bar 1} \mid B(s_{\mathfrak{g}}(x), s_{\mathfrak{g}}(y)) = 0 \quad \text{for any } y \in \mathfrak{g}_{\bar 1}\}.$$

Let $\mathfrak{z}(\mathfrak{g})_{\bar 1} \cap \mathscr{C}(\mathfrak{g}, B) \neq \{0\}$. Then $(\mathfrak{g}, B)$ is obtained from a nis-superalgebra $(\mathfrak{a}, B_{\mathfrak{a}})$ as either a $D_{\bar 0}$-extension, or a $D_{\bar 1}$-extension.

3.4 The case where $B_{\mathfrak{a}}$ is odd Here $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$ as spaces, where $\mathscr{K} := \mathbb{K}c$ and $\mathscr{K}^* := \mathbb{K}D$.

$D_{\bar 0}$-extensions

3.4.1. Theorem. *Let $(\mathfrak{a}, B_\mathfrak{a})$ be a nis-superalgebra, $p(B_\mathfrak{a}) = \bar 1$. Let $D \in \mathfrak{der}_{\bar 0}(\mathfrak{a})$ preserve $B_\mathfrak{a}$. Then there exists a nis-superalgebra structure on $\mathfrak{g}$, defined as follows. The squaring is given by*

$$s_\mathfrak{g}(a + \mu c) := s_\mathfrak{a}(a) \qquad \text{for any } a \in \mathfrak{a}_{\bar 1} \text{ and } \mu \in \mathbb{K}.$$

The bracket is given by eq. (24). *The NIS B on $\mathfrak{g}$ is given by formulas* (12).

We call the nis-superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.4.1 the $D_{\bar 0}$-*extension* of $(\mathfrak{a}, B_\mathfrak{a})$ by means of D.

3.4.1a. Proposition. *Let $(\mathfrak{g}, B)$ be an indecomposable nis-superalgebra, $p(B) = \bar 1$. If $s_\mathfrak{g}(\mathfrak{z}(\mathfrak{g})_{\bar 1}) \cap s_\mathfrak{g}(\mathfrak{g}_{\bar 1})^\perp \neq \{0\}$, then $(\mathfrak{g}, B)$ is obtained from a nis-superalgebra $(\mathfrak{a}, B_\mathfrak{a})$ as either a $D_{\bar 0}$-extension, or a $D_{\bar 1}$-extension.*

$D_{\bar 1}$-extensions. The quadratic form q, and the elements $A \in \mathfrak{a}_{\bar 0}$ and $m \in \mathbb{K}$

3.4.1b. Theorem. *Let $(\mathfrak{a}, B_\mathfrak{a})$ be a nis-superalgebra, $p(B_\mathfrak{a}) = \bar 1$. Let $A \in \mathfrak{a}_{\bar 0}$ and $D \in \mathfrak{der}_{\bar 1}(\mathfrak{a})$ preserving $B_\mathfrak{a}$ satisfy the conditions* (26)*:*

Let q be a quadratic form on $\mathfrak{a}_{\bar 1}$ such that $B_q(a, b) = B_\mathfrak{a}(D(a), b)$.

Then there exists a nis-superalgebra structure on $\mathfrak{g}$, defined as follows. The squaring is given by

$$\begin{gathered} s_\mathfrak{g}(a + \mu D) := s_\mathfrak{a}(a) + (\mu^2 m + q(a))c + \mu^2 A + \mu D(a) \\ \text{for any } a \in \mathfrak{a}_{\bar 1} \text{ and } \mu \in \mathbb{K}, \text{ and some } m \in \mathbb{K}. \end{gathered}$$

The bracket is given by eq. (24). *The NIS B on $\mathfrak{g}$ is given by formulas* (12).

We call the nis-superalgebra $(\mathfrak{g}, B)$ constructed in Theorem 3.4.1b a $D_{\bar 1}$-*extension* of $(\mathfrak{a}, B_\mathfrak{a})$ by means of D, q, A, and m.

3.4.1c. Proposition. *Let $(\mathfrak{g}, B)$ be an indecomposable nis-superalgebra, $p(B) = \bar 1$. If $\mathfrak{z}(\mathfrak{g})_{\bar 0} \neq \{0\}$, then $(\mathfrak{g}, B)$ is obtained as a $D_{\bar 1}$-extension from a nis-superalgebra.*

3.5 Summary: to construct a double extension of a nis-(super)algebra $\mathfrak{a}$, we need q, A, m

D \ $B_\mathfrak{a}$	even	odd
even	q	-
odd	A	q, A, m

(27)

4 Isomorphisms, and equivalence classes of derivations (after [BeB1])

For a nis-superalgebra $\mathfrak{a}$ with NIS $B_\mathfrak{a}$, denote by $(\mathfrak{g}, B_\mathfrak{g})$ (resp. $(\tilde{\mathfrak{g}}, \tilde B_\mathfrak{g})$) the double extension of $\mathfrak{a}$ by means of a derivation D (resp. $\tilde D$), i.e., $\mathfrak{g} := \mathscr{K} \oplus \mathfrak{a} \oplus \mathscr{K}^*$, where $\mathscr{K} = \mathbb{K}c$ and $\mathscr{K}^* = \mathbb{K}D$ (resp. $\tilde{\mathfrak{g}} := \tilde{\mathscr{K}} \oplus \mathfrak{a} \oplus \tilde{\mathscr{K}}^*$, $\tilde{\mathscr{K}} = \mathbb{K}\tilde c$ and $\tilde{\mathscr{K}}^* = \mathbb{K}\tilde D$).

If $p = 2$, we **additionally** require a quadratic form q (resp. $\tilde{q}$) if the center c is even, while in the case of $D_{\bar{1}}$-extensions, we need, moreover, an element $A \in \mathfrak{a}_{\bar{0}}$, and sometimes $m \in \mathbb{K}$ (resp. $\tilde{A}$ and $\tilde{m}$) described in the previous section.

An *isomorphism* of of nis-superalgebras $\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ is a nis-preserving isomorphism of Lie superalgebra structures:

$$\pi([f, g]_{\mathfrak{g}}) = [\pi(f), \pi(g)]_{\tilde{\mathfrak{g}}}, \quad B_{\tilde{\mathfrak{g}}}(\pi(f), \pi(g)) = B_{\mathfrak{g}}(f, g) \text{ for any } f, g \in \mathfrak{g}.$$

4.1 The case where $B_{\mathfrak{a}}$ is even

4.1.1. Theorem. *Let $p(D) = p(\tilde{D}) = \bar{0}$. Let $\pi_{\bar{0}}$ be an automorphism of $(\mathfrak{a}, B_{\mathfrak{a}})$. Let $\lambda \in \mathbb{K}^\times$ and let $y \in \mathfrak{a}_{\bar{0}}$ satisfy the following conditions:*

$$\begin{array}{l}
\tilde{q}(a) = \lambda q \circ \pi_0^{-1}(a) + B_{\mathfrak{a}}(y, s_{\mathfrak{a}} \circ \pi_0^{-1}(a)) \quad \textit{for any } a \in \mathfrak{a}_{\bar{1}};\\
\pi_0^{-1} \tilde{D} \pi_0 = \lambda D + \mathrm{ad}_y \quad \textit{for any } a \in \mathfrak{a}_{\bar{0}};\\
B_{\mathfrak{g}}(D, D) = \lambda^{-2}(B_{\mathfrak{a}}(y, y) + B_{\mathfrak{g}}(\tilde{D}, \tilde{D})).
\end{array}$$

Then there exists an isomorphism $\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ given by the formulas

$$\begin{array}{ll}
\pi(a) & = \pi_0(a) + B_{\mathfrak{a}}(y, a)\tilde{c} \quad \textit{for any } a \in \mathfrak{a};\\
\pi(c) & = \lambda \tilde{c};\\
\pi(D) & = \lambda^{-1}(\tilde{D} + \pi_0(y)) + \nu \tilde{c}, \ \textit{where } \nu \in \mathbb{K} \textit{ is arbitrary.}
\end{array}$$

If $[D] = [\tilde{D}]$ in $\mathrm{H}^1_{\bar{0}}(\mathfrak{a}; \mathfrak{a})$, then they define isomorphic double extensions of $\mathfrak{a}$.

4.1.2. Theorem. *Let $p(D) = p(\tilde{D}) = \bar{1}$. Let π_0 be an automorphism of $(\mathfrak{a}, B_{\mathfrak{a}})$. Let $\lambda \in \mathbb{K}^\times$ and $y \in \mathfrak{a}_{\bar{1}}$ satisfy the following condition:*

$$\pi_0^{-1} \tilde{D} \pi_0 = \lambda D + \mathrm{ad}_y \quad \textit{on } \mathfrak{a}.$$

Then there exists an isomorphism $\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ given by the formulas

$$\begin{array}{ll}
\pi(a) & = \pi_0(a) + B_{\mathfrak{a}}(y, a)\tilde{c} \quad \textit{for any } a \in \mathfrak{a}_{\bar{1}};\\
\pi & = \pi_0 \quad \textit{on } \mathfrak{a}_{\bar{0}};\\
\pi(c) & = \lambda \tilde{c};\\
\pi(D) & = \lambda^{-1}(\tilde{D} + \pi_0(y)) + \nu \tilde{c}, \ \textit{where } \nu \textit{ is arbitrary};\\
\tilde{A} & = \lambda^2 \pi_0(A) + s_{\mathfrak{a}}(\pi_0(y)) + \lambda \pi_0(D(y)).
\end{array}$$

If $[D] = [\tilde{D}]$ in $\mathrm{H}^1_{\bar{1}}(\mathfrak{a}; \mathfrak{a})$, then they define isomorphic double extensions of $\mathfrak{a}$.

4.2 The case where $B_{\mathfrak{a}}$ is odd

4.2.1. Theorem. *Let $p(D) = p(\tilde{D}) = \bar{1}$. Let π_0 be an automorphism of $(\mathfrak{a}, B_{\mathfrak{a}})$. Let $\lambda \in \mathbb{K}^\times$ and $y \in \mathfrak{a}_{\bar{1}}$ satisfy the following conditions:*

$$\begin{aligned}
\tilde{q}(a) = \lambda q \circ \pi_0^{-1}(a) + B_{\mathfrak{a}}(y, s_{\mathfrak{a}} \circ \pi_0^{-1}(a)) \quad \textit{for any } a \in \mathfrak{a}_{\bar{1}};\\
\pi_0^{-1} \tilde{D} \pi_0(a) = \lambda D(a) + \mathrm{ad}_y(a) \quad \textit{for any } a \in \mathfrak{a}_{\bar{0}};\\
\tilde{A} = \lambda^2 \pi_0(A) + \lambda \pi_0(D(y)) + s_{\mathfrak{a}}(\pi_0(y));\\
\tilde{m} = \lambda^2 q(y) + \lambda B_{\mathfrak{a}}(y, s_{\mathfrak{a}}(y) + \lambda A) + \lambda^3 m.
\end{aligned}$$

Then there exists an isomorphism $\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ *given by the formulas*

$$\begin{array}{ll} \pi(a) & = \pi_0(a) + B_{\mathfrak{a}}(y,a)\tilde{c} \quad \textit{for any } a \in \mathfrak{a}; \\ \pi(c) & = \lambda\tilde{c}; \\ \pi(D) & = \lambda^{-1}(\tilde{D} + \pi_0(y)). \end{array} \tag{28}$$

If $[D] = [\tilde{D}]$ *in* $\mathrm{H}^1_{\bar{1}}(\mathfrak{a};\mathfrak{a})$*, then they define isomorphic double extensions of* $\mathfrak{a}$*.*

4.2.2. Theorem. *Let* $p(D) = p(\tilde{D}) = \bar{0}$*. Let* π_0 *be an automorphism of* $(\mathfrak{a}, B_{\mathfrak{a}})$*. Let* $\lambda \in \mathbb{K}^\times$ *and* $y \in \mathfrak{a}_{\bar{0}}$ *satisfy the following conditions:*

$$\pi_0^{-1}\tilde{D}\pi_0(a) = \lambda D(a) + \mathrm{ad}_y(a) \quad \textit{for any } a \in \mathfrak{a}.$$

Then there exists an isomorphism $\pi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}}$ *given by the formulas* (28)*. If* $[D] = [\tilde{D}]$ *in* $\mathrm{H}^1_{\bar{0}}(\mathfrak{a};\mathfrak{a})$*, then they define isomorphic double extensions of* $\mathfrak{a}$*.*

5 The exceptional cases: $\mathfrak{a} = \mathfrak{h}^{(1)}_B(a|b)$ for $a + b = 4$ and $\mathfrak{a} = \mathfrak{le}^{(1)}(2|2)$

Recall that the $\mathbb{Z}$-grading of a given vectorial Lie superalgebra is called *standard* if the degree of each indeterminate X_i is equal to 1. This *grading* is given by the Euler operator $\sum X_i\partial_{X_i}$. The *roots* are given with respect to the maximal torus, spanned by q_ip_i, $\eta_i\xi_i$, and $q_i\pi_i$ in the cases $\mathfrak{po}_\Pi(a|0)$, $\mathfrak{po}_\Pi(0|a)$, and $\mathfrak{le}(a|a)$, respectively. According to Subsection 1.2 the degree and roots are considered over $\mathbb{R}$, assuming that the ith coordinate of the weight of ξ_i, π_i, and p_i is equal to 1, whereas the ith coordinate of the weight of η_i, and q_i is equal to -1.

The standard $\mathbb{Z}$-grading of $\mathfrak{a} = \mathfrak{h}^{(1)}_B(a|b)$ (resp. $\mathfrak{le}^{(1)}(a|a)$) is symmetric, i.e.,

$$\mathfrak{a} = \bigoplus_{-1\leq i\leq 1} \mathfrak{a}_i \text{ with } \mathfrak{a}_{-1} \simeq \mathfrak{a}_1 \text{ (resp. } \mathfrak{a}_{-1} \simeq \Pi(\mathfrak{a}_1)^*) \text{ as } \mathfrak{a}_0\text{-modules,} \tag{29}$$

only if $a + b = 4$ (resp. $a = 2$). That is why some of the double extensions in these cases have no analogs in the generic cases: they appear due to symmetry.

5.1 Outer derivations Let the superscript of the derivation D be its degree, let the subscript be its weight or a monomial (x or b) or a label θ.

$\mathfrak{h}^{(1)}_\Pi(0|4)$ One outer derivation in each of degrees ± 2 and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$, see [BeB1].

$\mathfrak{h}^{(1)}_\Pi(4|0)$ One outer derivation in each of degrees ± 2 and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$. This answer is the desuperization of the case $\mathfrak{h}^{(1)}_\Pi(0|4)$; the cocycles are identical in shape to those of $\mathfrak{h}^{(1)}_\Pi(0|4)$.

$\mathfrak{h}^{(1)}_{\Pi\Pi}(2|2)$ One outer derivation in each of degrees ± 2 and 5 derivations of degree 0, whose weights are $(\pm 2, 0)$, $(0, \pm 2)$ and $(0, 0)$. This answer is a partial desuperization of the case $\mathfrak{h}^{(1)}_\Pi(0|4)$; the cocycles are identical in shape to those of $\mathfrak{h}^{(1)}_\Pi(0|4)$.

$\mathfrak{h}^{(1)}_{II}(2|2)$, $\mathfrak{h}^{(1)}(1|3)$, $\mathfrak{h}^{(1)}(3|1)$, $\mathfrak{h}^{(1)}_{I\Pi}(2|2)$, $\mathfrak{h}^{(1)}_{\Pi I}(2|2)$ One outer derivation in each of degrees ± 2. The apparently missing Euler operator is in this case an inner derivation. The same reason causes absence of certain derivations in cases $\mathfrak{h}^{(1)}_I(0|4)$, and $\mathfrak{h}^{(1)}_I(4|0)$, and in this case as compared with the case $\mathfrak{h}^{(1)}_\Pi(0|4)$.

$\mathfrak{h}^{(1)}_I(0|4)$ One outer derivation in each of degrees ± 2 and 4 derivations of degree 0, whose weights are ± 2 and two derivations of weight 0.

$\mathfrak{h}^{(1)}_I(4|0)$ The answer is the desuperization of the case $\mathfrak{h}^{(1)}_I(0|4)$; derivations of the same shape.

$\mathfrak{le}^{(1)}(2|2)$ The shape of derivations is identical to those for $\mathfrak{h}^{(1)}_\Pi(0|4)$; but $D^{(0)}_b$ are odd.

5.2 The double extensions

Note: in all exceptional cases $p(B_\mathfrak{a}) = \bar{0}$.

Case $\mathfrak{h}$. For the proof for the cocycle $D^{(-2)}$ in case $\mathfrak{a} = \mathfrak{h}^{(1)}_\Pi(0|4)$, see [BeB1]; the result for D^2 is isomorphic due to the symmetry (29). The idea of the proof is identical for the other derivations. The condition $B^T_\mathfrak{a} D^{(2)} = D^{(2)} B_\mathfrak{a}$ is easily seen. Besides, since $D^{(2)}$ acts by zero on $\mathfrak{a}_{\bar{0}}$, it follows that

$$B_\mathfrak{a}(D^{(2)}(f), f) = 0 \quad \text{for any } f \in \mathfrak{a}_{\bar{0}}.$$

Tables (31) and (30) show existence of the quadratic form q (resp. element A) associated with each derivation D, and names of the respective double extension DE. In the 6th column of (31) $B = II$, or $I\Pi$, or ΠI.

Derivation	q	DE of $\mathfrak{h}^{(1)}_I(0\vert 4)$	DE of $\mathfrak{h}^{(1)}_I(4\vert 0)$
$D^{(0)}_0$	yes	$\widehat{\mathfrak{po}}_I(0\vert 4)$	$\widehat{\mathfrak{po}}_I(4\vert 0)$
$D^{(0)}_b$	yes	$\widetilde{\mathfrak{po}}_I(0\vert 4)$	$\widetilde{\mathfrak{po}}_I(4\vert 0)$
$D^{(0)}_\theta$	–	–	–
$D^{(\pm 2)}$	yes	$\mathfrak{po}_I(0\vert 4)$	$\mathfrak{po}_I(4\vert 0)$

(30)

D	q	$\mathfrak{h}^{(1)}_\Pi(0\vert 4)$	$\mathfrak{h}^{(1)}_\Pi(4\vert 0)$	$\mathfrak{h}^{(1)}_{\Pi\Pi}(2\vert 2)$	$\mathfrak{h}^{(1)}_B(2\vert 2)$	A	$\mathfrak{le}^{(1)}(2\vert 2)$
$D^{(0)}_b$	yes	$\widetilde{\mathfrak{po}}_\Pi(0\vert 4)$	$\widetilde{\mathfrak{po}}_\Pi(4\vert 0)$	$\widetilde{\mathfrak{po}}_{\Pi\Pi}(2\vert 2)$	–	0	$\tilde{\mathfrak{b}}(2\vert 2)$
$D^{(0)}_0$	yes	$\mathfrak{gl}(2\vert 2)$	$\mathfrak{gl}(4\vert 0)$	$\mathfrak{gl}(2\vert 2)$	–	–	–
$D^{(2)}$	yes	$\mathfrak{po}_\Pi(0\vert 4)$	$\mathfrak{po}_\Pi(4\vert 0)$	$\mathfrak{po}_{\Pi\Pi}(2\vert 2)$	$\mathfrak{po}_B(2\vert 2)$	0	$\mathfrak{b}(2\vert 2)$

(31)

Case $\mathfrak{le}$. Compare with the general case in Subsection 6.13.

5.2.1. Lemma. *Let the super-rank of the operator A in the superspace V be the super dimension of the superspace $V/\operatorname{Ker} A$.*

1) *Because the super-rank of $D^{(0)}_b$ is $(2,2)$ and no element in the corresponding $\mathfrak{po}$ and $\mathfrak{gl}$ has such rank, it follows that $\widetilde{\mathfrak{po}}$ is not isomorphic to the other two double extensions.*

2) *Because $\operatorname{rk} D^{(0)}_0 = 8$ in $\widehat{\mathfrak{po}}$ and no element in $\mathfrak{po}_I$ and in $\widetilde{\mathfrak{po}}_I$ has such rank, it follows that $\widehat{\mathfrak{po}}_I$ is not isomorphic to the other two double extensions.*

3) *Because $\operatorname{rk}\operatorname{ad}_{p_i} = \operatorname{rk}\operatorname{ad}_{\pi_i} = 7$ in $\tilde{\mathfrak{b}}(2|2)$ and no element in $\mathfrak{b}(2|2)$ has such rank, $\tilde{\mathfrak{b}}(2|2) \not\simeq \mathfrak{b}(2|2)$.*

6 The general cases. Outer derivations

Let $\bar{X}$ be the product of all indeterminates. Derivations of $\mathfrak{a}$ will be called *equivalent* if they lie on one orbit of the group of automorphisms of $\mathfrak{der}\ \mathfrak{a}$.

6.1 $\mathfrak{out}(\mathfrak{h}_{\Pi}^{(1)}(0|2n))$ Notation for convenience (tags "($\mathfrak{lc}$ odd)" are used in Subsection 6.13):

1. Basis: $B = \{\eta_1, \dots, \eta_n; \xi_1, \dots, \xi_n\}$.
2. The set of products of any k basis elements (Choose): $Ch_B(k) = \{x_1 \dots x_k \mid x_i \in B\}$; $Ch_\xi(k) = \{x_1 \dots x_k \mid x_i \in \{\xi_1, \dots, \xi_n\}\}$; $Ch_\eta(k) = \{x_1 \dots x_k \mid x_i \in \{\eta_1, \dots, \eta_n\}\}$. For example,
$$Ch_B(2) = \{\eta_i\eta_j, \xi_i\xi_j \mid \text{for all } \ i \neq j \leq n\} \cup \{\eta_j\xi_i \mid i, j \leq n\}.$$
3. Index of elements: $Ind(x) =$ The set of indices of x in terms of η, ξ. For example, $Ind(\eta_1\xi_3) = \{1, 3\}$, $Ind(\eta_2\xi_5\xi_7) = \{2, 5, 7\}$.
4. The switch symbol: $S(\xi_i) = \eta_i$ and $S(\eta_i) = \xi_i$.
5. Let O denote the set of all monomials of odd degree.

(a) Deg=0. There are $2n$ equivalent derivations: for any $b \in B$, we have
$$D_b^{(0)} := \sum_{0 \leq i \leq 2n-2} \sum_{x \in Ch(i)} (bx) \otimes (\widehat{S(b)x}). \quad (\mathfrak{lc}\text{: always odd}) \tag{32}$$

(b) Deg=0. One particular derivation — the Euler operator
$$D_0^{(0)} = \sum_{1 \leq i \leq n-1} \sum_{0 \leq j \leq 2i+1} \sum_{x \in Ch_\xi(2i)} \sum_{y \in Ch_\eta(j)} (xy) \otimes (\widehat{xy}). \tag{33}$$

(c)Deg$= 2n - 2$.
$$D^{(2n-2)} := \sum_{x \in B} \frac{\partial \bar{X}}{\partial x} \otimes (\widehat{S(x)}). \quad (\mathfrak{lc}\text{: odd with } n) \tag{34}$$

6.2 $\mathfrak{out}(\mathfrak{h}_{\Pi}^{(1)}(2n|0))$ The desuperization of the case $\mathfrak{out}(\mathfrak{h}_{\Pi}^{(1)}(0|2n))$; the same cocycles.

6.3 $\mathfrak{out}(\mathfrak{h}_{\Pi\Pi}^{(1)}(2a|2b))$ A partial desuperization of $\mathfrak{out}(\mathfrak{h}_{\Pi}^{(1)}(0|2a + 2b))$; the same cocycles.

6.4 $\mathfrak{out}(\mathfrak{h}_{I}^{(1)}(0|2n))$ Cocycles:

(a) Deg=0. There are $2n$ equivalent derivations: for any $b \in B$, we have
$$D_b^{(0)} := \sum_{0 \leq i \leq 2n-2} \sum_{x \in Ch(i)} (bx) \otimes (\widehat{S(b)x}). \tag{35}$$

(b) Deg=0. One particular derivation (the apparent asymmetry of θ_1 and θ_2 is due to SuperLie's aesthetic criteria)
$$D_\theta^{(0)} = \sum_{x \in \tilde{C}(\theta_1)} \theta_1 x \otimes (\widehat{\theta_1 x}). \tag{36}$$

(c) Deg=0. Another particular derivation — the Euler operator

$$D_0^{(0)} = \sum_{x\in O} x \otimes (\widehat{x}). \tag{37}$$

(d) Deg $= 2n-2$. See eq. (34).

6.5 $\mathfrak{out}(\mathfrak{h}_I^{(1)}(2n|0))$ The desuperization of the case $\mathfrak{out}(\mathfrak{h}_I^{(1)}(0|2n))$; the same cocycles.

6.6 $\mathfrak{out}(\mathfrak{h}_{II}^{(1)}(2a|2b))$ Cocycles:
(a) Deg $= 2a+2b-2$. See eq. (34).

6.7 $\mathfrak{out}(\mathfrak{h}^{(1)}(0|2n+1))$ Notation for convenience:

1. Basis: $B = \{\eta_1, \dots, \eta_n; \theta; \xi_1, \dots, \xi_n\}$ or $\tilde{B} = \{\eta_1, \dots, \eta_n; \xi_1, \dots, \xi_n\}$.
2. Let $C(x)$ denote the set of all monomials each of which is a multiple of x; let $\tilde{C}(x)$ denote the set of all monomials each not a multiple of x.
3. The switch symbol: $S(\xi_i) = \eta_i$ and $S(\eta_i) = \xi_i$, whereas $S(\theta) = \theta$.
4. Let O denote the set of all monomials of odd degree.

Cocycles:
(a) Deg=0. There are $2n$ equivalent derivations: of weight $2w$ for any $x \in \tilde{B}$ of weight w,

$$D_x^{(0)} := \sum_{y\in \tilde{C}(x)\bigcap \tilde{C}(S(x))} xy \otimes (\widehat{S(x)y}). \tag{38}$$

(b) Deg=0. One particular derivation — the Euler operator, see eq. (37).
(c) Deg $= 2n-1$: See eq. (34).

6.8 $\mathfrak{out}(\mathfrak{h}_{\Pi\Pi}^{(1)}(2a+1|2b+1))$ Cocycles:
(a) Deg=0. A particular derivation $D_\theta^{(0)}$, see eq. (36)
(b) Deg=0. Another particular derivation — the Euler operator, see eq. (37)
(c) Deg $= 2a+2b$. See eq. (34).

6.9 $\mathfrak{out}(\mathfrak{h}_{\Pi I}^{(1)}(2a|2b))$ Cocycles:
(a) Deg=0. There are $2b$ equivalent derivations, see eq. (38).
(b) Deg=0. A particular derivation $D_\theta^{(0)}$, see eq. (36)
(c) Deg=0. Another particular derivation — the Euler operator, see eq. (37)
(d) Deg $= 2a+2b-2$. See eq. (34).

6.10 $\mathfrak{out}(\mathfrak{h}_{I\Pi}^{(1)}(2a|2b))$ Same cocycles as for $\mathfrak{out}(\mathfrak{h}_{\Pi I}^{(1)}(2b|2a))$, but with p, q and ξ, η interchanged in all cocycles.

6.11 $\mathfrak{out}(\mathfrak{h}_{\Pi\Pi}^{(1)}(2a|2b+1))$ A partial desuperization of $\mathfrak{h}_{\Pi\Pi}^{(1)}(0|2n+1)$.

6.12 $\mathfrak{out}(\mathfrak{h}_{\Pi\Pi}^{(1)}(2a+1|2b))$ Same cocycles as for $\mathfrak{out}(\mathfrak{h}_{\Pi\Pi}^{(1)}(2b|2a+1))$, but with p, q and ξ, η interchanged in all cocycles.

6.13 $\mathfrak{out}(\mathfrak{le}^{(1)}(n|n))$ Same cocycles as for $\mathfrak{out}(\mathfrak{h}_{\Pi}^{(1)}(2n|0))$ with q, π replacing q, p; odd cocycles are marked in Subsection 6.1. Observe that $p(B) \equiv n \pmod 2$, so there are no $D_{\bar{1}}$-extensions for n odd, and no $D_{\bar{0}}$-extensions for n even, since the center c is always odd and $p(c) = p(B) + p(D)$.

7 The general cases. Double extensions

Note: in all cases of $\mathfrak{h}$ series, $p(B_{\mathfrak{a}})$ is congruent to the parity of the number of odd indeterminates; $p(B_{\mathfrak{a}}) \equiv a + \bar{1}$ for $\mathfrak{le}(a|a)$. Let $\bar{X}$ be the product of all indeterminates.

7.1 The double extensions of $\mathfrak{a} = \mathfrak{h}^{(1)}(0|2n+1)$, $\mathfrak{h}^{(1)}_{\Pi\Pi}(2a|2b+1)$, $\mathfrak{h}^{(1)}_{\Pi\Pi}(2a+1|2b)$ for $a+b=n>2$

The $D_{\bar{0}}$-extension The derivation $D_0^{(0)}$ does not preserve the NIS B. Indeed,

$$B_{\mathfrak{a}}(D_0^{(0)}(\theta), \frac{\partial \bar{X}}{\partial \theta}) = 1 \text{ while } B_{\mathfrak{a}}(\theta, D_0^{(0)}(\bar{X})) = 0.$$

Now, the derivations $D_x^{(0)}$ preserve the bilinear form $B_{\mathfrak{a}}$, and the proof is similar to that for $\mathfrak{h}^{(1)}_{\Pi}(0|4)$ in Subsection 5.2; all respective double extensions are isomorphic. The double extension by means of the derivation $D_x^{(0)}$ for any $x \in \tilde{B}$, see Subsection 6.7, is a Lie superalgebra denoted by $\widetilde{\mathfrak{po}}(0|5)$ in [BeB1] for $n=2$:

Derivation	preserves $B_{\mathfrak{a}}$	$s_{\mathfrak{g}}(c)$	Extension	Extension	Extension
$D_x^{(0)}$	Yes	0	$\widetilde{\mathfrak{po}}(0\|2n+1)$	$\widetilde{\mathfrak{po}}_{\Pi\Pi}(2a\|2b+1)$	$\widetilde{\mathfrak{po}}_{\Pi\Pi}(2a+1\|2b)$
$D_0^{(0)}$	No	–	–	–	–

(39)

The $D_{\bar{1}}$-extension In a lexicographically ordered basis on $\mathfrak{h}^{(1)}(0|5)$ the Gram matrix of NIS is $B = \text{antidiag}(1, ..., 1)$. The condition $B_{\mathfrak{a}}^T D^{(2n-1)} = D^{(2n-1)} B_{\mathfrak{a}}$ is easy to see. Besides, since $D^{(3)}|_{\mathfrak{g}_{\bar{0}}} = 0$, it follows that

$$B_{\mathfrak{a}}(D^{(2n-1)}(f), f) = 0 \quad \text{for any } f \in \mathfrak{a}_{\bar{0}}.$$

Now, since $0 = (D^{(2n-1)})^2 = \text{ad}_A$, it follows that $A = 0$ since $\mathfrak{a}$ has no center. We have, therefore, a parametric family of double extensions by means of $D^{(2n-1)}$, q, $A = 0$ and $m \in \mathbb{K}$ (see Theorem 3.4.1b). It is proved in [BeB1], that $\mathfrak{po}(0|5; m)$ for $m \neq 0$ is not isomorphic to $\mathfrak{po}(0|5; 0) := \mathfrak{po}(0|5)$, whereas $\mathfrak{po}(0|5; m) \simeq \mathfrak{po}(0|5; \tilde{m})$ for any pair $m\tilde{m} \neq 0$. The same arguments are true for any $n > 1$. Table (40) summarizes these results for any $m \in \mathbb{K}^{\times}$:

D	$q(a)$	$s_{\mathfrak{g}}(D)$	Extension	Extension	Extension
$D^{(2n-1)}$	yes	0	$\mathfrak{po}(0\|2n+1)$	$\mathfrak{po}_{\Pi\Pi}(2a\|2b+1)$	$\mathfrak{po}_{\Pi\Pi}(2a+1\|2b)$
$D^{(2n-1)}$	yes	mc	$\mathfrak{po}(0\|2n+1; m)$	$\mathfrak{po}_{\Pi\Pi}(2a\|2b+1; m)$	$\mathfrak{po}_{\Pi\Pi}(2a+1\|2b; m)$

(40)

7.2 The double extensions of $\mathfrak{a} = \mathfrak{h}^{(1)}_{\Pi}(0|2n)$ for $n > 2$ and its desuperizations

D	q	$\mathfrak{h}^{(1)}_{\Pi}(0\|2n)$	$\mathfrak{h}^{(1)}_{\Pi}(2n\|0)$	$\mathfrak{h}^{(1)}_{\Pi\Pi}(2a\|2b)$	$\mathfrak{h}^{(1)}_{\Pi\Pi}(2a+1\|2b+1)$
$D_b^{(0)}$	yes	$\widetilde{\mathfrak{po}}_{\Pi}(0\|2n)$	$\widetilde{\mathfrak{po}}_{\Pi}(2n\|0)$	$\widetilde{\mathfrak{po}}_{\Pi\Pi}(2a\|2b)$	$\widetilde{\mathfrak{po}}_{\Pi\Pi}(2a+1\|2b+1)$
$D_0^{(0)}$	no	–	–	–	–
$D^{(2n-2)}$	yes	$\mathfrak{po}_{\Pi}(0\|2n)$	$\mathfrak{po}_{\Pi}(2n\|0)$	$\mathfrak{po}_{\Pi\Pi}(2a\|2b)$	$\mathfrak{po}_{\Pi\Pi}(2a+1\|2b+1)$

(41)

7.3 The double extensions of $\mathfrak{a} = \mathfrak{h}_I^{(1)}(0|2n)$ for $n > 2$ and its desuperizations Clearly, the results in cases $I\Pi$ and ΠI are obtained from one another.

D	q	$\mathfrak{h}_I^{(1)}(0\vert 2n)$	$\mathfrak{h}_I^{(1)}(2n\vert 0)$	$\mathfrak{h}_{II}^{(1)}(2a\vert 2b)$	$\mathfrak{h}_{\Pi I}^{(1)}(2a\vert 2b)$
$D_0^{(0)}$	yes	$\widehat{\mathfrak{po}}_I(0\vert 2n)$	$\widehat{\mathfrak{po}}_{II}(2n\vert 0)$	$\widehat{\mathfrak{po}}_{II}(2a\vert 2b)$	$\widehat{\mathfrak{po}}_{\Pi I}(2a\vert 2b)$
$D_b^{(0)}$	yes	$\widetilde{\mathfrak{po}}_I(0\vert 2n)$	$\widetilde{\mathfrak{po}}_{II}(2n\vert 0)$	$\widetilde{\mathfrak{po}}_{II}(2a\vert 2b)$	$\widetilde{\mathfrak{po}}_{\Pi I}(2a\vert 2b)$
$D_\theta^{(0)}$	–	–	–	–	–
$D^{(2n-2)}$	yes	$\mathfrak{po}_I(0\vert 2n)$	$\mathfrak{po}_I(2n\vert 0)$	$\mathfrak{po}_{II}(2a\vert 2b)$	$\mathfrak{po}_{\Pi I}(2a\vert 2b)$

(42)

7.4 The double extensions of $\mathfrak{a} = \mathfrak{le}^{(1)}(n|n)$

D	A	$\mathfrak{le}^{(1)}(2n\vert 2n)$	A	$\mathfrak{le}^{(1)}(2n+1\vert 2n+1)$
$D_b^{(0)}$	0	$\tilde{\mathfrak{b}}(2n\vert 2n)$	–	–
$D_0^{(0)}$	–	–	–	–
$D^{(4n-2)}$	0	$\mathfrak{b}(2n\vert 2n)$	0	$\mathfrak{b}(2n+1\vert 2n+1)$

(43)

7.4.1. Lemma. 1) *We have* $\operatorname{rk} D^{(2n-2)} = 2n$ *in* $\mathfrak{po}$*, and there is no such element in both* $\widetilde{\mathfrak{po}}$ *and* $\widehat{\mathfrak{po}}$*. The rank of* $D_0^{(0)}$ *in* $\widehat{\mathfrak{po}}$ *is* $2^{2n-1} \neq 2n$ *for* $n > 1$*, and there is no such element in* $\widetilde{\mathfrak{po}}$*, so* $\widetilde{\mathfrak{po}}$ *and* $\widehat{\mathfrak{po}}$ *are non-isomorphic double extensions and each of them is not isomorphic to* $\mathfrak{po}$*.*

2) *Because* $\operatorname{rk}\operatorname{ad}_{p_i} = \operatorname{rk}\operatorname{ad}_{\pi_i} = 2^{4n-1} - 1$ *in* $\tilde{\mathfrak{b}}(2n|2n)$ *and no element in* $\mathfrak{b}(2n|2n)$ *has odd rank,* $\tilde{\mathfrak{b}}(2n|2n) \not\simeq \mathfrak{b}(2n|2n)$*.*

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