The reciprocal algebraic integers having small house

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Abstract. Let α be an algebraic integer of degree d, which is reciprocal. The house of α is the largest modulus of its conjugates. We proved that d-th power of the house of reciprocal α has a limit point. We presented a property of antireciprocal hexanomials. We compute the minimum of the houses of all reciprocal algebraic integers of degree d having the minimal polynomial which is a factor of a D-th degree reciprocal or antireciprocal polynomial with at most eight monomials, say mr(d), for d at most 180, $D \le 1.5d$ and $D \le 210$. We show that it is not necessary to take into account unprimitive polynomials. The computations suggest several conjectures.

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1 Introduction

Let α be an algebraic integer of degree d, with conjugates $\alpha = \alpha_1, \alpha_2, \dots, \alpha_d$ and minimal polynomial P. The house of α (and of P) is defined by:

$$\overline{\alpha} = \max_{1 \le i \le d} |\alpha_i|.$$

The Mahler measure of α is $M(\alpha) = \prod_{i=1}^{d} \max(1, |\alpha_i|)$. Clearly, $\overline{\alpha} > 1$, and a theorem of Kronecker tells us that $\overline{\alpha} = 1$ if and only if α is a root of unity. In 1965, Schinzel and Zassenhaus [5] have made the following conjecture:

Conjecture 1 [SZ]. There is a constant c > 0 such that if α is not a root of unity, then $\overline{\alpha} \ge 1 + c/d$.

If a polynomial has only eight non-zero coefficients then it is called an octanomial. Similarly, if the number of non-zero coefficients is five, six and seven, such polynomial is called pentanomial, hexanomial and heptanomial respectively. A polynomial P(x) of degree d is antireciprocal if it satisfies $P(x) = -x^d P(1/x)$. The height of a polynomial is defined to be the maximum of the moduli of its coefficients. Let m(d) denote the minimum of α over α of degree d which are not roots of unity. Let an α attaining m(d) be called extremal. We say that α is reciprocal if α^{-1} is a conjugate of α , i.e. $X^d P(1/X) = P(X)$. Let mr(d) denote the minimum of α over reciprocal α of degree d which are not roots of unity. Let an α attaining mr(d) be called extremal reciprocal. In 1985, D. Boyd [2] conjectured, using a result of C.J. Smyth [6], that c should be equal to $3/2 \log \theta$ where $\theta = 1.324717...$ is the smallest Pisot number, the real root of the polynomial $x^3 - x - 1$. Intending to verify his conjecture that extremal α are always nonreciprocal, Boyd has computed the smallest houses for reciprocal polynomials of even degrees ≤ 16 . Wu and Zhang [8] continued the Boyd's computation with even degrees ≤ 42 . They showed in their Table 5 that the minimal polynomial of extremal reciprocal algebraic integer can be written as a factor of a reciprocal polynomial with at most eight monomials. The same fact is valid for many polynomials having Mahler measure less than 1.3 and has been used for creation of the Mossinghoff's list of such polynomials [4]. We used here this idea to search for extremal reciprocals of degree d having the minimal polynomial which is a factor of a D-th degree reciprocal or antireciprocal hexanomial or octanomial, where d is at most 180 and D is at most 210.

2 Theorems and proofs

A polynomial P(x) is primitive if it cannot be expressed as a polynomial in x^k , for some $k \ge 2$. Clearly, if p is an odd prime number then any reciprocal polynomial of degree 2p with more than three monomials has to be primitive. It is easy to verify that

$$\boxed{P(x^k)} = \sqrt[k]{P(x)}.$$
(2.1)

Let mrp(d) denote the minimum of $\overline{\alpha}$ over reciprocal algebraic integer α of degree d which are not roots of unity and which have a primitive minimal polynomial. Let mrp(d) is attained for α_d with minimal reciprocal primitive polynomial $R_d(x)$. Let α_d be called extremal reciprocal primitive. Clearly

$$mr(d) \le mrp(d),\tag{2.2}$$

and the equality is strict if and only if the α attaining mr(d) is not a root of a primitive polynomial.

Lemma 1. Let k_i , k_j be integers and d_i , d_j be even integers such that $k_i d_i = k_j d_j = d$. If $\operatorname{mrp}^{d_i}(d_i) < \operatorname{mrp}^{d_j}(d_j)$ then the house of $R_{d_i}(x^{k_i})$ is less than the house of $R_{d_j}(x^{k_j})$.

Proof. Raising both sides of $\operatorname{mrp}^{d_i}(d_i) < \operatorname{mrp}^{d_j}(d_j)$ to the power 1/d we obtain $\operatorname{mrp}^{1/k_i}(d_i) < \operatorname{mrp}^{1/k_j}(d_j)$. It remains to recall that the house of $R_{d_i}(x^{k_i})$ is equal to $\operatorname{mrp}^{1/k_i}(d_i)$ and the house of $R_{d_j}(x^{k_j})$ is equal to $\operatorname{mrp}^{1/k_j}(d_j)$. \Box

Corollary 1. Let d be an even natural number and let d_0, d_1, \ldots, d_m be all even natural divisors of d. Let $mrp(d_j)$ is attained for a reciprocal α_{d_j} with minimal polynomial $R_{d_j}(x)$ where, $R_{d_j}(x)$ is a primitive reciprocal polynomial, $k_j = d/d_j$, $j = 0, 1, \ldots, m$. If

$$\operatorname{mrp}^{d_0}(d_0) < \operatorname{mrp}^{d_1}(d_1) < \dots < \operatorname{mrp}^{d_m}(d_m)$$

then $mr(d) = mrp^{1/k_0}(d_0).$

Proof. Straightforwardly from Lemma 1 it follows that polynomial $R_{d_0}(x^{k_0})$ has the house which is less than the house of any other polynomial of degree d. Therefore

$$\operatorname{mr}(d) = R_{d_0}(x^{k_0})$$

= $\operatorname{mrp}^{1/k_0}(d_0).$

The smallest limit point of the Mahler measure is believed to be 1.255433... [1] which appears from the sequence of reciprocal heptanomials $(Q_n(x))_{n\geq 1}$,

$$Q_n(x) = x^{2n} + x^{2n-1} + x^{n+1} + x^n + x^{n-1} + x + 1.$$

What is the smallest limit point of $\overline{P_d(x)}^d$ where $P_d(x)$ is a reciprocal polynomial of degree d arises as an interesting problem.

Lemma 2. The sequence $(mr^d(d))_{d\geq 1}$ is bounded. If $\alpha = 3/2 + \sqrt{5}/2 = 2.618 \dots$ then $\alpha^2 = 6.854 \dots$ is an upper bound.

Proof. If mr(d) is attained for α_d then

$$\operatorname{mr}(d) = [\overline{\alpha_d}] \le \overline{x^d + 3x^{d/2} + 1} = \sqrt[d]{2.618\dots}$$

The claim follows straightforwardly if we raise both sides of the inequality to the power d. \Box

It is much more difficult to prove that the sequence $(mrp^d(d))_{d\geq 1}$ is bounded. Let us introduce the following sequence of polynomials $(P_{2n}(x))_{n\geq 2}$

$$P_{2n}(x) = x^{2n} - x^{n+1} - x^n - x^{n-1} + 1.$$

They are obviously reciprocal and primitive.

Theorem 1. There is a unique real root $\alpha_n \in (\sqrt[n]{2}, \sqrt[n]{3})$ of $P_{2n}(x)$ such that the house of $P_{2n}(x)$ is equal to α_n .

Proof. There is a real root $\alpha_n \in (\sqrt[n]{2}, \sqrt[n]{3})$ of $P_{2n}(x)$ because $P(\sqrt[n]{2}) = 2^2 - 2^{(n+1)/n} - 2 - 2^{(n-1)/n} + 1 < 1 - 2^{(n-1)/n} < 0$ and $P(\sqrt[n]{3}) = 3^2 - 3^{(n+1)/n} - 3 - 3^{(n-1)/n} + 1 > 2^2 - 2^{(n+1)/n} > 0$. If we divide the equation $P_{2n}(x) = 0$ with x^n we get

$$x^n - x - 1 - x^{-1} + x^{-n} = 0.$$

Afterwards we use the trigonometric form of complex $x, x = r(\cos(\varphi) + i\sin(\varphi))$ and get

$$r^{n}(\cos n\varphi + i\sin n\varphi) - r(\cos \varphi + i\sin \varphi) - 1 - r^{-1}(\cos(-\varphi) + i\sin(-\varphi)) + r^{-n}(\cos(-n\varphi) + i\sin(-n\varphi)) = 0$$

Separating real and imaginary part of this equation we conclude that every root has to satisfy the following system:

$$\left(r^{n} + \frac{1}{r^{n}}\right)\cos(n\varphi) - \left(r + \frac{1}{r}\right)\cos(\varphi) - 1 = 0,$$
(2.3)

$$\left(r^n - \frac{1}{r^n}\right)\sin(n\varphi) - \left(r - \frac{1}{r}\right)\sin(\varphi) = 0.$$
(2.4)

If a root is unimodal i.e. r = 1 then (2.4) is clearly satisfied. Equation (2.3) then gives $2\cos(n\varphi) - 2\cos(\varphi) - 1 = 0$, and afterwards

$$\cos(n\varphi) = \cos(\varphi) + 0.5. \tag{2.5}$$

Since the left side of (2.5) $\cos(n\varphi) \in [-1, 1]$ it follows that on the right side $\cos \varphi \in [-1, 0.5]$) $\Leftrightarrow \varphi \in (\pi/3, 5\pi/3)$. If $\varphi \in (-\pi/3, \pi/3)$ then there are no unimodal roots because the left side of (2.5) can not be equal to the right side. If r > 1 we can solve (2.3) for $\cos(n\varphi)$ and (2.4) for $\sin(n\varphi)$ in terms of φ and r. Finally we get an implicit equation of a plane curve $r = r(\varphi)$ which contains all roots which are not unimodal when $\varphi \in (-\pi/3, \pi/3)$:

$$\left[\frac{\left(r-\frac{1}{r}\right)\sin(\varphi)}{r^{n}-\frac{1}{r^{n}}}\right]^{2} + \left[\frac{\left(r+\frac{1}{r}\right)\cos(\varphi)+1}{r^{n}+\frac{1}{r^{n}}}\right]^{2} = 1.$$
(2.6)

It is obvious that if $r(\varphi)$ satisfies (2.6) then $r_1(\varphi) = 1/r(\varphi)$ also satisfies (2.6). We can calculate θ such that $r(\theta) = 1$. If we use that $r^n - \frac{1}{r^n} = \left(r - \frac{1}{r}\right)\left(r^{n-1} + r^{n-3} + \cdots + r^{-(n-1)}\right)$ and replace r = 1 in (2.6) we obtain a quadratic equation

$$\frac{1 - \cos^2(\theta)}{n^2} + \frac{(2\cos(\theta) + 1)^2}{4} = 1$$

and solve it for $\cos(\theta)$. The solution $\cos(\theta) = \frac{\sqrt{4n^4 - 7n^2 + 4} - n^2}{2n^2 - 2} \in (0, \frac{1}{2})$ and tends to 0.5 when *n* tends to infinity. It follows that $\pi/3 < \theta < \pi/2$ and $r(\varphi) > 1$ on $(-\theta, \theta) \supset (-\pi/3, \pi/3)$ (see Figure 1). The claim will be proved if we show that $r(\varphi)$ increase on $(-\pi/3, 0)$ and decrease on $(0, \pi/3)$. If we denote the left



Figure 1. Roots of the reciprocal polynomial $P_{42}(x) = x^{42} - x^{22} - x^{21} - x^{20} + 1$ are represented with \circ . If modulus of a root is equal to one then its argument $\in (\pi/3, 5\pi/3)$, else the root is lying on the graph of $r(\varphi)$, the solution of the polar equation (2.6). The root having maximum modulus is the real root denoted with \bullet .

side of (2.6) by $F(r(\varphi), \varphi)$ then the implicit equation becomes $F(r(\varphi), \varphi) = 1$. It is well known that the first derivative of $r(\varphi)$ is

$$r'(\varphi) = -rac{rac{\partial F}{\partial \varphi}}{rac{\partial F}{\partial r}}.$$

Since

$$\frac{\partial F}{\partial \varphi} = \frac{2\sin(\varphi)\cos(\varphi)(r-\frac{1}{r})}{r^n - \frac{1}{r^n}} - \frac{2\sin(\varphi)(r+\frac{1}{r})((r+\frac{1}{r})\cos(\varphi)+1)}{(r^n + \frac{1}{r^n})^2}$$
(2.7)

can be simplified

$$-\frac{r^{4n}-2r^{2n}+1+r^{4n+2}-2r^{2n+2}+r^2+4r^{4n+1}\cos\varphi-4r^{2n+3}\cos\varphi-4r^{2n-1}\cos\varphi+4r\cos\varphi}{(2r^{2n-1}\sin\varphi)^{-1}(r^{2n}-1)^2(r^{2n}+1)^2}$$

and factored into

$$\frac{2r^{2n-1}\sin\varphi\left((r^{2n}-1)^2(r^2+1)+4r\cos\varphi(r^{2n+2}-1)(r^{2n-2}-1)\right)}{(r^{2n}-1)^2(r^{2n}+1)^2}$$

it follows that

$$\varphi \in (-\pi/3, 0) \Rightarrow \frac{\partial F}{\partial \varphi} > 0, \quad \varphi \in (0, \pi/3, 0) \Rightarrow \frac{\partial F}{\partial \varphi} < 0.$$
 (2.8)

so that the claim will be proved if we show that $-\frac{\partial F}{\partial r} > 0$ on $(-\pi/3, \pi/3)$. We derive that

$$\frac{\partial F}{\partial r} = \frac{2\sin^2\varphi\left(nr^{n-1} + \frac{n}{r^{n+1}}\right)\left(r - \frac{1}{r}\right)^2}{\left(\frac{1}{r^n} - r^n\right)^3} - \frac{2\left[\cos\varphi\left(r + \frac{1}{r}\right) + 1\right]^2\left(nr^{n-1} - \frac{n}{r^{n+1}}\right)}{\left(\frac{1}{r^n} + r^n\right)^3} + \frac{2\sin^2\varphi\left(r - \frac{1}{r}\right)\left(\frac{1}{r^2} + 1\right)}{\left(\frac{1}{r^n} - r^n\right)^2} - \frac{2\cos\varphi\left[\cos\varphi\left(r + \frac{1}{r}\right) + 1\right]\left(\frac{1}{r^2} - 1\right)}{\left(\frac{1}{r^n} + r^n\right)^2}.$$

can be expanded

$$\frac{\partial F}{\partial r} = -\frac{2r^{2n-3}\left(4r^2A(r)\cos^2\varphi + B(r)\cos\varphi + C(r)\right)}{\left(r^{4n} - 1\right)^3}$$

where

$$B(r) = (2n-1)r^{8n+3} + (2n+1)r^{8n+1} - (8n-2)r^{6n+3} - (8n+2)r^{6n+1} + 12nr^{4n+3} + 12nr^{4n+1} - (8n+2)r^{2n+3} - (8n-2)r^{2n+1} + (2n+1)r^3 + (2n-1)r$$
$$= r(r^{2n}-1)^3((2n-1)(r^{2n+2}-1) + (2n+1)r^2(r^{2n-2}-1))$$

so that we can conclude that B(r) is greater than 0 when r > 1. We intend to prove that also

$$A(r) = nr^{8n} - (2n-1)r^{6n+2} - (2n+1)r^{6n-2} + 6nr^{4n} - (2n+1)r^{2n+2} - (2n-1)r^{2n-2} + n$$

and

$$C(r) = (n-1)r^{8n+4} - nr^{8n+2} + (n+1)r^{8n} + (4n-2)r^{6n+4} - 12nr^{6n+2} + (4n+2)r^{6n} +$$

 $+6nr^{4n+4} - 6nr^{4n+2} + 6nr^{4n} + (4n+2)r^{2n+4} - 12nr^{2n+2} + (4n-2)r^{2n} + (n+1)r^4 - nr^2 + n - 1$ are greater than 0 when r > 1. If we substitute r^2 in A(r) with t + 1 then we have to prove that

$$D_{4n}(t) = n(t+1)^{4n} - (2n-1)(t+1)^{3n+1} - (2n+1)(t+1)^{3n-1} + 6n(t+1)^{2n} - (2n+1)(t+1)^{n+1} - (2n-1)(t+1)^{n-1} + n > 0$$

when t > 0. If we develop $D_{4n}(t)$ and denote its coefficients by a_k i.e. $D_{4n}(t) = a_{4n}t^{4n} + a_{4n-1}t^{4n-1} + \dots + a_1t + a_0$ then it is sufficiently to prove that $a_k = \frac{n\binom{4n}{k} - (2n-1)\binom{3n+1}{k} - (2n+1)\binom{3n-1}{k}}{(2n+1)\binom{n+1}{k}} + (6n)\binom{2n}{k} - (2n+1)\binom{n+1}{k} - (2n-1)\binom{n-1}{k} + n\binom{0}{k}$ are ≥ 0 where we use the convention that

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \ 0 < k \le n, \ \binom{n}{0} = 1, \ \binom{0}{0} = 1, \ \binom{n}{k} = 0, \ k > n.$$

We can verify that $a_k = 0$, k = 0, 1, 2, 3 and that $a_4 = n(n-1)(n+1)(3n^2-1)/3 > 0$. Let us denote underlined part of a_k with b_k and the remainder with c_k . For $k \ge 5$ it is sufficient to prove that $b_k > 0$ and

 $c_k > 0$. If $n \ge k > 3n + 1$ then, using the convention, $b_k = n \binom{4n}{k}$ so that $b_k > 0$. If $k \le 3n - 1$ then we prove that $b_k > 0$, using the principle of mathematical induction. For k = 5 and $n \ge 2$ we have

$$b_5 = n \binom{4n}{5} - (2n-1) \binom{3n+1}{5} - (2n+1) \binom{3n-1}{5} \\ = (n-1)(26n^5 + 366n^4 - 539n^3 - 39n^2 + 240n - 60)/60 \\ > 0.$$

Assume that the inequality is true for k = m, m < 3n - 1 and $m \ge 5$ and let us prove that it is true for k = m + 1. For k = m we have $0 < b_m$ i.e.

$$0 < n \binom{4n}{m} - (2n-1) \binom{3n+1}{m} - (2n+1) \binom{3n-1}{m}.$$

If we multiply both sides of this inequality by (3n + 1 - m)/(m + 1) > 0 then we have

$$\begin{aligned} 0 < n \binom{4n}{m} \frac{3n+1-m}{m+1} - (2n-1)\binom{3n+1}{m} \frac{3n+1-m}{m+1} - (2n+1)\binom{3n-1}{m} \frac{3n+1-m}{m+1} \\ < n \binom{4n}{m} \frac{4n-m}{m+1} - (2n-1)\binom{3n+1}{m} \frac{3n+1-m}{m+1} - (2n+1)\binom{3n-1}{m} \frac{3n-1-m}{m+1} \\ = n\binom{4n}{m+1} - (2n-1)\binom{3n+1}{m+1} - (2n+1)\binom{3n-1}{m+1} \\ = b_{m+1}, \end{aligned}$$

which coincides with the rewritten inequality for k = m + 1. It remains to be proved that $b_{3n} > 0$ and $b_{3n+1} > 0$. We have just proved that

$$0 < b_{3n-1} = n \binom{4n}{3n-1} - (2n-1) \binom{3n+1}{3n-1} - (2n+1) \binom{3n-1}{3n-1}$$

so that it follows

$$0 < n \binom{4n}{3n-1} - (2n-1) \binom{3n+1}{3n-1}.$$

If we multiply both sides of this inequality by 2/(3n) > 0 then we have

$$0 < n \binom{4n}{3n-1} \frac{2}{3n} - (2n-1) \binom{3n+1}{3n-1} \frac{2}{3n} < n \binom{4n}{3n-1} \frac{n+1}{3n} - (2n-1) \binom{3n+1}{3n-1} \frac{2}{3n} = n \binom{4n}{3n} - (2n-1) \binom{3n+1}{3n} = b_{3n}.$$

Similarly, if we multiply both sides of the inequality $0 < b_{3n}$ by 1/(3n+1) > 0 then we have

$$0 < n \binom{4n}{3n} \frac{1}{3n+1} - (2n-1) \binom{3n+1}{3n} \frac{1}{3n+1} < n \binom{4n}{3n} \frac{n}{3n+1} - (2n-1) \binom{3n+1}{3n} \frac{1}{3n+1} = b_{3n+1}.$$

Finally it remains to be proved that $c_k > 0$. Since

$$c_k = (3n)\binom{2n}{k} - (2n+1)\binom{n+1}{k} + (3n)\binom{2n}{k} - (2n-1)\binom{n-1}{k} + n\binom{0}{k},$$

and it is obviously that

$$(3n)\binom{2n}{k} > (2n+1)\binom{n+1}{k}, \quad (3n)\binom{2n}{k} > (2n-1)\binom{n-1}{k}$$

the claim follows.

If we substitute r^2 in C(r) with t + 1 then we have to prove that

$$E_{4n}(t) = (n-1)(t+1)^{4n+2} - n(t+1)^{4n+1} + (n+1)(t+1)^{4n} + (4n-2)(t+1)^{3n+2} - 12n(t+1)^{3n+1} + (4n+2)(t+1)^{3n} + 6n(t+1)^{2n+2} - 6n(t+1)^{2n+1} + 6n(t+1)^{2n} + (4n+2)(t+1)^{n+2} - 12n(t+1)^{n+1} + (4n-2)(t+1)^n + (n+1)(t+1)^2 - n(t+1) + n - 1 > 0$$

when t > 0. If we develop $E_{4n}(t)$ and denote its coefficients by d_k i.e. $E_{4n}(t) = d_{4n}t^{4n} + d_{4n-1}t^{4n-1} + \cdots + d_1t + d_0$ then it is sufficiently to prove that $d_k = e_k + f_k \ge 0$ where

$$e_{k} = (n-1)\binom{4n+2}{k} - n\binom{4n+1}{k} + (n+1)\binom{4n}{k} + (4n-2)\binom{3n+2}{k} - 12n\binom{3n+1}{k} + (4n+2)\binom{3n}{k} + (4n+2)\binom{3n}{k} + (4n+2)\binom{3n}{k} + (4n+2)\binom{n+2}{k} - 12n\binom{n+1}{k} + (4n-2)\binom{n}{k} + (n+1)\binom{2}{k} - n\binom{1}{k} + (n-1)\binom{0}{k} \ge 0$$

We can verify that $d_k = 0$, k = 0, 1, 2, 3 and that $d_4 = n(2n-1)(n+1)(5n^3 + n^2 + 12n + 12)/12 > 0$. Let us prove that $e_k > 0$, $3n \ge k \ge 5$. Since

$$e_{k} = G_{1}[(n-1)(4n+2)(4n+1) - n(4n+1)(4n-k+2) + (n+1)(4n-k+2)(4n-k+1)] + (2.9) + G_{2}[(4n-2)(3n+2)(3n+1) - 12n(3n+1)(3n-k+2) + (4n+2)(3n-k+2)(3n-k+1)] + (2.10)$$

where $G_1 = \frac{4n(4n-1)\cdots(4n-k+3)}{k!}$ and $G_2 = \frac{3n(3n-1)\cdots(3n-k+3)}{k!}$ so that if we divide both sides of the inequality $e_k > 0$ with G_1 we get the equivalent inequality

$$(n-1)(4n+2)(4n+1) - n(4n+1)(4n-k+2) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+2) + (n+1)(4n-k+2)(4n-k+2) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+2) + (n+1)(4n-k+2)(4n-k+1) + (n+1)(4n-k+2)(4n-k+2) + (n+1)(4n-k+2)($$

$$+M[(4n-2)(3n+2)(3n+1) - 12n(3n+1)(3n-k+2) + (4n+2)(3n-k+2)(3n-k+1)] > 0$$

where $M = \frac{3n(3n-1)\cdots(3n-k+3)}{4n(4n-1)\cdots(4n-k+3)}$. The left side of the inequality is quadratic in k thus we reorder it and then we have

$$(4Mn + n + 2M + 1)k^{2} + (12Mn^{2} - 4n^{2} - 12Mn - 10n - 6M - 3)k + -36Mn^{3} + 16n^{3} - 36Mn^{2} + 12n^{2} - 8Mn + 2n > 0$$

so that it is fulfilled if its discriminant $\Delta_1 < 0$ where

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$$\Delta_1 = (720n^4 + 576n^3 + 416n^2 + 208n + 36)M^2 + (-208n^4 - 176n^3 + 264n^2 + 208n + 36)M + (-208n^4 - 176n^2 + 208n^4 + 208n^4$$

$$-48n^4 - 32n^3 + 68n^2 + 52n + 9.$$

Since Δ_1 is quadratic in M it follows that inequality $\Delta_1 < 0$ is fulfilled if M is between roots M_1 , M_2 of Δ_1 where

$$M_{1,2} = \frac{104n^4 + 88n^3 - 132n^2 - 104n - 18}{720n^4 + 576n^3 + 416n^2 + 208n + 36} + \frac{\pm 4\sqrt{2}n\sqrt{1418n^6 + 2156n^5 - 946n^4 - 3068n^3 - 1905n^2 - 485n - 45}}{720n^4 + 576n^3 + 416n^2 + 208n + 36}$$

tend to

$$\frac{104 \pm 4\sqrt{2 \cdot 1418}}{720} = -0.151, \ 0.440$$

when n tends to ∞ . Since $k \ge 5$ and $\frac{3n-j}{4n-j} < \frac{3}{4}$, j > 0 it follows $0 < M < \frac{3^3}{4^3} = 0.422$ so that M is really between M_1 and M_2 when n is large enough. If we replace k = 3n + 1 in e_k and in G_1 we get

$$e_{3n+1} = G_1(13n^3 - 7n^2 - 10n - 2) + 2(6n^2 - 5n - 2)$$

which is > 0 when $n \ge 2$. Since (n-1)(4n+2)(4n+1) - n(4n+1)(4n-k+2) > 0 is equivalent with k > 4 + 2/n, which is obviously true for $n \ge 3$, it follows that $e_k > 0$ is also fulfilled for $k = 3n + 2, 3n + 3, \ldots, 4n$. Finally we completed the proof that $e_k > 0$ for $k \ge 5$ and n large enough.

It remains to be proved that $f_k > 0$ when $4n \ge k \ge 5$.

$$f_k = 6n\binom{2n+2}{k} - 6n\binom{2n+1}{k} + 6n\binom{2n}{k} + (4n+2)\binom{n+2}{k} - 12n\binom{n+1}{k} + (4n-2)\binom{n}{k} + (n+1)\binom{2}{k} - n\binom{1}{k} + (n-1)\binom{0}{k} \ge 0$$

Let us prove that $f_k > 0, n \ge k \ge 5$. Since

$$f_k = H_1[6n(2n+2)(2n+1) - 6n(2n+1)(2n-k+2) + 6n(2n-k+2)(2n-k+1)] + H_2[(4n+2)(n+2)(n+1) - 12n(n+1)(n-k+2) + (4n-2)(n-k+2)(n-k+1)]$$
(2.12)

where $H_1 = \frac{2n(2n-1)\cdots(2n-k+3)}{k!}$ and $H_2 = \frac{n(n-1)\cdots(n-k+3)}{k!}$ so that if we divide both sides of the inequality $f_k > 0$ with H_1 we get the equivalent inequality

$$6n(2n+2)(2n+1) - 6n(2n+1)(2n-k+2) + 6n(2n-k+2)(2n-k+1) +$$

$$+N[(4n+2)(n+2)(n+1) - 12n(n+1)(n-k+2) + (4n-2)(n-k+2)(n-k+1)] > 0$$

where $N = \frac{n(n-1)\cdots(n-k+3)}{2n(2n-1)\cdots(2n-k+3)}$ The left side of the inequality is quadratic in k thus we reorder it and then we have

$$(4Nn + 6n - 2N)k^{2} + (4Nn^{2} - 12n^{2} + 4Nn - 12n + 6N)k + -4Nn^{3} + 24n^{3} + 36n^{2} - 12Nn^{2} - 8Nn + 12n.$$

so that it is fulfilled if its discriminant $\Delta_2 < 0$ where

$$\Delta_2 = (20n^4 + 48n^3 + 24n - 4n + 9)N^2 + (12n^2 - 72n^3 - 96n^4 - 12n)N - (108n^4 + 144n^3 + 36n^2)$$

Since Δ_2 is quadratic in N it follows that inequality $\Delta_2 < 0$ is fulfilled if N is between roots N_1 , N_2 of Δ_2 where

$$N_{1,2} = \frac{48n^4 + 36n^3 - 6n^2 + 6n \pm 6\sqrt{2}n\sqrt{(n+1)(62n^5 + 98n^4 + 54n^3 + 14n^2 + 10n + 5)}}{20n^4 + 48n^3 + 24n^2 - 4n + 9}$$

tend to

$$\frac{48 \pm 6\sqrt{2 \cdot 62}}{20} = -0.941, \ 5.741$$

when n tends to ∞ . Since $k \ge 5$ and $\frac{n-j}{2n-j} < \frac{1}{2}$, j > 0 it follows $0 < M < \frac{1}{2^3} = 0.125$ so that M is really between M_1 and M_2 when n is large enough. If we replace k = n + 1 in f_k and in H_1 we get

$$f_{n+1} = H_1(18n^3 + 24n^2 + 6n) + 2(2n^2 - n + 2)$$

which is > 0 when $n \ge 2$. Since $6n\binom{2n+2}{k} - 6n\binom{2n+1}{k} = 6n\binom{2n+1}{k-1} > 0$ for k > 0, it follows that $f_k > 0$ is also fulfilled for $k = n + 2, n + 3, \ldots, 2n + 1$. Finally we completed the proof that $d_k = e_k + f_k > 0$ for k such that $4n + 2 \ge k \ge 4$ and n large enough.

Theorem 2. The house of $P_{2n}(x) = x^{2n} - x^{n+1} - x^n - x^{n-1} + 1$ raised to 2n-th power tends to $\alpha^2 = 6.854...$ where

$$\alpha = \frac{3+\sqrt{5}}{2} = 2.618\dots$$

is the greatest root of $x^2 - 3x + 1$.

Lemma 3. For a real number a > 1 and a natural number $n \ge 0$ the inequality

$$-2a^{2n+2} + a^2 < -a^{n+1} - a^n + 1$$

is valid.

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Proof. We rewrite the inequality as

$$a^2 - 1 < 2a^{2n+2} - a^{n+1} - a^n$$

and prove it using the principle of mathematical induction. For n = 0 we have $a^2 - 1 < 2a^2 - a - 1$ which is true. Assume that the inequality is true for $n = k, k \ge 0$ and let us prove that it is true for n = k + 1. For n = k we have

$$a^2 - 1 < 2a^{2k+2} - a^{k+1} - a^k.$$

Let us multiply both sides of this inequality by a. Then we obtain

$$a(a^{2}-1) < 2a^{2k+3} - a^{k+2} - a^{k+1}.$$

Since $a^2 - 1 < a(a^2 - 1)$ we deduce the inequality

$$a^2 - 1 < 2a^{2k+3} - a^{k+2} - a^{k+1}$$

which coincides with the rewritten inequality for n = k + 1. \Box *Proof of Theorem* 2 Using the Theorem 1 it is sufficient to prove that the sequence $(\alpha_n^{2n})_{n\geq 1}$ tends to α^2 . Let we denote $Q_{2n}(x) = x^{2n} - 3x^n + 1$. Using the inequality of arithmetic and geometric means we have

$$Q_{2n}(\sqrt[n]{\alpha}) = = 0$$

= $P_{2n}(\alpha_n)$
= $\alpha_n^{2n} - \alpha_n^{n+1} - \alpha_n^n - \alpha_n^{n-1} + 1$
< $\alpha_n^{2n} - \alpha_n^n - 2\sqrt{\alpha_n^{n+1}\alpha_n^{n-1}} + 1$
= $\alpha_n^{2n} - 3\alpha_n^n + 1$
= $Q_{2n}(\alpha_n)$.

If $x \in [\sqrt[n]{\alpha}, +\infty)$ then $Q'_{2n}(x) = 2nx^{2n-1} - 3nx^{n-1} = nx^{n-1}(2x^n - 3) \ge n\alpha^{(n-1)/n}(2 \cdot 2.6 - 3) > 0$ so that $Q_{2n}(x)$ increase. Hence from $Q_{2n}(\sqrt[n]{\alpha}) < Q_{2n}(\alpha_n)$ we deduce that

$$\sqrt[n]{\alpha} < \alpha_n. \tag{2.13}$$

On the other hand

$$Q_{2n}(\alpha_{n+1}) =$$

$$= \alpha_{n+1}^{2n} - 3\alpha_{n+1}^{n} + 1$$

$$< \alpha_{n+1}^{2}(\alpha_{n+1}^{2n} - 3\alpha_{n+1}^{n} + 1)$$

$$= \alpha_{n+1}^{2n+2} - \alpha_{n+1}^{n+2} - 2\alpha_{n+1}^{n+2} + \alpha_{n+1}^{2} \text{ (by Lemma 3)}$$

$$< \alpha_{n+1}^{2n+2} - \alpha_{n+1}^{n+2} - \alpha_{n+1}^{n+1} - \alpha_{n+1}^{n} + 1$$

$$= 0.$$

Since $Q_{2n}(x)$ increase on $[\sqrt[n]{\alpha}, +\infty)$ it follows that it is positive on $(\sqrt[n]{\alpha}, +\infty)$. Thus from $Q_{2n}(\alpha_{n+1}) < 0$ we deduce that $\alpha_{n+1} < \sqrt[n]{\alpha}$. Clearly then

$$\alpha_n < \sqrt[n-1]{\alpha} \tag{2.14}$$

is valid. Raising both sides of (2.13) and of (2.14) to the power 2n we have

$$\alpha^2 < \alpha_n^{2n} < \alpha^{\frac{2n}{n-1}}.$$

Finaly the claim follows by the squeeze (two policemen) theorem . \Box

Theorem 3. If P(x) is an antireciprocal, primitive hexanomial of degree d > 7 such that six of its non-zero coefficients are 1 or -1 then there is a natural number p < d such that $P(x)(x^p-1)$ is a reciprocal octanomial (having the house equal to the house of P(x)), such that eight of its non-zero coefficients are also 1 or -1.

Proof. Let a, b, d be integers such that

$$d - a > d - b > b > a > 0. (2.15)$$

Let $P_1(x) = x^d - x^{d-a} \pm x^{d-b} \mp x^b + x^a - 1$ then

$$P_1(x)(x^{d-a}-1) = x^{d+d-a} - x^{d-a+d-a} \pm x^{d-b+d-a} \mp x^{b+d-a} + \underline{x^{a+d-a}} - \underline{\underline{x^{d-a}}} - (\underline{x^d} - \underline{\underline{x^{d-a}}} \pm x^{d-b} \mp x^b + x^a - 1)$$

i.e. $P_1(x)(x^{d-a}-1) = x^{2d-a} - x^{2d-2a} \pm x^{2d-b-a} \mp x^{b+d-a} \mp x^{d-b} \pm x^b - x^a + 1$ which is written so that exponents of its eight monomials are strictly decreasing. Therefore it is reciprocal and has exactly eight monomials. Let $P_2(x) = x^d + x^{d-a} - x^{d-b} + x^b - x^a - 1$ and $b \neq 2a$ then

$$P_2(x)(x^{d-b}-1) = x^{d+d-b} + x^{d-a+d-b} - x^{d-b+d-b} + \underline{x^{b+d-b}} - x^{a+d-b} - \underline{x^{d-b}} - (\underline{x^d} + x^{d-a} - \underline{\underline{x^{d-b}}} + x^b - x^a - 1)$$

that is $P_2(x)(x^{d-b}-1) = x^{d+d-b} + x^{d-a+d-b} - x^{d-b+d-b} - x^{a+d-b} - x^{d-a} - x^b + x^a + 1$. Since $2a \neq b$ it follows that either a + d - b > d - a or a + d - b < d - a. In the first case $P_2(x)(x^{d-b} - 1)$ is written so that exponents of its eight monomials are strictly decreasing. In the second case we get such polynomial if we transpose middle two monomials.

If b = 2a then $P(x) = x^d + x^{d-a} - x^{d-2a} + x^{2a} - x^a - 1$. Since d - b > b it follows that d > 4a then for $p = d - 3a P(x)(x^{d-3a} - 1) =$

$$= x^{d+d-3a} + x^{d-a+d-3a} - x^{d-2a+d-3a} + \underline{x^{2a+d-3a}} - \underline{\underline{x^{a+d-3a}}} - x^{d-3a} - (x^d + \underline{x^{d-a}} - \underline{\underline{x^{d-2a}}} + x^{2a} - x^a - 1)$$

is equal to $x^{d+d-3a} + x^{d-a+d-3a} - x^{d-2a+d-3a} - x^d - x^{d-3a} - x^{2a} + x^a + 1$. Since P(x) is primitive and d > 5 it follows that $d \neq 5a$. Using d > 4a we conclude that in $P(x)(x^{d-3a}-1)$ none of two monomials has exponents equal to each other so that $P(x)(x^{d-3a}-1)$ is reciprocal and has exactly eight monomials. Let $P_3(x) = x^d + x^{d-a} + x^{d-b} - x^b - x^a - 1$ then

$$P_3(x)(x^b - 1) = x^{d+b} + x^{d-a+b} + \underline{x^{d-b+b}} - x^{b+b} - x^{a+b} - \underline{\underline{x^b}} - (\underline{x^d} + x^{d-a} + x^{d-b} - \underline{\underline{x^b}} - x^a - 1)$$

is equal to $x^{d+b} + x^{d-a+b} - x^{2b} - x^{a+b} - x^{d-a} - x^{d-b} + x^a + 1$. Since, using (2.15), d+b > d-a+b > d-a+bd-a > d-b > a > 1 and d+b > d-a+b > 2b > a+b > a > 1 it follows that all pairs of eventually equal exponents are: (2b, d-a), (2b, d-b), (a+b, d-a), (a+b, d-b). Finally we conclude that if

$$d \neq 2a + b, \quad d \neq a + 2b, \quad d \neq 3b \tag{2.16}$$

then $P_3(x)(x^b-1)$ is an octanomial. If we take p = b - a then

$$P_{3}(x)(x^{b-a}-1) = x^{d+b-a} + x^{d-a+b-a} + \underline{x^{d-b+b-a}} - x^{b+b-a} - \underline{\underline{x^{a+b-a}}} - x^{b-a} - (x^{d} + \underline{\underline{x^{d-a}}} + x^{d-b} - \underline{\underline{x^{b}}} - x^{a} - 1)$$

is equal to $x^{d+b-a} + x^{d+b-2a} - x^d - x^{2b-a} - x^{d-b} - x^{b-a} + x^a + 1$. Since, using (2.15), $d+b-a > \max(d+b-2a, d) \ge \min(d+b-2a, d) > \max(2b-a, d-b) \ge \min(2b-a, d-b) > \max(b-a, a) \ge \min(b-a, a) > 1$ it follows that all pairs of eventually equal exponents are: (d+b-2a, d), (2b-a, d-b), (b-a, a). Therefore if $d \ne 3b - a, b \ne 2a$ then $P_3(x)(x^{b-a} - 1)$ is an octanomial.

If d = 3b - a and $b \neq 2a$ then $d \neq a + 2b$ so that if $d \neq b + 2a$ then all conditions (2.16) for $P_3(x)(x^b - 1)$ to be an octanomial are fulfilled. If d = 3b - a and d = b + 2a it follows that 2b = 3a, b = 1.5a so that $P_3(x) = x^{3.5a} + x^{2.5a} + x^{2a} - x^{1.5a} - x^a - 1$. We conclude that a is even i.e. $a = 2a_1$ so that $P_3(x) = x^{7a_1} + x^{5a_1} + x^{4a_1} - x^{3a_1} - x^{2a_1} - 1$ and it is either of degree seven for $a_1 = 1$ or not primitive for $a_1 > 1$.

If d = 3b - a and b = 2a then $P_3(x) = x^{5a} + x^{4a} + x^{3a} - x^{2a} - x^a - 1$ is either of degree five or is not primitive. \Box

We created a procedure which generate all primitive reciprocal and antireciprocal polynomials of degree D with at most eight monomials. Then we use the standard procedures to find all roots of the polynomial, the root r_{max} with maximal modulus and for factoring the polynomial. Consequently, the degree d of r_{max} is determined so that we are able to decide whether r_{max} should be inserted in the list of d-th degree algebraic integers with small house. Finally for all even $d \leq 180$ we determine the smallest value of $\overline{\alpha}$ for reciprocal α having a primitive minimal polynomial $R_d(x)$. For $D \approx 100$ the computing took ten minutes while for $D \approx 200$ it spent two hours and the half on a 3.7 Ghz PC. So the whole calculation has taken about seventy days.

3 Results

In Table 1 we present the smallest house, mrp(d), of monic, irreducible, reciprocal, **primitive**, noncyclotomic polynomials with integer coefficients of even degree d, each of them is a factor of a reciprocal or antireciprocal D-th degree polynomial with at most eight monomials, for d at most 180, $D \leq 1.5d$ and $D \leq 210$. The minimum mrp(d) is attained for a polynomial $R_d(x)$ with ν conjugates outside the unit disc. A denominator is a product of cyclotomic polynomials Φ_n .

| d | ν_r | $\operatorname{mrp}(d)$ | $R_d(x)$ |
|----|---------|-------------------------|---|
| 2 | 1 | 2.618033989 | 13 |
| 4 | 2 | 1.539222338 | 113 |
| 6 | 2 | 1.321663156 | 1 2 2 1 |
| 8 | 2 | 1.169283030 | 100-11 |
| 10 | 2 | 1.125714822 | $(x^{15} + x^{13} - x^{12} - x^3 + x^2 + 1)/(\Phi_2 \Phi_{10})$ |
| 12 | 2 | 1.108054854 | $(x^{18} + x^{11} - x^9 + x^7 + 1)/(\Phi_3 \Phi_{12})$ |
| 14 | 4 | 1.093901686 | $(x^{19} + x^{15} - x^{12} - x^7 + x^4 + 1)/(\Phi_2 \Phi_{10})$ |
| 16 | 4 | 1.085689416 | $(x^{20} - x^{19} + x^{13} - x^{10} + x^7 - x + 1)/\Phi_{12}$ |
| 18 | 4 | 1.071850721 | $(x^{27} + x^{18} - x^{16} - x^{11} + x^9 + 1)/(\Phi_2 \Phi_{15})$ |
| 20 | 4 | 1.060442046 | $(x^{32} + x^{25} + x^{21} - x^{11} - x^7 - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_5 \Phi_8)$ |
| 22 | 4 | 1.066217585 | $(x^{29} + x^{25} - x^{17} - x^{12} + x^4 + 1)/(\Phi_2 \Phi_6 \Phi_{10})$ |
| 24 | 4 | 1.060034246 | $(x^{31} - x^{26} + x^{16} + x^{15} - x^5 + 1)/(\Phi_2 \Phi_6 \Phi_{10})$ |
| 26 | 8 | 1.057848469 | $(x^{37} + x^{25} + x^{24} - x^{23} - x^{14} + x^{13} + x^{12} + 1)/(\Phi_2 \Phi_{10} \Phi_{18})$ |
| 28 | 8 | 1.047786891 | $(x^{40} + x^{33} + x^{21} - x^{19} - x^7 - 1)/(\Phi_1 \Phi_2 \Phi_8 \Phi_{18})$ |
| 30 | 6 | 1.049786124 | $(x^{39} - x^{32} + x^{21} + x^{18} - x^7 + 1)/(\Phi_2 \Phi_{15})$ |
| 32 | 8 | 1.048455379 | $(x^{45} + x^{34} + x^{28} + x^{23} + x^{22} + x^{17} + x^{11} + 1)/(\Phi_2 \Phi_4 \Phi_8 \Phi_{18})$ |
| 34 | 10 | 1.047503370 | $(x^{42} - x^{28} + x^{25} - x^{17} + x^{14} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 36 | 8 | 1.045445455 | $x^{43} + x^{35} - x^{24} - x^{19} + x^8 + 1)/(\Phi_2 \Phi_{10})$ |
| 38 | 12 | 1.043402608 | $(x^{47} - x^{32} + x^{30} + x^{28} + x^{19} + x^{17} - x^{15} + 1)/(\Phi_2 \Phi_4 \Phi_{18})$ |
| 40 | 10 | 1.041409418 | $(x^{47} - x^{39} + x^{24} + x^{23} - x^8 + 1)/(\Phi_2 \Phi_{18})$ |

Table 1: The smallest house found of irreducible, reciprocal **primitive** algebraic integers

| 40 | 0 | 1 020052221 | $(51 43 27 24 8 1) / (\pi \pi)$ |
|----------|-----------------|-------------|---|
| 42 | ð 14 | 1.038052521 | $\frac{(x^{51} - x^{52} + x^{51} + x^{51} - x^{5} + 1)}{(92\Psi_{15})}$ |
| 44 | 14 | 1.038300334 | $(x^{00} - x^{00} + x^{01} + x^{00} + x^{20} + x^{20} + x^{22} - x^{10} + 1)/(\Psi_2 \Psi_4 \Psi_{18})$ |
| 46 | 10 | 1.034973093 | $(x^{30} + x^{43} + x^{33} - x^{23} - x^{11} - 1)/(\mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_5 \mathcal{Q}_8)$ |
| 48 | 10 | 1.033839781 | $(x^{57} + x^{40} + x^{53} - x^{24} - x^{11} - 1)/(\mathcal{Q}_1 \mathcal{Q}_{30})$ |
| 50 | 10 | 1.031791233 | $\frac{(x^{01} - x^{30} + x^{33} + x^{28} - x^{11} + 1)}{(\varphi_2 \Phi_6 \Phi_{10} \Phi_{12})}$ |
| 52 | 12 | 1.030630825 | $(x^{64} + x^{51} + x^{39} - x^{25} - x^{13} - 1)/(\Phi_1 \Phi_2 \Phi_8 \Phi_{12})$ |
| 54 | 12 | 1.030648009 | $(x^{54} + x^{55} - x^{37} + x^{27} - x^9 - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_8)$ |
| 56 | 12 | 1.030259738 | $(x^{64} + x^{53} + x^{53} - x^{31} - x^{11} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 58 | 12 | 1.029612538 | $(x_{56}^{66} + x_{53}^{53} - x_{39}^{39} - x_{27}^{27} + x_{13}^{13} + 1)/\Phi_{30}$ |
| 60 | 12 | 1.028423299 | $x^{68} + x^{57} - x^{35} + x^{33} - x^{11} - 1)/(\Phi_1 \Phi_2 \Phi_{18})$ |
| 62 | 12 | 1.028644239 | $(x_{0}^{67} + x_{0}^{55} - x_{0}^{36} - x_{0}^{31} + x_{1}^{12} + 1)/(\Phi_2 \Phi_{10})$ |
| 64 | 18 | 1.026826118 | $(x^{97} + x^{68} - x^{63} - x^{50} - x^{47} - x^{34} + x^{29} + 1)/(\Phi_1^2 \Phi_2 \Phi_3 \Phi_5 \Phi_7 \Phi_9 \Phi_{21})$ |
| 66 | 14 | 1.026395809 | $(x^{73} + x^{60} - x^{39} - x^{34} + x^{13} + 1)/(\Phi_2 \Phi_{18})$ |
| 68 | 20 | 1.024213262 | $(x^{83} + x^{59} + x^{48} - x^{46} - x^{37} + x^{35} + x^{24} + 1)/(\Phi_2 \Phi_{15} \Phi_{18})$ |
| 70 | 18 | 1.025005536 | $(x^{78} - x^{52} + x^{43} - x^{35} + x^{26} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 72 | 14 | 1.023289256 | $(x^{84} + x^{67} - x^{51} - x^{33} + x^{17} + 1)/(\Phi_8 \Phi_{30})$ |
| 74 | 16 | 1.023505081 | $(x^{86} + x^{75} - x^{53} + x^{33} - x^{11} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_{18})$ |
| 76 | 16 | 1.022682125 | $(x^{88} + x^{69} + x^{57} - x^{31} - x^{19} - 1)/(\Phi_1 \Phi_2 \Phi_8 \Phi_{18})$ |
| 78 | 16 | 1.022207266 | $(x^{87} - x^{71} + x^{48} + x^{39} - x^{16} + 1)/(\Phi_2 \Phi_{15})$ |
| 80 | 16 | 1.020969200 | $(x^{92} + x^{75} + x^{51} - x^{41} - x^{17} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_{18})$ |
| 82 | 18 | 1.021813323 | $(x^{89} - x^{75} + x^{47} + x^{42} - x^{14} + 1)/(\Phi_2 \Phi_{18})$ |
| 84 | 16 | 1.020986553 | $(x^{93} - x^{79} + x^{51} + x^{42} - x^{14} + 1)/(\Phi_2 \Phi_{15})$ |
| 86 | 15 | 1 021181880 | $(x^{92} + x^{49} - x^{46} + x^{43} + 1)/(\Phi_2 \Phi_{12})$ |
| 88 | 16 | 1.020725627 | $(x^{96} - x^{83} + x^{57} + x^{39} - x^{13} + 1)/(\Phi_{15})$ |
| 90 | 18 | 1.010367563 | $(x^{99} - x^{82} + x^{51} + x^{48} - x^{17} + 1)/(4^{15})$ |
| 02 | 18 | 1.012507505 | (x - x + x + x + x - x + 1)/((x - x - x - x - x - x - x - x - x - x |
| 92 | 22 | 1.010028307 | $(x + x + x - x - x - 1)/(\Psi_1\Psi_2\Psi_3\Psi_5\Psi_8)$ $(x^{102} - x^{68} + x^{55} - x^{47} + x^{34} - 1)/(\Phi_1\Phi_2\Phi_2\Phi_3\Phi_5)$ |
| 24 06 | 20 | 1.019026397 | $(x - x + x - x + x - 1)/(\Psi_1\Psi_2\Psi_6\Psi_8)$ $(m^{108} + m^{89} + m^{57} + m^{51} + m^{19} + 1)/(\Phi_1\Phi_2)$ |
| 90 | 20 | 1.01/020909 | $(x + x - x - x + x + 1)/(\Psi_8\Psi_{30})$ |
| 90 | 20 | 1.010130/01 | $(x^{-1} + x^{-1} + x^{-1} - x^{-1} - x^{-1})/(\Psi_1 \Psi_2 \Psi_{18})$ $(x^{112} + x^{95} - x^{61} + x^{51} - x^{17} - 1)/(\Phi - \Phi - \Phi - \Phi)$ |
| 100 | 20 | 1.01/5251/9 | $(x^{} + x^{} - x^{} + x^{} - x^{} - 1)/(\Psi_1\Psi_2\Psi_3\Psi_5\Psi_8)$ |
| 102 | 20 | 1.01/050/42 | $(x^{110} + x^{50} - x^{50} + x^{51} - x^{11} - 1)/(\Psi_1\Psi_2\Psi_{18})$ |
| 104 | 18 | 1.01/330618 | $(x^{112} + x^{50} + x^{50} + x^{50} + 1)/(\Psi_5\Psi_{12})$ |
| 100 | 20 | 1.016982801 | $(x^{110} - x^{51} + x^{51} + x^{50} - x^{10} + 1)/(\Psi_2\Psi_6\Psi_{12})$ |
| 108 | 24 | 1.016504509 | $(x^{117} + x^{75} - x^{51} - x^{50} + x^{55} + 1)/(\varphi_2 \varphi_{15})$ |
| 110 | 20 | 1.016401551 | $(x^{116} + x^{35} + x^{09} - x^{49} - x^{23} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_5)$ |
| 112 | 24 | 1.015669389 | $(x^{124} + x^{105} - x^{67} + x^{57} - x^{19} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_{18})$ |
| 114 | 26 | 1.015479967 | $(x^{126} + x^{84} + x^{73} + x^{53} + x^{42} + 1)/(\Phi_8\Phi_{30})$ |
| 116 | 24 | 1.015625196 | $(x^{128} + x^{111} - x^{77} + x^{51} - x^{17} - 1)/(\Phi_1 \Phi_2 \Phi_8 \Phi_{18})$ |
| 118 | 24 | 1.014982538 | $(x^{128} + x^{105} + x^{69} - x^{59} - x^{23} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_8)$ |
| 120 | 24 | 1.014911998 | $(x^{129} - x^{107} - x^{66} + x^{63} + x^{22} - 1)/(\varPhi_1 \varPhi_{30})$ |
| 122 | 24 | 1.014416023 | $(x^{133} - x^{110} + x^{69} + x^{64} - x^{23} + 1) / (\Phi_2 \Phi_6 \Phi_{10} \Phi_{12})$ |
| 124 | 24 | 1.014722774 | $(x^{131} - x^{106} + x^{75} + x^{56} - x^{25} + 1)/(\Phi_2 \Phi_6 \Phi_{12})$ |
| 126 | 26 | 1.014273084 | $(x^{136} + x^{115} - x^{73} + x^{63} - x^{21} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_8)$ |
| 128 | 24 | 1.014122887 | $(x^{136} + x^{113} + x^{69} - x^{67} - x^{23} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 130 | 30 | 1.014012153 | $(x^{138} - x^{92} + x^{73} - x^{65} + x^{46} - 1)/(\Phi_1 \Phi_2 \Phi_6 \Phi_8)$ |
| 132 | 38 | 1.013640050 | $(x^{149} + x^{107} - x^{88} + x^{84} + x^{65} - x^{61} + x^{42} + 1)/(\Phi_2 \Phi_4 \Phi_{15} \Phi_{18})$ |
| 134 | 26 | 1.013498976 | $(x^{144} + x^{115} + x^{87} - x^{57} - x^{29} - 1)/(\Phi_1 \Phi_2 \Phi_5 \Phi_8)$ |
| 136 | 30 | 1.013428352 | $(x^{144} + x^{96} - x^{77} - x^{67} + x^{48} + 1)/(\Phi_{15})$ |
| 138 | 26 | 1.013055754 | $(x^{147} - x^{121} + x^{78} + x^{69} - x^{26} + 1)/(\Phi_2 \Phi_{15})$ |
| 140 | 28 | 1.012795821 | $(x^{152} + x^{129} - x^{83} + x^{69} - x^{23} - 1)/(\Phi_1 \Phi_2 \Phi_8 \Phi_{18})$ |
| 142 | 28 | 1.012635977 | $(x^{152} + x^{125} + x^{81} - x^{71} - x^{27} - 1)/(\phi_1 \phi_2 \phi_5 \phi_8)$ |
| 144 | $\frac{-0}{28}$ | 1.012523350 | $(x^{156} - x^{133} + x^{87} + x^{69} - x^{23} + 1)/(\Phi_{8}\Phi_{15})$ |
| 146 | $\frac{-0}{28}$ | 1.012486423 | $(x^{154} + x^{125} + x^{87} - x^{67} - x^{29} - 1)/(\Phi_1 \Phi_2 \Phi_5)$ |
| 148 | 30 | 1.012389544 | $(x^{155} - x^{129} + x^{78} + x^{77} - x^{26} + 1)/(\Phi_0 \Phi_{10})$ |
| 0 | 20 | | (|

| 150 | 34 | 1.011974270 | $(x^{162} + x^{108} + x^{91} + x^{71} + x^{54} + 1)/(\Phi_8\Phi_{30})$ |
|-----|----|-------------|---|
| 152 | 30 | 1.012000260 | $(x^{160} + x^{131} + x^{87} - x^{73} - x^{29} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 154 | 30 | 1.011896494 | $(x^{162} - x^{131} + x^{93} + x^{69} - x^{31} + 1)/(\Phi_{15})$ |
| 156 | 30 | 1.011777895 | $(x^{163} - x^{134} + x^{87} + x^{76} - x^{29} + 1)/(\Phi_2 \Phi_6 \Phi_{12})$ |
| 158 | 30 | 1.011697187 | $(x^{164} + x^{135} + x^{87} - x^{77} - x^{29} - 1)/(\Phi_1 \Phi_2 \Phi_5)$ |
| 160 | 30 | 1.011417661 | $(x^{168} + x^{139} - x^{87} - x^{81} + x^{29} + 1)/(\Phi_{30})$ |
| 162 | 30 | 1.011256404 | $(x^{171} - x^{143} + x^{87} + x^{84} - x^{28} + 1)/(\Phi_2 \Phi_{15})$ |
| 164 | 32 | 1.010886959 | $(x^{176} + x^{145} + x^{93} - x^{83} - x^{31} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_5)$ |
| 166 | 32 | 1.011082816 | $(x^{173} - x^{142} + x^{93} + x^{80} - x^{31} + 1)/(\Phi_2 \Phi_6 \Phi_{12})$ |
| 168 | 32 | 1.010841122 | $(x^{177} - x^{146} + x^{93} + x^{84} - x^{31} + 1)/(\Phi_2 \Phi_{15})$ |
| 170 | 32 | 1.011010457 | $(x^{178} - x^{141} - x^{111} + x^{67} + x^{37} - 1)/(\Phi_1 \Phi_2 \Phi_9)$ |
| 172 | 34 | 1.010501552 | $(x^{184} + x^{155} - x^{97} + x^{87} - x^{29} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_5 \Phi_8)$ |
| 174 | 34 | 1.010497511 | $(x^{183} - x^{149} + x^{102} + x^{81} - x^{34} + 1)/(\Phi_2 \Phi_{15})$ |
| 176 | 34 | 1.010469386 | $(x^{184} + x^{149} + x^{105} - x^{79} - x^{35} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 178 | 38 | 1.010370370 | $(x^{186} - x^{124} + x^{97} - x^{89} + x^{62} - 1)/(\Phi_1 \Phi_2 \Phi_3 \Phi_8)$ |
| 180 | 34 | 1.010150047 | $(x^{189} - x^{157} + x^{96} + x^{93} - x^{32} + 1)/(\Phi_2 \Phi_{15})$ |
| | | | |

Assuming that Table 1 contains mrp(d) for every d, we create Table 2 using the following algorithm based on the corollary 1:

- 1. We calculate $mrp^d(d)$ and write it in the second column.
- 2. For every even divisor d_j of d we calculate $mrp^{d_j}(d_j)$ then find their minimum and write it in the third column. Let the minimum be attained for $d_j = d_0$.
- 3. For $k_0 = d/d_0$ we calculate $mr(d) = \sqrt[k_0]{mrp(d_0)}$ and write it in the fourth column. We can also calculate mr(d) as the *d*-th root of the minimum written in the third column.
- 4. The minimal polynomial $P_d(x)$ of the extremal reciprocal algebraic integer α whose house is denoted by mr(d) is equal to $R_{d_0}(x^{k_0})$. If $d = d_0$ so that $P_d(x)$ is primitive then we present the first half coefficients of $P_d(x)$ in the sixth column.
- 5. We calculate the number $\nu(d)$ of roots of $P_d(x)$ outside the unit disc as $\nu(d) = k_0 \nu_r(d_0)$ and write it in the fifth column.

| d | $\operatorname{mrp}^{d}(d)$ | $\min_{\delta d}(\mathrm{mrp}^{\delta}(\delta))$ | $\operatorname{mr}(d)$ | ν | $P_d(x)$ |
|----|-----------------------------|--|------------------------|---|---|
| 2 | 6.854101968 | 6.854101968 | 2.618033989 | 1 | 13 |
| 4 | 5.613133701 | 5.613133701 | 1.539222338 | 2 | 113 |
| 6 | 5.329970273 | 5.329970273 | 1.321663156 | 2 | 1 2 2 1 |
| 8 | 3.494275747 | 3.494275747 | 1.169283030 | 2 | $1 \ 0 \ 0 \ -1 \ 1$ |
| 10 | 3.268013514 | 3.268013514 | 1.125714822 | 2 | 101101 |
| 12 | 3.425587986 | 3.425587986 | 1.108054854 | 2 | $1\ 1\ 1\ 0\ -1\ -1\ -1$ |
| 14 | 3.513145071 | 3.513145071 | 1.093901686 | 4 | 10001101 |
| 16 | 3.726401663 | 3.494275747 | 1.081333912 | 4 | $R_8(x^2)$ |
| 18 | 3.486723207 | 3.486723207 | 1.071850720 | 4 | 1011121221 |
| 20 | 3.233990794 | 3.233990794 | 1.060442046 | 4 | $1\ 2\ 2\ 1\ -1\ -3\ -3\ -2\ 0\ 2\ 3$ |
| 22 | 4.098344884 | 4.098344884 | 1.066217585 | 4 | $1 - 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 - 1$ |
| 24 | 4.052075275 | 3.425587986 | 1.052641845 | 4 | $R_{12}(x^2)$ |
| 26 | 4.315290210 | 4.315290210 | 1.057848469 | 8 | $1 \ 0 \ 0 \ 1 \ 0 \ -1 \ 0 \ 0 \ -1 \ -1$ |
| 28 | 3.695242104 | 3.513145071 | 1.045897550 | 8 | $R_{14}(x^2)$ |

Table 2: The smallest values found of $\overline{\alpha}$ for reciprocal α having a minimal polynomial $P_d(x)$ of even degree $d \leq 180$.

| 30 | 4.295609952 | 3.268013514 | 1.040262145 | 6 | $R_{10}(x^3)$ |
|----------|--|-----------------------------|--------------------|----------|---|
| 32 | 4.545675907 | 3.494275747 | 1.039872065 | 8 | $R_8(x^4)$ |
| 34 | 4.844897357 | 4.844897357 | 1.047503370 | 10 | 1 1 1 0 -1 -1 0 1 2 1 0 -1 -1 |
| | 0000-1 | | | | |
| 36 | 4.952786876 | 3.425587986 | 1.034793646 | 6 | $R_{12}(x^3)$ |
| 38 | 5.025425981 | 5.025425981 | 1.043402608 | 12 | 1 - 1010 - 1010 - 211 - 1 |
| 20 | -1100001 | 01020120701 | 110.10.102000 | | 1 1010 1010 211 1 |
| 40 | 5 068273424 | 3 233990794 | 1 029777668 | 8 | $B_{\rm ac}(r^2)$ |
| 10 | 1 700635323 | 3 513145071 | 1.020768053 | 12 | $R_{20}(x^3)$ |
| 44 | 5 226500253 | 1 008311881 | 1.030508955 | 12 Q | $R_{14}(x)$ $R_{14}(x^2)$ |
| 44 | <i>J.220309233</i> <i>A</i> 861124201 | 4.090344004 | 1.032378123 | 0 | $n_{22}(x)$ |
| 40 | 4.001124291 | 2 01124291 | 1.034973093 | 10 | 1-11-101-110-12-1 |
| 19 | 01 - 220 - 1 | 2 - 2 1 1 - 2 5 | 1 025082255 | 0 | $D(m^4)$ |
| 40 50 | 4.940324029 | 2 262012514 | 1.023965333 | 0 | $D_{12}(x)$ |
| 50 | 4.781802892 | 5.208015514 | 1.025900551 | 10 | $n_{10}(x^*)$ |
| 52 | 4.801341912 | 4.315290210 | 1.02851/608 | 10 | $R_{26}(x^2)$ |
| 54 | 5.1045/8/24 | 3.486/2320/ | 1.023398481 | 12 | $R_{18}(x^3)$ |
| 56 | 5.309049530 | 3.494275747 | 1.022592979 | 14 | $R_8(x')$ |
| 58 | 5.433525446 | 5.433525446 | 1.029612538 | 12 | $1\ 1\ 1\ 0\ 0\ -1\ -1\ -1\ 0\ 0\ 0\ 0\ 0$ |
| 6.0 | | 10 - 1 - 1000 | 0100 | | |
| 60 | 5.374207407 | 3.233990794 | 1.019754537 | 12 | $R_{20}(x^3)$ |
| 62 | 5.760262620 | 5.760262620 | 1.028644239 | 12 | $1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0$ |
| | 10100101 | 001010011 | | | - () |
| 64 | 5.442545008 | 3.494275747 | 1.019741176 | 16 | $R_8(x^8)$ |
| 66 | 5.581891922 | 4.098344884 | 1.021602500 | 12 | $R_{22}(x^3)$ |
| 68 | 5.087997146 | 4.844897357 | 1.023476121 | 20 | $R_{34}(x_{-}^2)$ |
| 70 | 5.634232571 | 3.268013514 | 1.017060791 | 14 | $R_{10}(x^7)$ |
| 72 | 5.246695122 | 3.425587986 | 1.017248075 | 12 | $R_{12}(x^6)$ |
| 74 | 5.580334344 | 5.580334344 | 1.023505081 | 16 | 1 - 1 1 0 0 1 - 1 1 0 - 1 2 - 1 |
| | 01-12-10 | 2 - 2 2 0 - 1 2 - | -220 - 23 - 211 | -2.2 | -102-3 |
| 76 | 5.499086530 | 5.025425981 | 1.021470806 | 24 | $R_{38}(x^2)$ |
| 78 | 5.546757323 | 4.315290210 | 1.018922503 | 24 | $R_{26}(x^3)$ |
| 80 | 5.260309040 | 3.233990794 | 1.014779616 | 16 | $R_{20}(x^4)$ |
| 82 | 5.867701324 | 5.867701324 | 1.021813323 | 18 | 1 - 1 1 0 0 0 0 0 0 - 1 1 - 1 |
| | 00-11-10 | $1 - 1 \ 1 \ 0 \ 0 \ 1 - 1$ | 10-11-100- | 11- | $1\ 0\ 1\ -1\ 1\ 0\ 0\ 1$ |
| 84 | 5.723765933 | 3.425587986 | 1.014765969 | 14 | $R_{12}(x^7)$ |
| 86 | 6.065499920 | 6.065499920 | 1.021181880 | 15 | $1 \overline{1} \overline{1} 0 - 1 - 1 - 1 0 0 0 0 0$ |
| | $1 \ 1 \ 1 \ 0 \ -1 \ -1$ | -10000011 | 10-1-1-1000 | 000 | $1 \ 1 \ 1 \ 0 \ -1 \ -1 \ -1 \ -1$ |
| 88 | 6.081260895 | 3.494275747 | 1.014318889 | 22 | $R_8(x^{11})$ |
| 90 | 5.620473341 | 3.268013514 | 1.013244523 | 18 | $R_{10}(x^9)$ |
| 92 | 5.418615039 | 4.861124291 | 1.017336273 | 20 | $R_{46}^{10}(x^2)$ |
| 94 | 5.881809242 | 5.881809242 | 1.019028397 | 22 | 1 - 1 1 0 - 1 1 0 - 1 2 - 1 0 1 |
| | -101 - 110 | 0000001 - 1 | 10 - 110 - 12 - 12 | 1 - 12 | 2 - 202 - 211 - 210 - 11 - 1 |
| 96 | 5.453773907 | 3.425587986 | 1.012908365 | 16 | $R_{12}(x^8)$ |
| 98 | 5 833366397 | 3 513145071 | 1 012904096 | 28 | $R_{14}(x^7)$ |
| 100 | 5 571592570 | 3 233990794 | 1.011806320 | 20 | $R_{14}(x)$ $R_{20}(x^5)$ |
| 100 | 5 957620928 | 4 844897357 | 1.015590141 | 30 | $R_{20}(x^{3})$ |
| 102 | 5 971177596 | 3 494275747 | 1.012102711 | 26 | $R_{34}(x) = R_{2}(x^{13})$ |
| 104 | 5 959948109 | 5 950948100 | 1.012102711 | 20 | 101 - 10 - 10000001 |
| 100 | 0.1 = 1.0 = 1.0 | -10 - 11020 | 1.010902001 | _1 1 i | 0201 - 10 - 10 - 10 - 110 |
| | 201 - 10 - 10 | 10 11020 | 1 10 10-10 | 11 | 0201 10 10-10-110 |
| 108 | 5 858756079 | 3 425587086 | 1 011/65012 | 18 | $R_{12}(x^9)$ |
| 110 | 5.050750070 | 3.425567560 | 1 010002110 | 22 | $R_{12}(x)$ $R_{12}(x^{11})$ |
| 110 | 5 705110145 | 3.200013314 3.404275747 | 1.010023440 | 22 20 | $P_{10}(\omega)$ $P_{10}(\pi^{14})$ |
| 11Z | 5 761402267 | J.4742/J/4/ 5 025/25001 | 1.011233393 | 20 26 | $D(\pi^3)$ |
| 114 | 5./0149550/ | 5.023423981 | 1.014203132 | 30 24 | $n_{38}(x^2)$ |
| 110 | 0.040028208 | 5.455525446 | 1.014698230 | 24 | $n_{58}(x^{-})$ |

5.782442619 5.782442619 1.014982538 24 118 11110 - 1 - 1 - 101221 $0 - 1 - 2 - 1 \ 0 \ 1 \ 2 \ 2 \ 1 \ 0 - 2 - 2 - 2 - 1 \ 1 \ 2 \ 2 \ 2 \ 0 - 1 - 2 - 2 - 1 \ 0 \ 1 \ 2 \ 1 \ 1 \ 0 - 1 - 1 - 1$ -10011110-1-1-2-10115.907536277 3.233990794 24 $R_{20}(x^6)$ 120 1.009828964 122 5.732766447 24 1 1 1 0 - 1 - 2 - 2 - 1 0 1 2 25.732766447 1.014416023 $2\ 1\ 0\ -2\ -3\ -3\ -2\ 0\ 2\ 3\ 3\ 1\ 0\ -2\ -2\ -2\ -1\ 0\ 1\ 2\ 2\ 1\ 0\ -2\ -2\ -2\ 0\ 1\ 2\ 2\ 1\ 0\ -1\ -2$ -2 - 2 0 1 3 3 2 0 - 2 - 3 - 3 - 2 0 1 3 3124 6.124611460 5.613133701 1.014009395 62 $R_4(x^{31})$ $R_{18}(x^7)$ 28 126 5.963723281 3.486723207 1.009961690 $R_8(x^{16})$ 128 6.019976073 3.494275747 1.009822349 32 1.009150709 $R_{10}(x^{13})$ 130 6.103947784 3.268013514 26 132 5.979385283 3.425587986 1.009371467 22 $R_{12}(x^{11})$ 134 6.030094351 6.030094351 1.013498976 26 1 1 1 1 1 0 - 1 - 1 - 1 0 1 2 2 $1\ 0\ -1\ -2\ -1\ 0\ 1\ 2\ 2\ 1\ 0\ -1\ -1\ -1\ 0\ 1\ 1\ 0\ 0\ -1\ -1\ 0\ 1\ 1\ 1\ 0\ -1\ -2\ -1\ 0\ 1\ 2\ 2\ 0\ -1$ -2 -2 -1 1 2 2 1 0 -2 -2 -2 -1 0 1 1 1 0 0 -1 -1 -1 $R_8(x^{17})$ 136 6.135568540 3.494275747 1.009241902 34 $R_{46}(x^3)$ 138 5.989657253 4.861124291 1.011524376 30 140 5.930155706 3.233990794 1.008418934 28 $R_{20}(x^7)$ 142 5.948070694 5.948070694 1.012635977 28 1 - 1 1 - 1 0 1 - 1 1 0 - 1 2 $-2 \ 1 \ 0 \ -1 \ 2 \ -1 \ 0 \ 1 \ -2 \ 2 \ -1 \ 0 \ 1 \ -1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 1 \ 0 \ -1 \ 2 \ -2 \ 1 \ 1 \ -2 \ 3 \ -2 \ 0 \ 2$ $-3\ 3\ -1\ -1\ 3\ -3\ 2\ 0\ -2\ 3\ -2\ 1\ 1\ -2\ 2\ -1\ 0\ 1\ -1\ 1\ 0\ 0\ 0\ 0\ 0\ 1$ $R_{12}(x^{12})$ 24 144 6.002426057 3.425587986 1.008587168 1 - 2 2 - 1 0 1 - 1 0 1 - 1 1 - 1146 6.121028493 6.121028493 1.012486423 28 $0 - 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 - 1 \ 0 \ 1 - 1 \ 0 \ - 1 \ 2 \ - 2 \ 2 \ - 1 \ 0 \ 1 \ - 1$ 148 6.186604634 5.580334344 1.011684279 32 $R_{74}(x^2)$ $R_{10}(x^{15})$ 150 5.962392616 1.007925793 30 3.268013514 $R_8(x^{19})$ 6.129920758 152 3.494275747 1.008265062 38 $R_{14}(x^{11})$ 154 6.179566941 3.513145071 1.008192543 44 $R_{12}(x^{13})$ 156 6.212825396 3.425587986 1.007924007 26 158 6.280376894 6.280376894 1.011697187 30 111110000011111 $1\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ -1\ -1\ -2$ $R_{20}(x^8)$ 160 32 6.150143601 3.233990794 1.007362703 $R_{18}(x^9)$ 162 6.130955341 3.486723207 1.007739440 36 5.905074844 5.613133701 1.010574477 82 $R_4(x^{41})$ 164 6.231564151 6.231564151 1.011082816 32 101 - 10 - 100000010166 $R_{12}(x^{14})$ 168 6.119661348 3.425587986 1.007355930 28 $R_{10}(x^{17})$ 170 6.433689353 3.268013514 1.006990096 34 $R_4(x^{43})$ 172 6.030612145 5.613133701 1.010080170 86 $R_6(x^{29})$ 174 6.153655414 5.329970273 1.009663320 58 $R_8(x^{22})$ 6.252824150 176 3.494275747 1.007133998 44 6.274037304 6.274037304 1.010370370 38 1 1 1 0 - 1 - 1 0 1 2 1 0 - 1 - 1178 $1 \ 1 \ 0 \ -1 \ 2 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ -1 \ -2 \ -1 \ 1 \ 2 \ 2 \ 0 \ -2 \ -2 \ -1 \ 1 \ 2 \ 1 \ 0 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0 \ -1$ $R_{20}(x^9)$ 180 6.158286769 3.233990794 1.006541955 36

4 The old and new conjectures

The first, fourth and sixth column of our Table 2 represent the continuation of the Table 1 of Wu and Zhang [8] and these two tables definitely matches for $2 \le d \le 42$. Although we can not guarantee that, for d > 42, we

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have found a reciprocal polynomial with the smallest house we certainly have made a good approximation of mr(d). There are three reasons for our confidence. The first one is the following

Conjecture 2 [Wu, Zhang [8]]. Any extremal reciprocal algebraic integer α with degree $d \ge 6$ has minimal polynomial which is a factor of reciprocal polynomial with at most eight monomials with height 1.

This conjecture is proved for $6 \le d \le 42$ so if it is not true for all d it is reasonably to expect that it is correct for many d not too large. Although we involved antireciprocal polynomials in our research and spent as many CPU time as for reciprocal polynomials, we did not disprove the conjecture. As for antireciprocal hexanomials the Theorem 3 actually supports the conjecture. Using them we only succeeded to get simpler representation of many extremals in Table 1 than by using reciprocal octanomials. Also we did not find any antireciprocal octanomial such that the minimal polynomial of an extremal reciprocal algebraic integer is its factor and is not a factor of any reciprocal pentanomial, hexanomial, heptanomial or octanomial.

The second reason is our extensive computation. We compute the minimum of the houses of all reciprocal algebraic integers of degree d such that its minimal polynomial is a factor of a D-th degree reciprocal or antireciprocal polynomial with at most eight monomials for d at most 180 and D at most 210. As the factoring of a polynomial spends lot of processor time we reject a polynomial if its house is greater than $1 + c_1/D$. Experimenting with several values of c_1 we concluded that $c_1 = 2.5$ is ideal. If $c_1 > 2.5$ then we have too much unnecessary calculations, but if $c_1 < 2.5$ then an extremal reciprocal can be missed. For $d \approx 200$ the duration of computation with $c_1 = 2.8$ was approximately five hours, which is more than double the time spent for $c_1 = 2.5$ on a 3.7 Ghz PC. But our attempt to find polynomials with smaller houses increasing c_1 to $c_1 = 2.8$ failed. We discovered only few unknown polynomials with small house but no one decreased mr(d). Actually, many reciprocal α can be found in different ways, for example 1.013333049, the subextremal reciprocal of degree 138, as a root of the reciprocal octanomial $x^{191} - x^{168} + x^{145} + x^{115} + x^{76} + x^{46} - x^{23} + 1$, is rejected by our program because it is greater than $1 + 2.5/191 \approx 1.0131$. But this number, as a root of $x^{168} - x^{122} + x^{99} + x^{92} + x^{76} + x^{69} - x^{46} + 1$, is accepted because it is less than $1 + 2.5/168 \approx 1.0149$.

The third reason is statistical. If we plot

$$\frac{1}{\operatorname{nrp}(d) - 1}$$

versus degree we can notice that these points appear to fall very close to a straight line. If we model the line using the method of least squares [7] then for $12 \le d \le 40$ we get that $1/(mrp(d) - 1) \approx 0.51d + 4.3$ and for $12 \le d \le 180$ we get $1/(mrp(d) - 1) \approx 0.52d + 4.1$. Since it is almost the same line we conclude that our approximations are good. We remark that the coefficient of determination is 0.953 and 0.998 respectively, which means that there is almost perfect correlation. Using these calculations we establish the following

Conjecture 3. Let mrp(d) be the smallest house of monic, irreducible, reciprocal, primitive, noncyclotomic polynomials with integer coefficients of even degree d. Then points

$$\left(d, \frac{1}{\operatorname{mrp}(d) - 1}\right)$$

are very close to a straight line. If the least squares method is used then the line of best fit through these points is $\approx 0.52d + 4.1$, with the coefficient of determination close to 1.

If we analyse our Table 2 then we conclude that it supports the next

Conjecture 4 [Wu, Zhang [8]]. Let α be a reciprocal algebraic integer, not a root of unity, and let $d = \deg(\alpha) \ge 2$. Then

$$\operatorname{mr}^{d}(d) \ge \operatorname{mr}^{20}(20),$$

and if $10 \nmid d$ then

$$\operatorname{mr}^{d}(d) \ge \operatorname{mr}^{12}(12).$$

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If p is a prime number then it is obvious that the minimal polynomial of the extremal reciprocal of degree 2p is primitive or $R_2(x^p) = x^{2p} + 3x^p + 1$. Table 2 suggests that $P_8(x)$, $P_{12}(x)$, $P_{18}(x)$ and $P_{20}(x)$ are the only primitive minimal polynomials of an extremal reciprocal of a degree d such that d/2 is a composite number.

Conjecture 5. Let d be an even natural number such that d/2 is composite. If $d \notin \{8, 12, 18, 20\}$ then $P_d(x)$ is not primitive, where $P_d(x)$ is the minimal polynomial of an extremal reciprocal of degree d.

If the previous conjecture is true then we just need to determine mr(d) for d/2 > 10 is a prime number. If d/2 is a composite number we can easily calculate $mr(d) = mr^{p_1/d}(p_1)$ using the algorithm.

Proposition 1. An extremal reciprocal primitive of degree $d \le 180$ can not be a root of an reciprocal octanomial of degree D_1 such that $D_1 < 210$, $D_1 < 2d$ and all its inner monomials have minus sign. An extremal reciprocal primitive of degree $d \le 180$ can not be a root of an reciprocal octanomial of degree D_2 such that $D_2 < 210$, $D_2 < 1.5d$ and all its monomials have plus sign.

Proof Analysing our list of reciprocal octanomials, which are divisible by $R_d(x)$ from Table 1, we show that the claim is true.

The condition $D_1 < 2d$ in the previous proposition can not be omited because for degree 86 there is the octanomial $x^{181} - x^{132} - x^{95} - x^{92} - x^{89} - x^{86} - x^{49} + 1$ whose divisor $R_{86}(x)$ has the house 1.021181880 which is equal to mrp(86), see Table 1, but $D_1 = 181$ is not less than 2d = 172. Also, the condition $D_2 < 1.5d$ can not be omited because for degree 44 there is the octanomial $x^{68} + x^{46} + x^{45} + x^{37} + x^{31} + x^{23} + x^{22} + 1$ whose divisor $R_{44}(x)$ has the house 1.038300334 which is equal to mrp(44), see Table 1, but $D_2 = 68$ is not less than 1.5d = 66. For d = 38 there is the octanomial $x^{62} + x^{45} + x^{43} + x^{34} + x^{28} + x^{19} + x^{17} + 1$ whose divisor $R_{38}(x)$ has the house 1.043402608 which is equal to mrp(38), see Table 1, but $D_2 = 62$ is not less than $1.5 \cdot 38 = 57$ etc.

Conjecture 6. An extremal reciprocal primitive of degree d can not be a root, neither of an reciprocal octanomial of degree D_1 such that $D_1 < 2d$ and all its inner monomials have minus sign, nor of an reciprocal octanomial of degree D_2 such that $D_2 < 1.5d$ and all its monomials have plus sign.

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