FREE CONVOLUTION POWERS VIA ROOTS OF POLYNOMIALS

STEFAN STEINERBERGER

Abstract. Let μ be a compactly supported probability measure on the real line. Bercovici-Voiculescu and Nica-Speicher proved the existence of a free convolution power $\mu^{\boxplus k}$ for any real $k \geq 1$. The purpose of this short note is to give a formal proof of an elementary description of $\mu^{\boxplus k}$ in terms of of polynomials and roots of their derivatives. This bridge allows us to switch back and forth between free probability and the asymptotic behavior of polynomials.

1. Introduction

1.1. **Free Convolution.** The notion of free convolution $\mu \boxplus \nu$ of two compactly supported probability measures is due to Voiculescu [41]. A definition is as follows: for any compactly supported probability measure, we can consider its Cauchy transform $G_{\mu} : \mathbb{C} \setminus \mathrm{supp}(\mu) \rightarrow \mathbb{C}$ defined via

$$G_{\mu}(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x).$$

For a compactly supported measure, $G_{\mu}(z)$ tends to 0 as $|z| \rightarrow \infty$. Given G_{μ}, we define the $R-$transform $R_{\mu}(s)$ for sufficiently small complex s by demanding that

$$\frac{1}{G_{\mu}(z)} + R_{\mu}(G_{\mu}(z)) = z$$

for all sufficiently large z. The free convolution $\mu \boxplus \nu$ is then the unique compactly supported measure for which

$$R_{\mu \boxplus \nu}(s) = R_{\mu}(s) + R_{\nu}(s)$$

for all sufficiently small s. A fundamental result due to Voiculescu is the *free central limit theorem*: if μ is a compactly supported probability measure with mean 0 and variance 1, then suitably rescaled copies of $\mu^{\boxplus k}$ converge to the semicircular distribution. This notion can be extended to real powers.

Theorem (Fractional Free Convolution Powers exist, [6, 25])**.** *Let μ be a compactly supported probability measure on $\mathbb{R}$ and assume $k \geq 1$ is real. Then there exists a unique compactly supported probability measure $\mu^{\boxplus k}$ such that*

$$R_{\mu^{\boxplus k}}(s) = k \cdot R_{\mu}(s) \qquad \textit{for all } s \textit{ sufficiently small.}$$

This was first shown for k sufficiently large by Bercovici & Voiculescu [6] and then by Nica & Speicher [25] for all $k \geq 1$. We also refer to [1, 2, 3, 4, 5, 15, 17, 24, 26, 37, 43].

2020 *Mathematics Subject Classification.* 46L54, 26C10.
Key words and phrases. Free Probability, Roots of Polynomials, Free Additive Convolution.
S.S. is supported by the NSF (DMS-1763179) and the Alfred P. Sloan Foundation.

The purpose of this note is to (formally) prove an elementary description of $\mu^{\boxplus k}$ in terms of polynomials and the density of the roots of their derivatives.

1.2. **Polynomials.** Roots of polynomials are a classical subject and there are many results we do not describe here, see [7, 8, 9, 13, 14, 17, 18, 19, 20, 21, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 40]. Our problem will be as follows: let μ be a compactly supported probability measure on the real line and suppose $x_1, \dots, x_n$ are n independent random variables sampled from μ (which we assume to be sufficiently nice). We then associate to these numbers the random polynomial

$$p_n(x) = \prod_{k=1}^{n} (x - x_k)$$

having roots exactly in these points. What can we say about the behavior of the roots of the derivative p_n'? There is an interlacing phenomenon and the roots of p_n are also distributed according to μ as $n \to \infty$. The same is true for the second derivative p_n'' and any finite derivative. However, once the number of derivatives is proportional to the degree, the distribution will necessarily change.

Question. Fix $0 < t < 1$. How are the roots of $p_n^{\lfloor t \cdot n \rfloor}$ distributed?

The question was raised by the author in [39]. The answer, if it exists, should be another measure $u(t,x)dx$. Note that, since this measure describes the distribution of roots of polynomials of degree $(1-t) \cdot n$, as $n \to \infty$, we have

$$\int_{\mathbb{R}} u(t,x)dx = 1 - t.$$

Relatively little is known about the evolution of $u(t,x)$: [39] established, on a formal level, a PDE for $u(t,x)$. This PDE is given by

$$\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan\left(\frac{Hu}{u}\right) = 0 \qquad \text{on } \operatorname{supp}(u),$$

where

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy \qquad \text{is the Hilbert transform.}$$

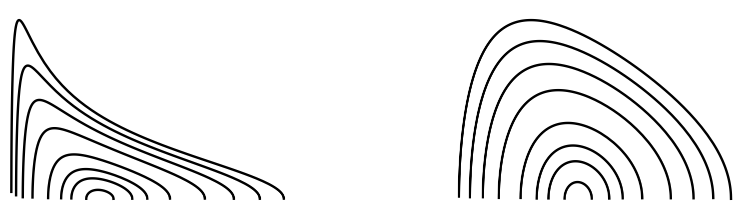

FIGURE 1. The densities of two evolving measures $u(t,x)dx$. They shrink and vanish at time $t = 1$.

It is also known that it has to satisfy the conservation laws

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(t,x)(x-y)^2 u(t,y) \ dxdy = (1-t)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} u(0,x)(x-y)^2 u(0,y) \ dxdy.$$

Hoskins and the authors [16] established a universality result for large derivatives of polynomials with random roots: such derivatives behave like random shifts of

Hermite polynomials. Hermite polynomials, in turn, have roots whose density is given by a semicircle and this leads one to believe that $u(t,x)$ should, for t close to 1, look roughly like a semicircle (and this has also been observed numerically). There are two explicit closed-form solutions, derived in [39], a shrinking semicircle and a one-parameter solution that lies in the Marchenko-Pastur family (see Fig. 1). Numerical simulations in [16] also suggested that the solution tends to become smoother. O'Rourke and the author [30] derived an analogous transport equation for polynomials with roots following a radial distribution in the complex plane.

2. The Result

2.1. **An Equivalence.** We can now state our main observation: both the free convolution of a measure with itself, $\mu^{\boxplus k}$, and the density of roots of derivatives of polynomials, $u(t,x)$, are described by the same underlying process.

Theorem. *At least formally, if $\mu = u(0,x)dx$ and $\{x : u(0,x) > 0\}$ is an interval, then for all real $k \geq 1$*

$$\mu^{\boxplus k} = u\left(1 - \frac{1}{k}, \frac{x}{k}\right) dx.$$

We first clarify the meaning of 'formally'. In a recent paper, Shlyakhtenko & Tao [37] derived, formally, a PDE for the evolution of the $\mu^{\boxplus k}$. This PDE happens to be the same PDE (expressed in a different coordinate system) that was formally derived by the author for the evolution of $u(t,x)$ [39]. The derivation in [39] is via a mean-field limit approach, the 'microscopic' derivation is missing. In particular, the derivation in [39] assumed the existence of $u(t,x)$ and a crystallization phenomenon for the roots; such a crystallization phenomenon has been conjectured for a while, there is recent progress by Gorin & Kleptsyn [11].

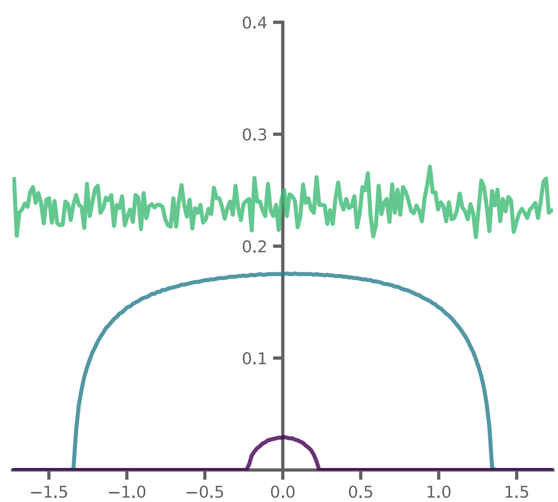

Figure 2. Evolution of $u(t,x)$ (from [16]) starting with random and uniformly distributed roots: the evolution smoothes and we see a semicircle before it vanishes.

Naturally, this has a large number of consequences since it allows us to go back and forth between results from free probability and results regarding polynomials and their roots. As an illustration, we recall that for the semicircle law $\mu_{sc}^{\boxplus k}$ is a another semicircle law stretched by a factor $k^{1/2}$, thus

$$\mu_{sc}^{\boxplus k} = \frac{2}{\pi}\sqrt{\frac{1}{k} - \frac{x^2}{k^2}}dx$$

Conversely, as was computed in [39], the evolution of densities of polynomials when beginning with a semicircle behaves as

$$u(t,x) = \frac{2}{\pi}\sqrt{1-t-x^2}.$$

One immediate consequence of the equivalence is that it provides us with a fast algorithm to approximate $\mu^{\boxplus k}$ when $\mu = f(x)dx$ and f is smooth. This may be useful in the study of the semigroup $\mu^{\boxplus k}$. Using the logarithmic derivative p_n'/p_n, it is possible quickly differentiate real-rooted polynomials p_n a large number of times, $t \cdot n$, even when the degree is as big as $n \sim 100.000$: this was done in [16] using a multipole method (a modification of an algorithm due to Gimbutas, Marshall & Rokhlin [10]). Fig. 2 shows an example computed using 80.000 roots: we observe the initial smoothing and the eventual convergence to a semicircle.

2.2. **Some Connections.** Some connections are as follows.

The Free Central Limit Theorem. Voiculescu [41] proved that $\mu^{\boxplus k}$ (suitably rescaled) approaches a semicircle distribution in the limit. Motivated by high-precision numerics, Hoskins and the author [16] conjectured that $u(t,x)$ starts looking like a semicircle for t close to 1 and proved a corresponding universality result for polynomials with random roots: if p_n is a polynomial with random roots (from a probability measure μ whose moments are all finite), then, for fixed $\ell \in \mathbb{N}$ and $n \to \infty$, we have for x in a compact interval

$$n^{\ell/2}\frac{\ell!}{n!} \cdot p_n^{(n-\ell)}\left(\frac{x}{\sqrt{n}}\right) \to He_\ell(x+\gamma_n),$$

where He_ℓ is the $\ell-$th probabilists' Hermite polynomial and γ_n is a random variable converging to the standard $\mathcal{N}(0,1)$ Gaussian as $n \to \infty$. Hermite polynomials have roots that are asymptotically distributed like a semicircle. A result in the deterministic setting has recently been provided by Gorin & Kleptsyn [11].

Conservation Laws. The author showed that the evolution $u(t,x)$ satisfies the algebraic relations

$$\int_{\mathbb{R}} u(t,x)\ dx = 1-t, \qquad \int_{\mathbb{R}} u(t,x)x\ dx = (1-t)\int_{\mathbb{R}} u(0,x)x\ dx,$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(t,x)(x-y)^2 u(t,y)\ dxdy = (1-t)^3\int_{\mathbb{R}}\int_{\mathbb{R}} u(0,x)(x-y)^2u(0,y)\ dxdy.$$

These are derived from Vieta-type formulas that express elementary symmetric polynomials in terms of power sums. Equivalently, we have $\kappa_n(\mu^{\boxplus k}) = k^n\kappa_n(\mu)$, where κ_n is the $n-$th free cumulant providing a large number of conservation laws.

Monotone Quantities. Voiculescu [42] introduced the free entropy
$$\chi(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} \log|s-t| d\mu(s) d\mu(t) + \frac{3}{4} + \frac{\log(2\pi)}{2}$$
and the free Fisher information
$$\Phi(\mu) = \frac{2\pi^2}{3} \int_{\mathbb{R}} \left(\frac{d\mu}{dx}\right)^3 dx.$$
Shlyakhtenko [36] proved that χ increases along free convolution of μ with itself whereas Φ decreases (both suitably rescaled). Shlyakhtenko & Tao [37] showed monotonicity along the entire flow $\mu^{\boxplus k}$ for real $k \geq 1$. Conversely, on the side of polynomials, it is known that
$$\frac{|\{x \in \mathbb{R} : u(t,x) > 0\}|}{1-t} \qquad \text{is non-decreasing in time.}$$
Another basic result for polynomials is commonly attributed to Riesz [8, 40]: denoting the smallest gap of a polynomial p_n having n real roots $\{x_1, \dots, x_n\}$ by
$$G(p_n) = \min_{i \neq j} |x_i - x_j|,$$
we have $G(p_n') \geq G(p_n)$: the minimum gap grows under differentiation. A simple proof is given by Farmer & Rhoades [8]. This would suggest that the maximal density cannot increase over time.

The Minor Process. Shlyakhtenko & Tao [37] connect the evolution to the minor process: trying to understand how the eigenvalues of the $n \times n$ minor of a large random Hermitian matrix $N \times N$ behave. This answers a question numerically verified by Hoskins and the author [16].

Related Results. There are several other papers in the literature that seem to be connected to this circle of ideas. We mention Gorin & Marcus [12], Marcus [22], Marcus, Spielman & Srivastava [23].

3. Proof

Proof. Shlyakhtenko & Tao [37] derive that if
$$d\mu^{\boxplus k} = f_k(x) dx$$
and if we substitute $k = 1/s$ (thus $0 < s < 1$) and $f := f_{1/s}$, then on a purely formal level
$$\left(-s\frac{\partial}{\partial s} + x\frac{\partial}{\partial x}\right) f = \frac{1}{\pi}\frac{\partial}{\partial x} \arctan\left(\frac{f}{Hf}\right).$$
On the other hand, the author derived [39], also on a formal level, that as long as $\{x : u(t,x) > 0\}$ is an interval
$$\frac{\partial u}{\partial t} + \frac{1}{\pi}\frac{\partial}{\partial x} \arctan\left(\frac{Hu}{u}\right) = 0 \qquad \text{on } \operatorname{supp}(u).$$
We note that Huang [17] showed that the number of connected components in the support of $\mu^{\boxplus k}$ is non-decreasing in k which shows that once the support is an interval, this property is preserved. We want to show that the solutions of these two PDEs are related via a change of variables: since both evolutions obey the same PDE, they must coincide. We observe that one nonlinear term seems to be

the reciprocal of the other, however, this compensates for the different sign. We compute

$$\frac{\partial}{\partial x}\arctan\left(\frac{f}{Hf}\right)=\frac{1}{1+\frac{f^2}{(Hf)^2}}\partial_x\frac{f}{Hf}=\frac{f_x(Hf)-f(Hf)_x}{f^2+(Hf)^2}$$

and compare it to

$$\frac{\partial}{\partial x}\arctan\left(\frac{Hf}{f}\right)=\frac{1}{1+\frac{(Hf)^2}{f^2}}\partial_x\frac{Hf}{f}=\frac{(Hf)_x f-f_x(Hf)}{f^2+(Hf)^2}$$

and see that it is the same term with opposite sign. This allows us to write

$$\frac{\partial u}{\partial t}=\frac{1}{\pi}\frac{\partial}{\partial x}\arctan\left(\frac{u}{Hu}\right).$$

We now claim that

$$f(s,x)=u(1-s,sx).$$

Note that the left-hand side transforms

$$(-s\partial_s+x\partial_x)f=-s\left(\frac{\partial u}{\partial t}(-1)+\frac{\partial u}{\partial x}x\right)+x\frac{\partial u}{\partial x}s=s\frac{\partial u}{\partial t}.$$

It remains to understand how the right-hand side transforms. The Hilbert transform commutes with dilations and thus

$$\arctan\left(\frac{f}{Hf}\right)=\arctan\left(\frac{u(1-s,sx)}{H\left[u(1-s,sx)\right]}\right)=\arctan\left(\frac{u(1-s,sx)}{\left[Hu(1-s,\cdot)\right](sx)}\right)$$

whose derivative scales exactly by a factor of s. $\square$

Remarks. We see that, both derivations being purely formal, many problems remain. Indeed, this connection suggests many interesting further avenues to pursue. Roots of polynomials seem to regularize under differentiation at the micro-scale: if one were to take a polynomial with random (or just relatively evenly spaced roots), then the roots of the $(\varepsilon\cdot n)-$th derivative are conjectured to behave locally like arithmetic progressions up to a small error. Results of this flavor date back to Polya [33] for analytic functions, see also Farmer & Rhoades [8] and Pemantle & Subramanian [32]. In the converse direction, it could be interesting to study the behavior of $u(t,x)$ when the initial conditions are close to a semi-circle: despite the equation being both non-linear and non-local, its linearization around the semicircle seems to diagonalize nicely under Chebychev polynomials – can PDE techniques be used to get convergence rates for the free central limit theorem?

References

[1] M. Anshelevich, The linearization of the central limit operator in free probability theory. Probability theory and related fields 115.3 (1999): 401–416.

[2] S. T. Belinschi, Some geometric properties of the subordination function associated to an operatorvalued free convolution semigroup, Complex Anal. Oper. Theory 13 (2019), 61–84.

[3] S. T. Belinschi, H. Bercovici, Atoms and regularity for measures in a partially defined free convolution semigroup, Mathematische Zeitschrift 248 (2004), 665–674.

[4] S. T. Belinschi, H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, International Mathematics Research Notices 2 (2005), 65–101.

[5] H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. Indiana Univ. Math. J. 42: p. 733–773, 1993.

[6] H. Bercovici and D. Voicolescu, Superconvergence to the central limit and failure of the Cramer theorem for free random variables, Prob. Theo. Related Fields 103 (1995), 215–222.

[7] S. S. Byun, J. Lee and T. R. Reddy, Zeros of random polynomials and its higher derivatives, arXiv:1801.08974

[8] D. Farmer and R. Rhoades, Differentiation evens out zero spacings. Trans. Amer. Math. Soc. 357 (2005), no. 9, 3789–3811.

[9] C.F. Gauss: Werke, Band 3, Göttingen 1866, S. 120:112

[10] Z. Gimbutas, N. Marshall, and V. Rokhlin, A fast simple algorithm for computing the potential of charges on a line, Appl. Comp. Harm. Anal. 49 (2020), p. 815–830

[11] V. Gorin and V. Kleptsyn, Universal Objects of the Infinite Beta Random Matrix Theory, arXiv:2009.02006

[12] V. Gorin and A. Marcus, Crystallization of Random Matrix Orbits, International Mathematics Research Notices 2020, Issue 3, p. 883–913,

[13] R. Granero-Belinchon, On a nonlocal differential equation describing roots of polynomials under differentiation, arXiv:1812.00082

[14] B. Hanin, Pairing of zeros and critical points for random polynomials, Ann. Inst. H. Poincaré, Probab. Statist. 53 (2017), p. 1498–1511.

[15] F. Hiai and P. Denes, The semicircle law, free random variables and entropy 77, American Mathematical Soc., 2000.

[16] J. Hoskins and S. Steinerberger, A Semicircle Law for Derivatives of Random Polynomials, arXiv:2005.09809

[17] H. W. Huang, Supports of measures in a free additive convolution semigroup, Int. Math. Res. Notices 2015 (2014), 4269–4292.

[18] Z. Kabluchko, Critical points of random polynomials with independent identically distributed roots. Proc. Amer. Math. Soc. 143 (2015), p. 695–702.

[19] Z. Kabluchko and H. Seidel, Distances between zeroes and critical points for random polynomials with i.i.d. zeroes, Electron. J. Probab. 24 (2019), paper no. 34, 25 pp.

[20] M. Kornik and G. Michaletzky, Wigner matrices, the moments of roots of Hermite polynomials and the semicircle law, Journal of Approximation Theory 211 (2016), p. 29–41.

[21] F. Lucas: Sur une application de la Mécanique rationnelle à la théorie des équations. in: Comptes Rendus de l'Académie des Sciences 89 (1879), S. 224–226

[22] A. Marcus, Polynomial convolutions and (finite) free probability. preprint (2016).

[23] A. Marcus, D. Spielman and N. Srivastava, Finite free convolutions of polynomials, arXiv:1504.00350

[24] J. A. Mingo and R. Speicher. Free probability and random matrices, volume 35 of Fields Institute Monographs. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.

[25] A. Nica, R. Speicher, On the multiplication of free N-tuples of noncommutative random variables, Amer. J. Math. 118 (1996), no. 4, 799–837.

[26] A. Nica and R. Speicher, Lectures on the combinatorics of free probability 13. Cambridge University Press, 2006.

[27] S. O'Rourke and N. Williams, Pairing between zeros and critical points of random polynomials with independent roots, Trans. Amer. Math. Soc. 371 (2019), p. 2343–2381

[28] S. O'Rourke and N. Williams, On the local pairing behavior of critical points and roots of random polynomials, Electron. J. Probab. Volume 25 (2020), paper no. 100, 68 pp

[29] S. O'Rourke and T.R. Reddy, Sums of random polynomials with independent roots, arXiv:1909.07939

[30] S. O'Rourke and S. Steinerberger, A Nonlocal Transport Equation Modeling Complex Roots of Polynomials under Differentiation, arXiv:1910.12161

[31] R. Pemantle, and I. Rivlin. The distribution of the zeroes of the derivative of a random polynomial. Advances in Combinatorics. Springer 2013. pp. 259–273.

[32] R. Pemantle and S. Subramanian, Zeros of a random analytic function approach perfect spacing under repeated differentiation. Trans. Amer. Math. Soc. 369 (2017), 8743–8764.

[33] G. Polya, Some Problems Connected with Fourier's Work on Transcendental Equations, The Quarterly Journal of Mathematics 1 (1930), p. 21–34.

[34] M. Ravichandran, Principal Submatrices, Restricted Invertibility, and a Quantitative Gauss–Lucas Theorem, IMRN, to appear

[35] T. R. Reddy, Limiting empirical distribution of zeros and critical points of random polynomials agree in general, Electron. J. Probab. 22 (2017), paper no. 74, 18 pp.

[36] D. Shlyakhtenko, A free analogue of Shannon's problem on monotonicity of entropy, Adv. Math. 208 (2007), no. 2, 824–833.

[37] D. Shlyakhtenko and T. Tao, Fractional free convolution powers, arXiv:2009.01882

[38] S. Subramanian, On the distribution of critical points of a polynomial, Electron. Commun. Probab. 17 (2012), paper no. 37, 9 pp.

[39] S. Steinerberger, A Nonlocal Transport Equation Describing Roots of Polynomials Under Differentiation, Proc. Amer. Math. Soc. 147 (2019), p. 4733–4744

[40] A. Stoyanoff, Sur un Theorem de M. Marcel Riesz, Nouv. Annal. de Math, 1 (1926), 97–99.

[41] D. Voiculescu, Addition of certain non-commuting random variables, J. Funct. Anal. 66 (1986), 323–346.

[42] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory I, Comm. Math. Phys. 155 (1993) p. 71–92.

[43] J. Williams, On the Hausdorff continuity of free Levy processes and free convolution semigroups, J. Math. Anal. Appl. 459 (2018), 604–613.

Department of Mathematics, University of Washington, Seattle
Email address: `steinerb@uw.edu`