

QUOTIENT GRAPHS AND AMALGAM PRESENTATIONS FOR UNITARY GROUPS OVER CYCLOTOMIC RINGS

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ABSTRACT. Suppose $4|n$, $n \geq 8$, $F = F_n = \mathbf{Q}(\zeta_n + \bar{\zeta}_n)$, and there is one prime $\mathfrak{p} = \mathfrak{p}_n$ above 2 in F_n . We study amalgam presentations for $\mathrm{PU}_2(\mathbf{Z}[\zeta_n, 1/2])$ and $\mathrm{PSU}_2(\mathbf{Z}[\zeta_n, 1/2])$ with the Clifford-cyclotomic group in quantum computing as a subgroup. These amalgams arise from an action of these groups on the Bruhat-Tits tree $\Delta = \Delta_{\mathfrak{p}}$ for $\mathrm{SL}_2(F_{\mathfrak{p}})$ constructed via the Hamilton quaternions. We explicitly compute the finite quotient graphs and the resulting amalgams for $8 \leq n \leq 48$, $n \neq 44$, as well as for $\mathrm{PU}_2(\mathbf{Z}[\zeta_{60}, 1/2])$.

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1. INTRODUCTION

This is the third in a series of papers devoted to the structure of unitary groups over cyclotomic rings. The first of these papers [IJK⁺19a] concerned the Euler-Poincaré characteristic of these groups. This invariant was sufficient, following Serre, to prove a conjecture of Sarnak [Sar15, p. 15^{IV}] on when these groups are generated by the Hadamard gate and the T-gate—two specific elements of finite order [IJK⁺19a, Theorem 1.2]. The second paper [IJK⁺19b] analyzed the corank of these groups, a more difficult invariant than the Euler-Poincaré characteristic, but only in the families $n = 2^s$ and $n = 3 \cdot 2^s$ where simplifications occur. In this paper we consider the case of general n , subject to the standing assumption that $n = 2^s d$, d odd, $s \geq 2$, $n \geq 8$, and Hypothesis 3 below: $\langle 2, -1 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$. Here we continue the method of [IJK⁺19b], analyzing an action of these groups on Bruhat-Tits trees Δ together with the resulting finite quotient graphs, with the emphasis on computing examples.

Set $\zeta_n = e^{2\pi i/n}$. The cyclotomic field $K_n := \mathbf{Q}(\zeta_n)$ has integers $\mathcal{O}_n := \mathbf{Z}[\zeta_n]$ and totally real subfield $F_n := K_n^+ = \mathbf{Q}(\zeta_n + \bar{\zeta}_n)$ with integers $\mathcal{O}_n^+ := \mathbf{Z}[\zeta_n]^+ = \mathbf{Z}[\zeta_n + \bar{\zeta}_n]$. We set $R_n := \mathcal{O}_n[1/2]$ and $\underline{R}_n := R_n^+ = \mathcal{O}_n^+[1/2]$. By our assumption on n , the cyclic group of roots of unity in K_n is generated by ζ_n and contains i . Also $F_n \neq \mathbf{Q}$ and the \mathcal{O}_n -ideal (2) is the square of an ideal of \mathcal{O}_n , which we will denote by $\mathfrak{q} = \mathfrak{q}_n$. Let \mathbf{H} be the Hamilton quaternions over \mathbf{Q} (the rational quaternion algebra ramified precisely at 2 and ∞), and put $\mathbf{H}_n = \mathbf{H} \otimes_{\mathbf{Q}} F_n$. We fix a \mathbf{Q} -basis $1, i, j, k$ of \mathbf{H} satisfying $i^2 = j^2 = k^2 = -1$, $ij = -ji$, $ik = -ki$, $jk = -kj$. The *standard* maximal \underline{R}_n -order of \mathbf{H}_n is

$$\widetilde{\mathcal{M}}_n := \underline{R}_n \langle 1, i, j, (1 + i + j + k)/2 \rangle.$$

Define the Hadamard matrix H and the matrix T_n by

$$H := \frac{1}{2} \begin{bmatrix} 1+i & 1+i \\ 1+i & -1-i \end{bmatrix} \quad \text{and} \quad T_n := \begin{bmatrix} 1 & 0 \\ 0 & \zeta_n \end{bmatrix}; \quad (1)$$

we have $H, T_n \in \mathrm{U}_2(R_n)$. The *Clifford-cyclotomic group* [FGKM15, Section 2.2] (resp., *special Clifford-cyclotomic group*) is

$$\mathcal{G}_n = \langle H, T_n \rangle \quad (\text{resp., } \mathrm{S}\mathcal{G}_n = \mathcal{G}_n \cap \mathrm{SU}_2(R_n)). \quad (2)$$

Put

$$\mathrm{U}_2^\zeta(R_n) = \{\gamma \in \mathrm{U}_2(R_n) \mid \det \gamma \in \langle \zeta_n \rangle\}; \quad (3)$$

we then have $\mathcal{G}_n \subseteq \mathrm{U}_2^\zeta(R_n) \subseteq \mathrm{U}_2(R_n)$. In general, $\mathrm{U}_2^\zeta(R_n) \subsetneq \mathrm{U}_2(R_n)$.

Various subgroups and quotient groups of $\mathrm{U}_2(R_n)$ and $\mathrm{SU}_2(R_n)$ occur throughout this paper. It is convenient to use the following notation:

Notation 1.

$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a <i>normal</i> subgroup of G
$H \ll G$	$H \leq G$ and $[G : H] = \infty$
$H \triangleleft\triangleleft G$	$H \trianglelefteq G$ and $[G : H] = \infty$
$H \lesssim G$	$H \leq G$ and $[G : H] < \infty$
$H \lesssim\triangleleft G$	$H \trianglelefteq G$ and $[G : H] < \infty$

For $H \leq U_2(R_n)$ denote by PH the image of H in $PU_2(R_n)$. For $H \leq \mathbf{H}_n^\times$, put $H_1 = \{h \in H \mid N_{\mathbf{H}_n/E_n}(h) = 1\}$; we have $H_1 \trianglelefteq H$. For a group G , denote by $G_f \trianglelefteq G$ the (normal) subgroup generated by the elements of G of finite order. We have the subgroup structure

$$\begin{aligned} \mathcal{G}_n \leq U_2(R_n)_f \trianglelefteq U_2^\zeta(R_n) \trianglelefteq U_2(R_n), \quad PU_2^\zeta(R_n) \lesssim PU_2(R_n), \quad \text{and} \\ S\mathcal{G}_n \leq SU_2(R_n)_f \trianglelefteq SU_2(R_n). \end{aligned} \quad (4)$$

If $U_2^\zeta(R_n) \neq U_2(R_n)$, then $U_2^\zeta(R_n) \ll U_2(R_n)$. The structure of $P\mathcal{G}_n$ is known from [RS99, Theorem 1]; see [IJK⁺19a, Theorem 4.1].

Theorem 2 (Radin and Sadun). *Let S_4 be the symmetric group on 4 letters and D_m be the dihedral group of order $2m$. Then $P\mathcal{G}_n \simeq S_4 *_{D_4} D_n$.*

For certain n there is a natural action of $U_2(R_n)$ and $SU_2(R_n)$ on a Bruhat-Tits tree Δ with finite stabilizers and finite quotient graph. The condition on n for these finite quotient graphs to exist is:

Hypothesis 3. $\langle 2, -1 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$.

Hypothesis 3 implies the following:

- (a) There is one prime $\mathfrak{p} = \mathfrak{p}_n$ of $F = F_n$ above 2 and $\mathbf{H} = \mathbf{H}_n$ is unramified at \mathfrak{p} .
- (b) There are explicit embeddings

$$\varphi_n : PSU_2(R_n) \xrightarrow{\sim} \Gamma_n \subseteq P\mathbf{H}_{n,1}^\times \quad \text{and} \quad \bar{\varphi}_n : PU_2(R_n) \xrightarrow{\sim} \bar{\Gamma}_n \subseteq P\mathbf{H}_n^\times$$

with $\bar{\varphi}_n|_{PSU_2(R_n)} = \varphi_n$ and $\Gamma_n = P\widetilde{\mathcal{M}}_{n,1}^\times = \widetilde{\mathcal{M}}_{n,1}/\langle \pm 1 \rangle$, see Section 4.2.

- (c) Let $\Delta = \Delta_{\mathfrak{p}}$ be the Bruhat-Tits tree for $SL_2(F_{\mathfrak{p}})$. Then $PGL_2(F_{\mathfrak{p}})$ acts on Δ . The identifications φ_n and $\bar{\varphi}_n$ above give an action of $PSU_2(R_n)$ and $PU_2(R_n)$ on Δ . There are finite quotient graphs $gr_n = \Gamma_n \backslash \Delta$ and $\bar{g}r_n = \bar{\Gamma}_n \backslash \Delta$. Moreover the stabilizers $\Gamma_{n,\mathbf{v}}$ and $\bar{\Gamma}_{n,\mathbf{v}}$ of a vertex $\mathbf{v} \in \text{Ver}(\Delta)$ in Γ_n and $\bar{\Gamma}_n$, respectively, are finite. Likewise the stabilizers $\Gamma_{n,\mathbf{e}}$ and $\bar{\Gamma}_{n,\mathbf{e}}$ are finite for an edge $\mathbf{e} \in \text{Ed}(\Delta)$. More generally there are quotient graphs-of-groups $Gr_n = (\Gamma_n, gr_n)$ and quotient h-graphs-of-groups $\bar{G}r_n = (\bar{\Gamma}_n, \bar{g}r_n)$. Knowing Gr_n and $\bar{G}r_n$ gives amalgam presentations of $PSU_2(R_n) \cong \pi_1(Gr_n)$ and $PU_2(R_n) \cong \pi_1(\bar{G}r_n)$ as in [Ser03] and Section 2 of this paper.

If Hypothesis 3 is *not* satisfied, then instead of quotient graphs one gets quotient regular cubical complexes of dimension $d \geq 2$ as in [JL00]. We do not treat these higher-dimensional quotients here. The first n for which Hypothesis 3 fails is $n = 68$.

The initial part of this paper, Sections 2–4.2, establishes the theoretical foundations for computing examples. Much of this material extends the results in [IJK⁺19b] for the specific families $n = 2^s$ and $n = 3 \cdot 2^s$ to general n . The highlights of the paper are in the second part, Sections 5–8, where we compute Gr_n and $\bar{G}r_n$ in MAGMA [BCP97] with corresponding amalgam presentations for $SU_2(R_n)$ and $PU_2(R_n)$ for $8 \leq n \leq 48$, $4|n$, $n \neq 44$. We give the quotient h-graph of groups $\bar{G}r_{60}$ and the corresponding amalgam presentation for $PU_2(R_{60})$.

A surprising feature of the examples is that we are able to identify $P\mathcal{G}_n$ as the fundamental group of a sub h-graph-of-groups of Gr_n . Subgroups of amalgamated products are not in

general sub amalgamated-products. But we get an amalgamated product presentation of $\pi_1(Gr_n) \cong \text{PU}_2(R_n)$ with $\text{PG}_n \cong S_4 *_{D_4} D_n$ as a sub amalgamated-product.

Here is part of the $n = 28$ example (see Section 5.2 for the definitions of the groups):

Example 4. (Section 7.2) Let A_m (resp., S_m) denote the alternating group (resp., symmetric group) on m letters, C_m the cyclic group of order m , D_m the dihedral group of order $2m$, and Q_{2m} the quaternion group of order $2m$. Denote the binary tetrahedral and octahedral groups by E_{24} and E_{48} , respectively.

(a) $\text{PU}_2(R_{28}) \cong D_{28} *_{C_{28}} D_{28} *_{D_4} S_4 *_{C_2^{*2}} = D_{28} *_{C_{28}} \text{PG}_{28} *_{C_2^{*2}}$.

(b) $\text{PG}_{28} \ll [\text{PU}_2(R_{28})]_f = \text{PU}_2(R_{28})$.

(c) $\text{PU}_2^\zeta(R_{28}) \cong S_4 *_{D_4} D_{28} *_{C_{28}} D_{28} *_{D_4} S_4 * \mathbf{Z}^{*2}$ and

$$\mathcal{G}_{28} \ll U_2(R_{28})_f \triangleleft U_2^\zeta(R_{28}) \triangleleft U_2(R_{28}).$$

(d) $\text{SU}_2(R_{28}) \cong E_{48} *_{Q_8} Q_{56} *_{C_{28}} Q_{56} *_{Q_8} E_{48} * \mathbf{Z}^{*4}$ and

$$\mathcal{SG}_{28} \ll \text{SU}_2(R_{28})_f \triangleleft \text{SU}_2(R_{28}).$$

Theorem 1.2 of [IJK⁺19a] already showed that $\mathcal{G}_{28} \ll U_2(R_{28})$ and $\mathcal{SG}_{28} \ll \text{SU}_2(R_{28})$. However, the explicit presentations and the further subgroup results above are new.

2. H-GRAPHS AND H-GRAPHS OF GROUPS

The standard reference for graphs constructed as quotients of trees by group actions is Serre's book [Ser03]. The generalization to h-graphs by Kurihara [Kur79] is treated in [IJK⁺19b, Section 1], which we use freely along with [Ser03]. Following [Ser03], a *graph* has oriented edges \mathbf{e} along with their opposites $\bar{\mathbf{e}}$, which are distinct. In an *h-graph*, the definition is relaxed to allow *half-edges*, edges \mathbf{e} with $\mathbf{e} = \bar{\mathbf{e}}$, as in [Kur79]. Edges \mathbf{e} with $\bar{\mathbf{e}} \neq \mathbf{e}$ are *regular edges*. Write $\text{Ed}_r(gr)$ and $\text{Ed}_h(gr)$ for the collection of regular and half-edges of gr respectively and $\text{Ed}(gr) := \text{Ed}_r(gr) \amalg \text{Ed}_h(gr)$ for the set of all edges. Half-edges \mathbf{e} originate and terminate at the same vertex $o(\mathbf{e}) = t(\mathbf{e})$. Every graph is also an h-graph.

Suppose gr is a finite connected h-graph with vertices $\text{Ver}(gr)$ and $v = v(gr) = \# \text{Ver}(gr)$. Set $e_r(gr) = \frac{1}{2} \# \text{Ed}_r(gr)$, $e_h = \frac{1}{2} \# \text{Ed}_h(gr)$, and $e = e(gr) := e_r(gr) + e_h(gr)$. The fundamental group $\pi_1(gr)$ has abelianization isomorphic to $H_1(gr, \mathbf{Z})$. The genus $\mathbf{g}(gr)$ of gr is the first Betti number $\text{rank } H_1(gr, \mathbf{Z})$. By Euler's formula $\mathbf{g}(gr) = 1 + e_r - v$.

Definition 5. A *graph of groups* [Ser03, Section 5] is a pair $Gr = (\Gamma, gr)$ with gr a graph and Γ an assignment $\mathbf{v} \mapsto \Gamma_{\mathbf{v}}$, $\mathbf{e} \mapsto \Gamma_{\mathbf{e}}$ of a group to each $\mathbf{v} \in \text{Ver}(gr)$, $\mathbf{e} \in \text{Ed}(gr)$ with $\Gamma_{\bar{\mathbf{e}}} = \Gamma_{\mathbf{e}}$ together with an injection $\Gamma_{\mathbf{e}} \hookrightarrow \Gamma_{t(\mathbf{e})}$ (denoted $g \mapsto g^{\mathbf{e}}$). For an edge $\mathbf{e} \in \text{Ed}(gr)$ we have injections $\Gamma_{\mathbf{e}} \hookrightarrow \Gamma_{t(\mathbf{e})}$ and $\Gamma_{\mathbf{e}} = \Gamma_{\bar{\mathbf{e}}} \hookrightarrow \Gamma_{t(\bar{\mathbf{e}})} = \Gamma_{o(\mathbf{e})}$ into the vertex groups of the origin and target vertices. The first sends $g \in \Gamma_{\mathbf{e}}$ to $g^{\mathbf{e}} \in \Gamma_{t(\mathbf{e})}$ and the latter to $g^{\bar{\mathbf{e}}} \in \Gamma_{o(\mathbf{e})}$.

A graph of groups $Gr = (\Gamma, gr)$ has a fundamental group $\pi_1(\Gamma, gr) = \pi_1(Gr)$ with a surjection

$$\pi_1(Gr) \twoheadrightarrow \pi_1(gr) \tag{5}$$

whose kernel is the normal closure of $\langle \Gamma_{\mathbf{v}} : \mathbf{v} \in \text{Ver}(gr) \rangle$.

If gr is connected and acyclic then $\pi_1(Gr) = \pi_1(\Gamma, gr)$ is the amalgamation of the vertex groups over the edge groups. If gr is connected but not acyclic, choose a spanning tree T by deleting a collection of edges $E \subset \text{Ed}(gr)$. Let (Γ, T) be the associated subgraph of groups. The fundamental group of Gr based at T , $\pi_1(Gr; T)$, is defined to be the group generated by

$\pi_1(\Gamma, T)$ together with generators $\{x_e : e \in E\}$ subject only to the relations that $g^e x_e = x_e g^{\bar{e}}$ for $g \in \Gamma_e$.

We denote the free product of the groups A and B by $A * B$ with $A^{*1} := A$ and $A^{*(n+1)} := A * A^{*n}$ for $n \geq 1$. So \mathbf{Z}^{*n} is the free group on n generators. Since $\#E = \mathbf{g}(gr)$ we have that

$$\pi_1(Gr, T) \simeq \pi_1(\Gamma, T) * \mathbf{Z}^{*\mathbf{g}(gr)} / \mathcal{R}. \quad (6)$$

with \mathcal{R} the relations on the x_e described above. It is a theorem [Ser03, Proposition I.20] that the isomorphism class of $\pi_1(Gr; T)$ does not depend on the choice of spanning tree T ; we therefore denote it by $\pi_1(Gr)$. There is a construction for $\pi_1(Gr)$ that does not require fixing a spanning tree, but for our purposes (explicit representations of $\pi_1(Gr)$ as amalgamated products) choosing a spanning tree is more convenient. Adjoining the generators x_e one at a time constructs $\pi_1(Gr)$ as a $\mathbf{g}(gr)$ -fold iterated HNN extension of the amalgam $\pi_1(Gr, T)$.

A group Γ acting on a graph gr determines a graph of groups $Gr := (\Gamma, gr)$ by assigning the stabilizer group Γ_e or Γ_v in Γ of an edge e or a vertex v . If Γ acts without inversions on gr there is an induced quotient graph of groups $\Gamma \backslash \backslash Gr$ with underlying graph $\Gamma \backslash gr$ defined as follows. Let $gr \xrightarrow{\pi} \Gamma \backslash gr$ be the quotient map. For $v \in \text{Ver}(\Gamma \backslash gr)$ choose $\tilde{v} \in \text{Ver}(gr)$ lying above v and set $\Gamma_v := \Gamma_{\tilde{v}}$. Similarly, for $e \in \text{Ed}(\Gamma \backslash gr)$ choose $\tilde{e} \in \text{Ed}(gr)$ with $\pi(\tilde{e}) = e$ set $\Gamma_e := \Gamma_{\tilde{e}}$. Also choose an element $g \in \Gamma$ with $t(g \cdot \tilde{e}) = t(\tilde{e})$ and define the injection $\Gamma_e \rightarrow \Gamma_{t(e)}$ as the composition

$$\Gamma_e = \Gamma_{\tilde{e}} \xrightarrow{g \cdot} \Gamma_{g \cdot \tilde{e}} \rightarrow \Gamma_{t(g \cdot \tilde{e})} = \Gamma_{\widetilde{t(\tilde{e})}} = \Gamma_{t(e)}.$$

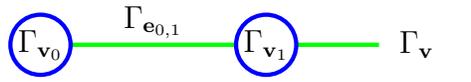
The choices of \tilde{e} , \tilde{v} , and g are arbitrary, subject to the above constraints, but once chosen are fixed. Notice that the maps $\Gamma_e \rightarrow \Gamma_{t(e)}$ are only well-defined up to conjugation by elements of $\Gamma_{t(e)}$.

Remark 6. Note that if $Gr' \subset Gr$ is a subgraph with all groups given by pullback, then there exists a natural injection $\pi_1(Gr') \rightarrow \pi_1(Gr)$.

The following is a key result in Bass-Serre theory:

Theorem 7 ([Ser03]). *Let Γ be a group which acts without inversions on a tree Δ and let $Gr := \Gamma \backslash \backslash (\Gamma, \Delta)$ be the associated quotient graph of groups. Then $\Gamma \simeq \pi_1(Gr)$.*

If Γ acts on a graph gr with inversions, let gr_Γ be the graph obtained from gr by subdividing exactly those edges that are inverted by Γ . By a *quotient h-graph of groups* for Γ acting on gr we mean $\Gamma \backslash \backslash (\Gamma, gr_\Gamma)$. When drawing h-graphs of groups we label each vertex and edge with its stabilizer group. We also draw all pairs $\{e, \bar{e}\}$ as a single undirected edge. In order to make clear the h-graph structure coming from $\Gamma \backslash gr$ we elide the extra vertices coming from the barycentric subdivision of inverted edges in an h-graph of groups. We only label the stabilizer subgroup of the corresponding half-edge if it differs from that of the elided vertex. An example is:



where v is a vertex lying under a vertex associated to the barycenter of an inverted edge in the tree.

Now suppose that Γ acts with inversions on a tree Δ . By Theorem 7, we know that $\Gamma \simeq \pi_1(\Gamma \backslash\backslash (\Gamma, \Delta_\Gamma))$. If $\Gamma_0 \subset \Gamma$ acts on Δ without inversions, then we have a cover

$$\Gamma_0 \backslash \Delta \rightarrow \Gamma \backslash \Delta$$

of an h-graph by a graph. We also have the cover of graphs

$$\Gamma_0 \backslash \Delta_\Gamma \rightarrow \Gamma \backslash \Delta_\Gamma$$

with the induced group injection

$$\pi_1(\Gamma_0 \backslash\backslash (\Gamma_0, \Delta_\Gamma)) \rightarrow \pi_1(\Gamma \backslash\backslash (\Gamma, \Delta_\Gamma)).$$

By the following theorem we also have an injection

$$\pi_1(\Gamma_0 \backslash\backslash (\Gamma_0, \Delta)) \rightarrow \pi_1(\Gamma \backslash\backslash (\Gamma, \Delta_\Gamma))$$

of the fundamental group of the quotient graph of groups for Γ_0 acting without inversions on Δ onto the fundamental group of the quotient h-graph of groups for Γ acting with inversions on Δ .

Theorem 8. *Let Γ act on a graph gr without inversions and let gr' be obtained from gr by subdividing all the edges in some set of edge orbits of Γ . Then*

$$\pi_1(\Gamma \backslash\backslash (\Gamma, gr)) \simeq \pi_1(\Gamma \backslash\backslash (\Gamma, gr')).$$

Proof. It suffices to consider a single edge in $\Gamma \backslash\backslash (\Gamma, gr)$



If subdivided in gr' this gives



in $\Gamma \backslash\backslash (\Gamma, gr')$. The fundamental group of the graph with the subdivided edge differs from that without only in that $G_0 *_G G_1$ is replaced by $G_0 *_G G *_G G_1$, which produces a canonically isomorphic group. \square

To compute amalgamated products for our examples we will need the following two theorems.

Theorem 9. *Suppose gr has a spanning tree T such that Γ_e is trivial for all $e \in \text{Ed}(gr) \setminus \text{Ed}(T)$. Then $\pi_1(\Gamma, gr) \simeq \pi_1(\Gamma, T) * \mathbf{Z}^{*\mathbf{g}}$, where $\mathbf{g} = \mathbf{g}(gr) = \#(\text{Ed}(gr) \setminus \text{Ed}(T))$.*

Proof. The additional generators $\{x_e : e \in \text{Ed}(gr) \setminus \text{Ed}(T)\}$ are subject only to the trivial relations $x_e = x_e$. \square

Theorem 10. *Let $Gr = (\Gamma, gr)$ be a graph of groups that consists of a single loop such that the stabilizer group of every edge and vertex is the same group G and the induced automorphism of G from the maps around the loop is inner. Then $\pi_1(Gr) \simeq G \oplus \mathbf{Z}$.*

Proof. Remove one edge e to form a spanning tree T . Now $\pi_1(Gr)$ is generated by $\pi_1(\Gamma, T) = G$ and an additional generator x_e subject to the constraint $g(x_e h) = x_e h g h^{-1} h = (x_e h) g$ for some h and all $g \in G = \pi_1(\Gamma, T)$. \square

It is clear that $\pi_1(\Gamma, T)$ lies in the kernel of (5). In the case that the $\Gamma_{\mathbf{v}}$ for $\mathbf{v} \in \text{Ver}(gr)$ are finite, then the kernel of (5) is the subgroup $\pi_1(\Gamma, gr)_f$ generated by all elements of $\pi_1(\Gamma, gr)$ of finite order. In particular, if Γ assigns the trivial group to each edge and vertex in gr , then (5) is an isomorphism.

We will use the following to show that $\text{PG}_n \ll (\text{PU}_{2n})_f$ in some cases.

Proposition 11. *Let (Γ, gr) be a connected graph of groups all of whose edge groups are finite, and let S a subtree of gr . Let T be a spanning tree of gr containing S with $\pi_1(\Gamma, T)$ infinite. Then either $\pi_1(\Gamma, T) = \pi_1(\Gamma, S)$ or else $\pi_1(\Gamma, S) \ll \pi_1(\Gamma, T)$. In the second case $\pi_1(\Gamma, S) \ll \pi_1(\Gamma, gr)_f$.*

Proof. If $\pi_1(\Gamma, S)$ is finite the result is trivial, so we will assume that it is infinite. Suppose that the natural map $\pi_1(\Gamma, S) \rightarrow \pi_1(\Gamma, T)$ is not surjective. Let T' be the tree obtained from collapsing S down to a single vertex \mathbf{s} : then $\text{Ver}(T') = (\text{Ver}(T) \setminus \text{Ver}(S)) \cup \{\mathbf{s}\}$. Edges between a vertex $\mathbf{v} \in \text{Ver}(T) \setminus \text{Ver}(S)$ and $\mathbf{w} \in \text{Ver}(S)$ now connect \mathbf{v} to \mathbf{s} . We make T' into a graph of groups (Γ', T') by defining $\Gamma_{\mathbf{s}} = \pi_1(\Gamma, S)$ and $\Gamma_{\mathbf{v}} = \Gamma_{\mathbf{v}}$ for $v \in \text{Ver}(T) \setminus \text{Ver} S$; the edge groups are the same as they were in (Γ, T) . We claim that $\pi_1(\Gamma', T') = \pi_1(\Gamma, T)$. Indeed, $\pi_1(\Gamma, T)$ is the amalgam of stabilizers of vertices of T over stabilizers of edges. As T is a tree, it does not matter in which order one amalgamates. Now $\pi_1(\Gamma', T')$ is obtained from (Γ, T) by first amalgamating over S to give (Γ', T') and then doing the remaining amalgamations in T . Since $\Gamma_{\mathbf{s}} = \pi_1(\Gamma, S) \neq \pi_1(\Gamma, T)$, there must be a vertex $\mathbf{v} \in \text{Ver}(T') \setminus \{\mathbf{s}\}$ with a path p between \mathbf{v} and \mathbf{s} such that all the intermediate vertices and edges along it have the same group Γ_p , but $\Gamma_{\mathbf{v}} \not\supseteq \Gamma_p$. We also have $\Gamma_{\mathbf{s}} = \pi_1(\Gamma, S)$ infinite, hence bigger than the finite group Γ_p . Thus, $\Gamma_{\mathbf{v}} *_{\Gamma_p} \Gamma_{\mathbf{s}}$ is a nontrivial amalgamation. Therefore, by the normal form for amalgams [LS01, Theorem IV.2.6], we have

$$\pi_1(\Gamma, S) = \Gamma_{\mathbf{s}} \ll \Gamma_{\mathbf{v}} *_{\Gamma_p} \Gamma_{\mathbf{s}} < \pi_1(\Gamma', T') = \pi_1(\Gamma, T) < \pi_1(\Gamma, T)_f.$$

□

3. UNITARY GROUPS OVER CYCLOTOMIC RINGS

Our notation will be consistent with that of [IJK⁺19b]. We assume $n = 2^s d$ with d odd and $s \geq 2$, $n \geq 8$; put $\zeta_n = e^{2\pi i/n}$. Let $K_n = \mathbf{Q}(\zeta_n)$. The ring of integers in K_n is $\mathcal{O}_n := \mathbf{Z}[\zeta_n]$ and its class group is $\text{Cl}(\mathcal{O}_n) = \text{Pic}(\mathcal{O}_n)$ with class number $h_n = \#\text{Cl}(\mathcal{O}_n)$. Put $R_n = \mathbf{Z}[\zeta_n, 1/2]$. If $H \leq K_n^\times$, put $H_1 := \{x \in H \mid x\bar{x} = 1\}$. Let $F_n = \mathbf{Q}(\zeta_n)^+$ with integers $\underline{\mathcal{O}}_n := \mathcal{O}_n^+ = \mathbf{Z}[\zeta_n + \bar{\zeta}_n]$, class group $\text{Cl}(\underline{\mathcal{O}}_n)$ with class number $h(\underline{\mathcal{O}}_n) = h_n^+$, and narrow class group $\tilde{\text{Cl}}(\underline{\mathcal{O}}_n)$ with narrow class number $\tilde{h}(\underline{\mathcal{O}}_n) = \tilde{h}_n^+$. Then $h_n = h_n^+ h_n^-$. Set $R_n = R_n^+ = \underline{\mathcal{O}}_n[1/2]$. For a subgroup $G \leq F_n^\times$, let G_+ be the subgroup of G consisting of totally positive elements: we have $G/G_+ \cong (\mathbf{Z}/2\mathbf{Z})^{c_G}$, where $0 \leq c_G \leq [F_n : \mathbf{Q}]$.

3.1. Cyclotomic Fields. Let \mathfrak{p}_i , $1 \leq i \leq r_+(n)$, be the $r_+(n)$ prime ideals in $\underline{\mathcal{O}}_n$ above the prime ideal (2) of \mathbf{Z} . If there is a unique prime above (2) in $\underline{\mathcal{O}}_n$, we denote it by $\mathfrak{p} = \mathfrak{p}(n)$. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_{r(n)}$ be the prime ideals of K_n above (2). If \mathfrak{p}_i splits in K_n , then $2r_+(n) = r(n)$; if \mathfrak{p}_i is inert or ramified in K_n , then $r_+(n) = r(n)$. If there is a unique prime above (2) in $\underline{\mathcal{O}}_n$, we denote it by $\mathfrak{P} = \mathfrak{P}(n)$.

Remark 12. We have $r_+(n) = r(n)$ if and only if $-1 \in \langle 2 \rangle \subseteq (\mathbf{Z}/d\mathbf{Z})^\times$.

We must determine various groups of units. It is well known that $\mathbf{Z}[\zeta_n]^\times \cong \mu_n \times \mathbf{Z}^{\phi(n)/2-1}$ and that $\mathbf{Z}[\zeta_n]_1^\times = \mu_n$. Further, R_n^\times is generated by $\mathbf{Z}[\zeta_n]^\times$ and one additional generator for each prime dividing 2 in K_n ; it is thus isomorphic to $\mu_n \times \mathbf{Z}^{\phi(n)/2-1+r(n)}$. Similarly, \underline{R}_n^\times is isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}^{\phi(n)/2-1+r_+(n)}$ and $\underline{R}_{n,+}^\times \cong \mathbf{Z}^{\phi(n)/2-1+r_+(n)}$. Recall that $r(n)$ is either $r_+(n)$ or $2r_+(n)$.

Now consider $R_{n,1}^\times := (R_n^\times)_1$. Let $N = \text{Norm}_{K_n/E_n}$. There is an exact sequence

$$1 \rightarrow R_{n,1}^\times \rightarrow R_n^\times \xrightarrow{N} \underline{R}_{n,+}^\times \rightarrow G \rightarrow 1,$$

where G is a finite group since $N(R_n^\times) \supseteq (\underline{R}_n^\times)^2$. Thus

$$R_{n,1}^\times \cong \begin{cases} \mu_n & r_+(n) = r(n), \\ \mu_n \times \mathbf{Z}^{r_+(n)} & 2r_+(n) = r(n); \end{cases} \quad (7)$$

a slightly weaker form of this statement is given in [FGKM15, Theorem 5.3]. It follows immediately that

$$R_{n,1}^\times / (R_{n,1}^\times)^2 \cong \begin{cases} \mu_n / \mu_n^2 & r_+(n) = r(n), \\ \mu_n / \mu_n^2 \times (\mathbf{Z}/2\mathbf{Z})^{r_+(n)} & 2r_+(n) = r(n). \end{cases} \quad (8)$$

Hence from (8) we get

$$R_{n,1}^\times / (R_{n,1}^\times)^2 \cong (\mathbf{Z}/2\mathbf{Z})^{1+r(n)-r_+(n)}. \quad (9)$$

We are interested in the groups $U_2(R_n)$ and $SU_2(R_n)$. The group $SU_2(\mathbf{Z}[\zeta_n])$ is finite: specifically it is the dihedral group of order $2n$. But $SU_2(R_n)$ (and *a fortiori* $U_2(R_n)$) is infinite. In fact, by strong approximation at the place 2 [Kne66, Main Theorem], $U_2(R_n)$ is a dense subgroup of $U_2(\mathbf{C})$.

We have natural inclusions

$$SU_2(R_n) \hookrightarrow U_2^\zeta(R_n) \hookrightarrow U_2(R_n).$$

For any complex unitary matrix A , the condition $A^{-1} = \overline{A}^t$ implies that $\alpha = \det(A)$ satisfies $\alpha\bar{\alpha} = 1$. Hence if $A \in U_2(R_n)$, then $\alpha = \det(A) \in R_{n,1}^\times$. Also, if $\alpha \in R_{n,1}^\times$, then $\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \in U_2(R_n)$. It follows that there is an exact sequence

$$1 \longrightarrow SU_2(R_n) \longrightarrow U_2(R_n) \xrightarrow{\det} R_{n,1}^\times \longrightarrow 1. \quad (10)$$

Proposition 13. ([FGKM15, Theorem 5.3]) *We have $U_2^\zeta(R_n) = U_2(R_n)$ if and only if $-1 \in \langle 2 \rangle \subseteq (\mathbf{Z}/d\mathbf{Z})^\times$.*

Proof. Combine the exact sequence (10) above with Remark 12 and (7). □

3.2. $PU_2(R_n)$, $PU_2^\zeta(R_n)$, and $PSU_2(R_n)$. We begin our study of unitary groups over cyclotomic rings by explaining the relationship between $PU_2(R_n)$, $PU_2^\zeta(R_n)$, and $PSU_2(R_n)$.

There is a commutative diagram with exact rows and columns (to save space we do not indicate the trivial groups on the sides):

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & R_{n,1}^\times & \xrightarrow{(\)^2} & (R_{n,1}^\times)^2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{SU}_2(R_n) & \longrightarrow & \mathrm{U}_2(R_n) & \xrightarrow{\det} & R_{n,1}^\times \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSU}_2(R_n) & \longrightarrow & \mathrm{PU}_2(R_n) & \xrightarrow{\det} & R_{n,1}^\times / (R_{n,1}^\times)^2.
\end{array} \tag{11}$$

The structure of $R_{n,1}^\times / (R_{n,1}^\times)^2$ is given in (8). In particular we have

- Proposition 14.** (a) $\mathrm{PU}_2(R_n) / \mathrm{PSU}_2(R_n) \cong (\mathbf{Z}/2\mathbf{Z})^{1+r(n)-r_+(n)}$.
(b) $\mathrm{PU}_2(R_n) / \mathrm{PU}_2^\zeta(R_n) \cong (\mathbf{Z}/2\mathbf{Z})^{r(n)-r_+(n)}$.
(c) $\mathrm{PU}_2^\zeta(R_n) / \mathrm{PSU}_2(R_n) \cong \mathbf{Z}/2\mathbf{Z}$.

Proof. By diagram (11) $\mathrm{PU}_2(R_n) / \mathrm{PSU}_2(R_n) \cong R_{n,1}^\times / (R_{n,1}^\times)^2$, hence (a) follows from (9).

Similarly, $\mathrm{PU}_2^\zeta(R_n) / \mathrm{PSU}_2(R_n) \cong \mu_n / \mu_n^2 \cong \mathbf{Z}/2\mathbf{Z}$ since by diagram (11) the determinant map is surjective. The claim (c) follows. \square

Assertion (b) now follows trivially. \square

If $r(n)_+ = r(n)$, i.e., if primes above 2 in F_n do not split in K_n , then the commutative diagram (11) becomes

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mu_n & \xrightarrow{(\)^2} & \mu_n^2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{SU}_2(R_n) & \longrightarrow & \mathrm{U}_2(R_n) & \xrightarrow{\det} & \mu_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{PSU}_2(R_n) & \longrightarrow & \mathrm{PU}_2(R_n) & \xrightarrow{\det} & \mu_n / \mu_n^2.
\end{array} \tag{12}$$

Proposition 15 below is elementary.

Proposition 15. *The following are equivalent:*

- (a) *There is a unique prime \mathfrak{p} of F_n above 2, i.e., $r_+(n) = 1$.*
(b) *We have $\langle 2, -1 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$.*

Proposition 16. *The following are equivalent:*

- (a) $r(n) = r_+(n)$.
(b) $-1 \in \langle 2 \rangle \subseteq (\mathbf{Z}/d\mathbf{Z})^\times$.
(c) $\mathrm{PU}_2(R_n) / \mathrm{PSU}_2(R_n) \cong \mathbf{Z}/2\mathbf{Z}$.
(d) $\mathrm{PU}_2(R_n) = \mathrm{PU}_2^\zeta(R_n)$.

Proof. The equivalence of (a) and (b) is elementary. The equivalence of (a) and (c) follows from diagram (11) and (8). The equivalence of (c) and (d) follows from Proposition 14(c). \square

3.3. **The Clifford-cyclotomic groups \mathcal{G}_n and $\mathcal{S}\mathcal{G}_n$.** The *Clifford group* \mathcal{C} can be defined as $\mathcal{C} = \mathrm{U}_2(R_4)$ [FGKM15, Section 2.1]. With T_n as in (1), define the *Clifford-cyclotomic group* [FGKM15, Section 2.2](resp., *special Clifford-cyclotomic group*) for $4|n$ by

$$\mathcal{G}_n = \langle \mathcal{C}, T_n \rangle \quad (\text{resp., } \mathcal{S}\mathcal{G}_n = \mathcal{G}_n \cap \mathrm{SU}_2(R_n)); \quad (13)$$

we have $\mathcal{G}_n \subseteq \mathrm{U}_2^\zeta(R_n)$. This definition agrees with (2) by [IJK⁺19a, Prop. 2.1]. For additional results on \mathcal{G}_n and $\mathcal{S}\mathcal{G}_n$ see [IJK⁺19a].

Proposition 17. *Suppose that $\mathrm{P}\mathcal{G}_n \ll [\mathrm{PU}_2(R_n)]_f$. Then $\mathcal{G}_n \ll \mathrm{U}_2(R_n)_f$, $\mathcal{S}\mathcal{G}_n \ll \mathrm{SU}_2(R_n)_f$, $\mathrm{PS}\mathcal{G}_n \ll [\mathrm{PSU}_2(R_n)]_f$.*

Proof. The subgroup of scalar matrices of \mathcal{G}_n and the image of the determinant homomorphism $\mathcal{G}_n \rightarrow \mathbf{C}$ are always finite; likewise for $\mathrm{U}_2(R_n)_f$. \square

4. THE HAMILTON QUATERNIONS AND UNITARY GROUPS

4.1. **The Hamilton quaternions.** Let \mathbf{H} be the Hamilton quaternions over \mathbf{Q} with a fixed \mathbf{Q} -basis $1, i, j, k$ satisfying $i^2 = j^2 = k^2 = -1$, $ij = -ji$, $ik = -ki$, $jk = -kj$. Put $\mathbf{H}_n = \mathbf{H} \otimes_{\mathbf{Q}} F_n$.

Proposition 18. *Let $n = 2^s d$ with d odd, $s \geq 2$, and $n \geq 8$. Then the quaternion algebra \mathbf{H}_n is unramified at the primes above 2 in F_n . Equivalently, \mathbf{H}_n is unramified at all finite primes of F_n .*

Proof. The quaternion algebra \mathbf{H}_n is unramified at \mathfrak{p}_i for $1 \leq i \leq r_+$ if and only if the order of the decomposition group

$$D(\mathfrak{p}_i) \subseteq \mathrm{Gal}(F_n/\mathbf{Q}) \simeq (\mathbf{Z}/n\mathbf{Z})^\times / \langle \pm 1 \rangle$$

is even. If $s > 2$, then F_n contains $\mathbf{Q}(\zeta_{2^s})^+$, in which $e(\mathfrak{p})$ is even. For $s = 2$, the extension $\mathbf{Q}(\zeta_n)$ has ramification index 2 above 2, with the inertia field being $\mathbf{Q}(\zeta_d) \not\subseteq F_n$ since $n \geq 8$. Thus F_n likewise has ramification index 2 above 2, so the decomposition group has even order as well. \square

The following assertion is elementary.

Proposition 19. *Let $n = 2^s d$ with d odd, $s \geq 2$, and $n \geq 8$. The following are equivalent:*

- (a) *There is a unique prime \mathfrak{p} of $F = F_n$ above 2 and the quaternion algebra \mathbf{H}_n is unramified at that prime \mathfrak{p} .*
- (b) *Hypothesis 3: $\langle 2, -1 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$.*

Proposition 20. *Let $n = 2^s d$ with d odd, $s \geq 2$, and $n \geq 8$. The following are equivalent:*

- (a) *There is a unique prime \mathfrak{p} of F_n above 2, the quaternion algebra \mathbf{H}_n is unramified at \mathfrak{p} , and \mathfrak{p} does not split in K_n .*
- (b) $\langle 2 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$.

Proof. Combine Proposition 19 with Proposition 16. \square

Obviously $n = 2^s$ satisfies the conditions in Proposition 20 for $s \geq 3$. However, $n = 8m$ also satisfies these conditions for $m \in \{3, 5, 6\}$, although not for $m = 7$. We will examine many of these graphs for n a small multiple of 4 explicitly in this paper.

Proposition 21. *Assume $n = 2^s d$ with d odd, $s \geq 2$, and $n \geq 8$ with $\langle 2, -1 \rangle = (\mathbf{Z}/d\mathbf{Z})^\times$. The following are equivalent:*

- (a) $H^1(\text{Gal}(K_n/F_n), R_n^\times) = 0$.
- (b) *If \mathfrak{p} splits as $\mathfrak{p} = \wp\bar{\wp}$ in K_n and r is the least positive integer such that $(\wp/\bar{\wp})^r = (\beta)$ is principal with $N_{K_n/F_n}(\beta) = 1$, then $(1 + \beta)\delta \in R_n^\times$ for some $\delta \in F_n$.*

Proof. Note that (a) is equivalent to the statement that every $\alpha \in R_{n,1}^\times$ is given by $\gamma/\bar{\gamma}$ for some $\gamma \in R_n^\times$. By (7), $R_{n,1}^\times \cong \mu_n$ if \mathfrak{p} does not split in K_n and $R_{n,1}^\times \cong \mu_n \times \mathbf{Z}$ if it does, where μ_n is generated by ζ_n and β is a generator of the \mathbf{Z} since it has norm 1 and is the “smallest” generator that does so.

Assume (b). By [IJK⁺19b, Lemma 3.9] $1 + \zeta_n \in R_n^\times$. Thus $\zeta_n = \gamma/\bar{\gamma}$ for $\gamma = 1 + \zeta_n \in R_n^\times$ and $\beta = \gamma/\bar{\gamma}$ for $\gamma = (1 + \beta)\delta \in R_n^\times$. Hence, since R_n^\times is generated by ζ_n and γ , we have $H^1(\text{Gal}(K_n/F_n), R_n^\times) = 0$ and (a) is true.

Conversely, assume (a). Then we have $\beta = \gamma/\bar{\gamma}$ for some $\gamma \in R_n^\times$. Let $\delta = \gamma/(1 + \beta)$. Then

$$\bar{\delta} = \frac{\bar{\gamma}}{1 + \bar{\beta}} = \frac{\gamma\beta}{1 + \bar{\beta}} = \frac{\gamma}{1 + \beta} = \delta.$$

Therefore, $\delta \in F_n$ and (b) follows. □

The *standard* maximal R_n -order of \mathbf{H}_n is

$$\widetilde{\mathcal{M}}_n := \underline{R}_n \langle 1, i, j, (1 + i + j + k)/2 \rangle.$$

Now for each n we choose an $\underline{\mathcal{O}}_n$ -maximal order

$$\mathcal{M}_n \supseteq \{1, i, j, (1 + i + j + k)/2\}.$$

The ideal (2) in $\underline{\mathcal{O}}_n$ is the square of an ideal $\mathfrak{q} = \mathfrak{q}_n$. Fix a set of generators $A = A(n)$ for \mathfrak{q} . For example, if $8|n$ we take $A = \{\sqrt{2}\}$; if not but $12|n$, then take $A = \{1 + \sqrt{3}\}$. Define the maximal $\underline{\mathcal{O}}_n$ -order $\mathcal{M}_n \subseteq \widetilde{\mathcal{M}}_n$ by

$$\mathcal{M}_n = \underline{\mathcal{O}}_n \langle 1, (1 + i)\alpha/2, (1 + j)\alpha/2, (1 + i + j + k)/2 \rangle, \quad (14)$$

where α runs over A . Observe that \mathcal{M}_n does not depend on the choice of generators $A = A(n)$ of $\mathfrak{q} = \mathfrak{q}_n$.

Remark 22. In general, $\underline{\mathcal{O}}_n \langle 1, i, j, (1 + i + j + k)/2 \rangle$ is not a maximal order of \mathbf{H}_n . Indeed, this order has discriminant (2); if \mathbf{H}_n is unramified at the primes above 2 (for example, if $n = 2^s$), then the discriminant of a maximal order of \mathbf{H}_n is the unit ideal. On the other hand, the order $\widetilde{\mathcal{M}}_n$ is a maximal \underline{R}_n -order, because 2 is a unit in \underline{R}_n .

Remark 23. In general the \underline{R}_n -type number of \mathbf{H}_n is not 1—there can be nonisomorphic \underline{R}_n -maximal orders of \mathbf{H}_n .

We now make definitions as in Kurihara [Kur79](who in turn follows Ihara [Iha66]):

Definition 24. Assume n satisfies Hypothesis 3 with \mathfrak{p} the unique prime of $F := F_n$ above 2. Note that $\mathbf{H}_n \otimes_F F_{\mathfrak{p}} = \text{Mat}_{2 \times 2}(F_{\mathfrak{p}})$. Set

$$\begin{aligned} \widetilde{\mathcal{M}}_{n,1}^\times &= \{m \in \widetilde{\mathcal{M}}_n^\times \mid \text{Norm}_{\mathbf{H}_n/F_n}(m) = 1\} \\ \widetilde{\mathcal{M}}_{n,+}^\times &= \{m \in \widetilde{\mathcal{M}}_n^\times \mid \text{val}_{\mathfrak{p}}(\text{Norm}_{\mathbf{H}_n/F_n}(m)) \text{ is even}\}. \end{aligned}$$

Define:

$$\begin{aligned}\Gamma_0 &= \Gamma_{0,n} = \Gamma_{0,n}(\widetilde{\mathcal{M}}_n) = \text{P}\widetilde{\mathcal{M}}_n^\times = \widetilde{\mathcal{M}}_n/\underline{R}_n^\times \\ \Gamma_+ &= \Gamma_{+,n} = \Gamma_{+,n}(\widetilde{\mathcal{M}}_n) = \text{P}\widetilde{\mathcal{M}}_{n,+}^\times = \widetilde{\mathcal{M}}_{n,+}^\times/\underline{R}_n^\times \\ \Gamma_1 &= \Gamma_{1,n} = \Gamma_{1,n}(\widetilde{\mathcal{M}}_n) = \text{P}\widetilde{\mathcal{M}}_{n,1}^\times = \widetilde{\mathcal{M}}_{n,1}^\times/\pm 1.\end{aligned}$$

Then $\Gamma_1 \subseteq \Gamma_+ \subseteq \Gamma_0$ are discrete, cocompact subgroups of $\text{PGL}_2(F_{\mathfrak{p}})$.

Recall that if $H \leq F_n^\times$, then H_+ is the subgroup of totally positive elements of H . Assume that n satisfies Hypothesis 3 with \mathfrak{p} the unique prime of $F := F_n$ above 2. Put

$$\underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times = \{x \in \underline{R}_{n,+}^\times \mid \text{val}_{\mathfrak{p}}(x) \text{ is even}\}.$$

The reduced norm map $N = \text{Norm}_{\mathbf{H}_n/F_n} : \mathbf{H}_n^\times \rightarrow F_n^\times$ induces maps

$$N_0 : \Gamma_0 \rightarrow \frac{\underline{R}_{n,+}^\times}{(\underline{R}_n^\times)^2}, \quad N_+ : \Gamma_+ \rightarrow \frac{\underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times}{(\underline{R}_n^\times)^2}, \quad N_1 : \Gamma_1 \rightarrow 1. \quad (15)$$

Let C_2 be the cyclic group of order 2, which we identify both with ± 1 and with \mathbf{F}_2 . For $1 \leq i \leq d = [F : \mathbf{Q}]$, let s_i be the map $F^\times \rightarrow C_2$ taking x to the sign of its image in the i -th real embedding of F . We then define the \mathfrak{p} -signature map $\text{sig}_{\mathfrak{p}} : \underline{R}_n^\times \rightarrow C_2^{d+1}$ by

$$\text{sig}_{\mathfrak{p}}(x) = (s_1(x), \dots, s_d(x), \text{val}_{\mathfrak{p}}(x) \bmod 2). \quad (16)$$

Proposition 25. (a) *The maps N_0, N_+, N_1 in (15) are surjective.*

(b) *There are isomorphisms*

$$\Gamma_0/\Gamma_+ \cong \frac{\underline{R}_{n,+}^\times}{\underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times} \quad \text{and} \quad \Gamma_+/\Gamma_1 \cong \frac{\underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times}{(\underline{R}_n^\times)^2}.$$

(c) *$\#\Gamma_0/\Gamma_+ \leq 2$, with equality if and only if the class $[\mathfrak{p}]$ of \mathfrak{p} in $\widetilde{\text{Cl}}(F)$ of F is of odd order.*

(d) *We have $\Gamma_+/\Gamma_1 \cong \text{Coker}(\text{sig}_{\mathfrak{p}}) \cong \mathbf{F}_2^r$ with $0 \leq r \leq d = [F : \mathbf{Q}]$.*

Proof. First we show that N_0 is surjective: pick any $x \in \underline{R}_{n,+}^\times$. Then by [IJK⁺19b, Lemma 3.19] there exists a $\gamma \in \widetilde{\mathcal{M}}$ of norm x . Observe that γ is a unit since its norm is. Thus γ gives an element of Γ_0 and N_0 is surjective. A similar argument holds for N_+ and N_1 .

To derive (b) from (a), note that all the definitions of the Γ 's are equivalent to the pullbacks under the reduced norm map of the groups in (15). Thus their quotients are the same as the quotients of the images of their norms.

With that done, (c) follows from the second isomorphism in (b). It is clear that $\#\Gamma_0/\Gamma_+ \leq 2$, and the class $[\mathfrak{p}]$ of \mathfrak{p} in $\widetilde{\text{Cl}}(F)$ is of odd order if and only if there is a totally positive element of F generating the ideal \mathfrak{p}^k for some odd k . If there is no such element, then $\underline{R}_{n,+}^\times = \underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times$ and the index is 1, whereas if there is such an element it generates the quotient and the index must be 2.

For (d), note that

$$\frac{\underline{R}_n^\times}{(\underline{R}_n^\times)^2} \cong \mathbf{F}_2^{d+1}.$$

The assertion then follows from (b) and the exact sequence

$$1 \longrightarrow \frac{\underline{R}_{n,+,\mathfrak{p}\text{-ev}}^\times}{(\underline{R}_n^\times)^2} \longrightarrow \frac{\underline{R}_n^\times}{(\underline{R}_n^\times)^2} \xrightarrow{\text{sig}_{\mathfrak{p}}} \mathbf{F}_2^{d+1} \longrightarrow \text{Coker}(\text{sig}_{\mathfrak{p}}) \longrightarrow 1$$

upon observing that $\dim_{\mathbf{F}_2} \text{Coker}(\text{sig}_{\mathfrak{p}}) \leq d$ since $\text{sig}_{\mathfrak{p}}(-1)$ is nontrivial. \square

Theorem 26. (a) *The groups $\Gamma_0, \Gamma_+, \Gamma_1$ are discrete cocompact subgroups of $\text{PGL}_2(F_{\mathfrak{p}})$. Let $\Delta = \Delta_{\mathfrak{p}}$. Then Γ_+, Γ_1 act on Δ without inversions and the quotients $gr_+ = \Gamma_+ \backslash \Delta$, $gr_1 = \Gamma_1 \backslash \Delta$ are finite bipartite graphs. The group Γ_0 acts on Δ possibly with inversions; the quotient $gr_0 = \Gamma_0 \backslash \Delta$ is a finite Kurihara graph.*

(b) *The natural covering $\pi: gr_+ \rightarrow gr_0$ is étale of degree 1 or 2. The degree is 2 if and only if the class $[\mathfrak{p}]$ in $\widetilde{\text{Cl}}(F)$ is of odd order.*

Proof. The assertion (a) follows from (c) and (d) of Proposition 25. A general discussion is in [Kur79, Section 4].

Part (b) follows from Proposition 25(c). \square

4.2. Connecting unitary groups to the Hamilton quaternions. Let $\mathbf{H}_{n,1}^{\times}$ be the subgroup of \mathbf{H}_n^{\times} of elements of norm 1. The following observation is standard and easy to check:

Proposition 27. *For all n , the map $\text{SU}_2(K_n) \rightarrow \mathbf{H}_{n,1}^{\times}$ defined by*

$$\begin{pmatrix} r + s\sqrt{-1} & t + u\sqrt{-1} \\ -t + u\sqrt{-1} & r - s\sqrt{-1} \end{pmatrix} \mapsto r - ui - tj - sk$$

is an isomorphism.

The map in Proposition 27 restricts to an isomorphism

$$\Psi_n: \text{SU}_2(R_n) \xrightarrow{\cong} \widetilde{\mathcal{M}}_{n,1}^{\times},$$

with an induced isomorphism

$$\overline{\Psi}_n: \text{PSU}_2(R_n) = \text{SU}_2(R_n) / \langle \pm 1 \rangle \xrightarrow{\cong} \text{P}\widetilde{\mathcal{M}}_{n,1}^{\times} := \widetilde{\mathcal{M}}_{n,1}^{\times} / \langle \pm 1 \rangle. \quad (17)$$

We now ask whether there is an isomorphism for PU_2 compatible with the isomorphism (17) for PSU_2 .

First we define a map $\varphi_n: \text{PU}_2(K_n) \mapsto \text{PH}_n^{\times}$.

Definition 28. For $A \in \text{U}_2(K_n)$, denote by $[A]$ its class in $\text{PU}_2(K_n)$. Similarly, for $a \in \mathbf{H}_n^{\times}$, denote by $[a]$ its class in PH_n^{\times} .

Suppose $A \in \text{U}_2(K_n)$ and $\alpha = \det(A)$ where $\alpha\bar{\alpha} = 1$. By Hilbert's Theorem 90 there is $\beta \in K_n^{\times}$ such that $\alpha = \bar{\beta}/\beta$. Let $A' = \beta A$. We have $\det A' = \beta^2 \alpha = \beta \bar{\beta} \in F_n$. Hence A' is of the form

$$A' = \begin{pmatrix} r + s\sqrt{-1} & t + u\sqrt{-1} \\ -t + u\sqrt{-1} & r - s\sqrt{-1} \end{pmatrix}$$

and we then define, for $[A] \in \text{PU}_2(R_n)$,

$$\overline{\varphi}_n([A]) = [r - ui - tj - sk] \in \text{PH}_n^{\times}. \quad (18)$$

Note that on $\text{PSU}_2(R_n)$ our map $\overline{\varphi}_n$ agrees with $\overline{\Psi}_n$.

Remark 29. Under the equivalent conditions of Proposition 19, the map $\overline{\varphi}_n$ makes $\text{PU}_2(R_n)$ a discrete subgroup of $\text{PGL}_2(F_{\mathfrak{p}})$, since $\text{PSU}_2(R_n)$ has finite index in $\text{PU}_2(R_n)$.

Applying $\bar{\varphi}_n$ to H and T_n given in (1) using $\beta = 1 + \sqrt{-1}, 1 + \zeta_n^{-1}$, respectively, we obtain

$$\bar{\varphi}_n([H]) = [h], \quad \bar{\varphi}_n([T_n]) = [t_n],$$

where

$$h := -i - k, \quad t_n = 1 + e^{2\pi k/n} := 1 + \frac{\zeta_n + \zeta_n^{-1}}{2} - \frac{(\zeta_n - \zeta_n^{-1})\sqrt{-1}k}{2}. \quad (19)$$

Remark 30. Clearly, $h \in \mathcal{M}_n$ when \mathcal{M}_n is of the form (14).

Theorem 31. *Recall that $4|n$ and $n > 4$. Let \mathcal{T}_n be the \mathcal{O}_n -order in \mathbf{H}_n generated by t_n and j .*

- (a) *If n is not a power of 2, then \mathcal{T}_n is maximal and $\mathcal{T}_n^\times / \mathcal{O}_n^\times \simeq D_n$, the dihedral group of order $2n$.*
- (b) *If n is a power of 2, then \mathcal{T}_n is not maximal. Its discriminant is \mathfrak{p}^2 and it is contained in exactly two maximal orders, the order generated by t_n and $(1+i)/(\zeta_n + \zeta_n^{-1})$ and its conjugate by t_n . The intersection of these two orders contains \mathcal{T}_n with quotient \mathfrak{p} , and conjugation by t_n exchanges them. In particular \mathcal{T}_n is not an Eichler order. Further, we have $\mathcal{T}_n^\times / \mathcal{O}_n^\times \simeq D_{n/2}$, the dihedral group of order n , unless $n = 8$, and likewise for \mathcal{M} a maximal order containing \mathcal{T}_n we have $\mathcal{M}^\times / \mathcal{O}_n^\times \simeq D_{n/2}$. Additionally, both $\mathcal{M} \cap \mathcal{M}^{t_n}$ and \mathcal{T}_n have stabilizer D_n .*

Proof. For this proof only, let $z = \zeta_n + \zeta_n^{-1}$. Consider the \mathcal{O}_n -submodule S_n generated by $1, t_n, j, t_n j$. Our first claim is that S_n is an order of discriminant $(z^2 - 4)$. To show that it is an order, we check that it is closed under multiplication. Indeed, t_n, j are integral, so their squares are still in the order, and once we show that $jt_n \in S_n$ the remaining products will follow by associativity. In fact

$$jt_n = (z + 2)j - t_n j. \quad (20)$$

To evaluate $\text{Disc } S_n$, we compute the matrix of traces of products of the basis vectors, obtaining

$$\begin{bmatrix} 2 & z+2 & 0 & 0 \\ z+2 & z^2+2z & 0 & 0 \\ 0 & 0 & -2 & -z-2 \\ 0 & 0 & -z-2 & -2z-4 \end{bmatrix}.$$

The top left 2×2 block has determinant $z^2 - 4$, the bottom right $-(z^2 - 4)$, so our claim follows.

This proves that $S_n = \mathcal{T}_n$ is a maximal order in the case where n is not a power of 2, since in that case $z^2 - 4 = (\zeta_n^2 - 1)^2 / \zeta_n^2$ is a unit. When n is a power of 2, the determinant generates the 4th power of the prime \mathfrak{p} of \mathcal{O}_n above 2, so the discriminant is \mathfrak{p}^2 and an order that contains S_n with quotient $(\mathcal{O}_n / \mathfrak{p})^2$ is maximal. We may enlarge the order by adjoining any of $1 + j, (t_n - 1)(1 + j), t_n(1 + j)$ divided by z , or their conjugates by t_n . It is easily checked that all of these orders contain $t_n(1 + j)/z$, and that $t_n(1 + j)/z$ generates an Eichler order of discriminant \mathfrak{p} as described.

To prove that the unit group is as claimed, we notice that by construction t_n is the image of the matrix T_n , whose order in PGL_2 is exactly n . It also holds that $jt_n j^{-1} = t_n^{-1}$ up to scalars, so in the first case where t_n is a unit we obtain the dihedral group D_n as a subgroup of $\mathcal{T}_n^\times / \mathcal{O}_n^\times$. In the second case t_n is not a unit; however, t_n^2/z is a unit, and this gives a unit

group of $D_{n/2}$ as claimed. To show that $\mathcal{M}^\times/\mathcal{Q}_n^\times$ is no larger than this, we consider the list of maximal subgroups of $\mathrm{PGL}_2(\mathbf{C})$. The tetrahedral, octahedral, and icosahedral groups have no dihedral subgroups larger than D_5 , so these are excluded except in the case $n = 8$, in which we find \mathcal{S}_4 ; the only other possibility is that the unit group is D_{kn} for some $k > 1$. In the first case, that is not possible, because the subring of \mathbf{H}_n obtained by adjoining an element of order kn to F_n would be a subfield of degree $\phi(kn)/[F_n : \mathbf{Q}] > 2$, a contradiction. In the second case it also does not occur. The same argument shows that we would have to have $k = 2$, and the only possible unit of order n would be a scalar multiple of t_n . However, no such multiple has unit norm, so we cannot obtain an automorphism of a maximal order this way. On the other hand, conjugation by t_n is of order 2, since t_n^2 is a scalar. Thus the Eichler order $\mathcal{M} \cap \mathcal{M}^{t_n}$ and its canonically defined suborder \mathcal{T}_n are preserved by conjugation by t_n ; this has order n and, together with the $D_{n/2}$, generates a group isomorphic to D_n . The same argument as in case 1 shows that this is the stabilizer of $\mathcal{M} \cap \mathcal{M}^{t_n}$ and \mathcal{T}_n . \square

Theorem 32. *Assume Hypothesis 3 and use the notation of Definition 24. For the map $\bar{\varphi}_n$ in (18)*

$$\bar{\varphi}_n(\mathrm{PU}_2(R_n)) \subset \mathrm{P}\widetilde{\mathcal{M}}_n^\times = \Gamma_0(\mathcal{M}_n, \mathfrak{p}) := \Gamma_{0,n}$$

if and only if the equivalent conditions of Proposition 21 are satisfied.

Proof. Using the notation above, notice that $H^1(\mathrm{Gal}(K_n/F_n), R_n^\times) = 0$ implies that $\beta \in R_n^\times$. Hence we end up with $\bar{\varphi}_n([A]) \in \mathrm{P}\widetilde{\mathcal{M}}_n^\times$.

Conversely, suppose $\bar{\varphi}_n(\mathrm{PU}_2(R_n)) \subset \mathrm{P}\widetilde{\mathcal{M}}_n^\times$. Pick any $\alpha \in R_n^\times$ with $\alpha\bar{\alpha} = 1$. Let $A \in \mathrm{U}_2(R_n)$ have $\det(A) = \alpha$. Then $\varphi_n([A]) = [r - ui - tj - sk] \in \mathrm{P}\widetilde{\mathcal{M}}_n^\times$; set $w := r - ui - tj - sk \in \widetilde{\mathcal{M}}_n^\times$. Now let

$$A' = \begin{pmatrix} r + s\sqrt{-1} & t + u\sqrt{-1} \\ -t + u\sqrt{-1} & r - s\sqrt{-1} \end{pmatrix} \in \mathrm{SU}_2(R_n).$$

By the construction of φ_n we have $A' = \beta A$ for some $\beta \in F_n^\times$ and by the above inclusion we must have $\beta \in R_n^\times$. Comparing determinants we get $\beta^2\alpha = \det(A') = N(w)$. Taking norms down to \underline{R}_n gives $(\beta\bar{\beta})^2 = N(w)^2$. Hence $\beta\bar{\beta} = N(w)$ since both quantities are totally positive. Therefore, $\alpha = N(w)/\beta^2 = \bar{\beta}/\beta$ with $\beta \in R_n^\times$. Since $[K_n : F_n] = 2$, this shows that $H^1(\mathrm{Gal}(K_n/F_n), R_n^\times) = 0$, which implies the equivalent conditions of Proposition 21. \square

We now ask when the map $\bar{\varphi}_n : \mathrm{PU}_2(R_n) \rightarrow \mathrm{P}\widetilde{\mathcal{M}}_n^\times = \Gamma_0(\widetilde{\mathcal{M}}_n)$ is an isomorphism.

Theorem 33. *Suppose the equivalent conditions of Proposition 21 are satisfied; retain the notation of Theorem 32. For $\alpha \in \underline{R}_{n,+}^\times$, denote by $[\alpha]$ its class in $\underline{R}_{n,+}^\times/(\underline{R}_n^\times)^2$.*

(a) *If \mathfrak{p} does not split in K_n , then $\bar{\varphi}_n$ is an isomorphism onto $\mathrm{P}\widetilde{\mathcal{M}}_n^\times = \Gamma_0(\widetilde{\mathcal{M}}_n)$ if and only if the map below with a ? is an isomorphism:*

$$\Gamma_{0,n}/\Gamma_{1,n} = \langle [(1 + \zeta_n)(1 + \bar{\zeta}_n)] \rangle \simeq \frac{\underline{R}_{n,+}^\times}{(\underline{R}_n^\times)^2} \stackrel{?}{\simeq} \mathbf{Z}/2\mathbf{Z} \simeq \frac{\mathrm{PU}_2(R_n)}{\mathrm{PSU}_2(R_n)} \xrightarrow{\simeq} \frac{R_{n,1}^\times}{(R_{n,1}^\times)^2}.$$

(b) *If \mathfrak{p} splits in K_n , then $\bar{\varphi}_n$ is an isomorphism onto $\mathrm{P}\widetilde{\mathcal{M}}_n^\times = \Gamma_0(\widetilde{\mathcal{M}}_n)$ if and only if the map below with a ? is an isomorphism:*

$$\Gamma_{0,n}/\Gamma_{1,n} \simeq \frac{\underline{R}_{n,+}^\times}{(\underline{R}_n^\times)^2} \stackrel{?}{\simeq} \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \simeq \frac{\mathrm{PU}_2(R_n)}{\mathrm{PSU}_2(R_n)} \xrightarrow{\simeq} \frac{R_{n,1}^\times}{(R_{n,1}^\times)^2}.$$

In this case note that

$$\langle [(1 + \zeta_n)(1 + \bar{\zeta}_n)] \rangle \simeq \mathbf{Z}/2\mathbf{Z} \subseteq \frac{R_{n,+}^\times}{(R_n^\times)^2} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

Proof. From the diagram (which omits 1's on the right/left and top/bottom) with Norm map $N_{0,n} : \Gamma_{0,n} \rightarrow \underline{R}_{n,+}^\times$

$$\begin{array}{ccccc} \mathrm{PSU}_2(R_n) & \xrightarrow{\simeq} & \Gamma_{1,n} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PU}_2(R_n) & \xrightarrow{\phi_n} & \Gamma_{0,n} & \xrightarrow{N_{0,n}} & \mathrm{coker} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PU}_2(R_n)/\mathrm{PSU}_2(R_n) \simeq R_{n,1}^\times/(R_{n,1}^\times)^2 & \longrightarrow & \Gamma_{0,n}/\Gamma_{1,n} \simeq \underline{R}_{n,+}^\times/(R_n^\times)^2 & \xrightarrow{N_{0,n}} & \mathrm{coker}, \end{array}$$

we see that $\bar{\varphi}_n$ is an isomorphism if and only if

$$\frac{R_{n,1}^\times}{(R_{n,1}^\times)^2} \xrightarrow{\simeq} \frac{R_{n,+}^\times}{(R_n^\times)^2} \quad (21)$$

is an isomorphism. The map in (21) is induced by $[\zeta_n] \mapsto [(1 + \zeta_n)(1 + \bar{\zeta}_n)]$. \square

Remark 34. Note in particular that by Weber's theorem [Web99] the hypotheses of Theorem 33 are always satisfied when n is a power of 2.

We now look at the group $\mathrm{PU}_2^\zeta(R_n)$ defined in the introduction.

Theorem 35. *Assume that n is not twice a prime power and use the notation of Definition 24. Then*

$$\bar{\varphi}_n(\mathrm{PU}_2^\zeta(R_n)) \subset \Gamma_{+,n}(\widetilde{\mathcal{M}}_n) := \Gamma_{+,n}.$$

Proof. Let $A \in \mathrm{U}_2^\zeta$ and $\zeta_n^k = \det(A)$. Since n is not twice a prime power, $N_{K_n/\mathbf{Q}}(\zeta_n + 1) = 1$; see [IJK⁺19b, Lemma 3.9]. So in Definition 28 we can take $\beta = (\zeta_n + 1)^{-k}$ since that $\zeta_n^k = \bar{\beta}/\beta$. Then with $A' = \beta A$ we have $\det A' = \beta^2 \zeta_n^k = \beta \bar{\beta} \in \mathcal{O}_{F_n}^\times$. Hence A' is of the form

$$A' = \begin{bmatrix} r + s\sqrt{-1} & t + u\sqrt{-1} \\ -t + u\sqrt{-1} & r - s\sqrt{-1} \end{bmatrix}$$

and

$$\bar{\varphi}_n([A]) = [r - ui - tj - sk] \in \Gamma_+(\widetilde{\mathcal{M}}_n).$$

\square

By Proposition 14(c), we always have $[\mathrm{PU}_2^\zeta(R_n) : \mathrm{PSU}_2(R_n)] = 2$, and thus $\bar{\varphi}_n|_{\mathrm{PU}_2^\zeta(R_n)}$ is an isomorphism precisely when $[\Gamma_{+,n} : \Gamma_{1,n}] = 2$. The condition for this is described in Proposition 25(d). Also notice that when n is a power of 2, we have $\mathrm{PU}_2^\zeta(R_n) = \mathrm{PU}_2(R_n)$ from Proposition 16.

4.3. The tree for $\mathrm{SL}_2(F_{n,\mathfrak{p}})$ via maximal orders in \mathbf{H}_n . Throughout this subsection and the next we assume Hypothesis 3.

Fix n and let $\Delta = \Delta_{\mathfrak{p}_n}$ be the Bruhat-Tits tree for $\mathrm{SL}_2(F_{n,\mathfrak{p}})$ with $F_{n,\mathfrak{p}}$ the completion of F_n at the prime $\mathfrak{p} = \mathfrak{p}_n$. Generalizing the discussion in [Kur79, Sect. 4], we may describe Δ in the following manner. Let \mathcal{M}_n be the maximal \mathcal{O} -order (14) of $\mathbf{H} = \mathbf{H}_n$. The vertices $\mathrm{Ver}(\Delta)$ are identified with the maximal \mathcal{O} -orders \mathcal{M} for which $\mathcal{M}_v := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}_v = \mathcal{M}_{n,v} := \mathcal{M}_n \otimes_{\mathcal{O}} \mathcal{O}_v$ for every place $v \neq \mathfrak{p}$ of \mathcal{O} . In this section we will identify the vertex $\mathbf{v} \in \mathrm{Ver}(\Delta)$ with its corresponding maximal order $\mathcal{M}_{\mathbf{v}}$. The edges originating from a vertex $\mathcal{M} \in \mathrm{Ver}(\Delta)$ correspond to left \mathcal{M} -ideals of norm \mathfrak{p} . The edge corresponding to an ideal I terminates at the right order of I . There are $\mathrm{Norm}(\mathfrak{p}) + 1$ edges originating from each vertex.

Let \mathbf{e} be an edge originating at a vertex \mathcal{M} and terminating at \mathcal{M}' , corresponding to the ideal I . The opposite edge $\bar{\mathbf{e}}$ then corresponds to the left \mathcal{M}' -ideal $\bar{I} = \mathfrak{p}I^{-1}$ where $I^{-1} = \{\alpha \in \mathbf{H} : I\alpha \subset \mathcal{M}\}$ and we have $\mathrm{Norm}(\bar{I}) = \mathfrak{p}$. Clearly $\mathcal{M}', \mathcal{M}$ are the left and right orders of \bar{I} and $\bar{I} = I$.

The undirected edge between \mathcal{M} and \mathcal{M}' is identified with the Eichler order $\mathcal{E} = \mathcal{M} \cap \mathcal{M}'$ of level \mathfrak{p} . The connection between the pair of directed edges I, \bar{I} and \mathcal{E} is that $\mathcal{E} = I + \bar{I}$ while I , respectively \bar{I} , is the unique maximal left \mathcal{M} -, respectively \mathcal{M}' -, ideal in \mathcal{E} . To see this, conjugate and choose a basis locally at \mathfrak{p} so that

$$\mathcal{E}_{\mathfrak{p}} = \begin{bmatrix} \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \\ \mathfrak{p}\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{bmatrix},$$

where $\mathcal{O}_{\mathfrak{p}}$ is the ring of integers in $F_{n,\mathfrak{p}}$. (Recall that \mathbf{H} is unramified at \mathfrak{p} , so $\mathbf{H} \otimes_{F_n} F_{n,\mathfrak{p}} \cong \mathrm{Mat}_{2 \times 2}(F_{n,\mathfrak{p}})$.) Switching labels if necessary we see that

$$\mathcal{M}_{\mathfrak{p}} = \begin{bmatrix} \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \\ \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{bmatrix}, \quad \mathcal{M}'_{\mathfrak{p}} = \begin{bmatrix} \mathcal{O}_{\mathfrak{p}} & \mathfrak{p}^{-1}\mathcal{O}_{\mathfrak{p}} \\ \mathfrak{p}\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{bmatrix},$$

are the two maximal orders containing $\mathcal{E}_{\mathfrak{p}}$.

Simple calculations show that

$$I_{\mathfrak{p}} = \begin{bmatrix} \mathfrak{p}\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \\ \mathfrak{p}\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{bmatrix}, \quad \bar{I}_{\mathfrak{p}} = \begin{bmatrix} \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \\ \mathfrak{p}\mathcal{O}_{\mathfrak{p}} & \mathfrak{p}\mathcal{O}_{\mathfrak{p}} \end{bmatrix}.$$

Note that $\mathcal{E}_{\mathfrak{p}} = I_{\mathfrak{p}} + \bar{I}_{\mathfrak{p}}$ and that $\mathcal{M}_{\mathfrak{p}}, \mathcal{M}'_{\mathfrak{p}}$ are the left and right orders of $I_{\mathfrak{p}}$, while the reverse holds for $\bar{I}_{\mathfrak{p}}$. Moreover, $\mathrm{Norm}(I_{\mathfrak{p}}) = \mathrm{Norm}(\bar{I}_{\mathfrak{p}}) = \mathfrak{p}$ and

$$I_{\mathfrak{p}}^{-1} = \begin{bmatrix} \mathfrak{p}^{-1}\mathcal{O}_{\mathfrak{p}} & \mathfrak{p}^{-1}\mathcal{O}_{\mathfrak{p}} \\ \mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}} \end{bmatrix},$$

so that $\bar{I}_{\mathfrak{p}} = \mathfrak{p}I_{\mathfrak{p}}^{-1}$.

Each edge in the tree is given length 1. As usual, the distance $\mathrm{dist}(\mathbf{v}, \mathbf{w})$ between two vertices \mathbf{v}, \mathbf{w} is the length of the shortest path between them.

4.4. The Clifford-cyclotomic group in $\bar{\Gamma}_n \subseteq \mathrm{PH}^{\times}$. Let the vertex $\mathbf{v} \in \Delta$ be such that $\mathcal{M}_{\mathbf{v}} = \mathcal{M}$. In what follows we assume that the ideal (2) in F_n is the square of a principal ideal: $(2) = (\alpha := \alpha_n)^2$. Let $[h], [t_n]$ be as in (19). For $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{Ver}(\Delta)$, let $P(\mathbf{v}_1, \mathbf{v}_2)$ be the path in Δ between \mathbf{v}_1 and \mathbf{v}_2 of length $\mathrm{dist}(\mathbf{v}_1, \mathbf{v}_2)$.

Proposition 36. Put $[t] := [t_n]$ and let \mathbf{w} be the midpoint of the path $P(\mathbf{v}, [t]\mathbf{v})$. If $\text{dist}(\mathbf{v}, [t]\mathbf{v})$ is even, then $\mathbf{w} \in \text{Ver}(\Delta)$. If $\text{dist}(\mathbf{v}, [t]\mathbf{v})$ is odd, then \mathbf{w} is a vertex in the barycentric subdivision of Δ . Then $[t]$ fixes \mathbf{w} .

Proof. If $v = [t]v$, then the statement holds trivially. If not, let w' be a vertex fixed by $[t]$. Then $\text{dist}(w', v) = \text{dist}(w', [t]v)$. Let P_v and $P_{[t]v}$ be the shortest paths from w' to v , $[t]v$ respectively, and let x be the last vertex that is in both. Then the paths from x to v , $[t]v$ obtained from P_v , $P_{[t]v}$ by deleting the path from w' to x are the shortest paths from x to v , $[t]v$, and the reverse of the path from x to v followed by the path from x to $[t]v$ is the shortest path from v to $[t]v$. Since $\text{dist}(x, v) = \text{dist}(x, v')$, the claim follows. \square

Proposition 37. Assume that n is not a power of 2. Let $\mathbf{w}'' \in \text{Ver}(\Delta)$ be such that $\mathcal{M}_{\mathbf{w}''} = \mathcal{T}_n$ as in Theorem 31 and let \mathbf{w} be as in Proposition 36. Then $\mathbf{w}'' = \mathbf{w}$.

Proof. By Proposition 36, $[t] \in \bar{\Gamma}_n$ fixes \mathbf{w} . But $\text{Norm}_{\mathbf{H}_n/F_n}(t_n)$ is a unit by [IJK⁺19b, Lemma 3.9], so $t_n \in \mathcal{M}_{\mathbf{w}}^\times$. Hence $e^{2\pi k/n} = t_n - 1 \in \mathcal{M}_{\mathbf{w}}$. We also know j is in both maximal orders $\mathcal{M}_{\mathbf{w}''} = \mathcal{T}_n$ and $\mathcal{M}_{\mathbf{v}} = \mathcal{M}$. Therefore j is in each maximal order corresponding to vertices in the path $P(\mathbf{w}'', \mathbf{v})$. In particular, applying the same argument as in the proof of Proposition 36 with \mathbf{w}'' replacing \mathbf{w}' , we get that \mathbf{w} is on $P(\mathbf{w}'', \mathbf{v})$. Therefore both j and $e^{2\pi k/n}$ are in $\mathcal{M}_{\mathbf{w}}$ and hence $\mathcal{M}_{\mathbf{w}} = \mathcal{M}_{\mathbf{w}''}$. \square

Proposition 38. We have

(a) $(t^{n/2}h)^2 = u \cdot j$ for $u \in \mathcal{Q}_n^\times$. Hence

$$\bar{\Gamma}_{n, \mathbf{w}} \cong D_n \subseteq \langle [h], [t] \rangle \cong \text{PG}_n.$$

(b) Conjugation by $(1+k)/\sqrt{2} \in \text{PM}^\times \cong \mathcal{S}_4$ is a 4-cycle and conjugation by $(-i-k)/\sqrt{2}$ is a transposition such that the product has order 3, and

$$\bar{\Gamma}_{n, \mathbf{v}} \cong \mathcal{S}_4 \subseteq \langle [h], [t] \rangle \cong \text{PG}_n.$$

Proof. Statement (a) is already contained in the proof of Theorem 31.

As for (b), the identification with \mathcal{S}_4 is a standard fact for the Hamilton quaternions over \mathbf{Q} . In particular, the conjugations by $(\pm r \pm s)$, where $r \neq s$ and $\{r, s\} \subset \{i, j, k\}$, are of order 2, since $(\pm r \pm s)^2 = -2$ is central. There are 6 of them, so these form a conjugacy class, which must be that of the transpositions. On the other hand, the conjugations by $1 \pm q$, for $q \in \{i, j, k\}$, are of order 4, since $(1 \pm q)^4$ is a scalar but $(1 \pm q)^2$ is not. It is easily checked that the product of one of each type is of order 2 if and only if $\{r, s, t\} = \{i, j, k\}$. Since one element of order 4 and one of order 2 generate \mathcal{S}_4 if and only if their product has order 3, it follows that conjugation by $(1+k)/\sqrt{2}$ and $(-i-k)/\sqrt{2}$ generate \mathcal{S}_4 . Since $t^{n/4} = (1+k)/\sqrt{2}$ up to scalars, conjugation by $t^{n/4}$ is the same as conjugation by $(1+k)/\sqrt{2}$, while conjugation by $h = -i-k$ is the same as conjugation by $(-i-k)/\sqrt{2}$, so the claim that $\bar{\Gamma}_{n, \mathbf{v}} \cong \mathcal{S}_4 \subseteq \langle [h], [t] \rangle$ follows as well. \square

5. INTRODUCTION TO THE EXAMPLES

Using Theorem 7 and the description of the Bruhat-Tits tree $\Delta = \Delta_n$ in terms of maximal orders in \mathbf{H}_n in [Kur79, Section 4], we compute $\text{PSU}_2(R_n)$, $\text{PU}_2^{\zeta}(R_n)$, and $\text{PU}_2(R_n)$ as the fundamental group of explicit graphs of groups for $n = 8, 12, 16, 20, 24, 28, 32, 36, 40, 48$ arising from the actions of Γ_1, Γ_+ , and Γ_0 (respectively) on Δ . Our computations, which are done

using Magma [BCP97], do not give enough information to present $\text{PSU}_2(R_{60})$ or $\text{PU}_2^\zeta(R_{60})$ in this manner, although they do for $\text{PU}_2(R_{60})$. Our attempted computations for $n = 52, 56$ did not finish in the small amount of time we allotted and we deemed all $n > 60$ too costly to try. These computations give a presentation of these groups as amalgamated products.

All computations are done using Magma's quaternion algebra machinery in \mathbf{H}_n by exploring the neighborhood around the standard maximal order \mathcal{M}_n (Section 4.1). This order has $\mathcal{M}_n^*/(\mathcal{O}_n^+)^* \simeq S_4$, thus rooting our graphs at a vertex common (in all the computed examples) to PG_n . This is important for $n = 28$ and 60 . These are the two cases where there exist some types of maximal orders that are not connected to \mathcal{M}_n . In our graphs, the vertex corresponding to \mathcal{M}_n is outlined in black. It has stabilizer subgroup S_4 .

In all our examples \mathcal{T}_n (see statement of Theorem 31) represents a vertex in the h-graph of groups for $\text{PU}_2(R_n)$ with stabilizer subgroup isomorphic to D_n . There is also a path (without backtracking) from the node represented by \mathcal{M}_n to that represented by \mathcal{T}_n whose fundamental group G telescopes to $G \simeq S_4 *_{D_4} D_n$.

Let h be as in (30) and t_n be as in the statement of Theorem 31. These generate PG_n . Since $h \in \mathcal{M}_n$ and $t_n \in \mathcal{T}_n$, PG_n embeds into G . This latter group is isomorphic to $\text{PG}_n(R_n)$ by Theorem 2. Theorem [IJK⁺19a, Theorem 1.2(1)] tells us that these two groups must be equal for $n = 8, 12, 16, 24$. We do not know if this holds for the remaining examples $n = 20, 28, 32, 36, 40, 48, 60$. In any, case, using Proposition 11 we have the new result $\text{PG}_n < G \ll [\text{PU}_2^\zeta(R_n)]_f$ for $n = 20, 28, 32$.

We can also compute the Euler-Poincaré characteristics from the associated graph of groups.

Definition 39. Let $Gr = (\Gamma, gr)$ be an h-graph of groups with finite vertex and edge isotropy groups. Define the mass of a vertex $\mathbf{v} \in gr$ to be $m(\mathbf{v}) = 1/\#\Gamma_{\mathbf{v}}$. The mass of an edge $\mathbf{e} \in \text{Ed}(gr)$ is $m(\mathbf{e}) = 1/(2\#\Gamma_{\mathbf{e}})$. The *vertex mass* of Gr is

$$\text{VM}(Gr) := \text{VM}(\Gamma, gr) := \sum_{\mathbf{v} \in \text{Ver}(gr)} m(\mathbf{v}).$$

Its *edge mass* is

$$\text{EM}(Gr) := \text{EM}(\Gamma, gr) := \sum_{\mathbf{e} \in \text{Ed}(gr)} m(\mathbf{e}).$$

Remark 40. Recall that, in our graphs, edges \mathbf{e} have opposite edges $\bar{\mathbf{e}}$ so the mass attached to the geometric edge $\{\mathbf{e}, \bar{\mathbf{e}}\}$ with $\mathbf{e} \neq \bar{\mathbf{e}}$ is $1/\#\Gamma_{\mathbf{e}} = 1/\#\Gamma_{\bar{\mathbf{e}}}$. Our h-graphs of groups are actually graphs of groups (being a quotient of the tree after subdividing inverted edges). In the pictures we draw we elide the inserted vertices to make clear the associated h-graph. These elided vertices still contribute to the vertex mass of the graph in the usual way.

By [IJK⁺19b, Theorem 2.20], the Euler-Poincaré characteristic of the fundamental group Gr of a graph of groups is given by

$$\chi(\pi_1(Gr)) = \text{VM}(Gr) - \text{EM}(Gr). \quad (22)$$

On the other hand, if we have a group presented as an amalgamated product, it is easy to compute its Euler-Poincaré characteristic: If A, B are finite groups with $C \leq A$ and $C \leq B$, then

$$\chi(A *_C B) = \frac{1}{\#A} + \frac{1}{\#B} - \frac{1}{\#C} \quad (23)$$

by [Ser71, Corollaire 1, p. 104].

5.1. **Tables.** Before treating examples in detail, we arrange in tabular form the results of some Magma computations. We first summarize the notation used in the tables, using the notation of Section 3. All of our examples will satisfy the equivalent conditions of Proposition 19 with one prime \mathfrak{p} of $F = F_n$ lying above 2. In addition, all but $n = 28, 60$ satisfy the equivalent conditions Proposition 20 having a unique prime above 2 in $K = K_n$. Let $q_n = 2^{f(\mathfrak{p})}$ be the order of the residue field of \mathfrak{p} and let Δ_n be the Bruhat-Tits tree associated to $\mathrm{SL}_2(F_{\mathfrak{p}})$. It is a regular tree of degree $q_n + 1$. We compute with Δ_n using the maximal orders in \mathbf{H}_n as described in §4.3.

The table below gives the results on F_n we will need in analyzing our examples.

TABLE 1. Data on the fields $F_n = \mathbf{Q}(\zeta_n)^+$

n	$\varphi(n)$	$q_n + 1$	$\tilde{h}(\mathcal{O}_n)$	$\tilde{h}(\mathcal{O}_n^{\mathfrak{p}})$	$\#\frac{\tilde{\mathrm{Cl}}(\mathcal{O}_n^{\mathfrak{p}})[2]}{\mathrm{Prin}(\mathcal{O}_n^{\mathfrak{p}})}$	$[\mathfrak{p}] \in \tilde{\mathrm{Cl}}(\mathcal{O}_n)^{2?}$
8	4	3	1	1	1	yes
12	4	3	2	1	1	no
16	8	3	1	1	1	yes
20	8	5	2	1	1	no
24	8	3	2	1	1	no
28	12	9	2	2	1	yes
32	16	3	1	1	1	yes
36	12	9	2	1	1	no
40	16	5	2	1	1	no
48	16	3	2	1	1	no
60	16	17	2	2	1	yes

The table below gives the information about \mathbf{H}_n we will need in analyzing our examples.

TABLE 2. Data on the quaternion algebras $\mathbf{H}_n = \mathbf{H} \otimes_{\mathbf{Q}} F_n$

n	$t(\mathbf{H}_n) = h_{[\mathfrak{p}]\text{-rel}}(\mathbf{H}_n)$ $= h_{\mathrm{rel}}(\mathbf{H}_n) = h(\mathbf{H}_n)$	$\#\frac{\Gamma_+(\mathcal{M}_n)}{\Gamma_1(\mathcal{M}_n)}$	$\#\frac{\Gamma_0(\mathcal{M}_n)}{\Gamma_+(\mathcal{M}_n)}$
8	1	1	2
12	2	2	1
16	2	1	2
20	3	2	1
24	3	2	1
28	5	2	2
32	58	1	2
36	6	2	1
40	25	2	1
48	39	2	1
60	9	2	2

We now explain the equalities in the column headings of the table.

Proposition 41. *For all n in the range of the table we have $t(\mathbf{H}_n) = h(\mathbf{H}_n) = h_{\text{rel}}(\mathbf{H}_n) = h_{[\mathfrak{p}]\text{-rel}}(\mathbf{H}_n)$.*

Proof. Within the range of the table, the class group of R_n is always trivial. So $h_{\text{rel}}(\mathbf{H}_n)$, the number of orbits of the class group on $\text{Cl}(\mathbf{H}_n)$, is equal to $\#\text{Cl}(\mathbf{H}_n) = h(\mathbf{H}_n)$, and likewise for $h_{[\mathfrak{p}]\text{-rel}}(\mathbf{H}_n)$, the number of orbits of the subgroup generated by \mathfrak{p} .

By [Voi19, Exercise 17.3, Lemma 17.4.8], the set of right orders of the I_i contains a representative of every isomorphism class of maximal orders, where $I_1, \dots, I_{h(\mathbf{H}_n)}$ is a set of representatives for the left ideal classes of a fixed maximal order. Thus $t(\mathbf{H}_n) \leq h(\mathbf{H}_n)$. In the range of the table it can be computed that $t(\mathbf{H}_n) = h(\mathbf{H}_n)$. \square

The following table shows the identifications of $\text{PSU}_2(R_n)$, $\text{PU}_2^\zeta(R_n)$, and $\text{PU}_2(R_n)$ with $\Gamma_{1,n}$, $\Gamma_{+,n}$, and $\Gamma_{0,n}$.

TABLE 3. Arithmetic discrete subgroups

n	$\text{PSU}_2(R_n)$	$\text{PU}_2^\zeta(R_n)$	$\text{PU}_2(R_n)$
8	$\Gamma_1 = \Gamma_+$	Γ_0	Γ_0
12	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
16	$\Gamma_1 = \Gamma_+$	Γ_0	Γ_0
20	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
24	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
28	Γ_1	Γ_+	Γ_0
32	$\Gamma_1 = \Gamma_+$	Γ_0	Γ_0
36	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
40	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
48	Γ_1	$\Gamma_+ = \Gamma_0$	$\Gamma_+ = \Gamma_0$
60	Γ_1	Γ_+	Γ_0

Remark 42. The quotient graph of groups $\text{SU}_2(R_n) \backslash \backslash (\text{SU}_2(R_n), \Delta_n)$, and hence its fundamental group $\text{SU}_2(R_n)$, can be constructed from

$$\text{PSU}_2(R_n) \backslash \backslash (\text{PSU}_2(R_n), \Delta_n)$$

by inflating each edge and vertex group by a central ± 1 . Thus, amalgam presentations of $\text{SU}_2(R_n)$ can be constructed from those we give for $\text{PSU}_2(R_n)$.

5.2. Notation and Basic Results for Groups. For a group Γ acting on a tree Δ we write $Gr(\Gamma) := \Gamma \backslash \backslash (\Gamma, \Delta_\Gamma)$, where, as in Section 2, Δ_Γ is the graph obtained from Δ by subdividing exactly those edges which are inverted by Γ .

Definition 1. We use the following conventions to label the vertex and edge groups of a graph of groups. For an integer n , we use C_n , or simply n , to denote the cyclic group and D_n for the dihedral group of order $2n$. For edge groups C_1 we elide the label altogether. For an even integer n , the *quaternion group* Q_{2n} of order $2n$ is the subgroup of $\mathbf{H}(\mathbf{R})$ generated by $e^{2\pi i/n}$ and j . It is easy to show that Q_{2n} is the unique extension of $\mathbf{Z}/n\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$ acting by $x \rightarrow x^{-1}$ that is not a semidirect product, and that $Q_{2n}/\{\pm 1\} \cong D_{n/2}$. In addition, we

denote the binary tetrahedral, octahedral, and icosahedral groups [Vig80, Théorème I.3.7] by E_{24} , E_{48} , and E_{120} respectively. Note that $|Q_n| = n$ and $|E_n| = n$ but $|D_n| = 2n$. Recall that \mathcal{G} or \mathcal{G}_n denotes the subgroup of U_2 generated by H and T'_n .

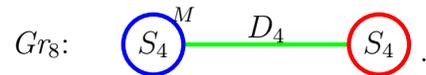
Below we give the key to our notation and conventions in the pictures of the quotient graphs in Sections 6 and 7. For $n < 32$ we draw the graphs of the quotients of Δ by $PU_2(R_n)$, $PU_2^\zeta(R_n)$, and $PSU_2(R_n)$. For $n \geq 32$ we only draw the graph of $\overline{gr}_n = \Gamma_{n,0} \backslash \Delta = PU_2(R_n) \backslash \Delta$ due to space constraints.

Example Key 43. Vertices and edges of quotient graphs are labeled by their corresponding stabilizer groups. A (nonelided) vertex of the quotient graph of $PU_2(R_n)$ or $PU_2^\zeta(R_n)$ is indicated by a square if it is ramified in the cover from $PSU_2(R_n)$ and by a circle if it is unramified there. The graphs $\Gamma_+ \backslash \Delta$ and $\Gamma_1 \backslash \Delta$ are bipartite; the bipartition on vertices is shown using red and blue. The vertex of $PU_2(R_n) \backslash \Delta$ lying below $\mathbf{v} \in \text{Ver}(\Delta)$ with $\mathcal{M}_{\mathbf{v}} = \mathcal{M}$ as in Section 4.4 is marked with an M. Likewise the vertex or elided half-edge vertex of $PU_2(R_n) \backslash \Delta$ lying below $\mathbf{w} \in \text{Ver}(\Delta)$ as in Proposition 36 is marked with a T. The subgraph-of-groups \overline{P}_n in $PU_2(R_n) \backslash \Delta$ with $\pi_1(\overline{P}_n) \cong P\mathcal{G}_n \cong S_4 *_{D_4} D_n$ as in Remark 6 is shown in the picture of $PU_2(R_n) \backslash \Delta$ with magenta edges.

6. THE $n = 8, 12, 16, 24$ EXAMPLES : $P\mathcal{G}_n = PU_2^\zeta(R_n) = PU_2(R_n)$

For $n = 8, 12, 16, 24$ it is known that $P\mathcal{G}_n = PU_2(R_n)$ (see [IJK⁺19a, Theorems 1.1, 1.2] for references) and that $P\mathcal{G}_n \cong S_4 *_{D_4} D_n$ by [RS99] (see [IJK⁺19a, Theorem 5.1]). We establish via quotient graphs in this section that $P\mathcal{G}_n = PU_2(R_n)$ and $P\mathcal{G}_n \cong S_4 *_{D_4} D_n$ for $n = 8, 12, 16, 24$. We also compute the Euler-Poincaré characteristics $\chi(PU_2(R_n))$, $\chi(PSU_2(R_n))$, and $\chi(PU_2^\zeta(R_n))$ from our Gr_n and \overline{Gr}_n for $n = 8, 12, 16, 24$. In all cases the answers agree with [IJK⁺19a, Theorem 6.6], giving a good check on our quotient graphs.

6.1. **$n = 8$.** We have $PSU_2(R_8) = \Gamma_{8,1} = \Gamma_{8,+}$ and $PU_2^\zeta(R_8) = PU_2(R_8) = \Gamma_{8,0}$. The quotient graph of groups Gr_8 for $PSU_2(R_8)$ is

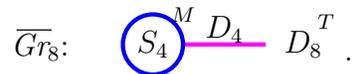


From Gr_8 we compute the Euler-Poincaré characteristic

$$\chi(PSU_2(R_8)) = 1/24 + 1/24 - 1/8 = -1/24$$

and the amalgam $PSU_2(R_8) = \pi_1(Gr_8) = S_4 *_{D_4} S_4$.

The quotient h-graph of groups \overline{Gr}_8 for $PU_2^\zeta(R_8) = PU_2(R_8)$ is



From \overline{Gr}_8 we compute using (22)

$$\chi(PU_2^\zeta(R_8)) = \chi(PU_2(R_8)) = 1/24 + 1/16 - 1/8 = -1/48$$

as well as the amalgam $PU_2(R_8) = \pi_1(\overline{Gr}_8) = S_4 *_{D_4} D_8$. We see that $P\mathcal{G}_8 = PU_2(R_8)$ and we hence recover $P\mathcal{G}_8 \cong S_4 *_{D_4} D_8$.

6.2. $n = 12$. We have $\text{PSU}_2(R_{12}) = \Gamma_{12,1}$ and $\text{PU}_2^\zeta(R_{12}) = \text{PU}_2(R_{12}) = \Gamma_{12,+} = \Gamma_{12,0}$. In this case the double cover

$$\Gamma_{12,1} \backslash \Delta \longrightarrow \Gamma_{12,+} \backslash \Delta = \Gamma_{12,0} \backslash \Delta$$

is not étale. The quotient graph of groups Gr_{12} for $\text{PSU}_2(R_{12})$ is

$$Gr_{12}: \quad \textcircled{A_4}^M \xrightarrow{D_2} \textcircled{D_6},$$

giving $\text{PSU}_2(R_{12}) \cong \pi_1(Gr_{12}) \cong A_4 *_{D_2} D_6$ and $\chi(\text{PSU}_2(R_{12})) = -1/12$.

The quotient h-graph of groups for $\text{PU}_2^\zeta(R_{12}) = \text{PU}_2(R_{12})$ is

$$\overline{Gr}_{12}: \quad \textcircled{S_4}^M \xrightarrow{D_4} \textcircled{D_{12}}^T,$$

from which we derive $\text{PU}_2^\zeta(R_{12}) = \text{PU}_2(R_{12}) \cong \pi_1(\overline{Gr}_{12}) \cong S_4 *_{D_4} D_{12}$,

$$\chi(\text{PU}_2^\zeta(R_{12})) = \chi(\text{PU}_2(R_{12})) = -1/24,$$

and $\text{PG}_n = \text{PU}_2(R_{12}) \cong S_4 *_{D_4} D_{12}$.

6.3. $n = 16$. Here $\text{PSU}_2(R_{16}) = \Gamma_{16,1} = \Gamma_{16,+}$ and $\text{PU}_2^\zeta(R_{16}) = \text{PU}_2(R_{16}) = \Gamma_{16,0}$.

The graph of groups Gr_{16} and the corresponding amalgam for $\text{PSU}_2(R_{16}) \cong \pi_1(Gr_{16})$ are

$$Gr_{16}: \quad \textcircled{S_4}^M \xrightarrow{D_4} \textcircled{D_8} \xrightarrow{D_8} \textcircled{D_8} \xrightarrow{D_4} \textcircled{S_4}$$

$$\text{PSU}_2(R_{16}) \cong S_4 *_{D_4} D_8 *_{D_4} S_4.$$

We compute $\chi(\text{PSU}_2(R_{16})) = -5/48$ using (22).

The quotient h-graph of groups \overline{Gr}_{16} for $\text{PU}_2^\zeta(R_{16}) = \text{PU}_2(R_{16})$ is

$$\overline{Gr}_{16}: \quad \textcircled{S_4}^M \xrightarrow{D_4} \textcircled{D_8} \xrightarrow{D_8} D_{16}^T,$$

giving $\text{PU}_2^\zeta(R_{16}) = \text{PU}_2(R_{16}) \cong \pi_1(\overline{Gr}_{16}) \cong S_4 *_{D_4} D_{16}$ and

$$\chi(\text{PU}_2^\zeta(R_{16})) = \chi(\text{PU}_2(R_{16})) = -5/96.$$

As before we have $\text{PU}_2(R_{16}) = \text{PG}_{16}$ from Proposition 38.

6.4. $n = 24$. In this case $\text{PSU}_2(R_{24}) = \Gamma_{24,1}$ while $\text{PU}_2^\zeta(R_{24}) = \text{PU}_2(R_{24}) = \Gamma_{24,+} = \Gamma_{24,0}$. Again the double cover $\text{PSU}_2(R_n) \backslash \Delta \longrightarrow \text{PU}_2(R_n) \backslash \Delta$ is not étale.

The quotient graph of groups Gr_{24} for $\text{PSU}_2(R_{24})$ is

$$Gr_{24}: \quad \begin{array}{c} \textcircled{S_4}^M \\ \textcircled{S_4} \end{array} \begin{array}{l} \xrightarrow{D_4} \\ \xrightarrow{D_4} \end{array} \textcircled{D_4} \xrightarrow{D_4} \textcircled{D_{12}}.$$

From this we compute $\chi(\mathrm{PSU}_2(R_{24})) = -1/8$ and $\mathrm{PSU}_2(R_{24})$ is the amalgam

$$\mathrm{PSU}_2(R_{24}) \cong \pi_1(\mathrm{Gr}_{24}) \cong *_{D_4}\{S_4, S_4, D_{12}\}$$

of D_{12} and the two copies of S_4 over their common subgroup D_4 .

Since $\Gamma_{24,0}$ acts without inversions we get the following quotient graph of groups for $\mathrm{PU}_2^\zeta(R_{24}) = \mathrm{PU}_2(R_{24}) = \Gamma_{24,+} = \Gamma_{24,0}$:

$$\overline{\mathrm{Gr}}_{24}: \quad \textcircled{S_4}^M \text{---}^{D_4} \text{---} \boxed{D_8} \text{---}^{D_8} \text{---} \boxed{D_{24}}^T.$$

Hence $\mathrm{PU}_2^\zeta(R_{24}) = \mathrm{PU}_2(R_{24}) \cong \pi_1(\overline{\mathrm{Gr}}_{24}) \cong S_4 *_{D_4} D_{24} \cong \mathrm{PG}_{24}$ and

$$\chi(\mathrm{PU}_2^\zeta(R_{24})) = \chi(\mathrm{PU}_2(R_{24})) = -1/16.$$

7. THE $n = 20, 28, 32, 36, 40, 48, 60$ EXAMPLES: $\mathrm{PG}_n \neq \mathrm{PU}_2^\zeta(R_n)$

Now $\mathcal{G}_n \neq \mathrm{U}_2^\zeta(R_n)$, $\mathrm{PG}_n \neq \mathrm{PU}_2^\zeta(R_n)$, $\mathrm{SU}_2(R_n) \neq \mathrm{S}\mathcal{G}_n$, and $\mathrm{PS}\mathcal{G}_n \neq \mathrm{PSU}_2(R_n)$ when $n \notin \{8, 12, 16, 24\}$ (see [IJK⁺19a, Theorems 1.1, 1.2] for references). In fact, in our examples PG_n can be seen as a proper subtree of the quotient h-graph of groups for $\mathrm{PU}_2^\zeta(R_n)$. Theorem 11 shows that $\mathrm{PG}_n \ll [\mathrm{PU}_2^\zeta(R_n)]_f$ for these n .

7.1. **$n = 20$.** We have the identifications $\mathrm{PSU}_2(R_{20}) = \Gamma_{20,1}$ and $\mathrm{PU}_2^\zeta(R_{20}) = \mathrm{PU}_2(R_{20}) = \Gamma_{20,+} = \Gamma_{20,0}$. The double cover $gr_n = \Gamma_{20,1} \setminus \Delta \rightarrow \overline{gr}_n = \Gamma_{20,0} \setminus \Delta$ is not étale. The quotient graph of groups Gr_{20} for $\mathrm{PSU}_2(R_{20})$ is

$$\mathrm{Gr}_{20}: \quad \begin{array}{c} \textcircled{A_5} \\ \textcircled{A_5} \end{array} \begin{array}{l} \text{---}^{A_4} \\ \text{---}^{A_4} \end{array} \textcircled{A_4}^M \text{---}^{D_2} \textcircled{D_{10}}.$$

So we obtain $\mathrm{PSU}_2(R_{20}) \cong \pi_1(\mathrm{Gr}_{20}) \cong A_5 *_{A_4} A_5 *_{D_2} D_{10}$ and $\chi(\mathrm{PSU}_2(R_{20})) = -1/4$.

Since $\Gamma_{20,0}$ acts without inversions we get the quotient graph of groups $\overline{\mathrm{Gr}}_{20}$ for $\mathrm{PU}_2^\zeta(R_{20}) = \mathrm{PU}_2(R_{20}) = \Gamma_{20,+} = \Gamma_{20,0}$ below:

$$\overline{\mathrm{Gr}}_{20}: \quad \textcircled{A_5} \text{---}^{A_4} \text{---} \boxed{S_4}^M \text{---}^{D_4} \text{---} \boxed{D_{20}}^T.$$

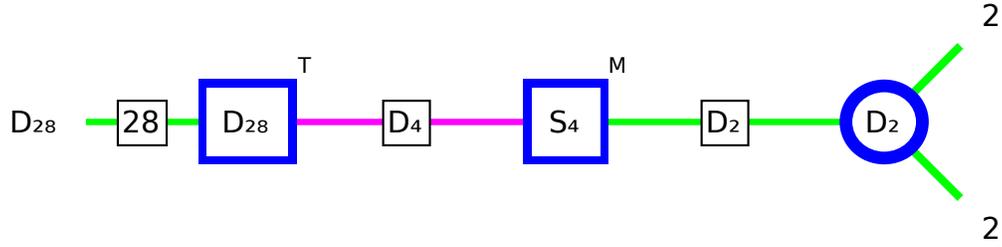
Thus $\mathrm{PU}_2^\zeta(R_{20}) = \mathrm{PU}_2(R_{20}) \cong \pi_1(\overline{\mathrm{Gr}}_{20}) \cong A_5 *_{A_4} S_4 *_{D_4} D_{20}$ and

$$\chi(\mathrm{PU}_2^\zeta(R_{20})) = \chi(\mathrm{PU}_2(R_{20})) = -1/8.$$

Proposition 11 shows that $\mathrm{PG}_{20} \ll [\mathrm{PU}_2(R_{20})]_f$, so Corollary 17 applies. On the other hand, it is clear that $[\mathrm{PU}_2(R_{20})]_f = \mathrm{PU}_2(R_{20})$.

7.2. **$n = 28$.** This is one of the two examples (the other is $n = 60$) for which the unique prime above 2 in F_n splits in K_n . Hence $\mathrm{PSU}_2(R_{28}) = \Gamma_{28,1}$, $\mathrm{PU}_2^\zeta(R_{28}) = \Gamma_{28,+}$, and $\mathrm{PU}_2(R_{28}) = \Gamma_{28,0}$ are all distinct with $\mathrm{PU}_2(R_{28})/\mathrm{PU}_2^\zeta(R_{28}) \cong \mathbf{Z}/2\mathbf{Z}$ and $\mathrm{PU}_2^\zeta(R_{28})/\mathrm{PSU}_2(R_{28}) = \mathbf{Z}/2\mathbf{Z}$.

The class number of \mathbf{H}_{28} is 5, but only three types of orders are connected to \mathcal{M} . The quotient h-graph of groups $\overline{\mathrm{Gr}}_{28}$ for $\mathrm{PU}_2(R_{28}) = \Gamma_{28,0}$ is given below:

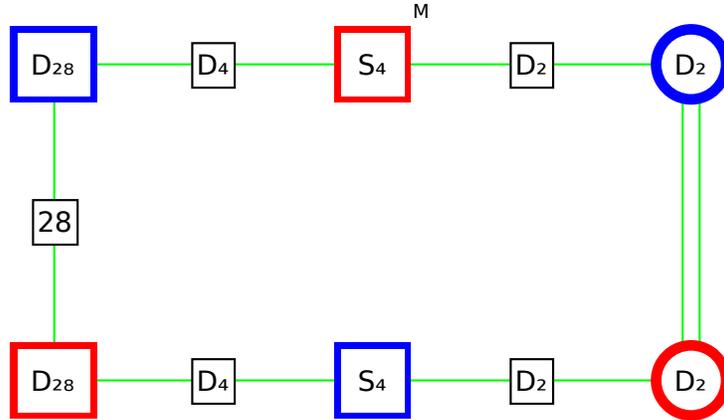


Thus

$$\mathrm{PU}_2(R_{28}) \cong \Gamma_{28,0} \cong \pi_1(\overline{Gr}_{28}) = D_{28} *_{C_{28}} D_{28} *_{D_4} S_4 *_{C_2^{*2}}$$

and $\chi(\mathrm{PU}_2(R_{28})) = -13/12$.

The quotient graph of groups for $\mathrm{PU}_2^\zeta(R_{28}) = \Gamma_{28,+}$, which is the bipartite double cover of \overline{Gr}_{28} , is:



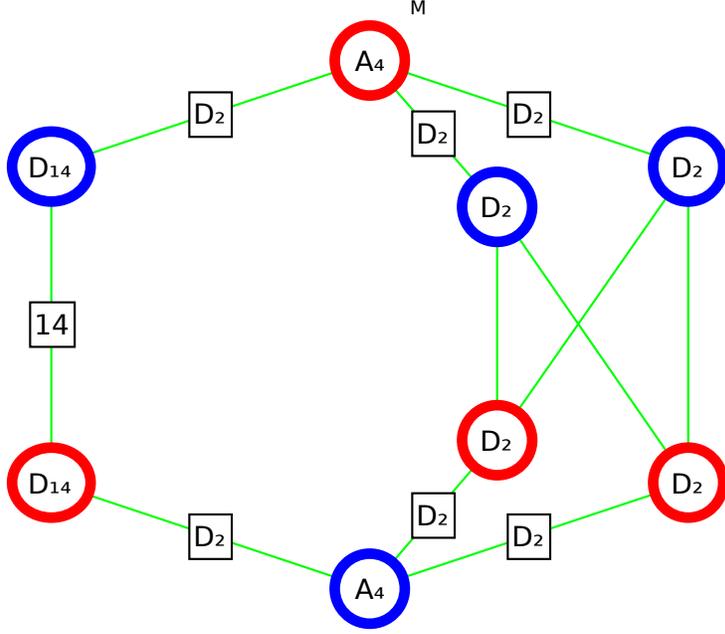
Notice that, unlike all the graphs we have dealt with above, this graph is not a tree; in fact it has genus 2.

By Theorem 9,

$$\mathrm{PU}_2^\zeta(R_{28}) \cong \Gamma_{28,+} = S_4 *_{D_4} D_{28} *_{C_{28}} D_{28} *_{D_4} S_4 *_{\mathbf{Z}^{*2}}$$

and $\chi(\mathrm{PU}_2^\zeta(R_{28})) = -13/6$.

The quotient graph of groups Gr_{28} for $\Gamma_{28,1}$ is the double cover of the quotient graph of groups for $\Gamma_{28,+}$ ramified at the vertices of that graph marked with a square:



This graph has genus 4. By Theorem 9

$$\mathrm{PSU}_2(R_{28}) \cong \Gamma_{28,0} \cong \pi_1(\mathrm{Gr}_{28}) = A_4 *_{D_2} D_{14} *_{C_{14}} D_{14} *_{D_2} A_4 * \mathbf{Z}^{*4} \quad (24)$$

and $\chi(\mathrm{PSU}_2(R_{28})) = -13/3$. We can summarize the $n = 28$ example with:

$$\mathcal{G}_{28} \ll \mathrm{U}_2(R_{28})_f \ll \mathrm{U}_2^\zeta(R_{28}) \ll \mathrm{U}_2(R_{28}) \quad (25)$$

$$\mathcal{PG}_{28} \ll [\mathrm{PU}_2(R_{28})]_f \ll \mathrm{PU}_2(R_{28}) \quad (26)$$

For any n we have $\mathrm{PU}_2^\zeta(R_n) \lesssim \mathrm{PU}_2(R_n)$, so in particular this is true for $n = 28$. Note that here $[\mathrm{PU}_2(R_n)]_f$ is not a subgroup of $\mathrm{PU}_2^\zeta(R_n)$: the cyclic group of order 28 is contained in $[\mathrm{PU}_2(R_n)]_f$ but not in $\mathrm{PU}_2^\zeta(R_n)$.

7.3. $n = 32$. We have $\mathrm{PSU}_2(R_{32}) = \Gamma_{32,1} = \Gamma_{32,+}$ and $\mathrm{PU}_2^\zeta(R_{32}) = \mathrm{PU}_2(R_{32}) = \Gamma_{32,0}$. The quotient graph of groups $\overline{\mathrm{Gr}}_{32}$ for $\mathrm{PU}_2^\zeta(R_{32}) = \mathrm{PU}_2(R_{32}) = \Gamma_{32,0}$ is shown below broken into two subgraphs. These subgraphs are to be glued together by identifying vertices with a label such as A or γ in Subgraph 1 with those with the same label in Subgraph 2. The vertices are also marked (in the interior of the circle representing the vertex) with their stabilizers in $\mathrm{PU}_2^\zeta(R_{32}) = \mathrm{PU}_2(R_{32})$. Recall that an integer n should be read as the cyclic group of order n .

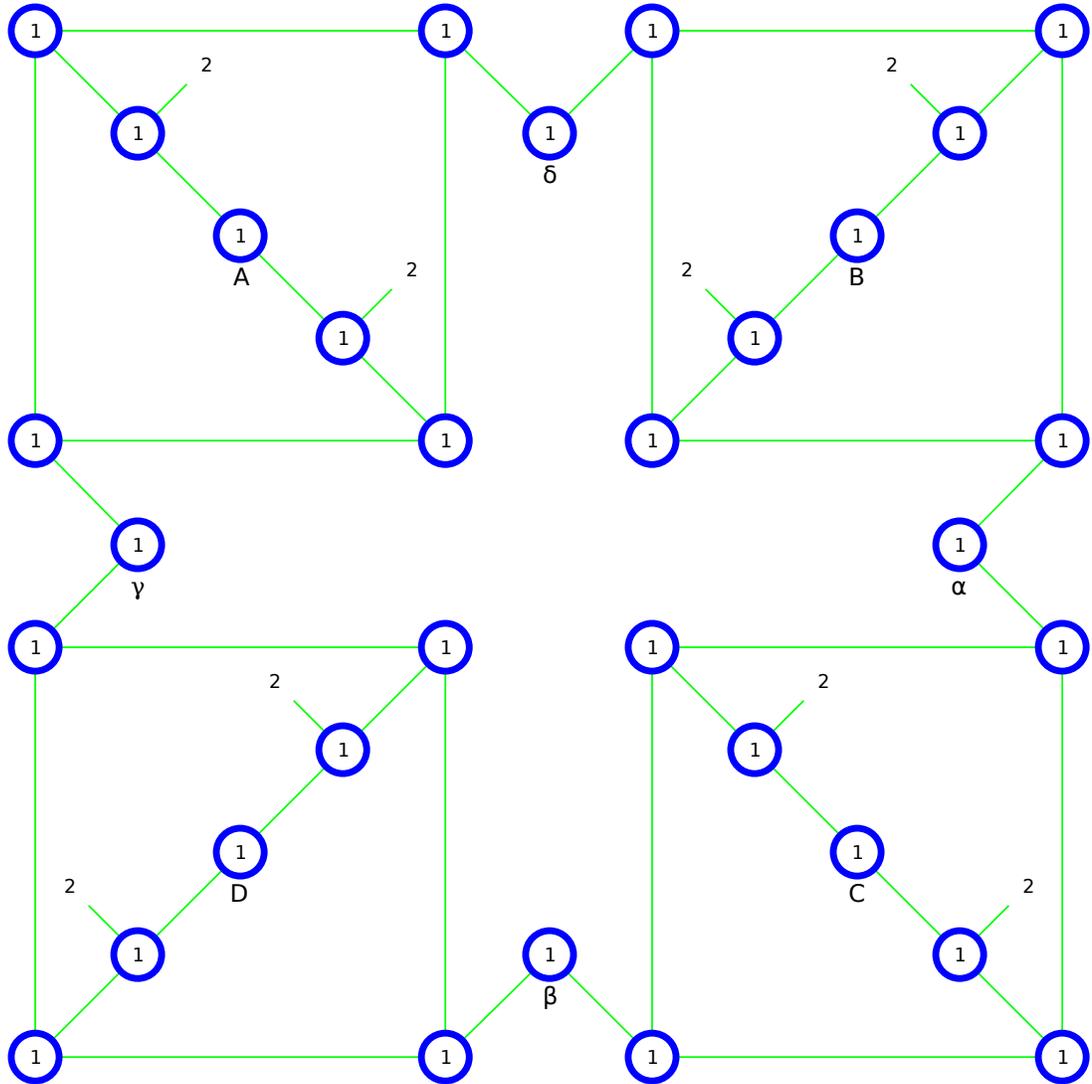


FIGURE 1. Subgraph 1 for $\text{PU}_2^{\zeta}(R_{32}) = \text{PU}_2(R_{32}) = \Gamma_{32,0}$.

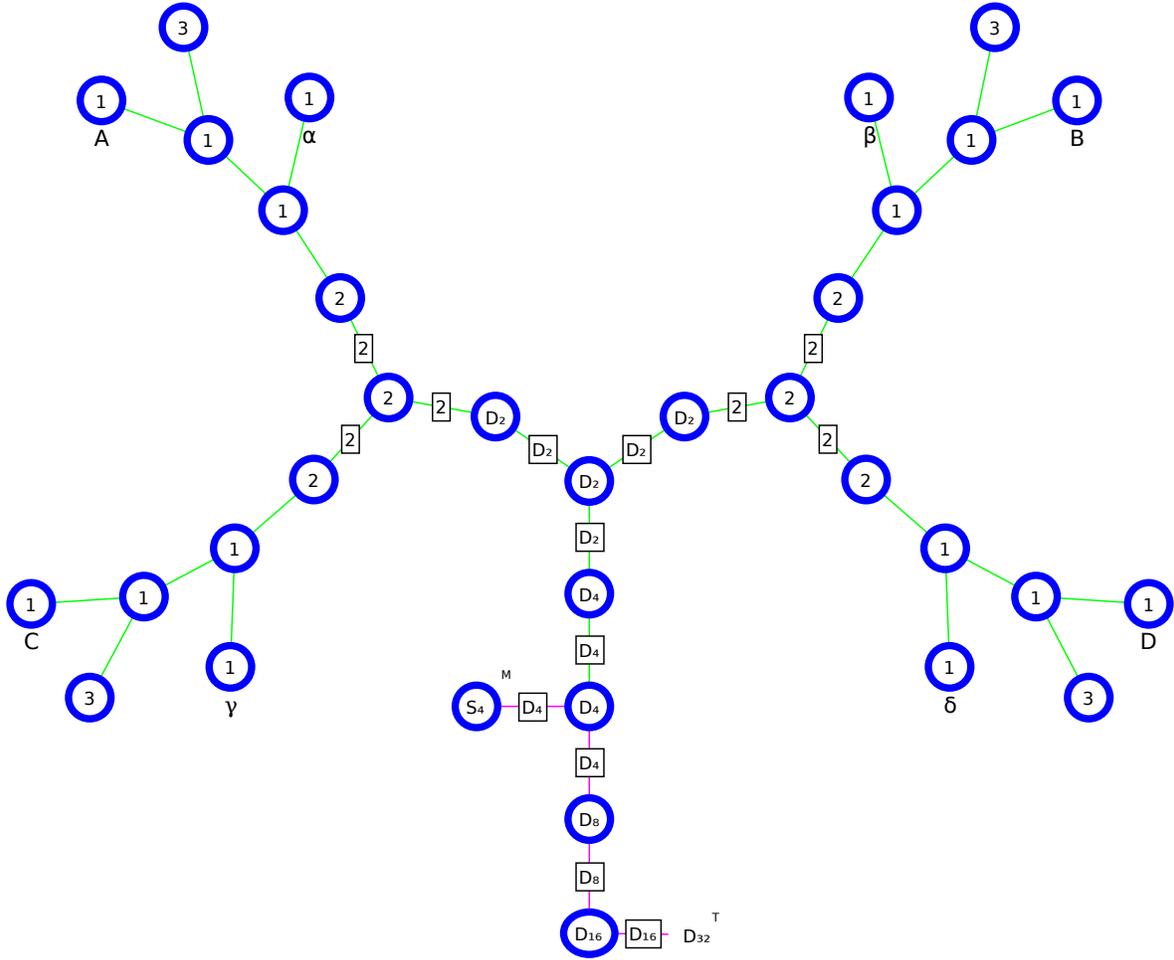


FIGURE 2. Subgraph 2 for $\text{PU}_2^z(R_{32}) = \text{PU}_2(R_{32}) = \Gamma_{32,0}$.

This case has the largest graph, since there are 58 maximal orders. On the other hand, 40 of the maximal orders have only ± 1 as units. It also has some edges that join a maximal order to an isomorphic one, which does not occur for $n = 40, 48$. The graph has genus 16. By Theorem 9 we have.

$$\text{PU}_2(R_{32}) = \text{PU}_2^z(R_{32}) \cong \pi_1(\overline{Gr}_{32}) \cong D_{32} *_{D_4} S_4 * C_3^{*4} * C_2^{*8} * \mathbf{Z}^{*16}$$

and $\chi(\mathrm{PU}_2(R_{32})) = -1455/64$. Again we see that $\mathrm{PG}_{32} \ll \mathrm{PU}_2(R_{32})_f$, but this time $\mathrm{PU}_2(R_{32})_f \ll \mathrm{PU}_2(R_{32})$.

The quotient graph of groups Gr_{32} for $\mathrm{PSU}_2(R_{32}) = \Gamma_{32,1} = \Gamma_{32,+}$ is the bipartite double cover of this graph, the vertex labels being the same. The maximal orders with nontrivial unit group again form a forest, and counting vertices and edges we see that the graph has genus 40. Thus, by Theorem 9, the group is

$$\mathrm{PSU}_2(R_{32}) = \Gamma_{32,1} = \Gamma_{32,+} \cong \pi_1(Gr_{32}) \cong S_4 *_{D_4} D_{16} *_{D_4} S_4 * C_3^{*8} * \mathbf{Z}^{*40}.$$

7.4. **$n = 36$.** In this case, $\mathrm{PSU}_2(R_{36}) = \Gamma_{36,1}$ while $\mathrm{PU}_2^\zeta(R_{36}) = \mathrm{PU}_2(R_{36}) = \Gamma_{36,+} = \Gamma_{36,0}$. The quotient graph of groups \overline{Gr}_{36} for $\mathrm{PU}_2^\zeta(R_{36}) = \mathrm{PU}_2(R_{36}) = \Gamma_{36,+} = \Gamma_{36,0}$ is shown below. Notice the doubled edge.



FIGURE 3. Graph of groups for $\mathrm{PU}_2^\zeta(R_{36}) = \mathrm{PU}_2(R_{36}) = \Gamma_{36,+} = \Gamma_{36,0}$.

In this case the residue field of the prime \mathfrak{p} above 2 has order 8, so each maximal order contains 9 Eichler orders of level \mathfrak{p} , rather than 3 or 5 as in the other examples. The graph has genus 1. By Theorem 9 we have

$$\begin{aligned} \mathrm{PU}_2^\zeta(R_{36}) = \mathrm{PU}_2(R_{36}) = \Gamma_{36,0} = \Gamma_{36,+} &\cong \\ \pi_1(\overline{Gr}_{36}) &\cong D_9 *_{C_2} D_3 * C_3 * S_4 *_{D_4} D_{36} * \mathbf{Z} \end{aligned}$$

and $\chi(\mathrm{PU}_2(R_{36})) = -217/72$.

As before, the presence of additional factors of finite order implies that $\mathrm{PG}_{36} \ll [\mathrm{PU}_2(R_{36})]_f$, while $[\mathrm{PU}_2(R_{36})]_f \ll \mathrm{PU}_2(R_{36})$ because of the \mathbf{Z} in the list of factors.

The quotient graph of groups Gr_{36} for $\mathrm{PSU}_2(R_{36}) = \Gamma_{36,1}$ is a double cover of \overline{Gr}_{36} ramified at the four vertices indicated by squares in Figure 3. The vertex labels for the unramified nodes are the same, and the ramified nodes have vertices labeled with subgroups of index 2. The graph has genus 3. By Theorem 9 we see that

$$\mathrm{PSU}_2(R_{36}) = \Gamma_{36,1} \cong \pi_1(Gr_{36}) \cong C_9 * C_3^{*3} * A_4 *_{D_2} D_{18} * \mathbf{Z}^{*3}.$$

7.5. **$n = 40$.** Here $\mathrm{PSU}_2(R_{40}) = \Gamma_{40,1}$, while $\mathrm{PU}_2^\zeta(R_{40}) = \mathrm{PU}_2(R_{40}) = \Gamma_{40,+} = \Gamma_{40,0}$. The quotient graph of groups \overline{Gr}_{40} for $\mathrm{PU}_2^\zeta(R_{40}) = \mathrm{PU}_2(R_{40}) = \Gamma_{40,+} = \Gamma_{40,0}$ is shown below broken into two subgraphs. The two subgraphs are to be glued by identifying vertices with the same label, e.g., Vertex A in Subgraph 1 is identified with Vertex A in Subgraph 2.

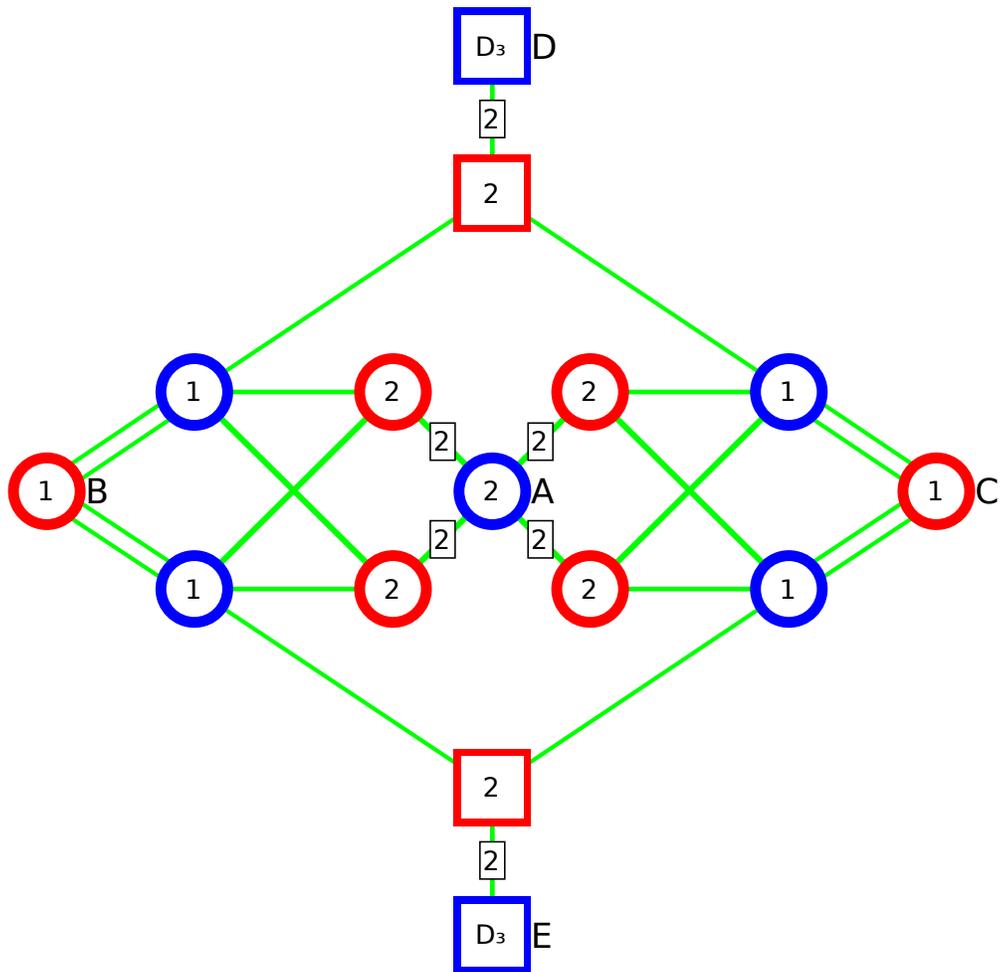


FIGURE 4. Subgraph 1 for $\text{PU}_2^\zeta(R_{40}) = \text{PU}_2(R_{40}) = \Gamma_{40,+} = \Gamma_{40,0}$.

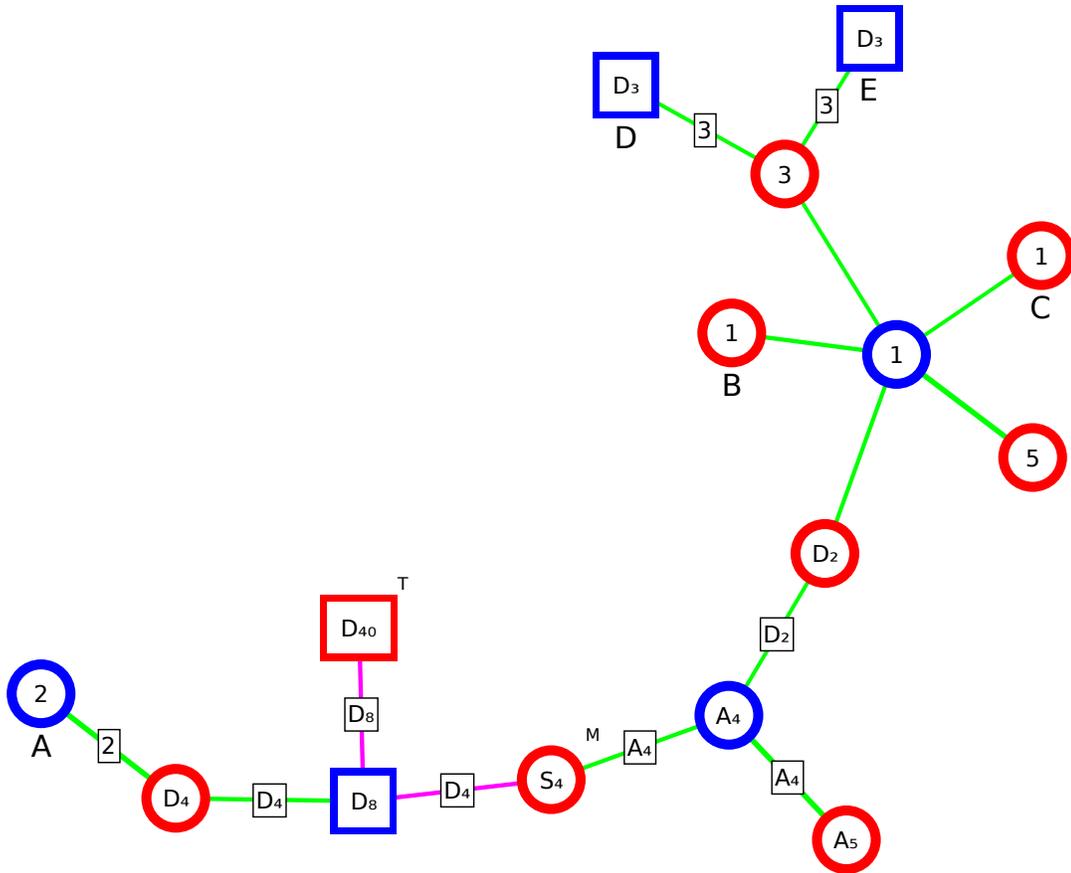


FIGURE 5. Subgraph 2 for $\text{PU}_2^\zeta(R_{40}) = \text{PU}_2(R_{40}) = \Gamma_{40,+} = \Gamma_{40,0}$.

In this case the residue field has order 4, so each maximal order contains five Eichler orders rather than three. This means that the graph is more highly connected than in the other cases with $\phi(n) = 16$.

The automorphism group of each edge is the same as the smaller automorphism group of its nodes except for the edge between the nodes with automorphism groups D_8 and S_4 which has automorphism group D_4 .

The graph again has genus 16. Theorem 9 gives

$$\begin{aligned} \mathrm{PU}_2(R_{40}) &= \mathrm{PU}_2^\zeta(R_{40}) \cong \pi_1(\overline{Gr}_{40}) \cong \\ &D_{40} *_{D_4} S_4 *_{A_4} A_5 * D_3 *_{C_3} D_3 * C_5 * \mathbf{Z}^{*16} \end{aligned}$$

and $\chi(\mathrm{PU}_2(R_{40})) = -287/16$. Once again we have

$$\mathrm{PG}_{40} \ll [\mathrm{PU}_2(R_{40})]_f \ll \mathrm{PU}_2(R_{40}).$$

The graph Gr_{40} for $\mathrm{PSU}_2(R_{40}) = \Gamma_{40,1}$ is a double cover of \overline{Gr}_{40} ramified at the six vertices in Figures 4, 5 indicated by squares. The vertex labels for the unramified nodes are the same, and the ramified nodes have vertices labeled with subgroups of index 2. This case is different in that the subgraph of nontrivial unit groups has a loop, specifically a square all whose vertices have group C_3 (the remaining components are all trees). Since $\mathrm{PU}_2(R_{40})/\mathrm{PSU}_2(R_{40})$ acts by reflection on this square, the monodromy is trivial.

The whole graph has genus 34. Theorems 10 and 9 show

$$\begin{aligned} \mathrm{PSU}_2(R_{40}) &\cong \Gamma_1 \cong \pi_1(Gr_{40}) \\ &\cong A_5 *_{A_4} S_4 *_{D_4} D_{20} *_{D_4} S_4 *_{A_4} A_5 * C_5^{*2} * (C_3 \oplus \mathbf{Z}) * \mathbf{Z}^{*33}. \end{aligned}$$

7.6. $n = 48$. Again $\mathrm{PSU}_2(R_{48}) = \Gamma_{48,1}$ while $\mathrm{PU}_2^\zeta(R_{48}) = \mathrm{PU}_2(R_{48}) = \Gamma_{48,+} = \Gamma_{48,0}$. The quotient graph of groups \overline{Gr}_{48} for $\mathrm{PU}_2^\zeta(R_{48}) = \mathrm{PU}_2(R_{48}) = \Gamma_{48,+} = \Gamma_{48,0}$ is shown in the same format as for $n = 40$.

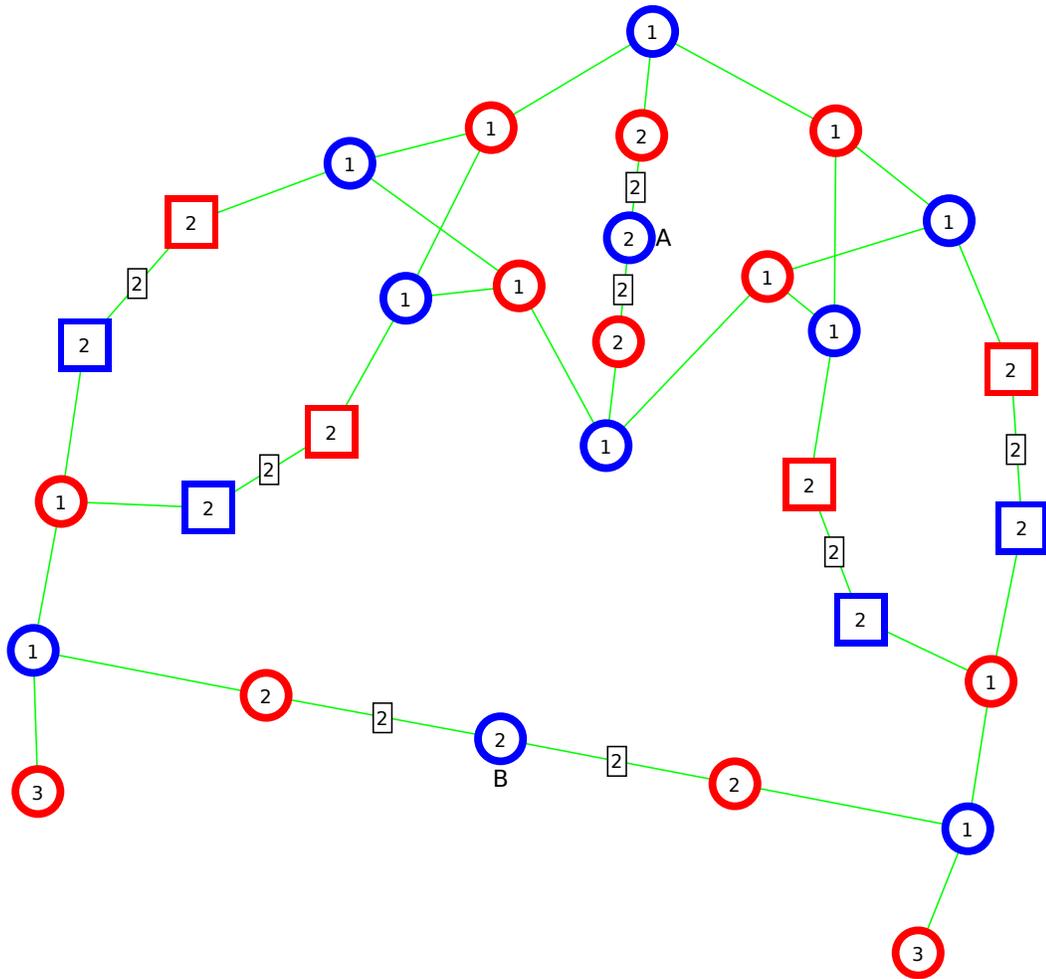


FIGURE 6. Subgraph 1 for $\text{PU}_2^{\zeta}(R_{48}) = \text{PU}_2(R_{48}) = \Gamma_{48,+} = \Gamma_{48,0}$.

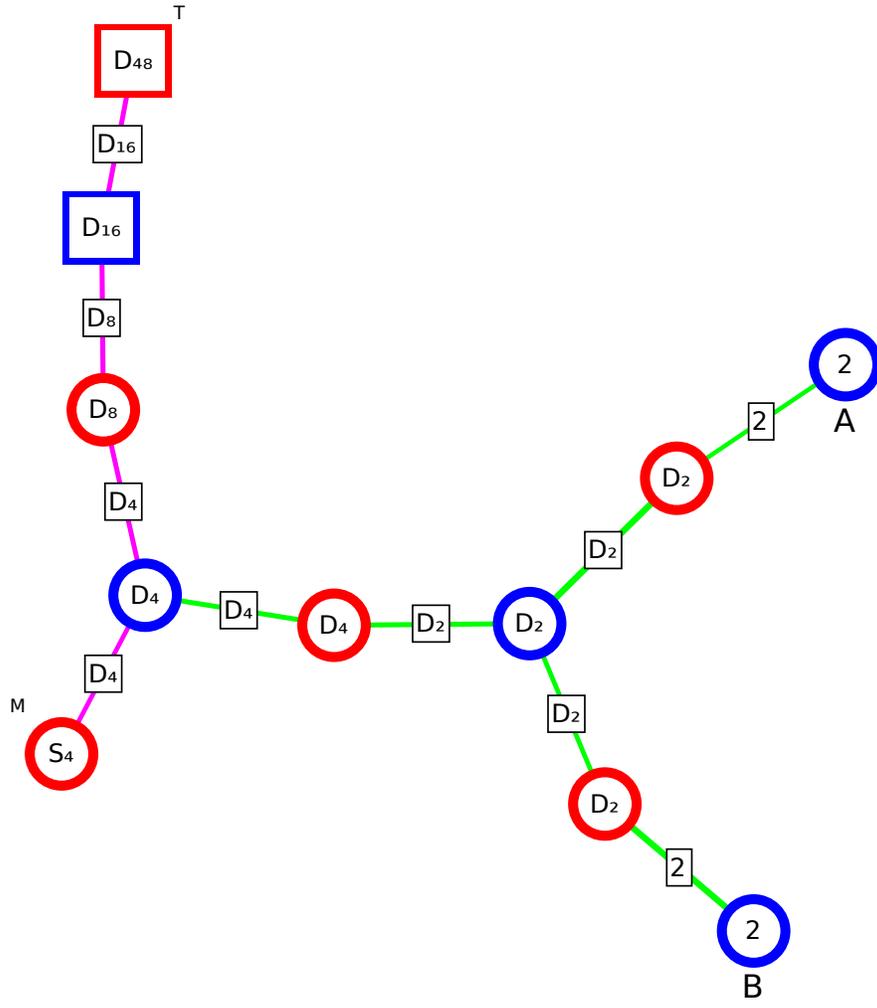


FIGURE 7. Subgraph 2 for $\text{PU}_2^{\zeta}(R_{48}) = \text{PU}_2(R_{48}) = \Gamma_{48,+} = \Gamma_{48,0}$.

There are 39 classes of maximal orders, but of these 14 have only ± 1 as units. Modulo ± 1 , there are also 14 orders with 2 units, three with 4, two each with 3 and 8, and one each with 16, 24, 32, 96. The automorphism group of each edge is the same as the smaller of the two automorphism groups of its incident nodes.

The graph has genus 8, and Theorem 9 and the diagram indicate that

$$\mathrm{PU}_2^\zeta(R_{48}) = \mathrm{PU}_2(R_{48}) \cong \pi_1(\overline{Gr}_{48}) \cong D_{48} *_{D_4} S_4 * C_3^{*2} * C_2^{*4} * \mathbf{Z}^{*8}$$

and $\chi(\mathrm{PU}_2(R_{48})) = -365/32$. For the same reason as in the cases $n = 32, 36, 40$, this presentation shows that

$$\mathrm{PG}_{48} \ll [\mathrm{PU}_2(R_{48})]_f \ll \mathrm{PU}_2(R_{48}). \quad (27)$$

The graph of groups Gr_{48} for $\mathrm{PSU}_2(R_{48}) = \Gamma_{48,1}$ is a double cover of \overline{Gr}_{48} ramified at the ten vertices indicated by squares in Figures 6, 7. As before the vertex labels for the unramified nodes are the same, and the ramified nodes have vertices labeled with subgroups of index 2. Counting vertices and edges we see that the graph has genus 20. Thus by Theorem 9 the group is

$$\mathrm{PSU}_2(R_{48}) \cong \pi_1(Gr_{48}) \cong S_4 *_{D_4} D_{48} *_{D_4} S_4 * C_3^{*4} * \mathbf{Z}^{*20}.$$

7.7. $n = 60$. As with $n = 28$, the unique prime above 2 in F_n splits in K_n . Hence $\mathrm{PSU}_2(R_{60}) = \Gamma_{60,1}$, $\mathrm{PU}_2^\zeta(R_{60}) = \Gamma_{60,+}$, and $\mathrm{PU}_2(R_{60}) = \Gamma_{60,0}$ are all distinct with

$$\frac{\mathrm{PU}_2(R_{60})}{\mathrm{PU}_2^\zeta(R_{60})} \cong \mathbf{Z}/2\mathbf{Z} \quad \text{and} \quad \frac{\mathrm{PU}_2^\zeta(R_{60})}{\mathrm{PSU}_2(R_{60})} \cong \mathbf{Z}/2\mathbf{Z}.$$

The class number of \mathbf{H}_{60} is 9, but as in the $n = 28$ case not all types are connected to \mathcal{M} . In this case, 7 of the 9 types occur in this (genus 5) quotient h-graph of groups \overline{Gr}_{60} for $\mathrm{PU}_2(R_{60}) = \Gamma_{60,0}$.

Let Gr_u be the graph of groups that is a single loop in the upper right whose vertex and edge groups are both D_{60} . As it turns out the induced automorphism on D_{60} is inner, so by Theorem 10, $\pi_1(Gr_u) = D_{60} \oplus \mathbf{Z}$. Let Gr_l be the graph of groups which remains after deleting Gr_u and the D_4 edge incident upon it. Clearly,

$$\mathrm{PU}_2(R_{60}) = \pi_1(Gr_u) *_{D_4} \pi_1(Gr_l).$$

Now, let T be the spanning tree for Gr_l obtained by eliminating the four edges with trivial stabilizer groups incident upon the vertex with vertex group C_3 . Two each of these edges are incident upon each of the two vertices with vertex groups D_2 . Theorem 9 then tells us that

$$\pi_1(Gr_l) \cong S_4 *_{A_4} A_5 *_{C_5} D_5 *_{C_2} D_2 * D_3 *_{C_3} * D_3 * C_2^{*4} * \mathbf{Z}^{*4},$$

so that

$$\mathrm{PU}_2(R_{60}) \cong (D_{60} \oplus \mathbf{Z}) *_{D_4} S_4 *_{A_4} A_5 *_{C_5} D_5 *_{C_2} D_2 * D_3 *_{C_3} * D_3 * C_2^{*4} * \mathbf{Z}^{*4}$$

and $\chi(\mathrm{PU}_2(R_{60})) = -15/2$.

The graph for $\mathrm{PU}_2^\zeta(R_{60}) = \Gamma_{60,+}$ is the bipartite double cover of the graph \overline{Gr}_{60} for $\mathrm{PU}_2(R_{60}) = \Gamma_{60,0}$. The graph Gr_{60} for $\mathrm{PSU}_2(R_{60}) = \Gamma_{60,1}$ is the double cover of the graph for $\mathrm{PU}_2^\zeta(R_{60}) = \Gamma_{60,+}$ ramified at the vertices lying above those in \overline{Gr}_{60} marked with a square in Figure 8.

8. SUMMARY

We summarize our results in the following table, which shows, for each group at each level, whether it is generated by torsion.

n	PSU_2	PU_2^ζ	PU_2	Γ_1	Γ_+	Γ_0
8	yes	yes	yes	yes	yes	yes
12	yes	yes	yes	yes	yes	yes
16	yes	yes	yes	yes	yes	yes
20	yes	yes	yes	yes	yes	yes
24	yes	yes	yes	yes	yes	yes
28	no	no	yes	no	no	yes
32	no	no	no	no	no	no
36	no	no	no	no	no	no
40	no	no	no	no	no	no
48	no	no	no	no	no	no
60	no	no	no	no	no	no

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