Redundancy of triangle groups in spherical CR representations

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Abstract

Falbel, Koseleff and Rouillier computed a large number of boundary unipotent CR representations of fundamental groups of non compact three-manifolds. Those representations are not always discrete. By experimentally computing their limit set, one can determine that those with fractal limit sets are discrete. Many of those discrete representations can be related to $(3,3,n)$ complex hyperbolic triangle groups. By exact computations, we verify the existence of those triangle representations, which have boundary unipotent holonomy. We also show that many representations are redundant: for n fixed, all the $(3,3,n)$ representations encountered are conjugate and only one among them is uniformizable.

Keywords

complex hyperbolic triangle groups; spherical CR representations; boundary unipotent representations; knot link complements; limit sets

Contents

1 Introduction **2**

2 Elements of complex hyperbolic geometry **6**
2.1 Limit sets 6
2.2 Complex hyperbolic triangle groups 7

3 Experimental approach **11**
3.1 Computing limit sets 11
3.2 Complex hyperbolic $(3,3,n)$ triangle groups 12

4 Morphisms and redundancy **12**
4.1 $\Lambda_2(3,3,4)$ – m004, m022, m029, m034, m081 and m117 15
4.2 $\Lambda_2(3,3,5)$ – m009, m015, m035, m142 and m146 17
4.3 $\Lambda_2(3,3,6)$ – m023 17
4.4 $\Lambda_2(3,3,7)$ – m032 and m045 18
4.5 $\Lambda_2(3,3,\infty)$ – m129 and m203 18
4.6 m039, s000 and m053 18

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5 More diverse examples **19**
5.1 A homological obstruction 19
5.2 A Lagrangian triangle group 20

6 Appendix: figures and code **22**

1 Introduction

Falbel, Koseleff and Rouillier [FKR15] explicitly computed a large number of fundamental group representations of knot and link complements in $\mathrm{PGL}(3,\mathbf{C})$ and in particular in $\mathrm{PU}(2,1)$ (it is "a great portion" among all, according to [FKR15]). With those numerical methods we have *all* the boundary unipotent representations in $\mathrm{PU}(2,1)$ for the complements described by four or less tetrahedra. They were made available in [Gör] with the collaboration of [GGZ15a; GGZ15b].

A boundary unipotent representation of M is a representation of the fundamental group $\pi_1(M)$ such that each subgroup corresponding to each cusp is sent to an abelian subgroup of $\mathrm{PU}(2,1)$ generated by one or two unipotent transformations.

With those representations, one can raise some delicate questions. *Which are discrete? Which have an image that is a complex hyperbolic triangle group up to finite index?*

The group $\mathrm{PU}(2,1)$ is the holomorphic isometry group of the *complex hyperbolic plane* $\mathbf{H}^2_{\mathbf{C}}$. Complex hyperbolic triangle groups will be defined and discussed in section 2.2. For the moment, we see them as the representations of the *abstract triangle groups*

$$\Lambda(p,q,r) = \left\langle a,b,c \;\middle|\; \begin{array}{c} a^2 = b^2 = c^2 = e, \\ (ab)^p = (bc)^q = (ca)^r = e \end{array} \right\rangle,\ 2 \le p \le q \le r \le \infty, \tag{1}$$

into $\mathrm{PU}(2,1)$ by complex reflections, with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1$. Such a representation will be called a *complex hyperbolic triangle group*, denoted by $\Delta(p,q,r;\theta)$, and the images of a, b and c are denoted by I_1, I_2 and I_3. There is a parameter θ describing the space of the representations, it is the *angular* (or *Cartan*) *invariant*.

There is an important example of a three-manifold having a representation of its fundamental group with a triangular image. Falbel [Fal08] constructed the boundary unipotent representations of the fundamental group of the figure-eight knot complement. There are three such representations and they are discrete. Among them, two are also (index two) complex hyperbolic triangle groups by [DF15]. To be more specific, those two representations can be identified with the normal subgroup of the even-length words of a complex hyperbolic triangle group $\Delta(3,3,4;\theta)$. We will denote index two normal subgroup by $\Delta_2(p,q,r;\theta) \subset \Delta(p,q,r;\theta)$.

Complex hyperbolic triangle groups furnish a tiny subset of infinite covolume discrete subgroups of $\mathrm{PU}(2,1)$. They are Coxeter groups and are usually recognized as surface groups. But we expect it to be a very rich class of representations even in the world of three-manifolds.

In our study, we take a boundary unipotent representation ρ and ask wether it is discrete and if its image is in a complex hyperbolic triangle group.

To study the discreteness of a representation we chose to numerically approximate its limit set. This set is an attractor for the iteration dynamic and a simple argument allows us to support the discreteness: if the limit set is fractal (that is to say, is neither dense in S^3 nor a smooth circle) then the representation is discrete.

Stored in SnapPy [Gör] by a joint work of [FKR15] with [GGZ15a; GGZ15b], we have at our disposition 1653 boundary unipotent representations. By inspection of all the approximated limit sets, we found 35 pairs of representations (paired by complex conjugation in $\mathrm{PU}(2,1)$) that have a fractal limit set.[1] They concern 20 manifolds of the census. In figures 1 and 2 (in the appendix) we exposed all those fractal limit sets.

Among those 35 pairs of representations, 21 are in fact derived from complex hyperbolic triangle groups $(3,q,r)$. They come from 19 manifolds. And among those 21 triangle representations, 19 are derived from $(3,3,n)$ complex hyperbolic triangles.

On the remaining 14 pairs, we show that 3 are not surjective morphisms into an index two $(3,3,n)$-triangle group (see proposition 5.2) and that an additional one is a triangular representation, but different from the others (see proposition 5.4): it is a *Lagrangian* triangle group (as defined in [Wil07]).

We will (mathematically) prove that those 19 triangle representations do in fact come from index two $(3,3,n)$ complex hyperbolic triangle groups. All those representations are discrete.

We show that the conjugacy class can be chosen so that $I_3I_2I_1I_2$ generates the boundary holonomy of the fundamental group. This phenomenon is not true in general (see proposition 5.4). For a fixed abstract triangle group, the transformation $I_3I_2I_1I_2$ is unipotent for a unique value of the angular invariant θ that we will denote θ_∞.

For each of those triangle representations, we find that the image group is the even-length words subgroup $\Delta_2(p,q,r;\theta_\infty) \subset \Delta(p,q,r;\theta_\infty)$. As it is suggested by its importance, we consider $\Lambda_2(p,q,r)$ (the abstract version of $\Delta_2(p,q,r;\theta_\infty)$) defined by:

$$\Lambda_2(p,q,r) = \langle x,y \mid x^p = y^q = (xy)^r = e \rangle. \tag{2}$$

The inclusion $\Delta_2(p,q,r;\theta) \subset \Delta(p,q,r;\theta)$ corresponds to an inclusion morphism $\Lambda_2(p,q,r) \to \Lambda(p,q,r)$ defined by $x \mapsto ab$ and $y \mapsto bc$.

Redundancy We call *redundancy* the phenomenon of having several manifold fundamental groups represented in a same $\Delta_2(p,q,r;\theta_\infty)$, or more generally in a same $\Lambda_2(p,q,r)$. To assert the equality of the images, we allow conjugacy in $\mathrm{PU}(2,1)$ and complex conjugation.

Deraux [Der15] first showed that m009 and m015 are two manifolds with representations in a same $\Delta_2(3,3,5;\theta_\infty)$. But only the representation of m009 is a *spherical CR uniformization.*

The *spherical CR structure* is the pair $(\mathrm{PU}(2,1), \partial\mathbf{H}^2_{\mathbf{C}})$. There is a delicate relationship between a representation of a manifold M (of its fundamental group) in $\mathrm{PU}(2,1)$ and a *uniformizable* spherical CR representation of M. In the latter case, the image group Γ completely determines M: if $U \subset \partial\mathbf{H}^2_{\mathbf{C}}$ is its discontinuity domain then U/Γ is diffeomorphic to M (that is our definition of being uniformizable, as in [Der15]).

It is very hard in general to determine if a representation in $\mathrm{PU}(2,1)$ is the holonomy of a spherical CR uniformization. In our study, we relied on [Aco19] to identify uniformizations.

[1] We also observed that every representation from the census that gave a fractal had each boundary component (a copy of $\mathbf{Z}^2$) sent to a *rank one* parabolic subgroup.

Theorem (4.1)**.** *Consider the following table. Each row consists of manifolds having a boundary unipotent representation with the group $\Delta_2(3,3,n;\theta_\infty)$ denoted in the first column for image (up to conjugation in* $\mathrm{PU}(2,1)$ *and complex conjugation).*

The manifolds in the column "Uniformization" are (spherical CR) uniformized by their representation and the others in the column "Redundant" are not.

Group	Uniformization	Redundant
$\Delta_2(3,3,4;\theta_\infty)$	m004	m022, m029, m034, m053, m081, m117
$\Delta_2(3,3,5;\theta_\infty)$	m009	m015, m035, m142, m146
$\Delta_2(3,3,6;\theta_\infty)$	m023	
$\Delta_2(3,3,7;\theta_\infty)$	m039	m032, m045
$\Delta_2(3,3,8;\theta_\infty)$	s000	m053
$\Delta_2(3,3,\infty;\theta_\infty)$	m129	m203

Notes We use the nomenclature of SnapPy for the three-manifolds "mxxx".

Our proof relies on the computation of a surjective morphism from $\pi_1(M)$ to $\Lambda_2(3,3,n)$. Indeed, we can observe that the data of a surjective morphism into a $\Lambda_2(p,q,r)$ always furnishes a surjective representation into the corresponding $\Delta_2(p,q,r;\theta_\infty)$ since $\Lambda_2(p,q,r) \to \Delta_2(p,q,r;\theta_\infty)$ is always surjective. It will remain to check that the peripheral holonomy is indeed unipotent.

We cannot *mathematically* prove that the representations appearing in the theorem are the only ones that can be found in the census [Gör]. There might be other representations in the census that are triangular by a $\Delta_2(3,3,n;\theta_\infty)$ triangle group. But we are strongly confident that we found all of them by the approximation of the limit sets.

A consequence of our work is that the property of having a surjective morphism onto a $\Lambda_2(p,q,r)$ is not a clearly understood property: some manifolds possess several (*e.g.* m053) and redundancy frequently occurs.

More recently, Will [Wil21] produced a list of more manifolds admitting a representation into $\Lambda_2(3,3,\infty) \cong \mathbf{Z}_3 * \mathbf{Z}_3$ and even $\Lambda_2(p,q,\infty) \cong \mathbf{Z}_p * \mathbf{Z}_q$. Most manifolds appearing in his list are not contained in the census [Gör].

Deformations Let M have a surjective morphism $\rho\colon \pi_1(M) \to \Lambda_2(p,q,r)$. Then this (abstract) representation gives various representations $\pi_1(M) \to \Lambda_2(p,q,r) \to \mathrm{PU}(2,1)$. The space of all those representations up to conjugation in $\mathrm{PU}(2,1)$ has real dimension 2. The principal invariant to distinguish two representations is the trace of the image of $xy^{-1} \in \Lambda_2(p,q,r)$. (See the proposition 4.2 and the following discussion for details.)

The manifolds considered in theorem 4.1 that originally had a few discrete unipotent representations now automatically get a 2-dimensional subset in their character variety. In general, the character variety has dimension at least two. When it is minimal, as with the figure-eight knot complement, then this construction gives an open component (compare with [Fal+16]).

Numerical approximations of deformations of some triangle groups are available on the webpage of the author [Aleb].

In most cases, a surjective morphism $\pi_1(M) \to \Lambda_2(3,3,n)$ is represented in $\mathrm{PU}(2,1)$ by $\Delta_2(3,3,n;\theta_\infty)$ and gives a boundary unipotent representation of M. But this phenomenon is not always true and reveals that complex hyperbolic triangles are not the only ones to be considered.

Proposition (5.4)**.** *There exists a surjective morphism*

$$\xi\colon \pi_1(\mathrm{m023}) \to \Lambda_2(3,3,4), \tag{3}$$

and a composition $\pi_1(\mathrm{m023}) \to \Lambda_2(3,3,4) \to \mathrm{PU}(2,1)$ *having boundary unipotent holonomy. The boundary unipotent representation in* $\mathrm{PU}(2,1)$ *is not induced by any representation* $\Lambda_2(3,3,4) \to \Delta_2(3,3,4;\theta)$.

In this case, the boundary holonomy is controlled by the element $[x,y] \in \Lambda_2(3,3,4)$. (In theorem 4.1 the boundary holonomy is always controlled by $xy^{-1} \in \Lambda_2(3,3,n)$.) But for any of representation $\Lambda_2(3,3,4) \to \Delta_2(3,3,4;\theta)$, the word $[x,y]$ is never sent to a unipotent element of $\mathrm{PU}(2,1)$. This will justify the proposition.

Outline of the paper In section 2, we succinctly expose a few elements of complex hyperbolic geometry that are needed in the paper. In section 3, we explain how the numerical experiments were driven. We also propose visual clues in order to recognize the limit sets of the various triangle representations $\Delta_2(3,3,n;\theta_\infty)$. In section 4 we state and show the main result, theorem 4.1. For this, we only employ formal computations, so the result is certain. In section 5 we give other examples. Three examples for which no surjective morphism to $\Lambda_2(3,3,n)$ does exist. We also show the proposition 5.4, stated earlier.

Methodology In this paper we deal with two kinds of evidence. What we call *experimental* arguments are the one arising from the numerical approximations of the limit sets. Those explain how we selected the candidates for triangle representations.

But the results and in particular theorem 4.1 and proposition 5.4 are entirely based on pure mathematical arguments. In fact, they only depend on the description of the fundamental groups of the cited manifolds. Proofs can be checked by hand.

The identification of a representation arising from theorem 4.1 with a corresponding one in the census [Gör] is made by an exact computation with the formal tools provided by Python and SageMath. The census has its representations stored as matrices with entrees that are polynomial or rational with rational coefficients in an algebraic field. Therefore the representations can be manipulated with exactness. It is only when we numerically approximate the limit sets that we numerically approximate the representations.

We will respect the following rule. *The stated theorems, propositions and lemmas are always mathematical.* That is to say, proven by hand or by formal (exact) computations.

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2 Elements of complex hyperbolic geometry

In this first section, we expose the main tools and notions in use. One can compare with [Wil16], [Gol99], [Pra05] and [CG74].

We consider the space $\mathbf{C}^{2,1}$: it is $\mathbf{C}^3$ equipped with the Hermitian product of signature $(2,1)$

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} - z_3\overline{w_3}. \tag{4}$$

The subspace of the vectors verifying $\langle z, z\rangle < 0$ can be projectivised in $\mathbf{CP}^2$ and is identified to the *complex hyperbolic plane*, $\mathbf{H}^2_{\mathbf{C}}$. In the affine chart $z_3 = 1$, one can identify $\mathbf{H}^2_{\mathbf{C}}$ with the set of vectors verifying $|z_1|^2 + |z_2|^2 < 1$. Its boundary in the complex projective plane is a differentiable sphere S^3 and is given by $|z_1|^2 + |z_2|^2 = 1$. Those points are in correspondance with the non-zero vector lines in $\mathbf{C}^{2,1}$ verifying $\langle z, z\rangle = 0$.

The unitary group of $\mathbf{C}^{2,1}$ is $\mathrm{U}(2,1)$ and its projectivised version is $\mathrm{PU}(2,1)$. The geometrical structure $(\mathrm{PU}(2,1), S^3)$ is called the *(holomorphic) spherical CR structure*.

Let $A \in \mathrm{SU}(2,1)$. If A has a fixed point in $\mathbf{H}^2_{\mathbf{C}}$ then A is *elliptic*. If $\inf\{d(x, A(x))\} > 0$ with d the associated distance function of $\mathbf{H}^2_{\mathbf{C}}$ then A is *loxodromic* (or *hyperbolic*). Otherwise, A is *parabolic*. One can determine the type of A by looking at its trace. We follow Goldman [Gol99] and let

$$f(\tau) = |\tau|^4 - 8\mathrm{Re}(\tau^3) + 18|\tau|^2 - 27. \tag{5}$$

If $f(\operatorname{tr} A) > 0$ then A is loxodromic, if $f(\operatorname{tr} A) < 0$ then A is elliptic (in fact regular elliptic: all its eigenvalues are different).

When $f(\operatorname{tr} A) = 0$ there are three cases: if $\operatorname{tr}(A)^3 = 27$ then A is (parabolic) unipotent (all its eigenvalues are 1), otherwise it is either elliptic (and therefore a reflection with respect to a point or a complex geodesic) or ellipto-parabolic (a screw transformation along a complex geodesic). Note that when τ is real:

$$f(\tau) = (\tau + 1)(\tau - 3)^3, \tag{6}$$

and (under the hypothesis that $\operatorname{tr}(A)$ is real) A is therefore regular elliptic if $\operatorname{tr}(A) \in]-1, 3[$, is loxodromic if $\operatorname{tr}(A) \notin [-1, 3]$ and is unipotent if $\operatorname{tr}(A) = 3$. Note that unipotent transformations are a special kind of parabolic elements.

Let M be a smooth manifold and $\pi_1(M)$ its fundamental group. A representation $\rho\colon \pi_1(M) \to \mathrm{PU}(2,1)$ is the holonomy of a (CR) *uniformization* of M if, with $U \subset \partial\mathbf{H}^2_{\mathbf{C}}$ the domain of discontinuity of $\rho(\pi_1(M))$, the quotient $U/\rho(\pi_1(M))$ (that is generally only an orbifold) is a manifold diffeomorphic to M. When ρ is discrete, $U = \partial\mathbf{H}^2_{\mathbf{C}} - L(\rho(\pi_1(M)))$, where $L(\rho(\pi_1(M)))$ is the *limit set* of $\rho(\pi_1(M))$. The next section will describe this set. A manifold admitting such a representation is said to be (CR) *uniformizable*. Those manifolds are central in the study of spherical CR structures and are determined by the algebraic data of ρ. In general, even when $U/\rho(\pi_1(M))$ is a smooth manifold, it is too hard to identify it. It remains unknown which three-manifolds are CR uniformizable.

2.1 Limit sets

Let $\Gamma \subset \mathrm{PU}(2,1)$ be a subgroup. Its *limit set* $L(\Gamma)$ is given by:

$$L(\Gamma) = \overline{\Gamma \cdot p} \cap \partial\mathbf{H}^2_{\mathbf{C}}, \tag{7}$$

where $p \in \mathbf{H}^2_{\mathbf{C}}$ is any point ($L(\Gamma)$ is independent of this choice) and where the overline denotes the topological closure in $\overline{\mathbf{H}^2_{\mathbf{C}}} \subset \mathbf{CP}^2$.

Lemma 2.1. *The main properties of this set are the following. (Compare with [CG74].)*

1. *The limit set $L(\Gamma)$ is compact and Γ-invariant.*
2. *If $A \subset \partial\mathbf{H}^2_{\mathbf{C}}$ is compact, Γ-invariant and is constituted of at least two points, then $L(\Gamma) \subset A$.*
3. *If $L(\Gamma) = \emptyset$ then Γ fixes a point in $\mathbf{H}^2_{\mathbf{C}}$.*
4. *If $L(\Gamma)$ has at most two points, then $L(\Gamma)$ (or Γ) is said to be* elementary, *otherwise it has an infinite number of points and is perfect (each point is an accumulation point).*

An important result is the following.

Proposition ([CG74])**.** *If Γ is not discrete then $L(\Gamma)$ is either elementary, or equal to $\partial\mathbf{H}^2_{\mathbf{C}}$, or equal to the boundary of a totally geodesic proper subspace, that is to say a smooth circle.*

By consequence, if $L(\Gamma)$ is a fractal (none of the above cases), then Γ is discrete. It is a powerful experimental way to investigate if Γ is discrete. Note that there is no general mathematical procedure to prove with certainty that a subgroup is discrete or not.

The self-similarity property of limit sets can be justified by the following lemma.

Lemma 2.2. *Let Γ be a discrete subgroup of $L(\Gamma)$ and suppose that $L(\Gamma)$ is not elementary. Let $a \in L(\Gamma)$ be any point and V be any open neighborhood of a. Then there exists $\gamma_1, \dots, \gamma_n \in \Gamma$ such that*

$$L(\Gamma) = \bigcup_{i=1}^{n} \gamma_i \cdot (V \cap L(\Gamma)) . \tag{8}$$

Proof. Let $W = \partial\mathbf{H}^2_{\mathbf{C}} - \bigcup \Gamma \cdot V$. It is compact and Γ-invariant. By construction, W cannot have more than one point. If $W = \{b\}$ then Γ fixes b and $L(\Gamma)$ must be elementary since it is discrete. Therefore $W = \emptyset$ and it follows that $L(\Gamma) \subset \bigcup \Gamma \cdot V$. By compactness of $L(\Gamma)$, only a finite number of $\gamma_i \in \Gamma$ are necessary. □

2.2 Complex hyperbolic triangle groups

We will now describe more precisely the *complex hyperbolic triangle groups*

$$\Delta = \left\langle I_1, I_2, I_3 \;\middle|\; \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^p = (I_2 I_3)^q = (I_3 I_1)^r = e \end{array} \right\rangle \subset \mathrm{PU}(2,1), \tag{9}$$

with I_1, I_2 and I_3 all three being complex reflections. The numbers p, q, r are integers possibly infinite. If one of those is infinite, then the corresponding relation is omitted.

We will prove that Δ only depends on p, q, r and an additional real parameter θ, up to conjugation and complex conjugation. This parameter θ will be determined by the value of the trace of $I_3 I_2 I_1 I_2$. In particular, for any triplet $(p, q, r) = (3, 3, r)$, we obtain a unique complex hyperbolic triangle group, up to conjugation and complex conjugation, with $I_3 I_2 I_1 I_2$ unipotent. This will justify the notation $\Delta(p, r, q; \theta)$ to identify a complex hyperbolic triangle group.

Recall from the introduction that we can see a complex hyperbolic triangle group as a representation in $\mathrm{PU}(2,1)$ by complex reflection of the *abstract* triangle group

$$\Lambda(p,q,r) = \left\langle a,b,c \;\middle|\; \begin{array}{c} a^2 = b^2 = c^2 = e, \\ (ab)^p = (bc)^q = (ca)^r = e \end{array} \right\rangle. \tag{10}$$

When complex hyperbolic triangle groups were exposed by Schwartz [Sch02], he asked the following: when is a complex hyperbolic triangle group a discrete and injective representation of the corresponding abstract triangle group? He proposed the following important conjecture.

Conjecture 2.3 (Schwartz)**.** *A complex hyperbolic triangle group $\Delta(p,q,r;\theta)$ is a discrete and injective representation of $\Lambda(p,q,r)$ in $\mathrm{PU}(2,1)$, if and only if $I_3I_2I_1I_2$ and $I_1I_2I_3$ are both not elliptic.*

Note that, in some rare cases, $\Delta(p,q,r;\theta)$ can remain discrete (but not injective) with $I_3I_2I_1I_2$ and $I_1I_2I_3$ elliptic of finite order. For example, Thompson [Tho10] found a representation $\Delta(3,3,4;\theta_7)$ with $I_3I_2I_1I_2$ of order 7 and a representation $\Delta(3,3,5;\theta_5)$ with $I_3I_2I_1I_2$ of order 5, and both representations are lattices of $\mathrm{PU}(2,1)$.

Schwartz [Sch07] has shown his conjecture when $\min(p,q,r)$ is large enough. A first step toward the general case is a result due to Grossi [Gro07]. In particular for $(p,q,r) = (3,3,n)$, Grossi shows that if $I_3I_2I_1I_2$ is not elliptic then $I_1I_2I_3$ is not either. A proof of Schwartz' conjecture in the case of $(3,3,n)$ has been given by Parker, Wang and Xie [PWX16] and the case of $(3,3,\infty)$ has been studied by Parker and Will [PW17a].

Theorem 2.4 ([PWX16],[PW17a])**.** *Let $4 \le n \le \infty$. Let $\Delta(3,3,n;\theta)$ be a hyperbolic $(3,3,n)$ triangle group. Then $\Delta(3,3,n;\theta)$ is discrete and faithful if and only if $I_3I_2I_1I_2$ is not elliptic.*

Now we proceed to a description of complex hyperbolic triangle groups. Let Δ be a complex hyperbolic triangle group, with $2 \le p \le q \le r \le \infty$ and $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. Each reflection I_k fixes a complex line (isomorphic to $\mathbf{H}^1_{\mathbf{C}}$) in $\mathbf{H}^2_{\mathbf{C}} \subset \mathbf{CP}^2$. Let H_1, H_2 and H_3 be the vector hyperplanes of $\mathbf{C}^3$ covering those complex lines.

Let L_1, L_2 and L_3 be the dual complex lines of those hyperplanes defined by $\langle H_k, L_k\rangle = 0$. The group Δ is fully described by them. Since H_k is a complex line of $\mathbf{H}^2_{\mathbf{C}}$, it is a negative-type complex plane of $\mathbf{C}^3$ ($\langle h,h\rangle < 0$), and L_k is a positive-type complex line of $\mathbf{C}^3$ ($\langle l,l\rangle > 0$).

We only need to choose a basis vector for each L_i in order to describe those lines. We say that the triangle group is *non-degenerate* when such basis vectors form a basis of $\mathbf{C}^3$. We will make this assumption from now on.

Let v_k be a basis vector of L_k, then

$$I_k(x) = -x + \frac{2\langle x, v_k\rangle}{\langle v_k, v_k\rangle} v_k \tag{11}$$

is a complex reflection and verifies $I_k(h_k) = -h_k$ for any $h_k \in H_k$. That is to say, in $\mathbf{CP}^2$, $[I_k(h_k)] = [h_k]$. Therefore I_k indeed defines the reflection fixing H_k. Because $\langle v_k, v_k\rangle > 0$, one can normalize v_k so that $\langle v_k, v_k\rangle = 1$.

The last free parameters describing the v_k's are an angle $z_k \in S^1$ for each v_k. One can set z_1 and then modify z_2 and z_3 so that $\langle v_1, v_2\rangle$ and $\langle v_2, v_3\rangle$ are real and positive. In general, $\langle v_1, v_3\rangle$ is not real and this lack can be measured by $\arg(\langle v_1, v_3\rangle)$. From an intrinsic point of view, that is to say without choosing the z_k's, the default for the vertices to be in a same real plane can be measured by

$$\theta = -\arg(\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle). \tag{12}$$

The value of θ is also known under the name of the *angular invariant*.

Once $\langle v_i, v_j\rangle = c_{ij}$ are known, it is easy to evaluate the matrices of I_1, I_2 and I_3 in the basis (v_1, v_2, v_3).

$$I_1 = \begin{pmatrix} 1 & 2c_{21} & 2c_{31} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{13}$$

$$I_2 = \begin{pmatrix} -1 & 0 & 0 \\ 2c_{12} & 1 & 2c_{32} \\ 0 & 0 & -1 \end{pmatrix} \tag{14}$$

$$I_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2c_{13} & 2c_{23} & 1 \end{pmatrix} \tag{15}$$

We still have to see how p, q, r and θ determine $\langle v_i, v_j\rangle = c_{ij}$. For the time being, we suppose $r < \infty$. In fact, we will prove that the matrix given by the c_{ij}'s is equal to:

$$H = \begin{pmatrix} 1 & \cos\frac{\pi}{p} & \cos\frac{\pi}{r} e^{i\theta} \\ \cos\frac{\pi}{p} & 1 & \cos\frac{\pi}{q} \\ \cos\frac{\pi}{r} e^{-i\theta} & \cos\frac{\pi}{q} & 1 \end{pmatrix}. \tag{16}$$

And this shows that the c_{ij}'s fully determine p, q, r and θ in return. This matrix is an Hermitian form preserved by I_1, I_2 and I_3. The determinant of this matrix is given by

$$1 + 2\cos(\theta)\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r} - \cos\left(\frac{\pi}{p}\right)^2 - \cos\left(\frac{\pi}{q}\right)^2 - \cos\left(\frac{\pi}{r}\right)^2. \tag{17}$$

This determinant allows us to decide when H has $(2,1)$ for signature. Since the trace of H is 3, it implies that at least one eigenvalue is positive. Therefore, its determinant is negative if and only if H has $(2,1)$ for signature. That is equivalent to:

$$\cos(\theta) < \frac{-1 + \cos\left(\frac{\pi}{p}\right)^2 + \cos\left(\frac{\pi}{q}\right)^2 + \cos\left(\frac{\pi}{r}\right)^2}{2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r}}. \tag{18}$$

That must be the case since the original Hermitian form $\langle\cdot,\cdot\rangle$ has $(2,1)$ for signature.

If $p = 2$ then c_{12} vanishes and one can make both c_{23} and c_{13} real. Therefore, $(2, q, r)$ complex hyperbolic triangle groups are rigid: θ is always zero.

We now justify the expression of H by computing the c_{ij}'s. Up to conjugation, we can suppose that $H_1 \cap H_2$ is generated by $(0,0,1)$. This implies that v_1 and v_2 are both of the form $(x, y, 0)$. Therefore, every c_{ij} is given by $v_i^1\overline{v_j^1} + v_i^2\overline{v_j^2}$ since at least one of v_i or v_j has a vanishing third coordinate.

Therefore, geometrically speaking, c_{ij} is the cosine of the angle in $\mathbf{C}^2$ formed by the complex lines generated by the first two coordinates of v_i and v_j. It is the real part of c_{ij} that is equal to the cosine of the angle formed by the vectors given by the first two coordinates of v_i and v_j (see [Gol99, p. 36]). In other terms, if $\langle v_i, v_j\rangle$ is real and v_i, v_j are unitary then $\langle v_i, v_j\rangle = \cos\theta$ with $\theta \in [0,\pi[$. If additionally $\langle v_i, v_j\rangle > 0$ then $\theta \in [0,\pi/2]$ (that will be the case for us).

Note that c_{13} is non real in general, but of course $\langle v_1, e^{i\theta} v_3\rangle = e^{-i\theta}\langle v_1, v_3\rangle = e^{-i\theta} c_{13}$ is real and can even be assumed to be positive. The angle formed by H_1 and H_2 is equal to $\frac{\pi}{p}$ since $(I_1 I_2)^p = e$. By taking the duals $v_1 \in L_1$ and $v_2 \in L_2$, we get $c_{12} = \cos\frac{\pi}{p}$. Likewise, $c_{23} = \cos\frac{\pi}{q}$ and $e^{-i\theta} c_{13} = \cos\frac{\pi}{r}$.

Finally, one can compute, with $i \neq j \neq k \neq i$:

$$\mathrm{tr}(I_i I_j I_k I_j) = 16|c_{ij} c_{kj}|^2 - 16\mathrm{Re}(c_{12} c_{23} c_{31}) + 4|c_{ik}|^2 - 1, \tag{19}$$

and note that in our conventions, we have $\mathrm{Re}(c_{12} c_{23} c_{31}) = c_{12} c_{23} c_{13} \cos\theta$. It shows that $\mathrm{tr}(I_i I_j I_k I_j)$ determines $\pm\theta$ once (p,q,r) is known. Since the complex conjugation changes θ into $-\theta$, we deduce from our discussion the following results. In the case where $(p,q,r) = (3,3,r)$ we have in fact:

$$\mathrm{tr}(I_i I_j I_k I_j) = 4\cos\left(\frac{\pi}{r}\right)^2 - 4\cos\frac{\pi}{r}\cos\theta. \tag{20}$$

Proposition ([Pra05])**.** *Let $3 \le p \le q \le r < \infty$ be such that $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. A representation of the triangle group*

$$\Delta(p,q,r;\theta) = \left\langle I_1, I_2, I_3 \;\middle|\; \begin{array}{c} I_1^2 = I_2^2 = I_3^2 = e, \\ (I_1 I_2)^p = (I_2 I_3)^q = (I_3 I_1)^r = e \end{array} \right\rangle \tag{21}$$

into $\mathrm{PU}(2,1)$ is determined by $\theta = \arg(\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_1, v_3\rangle)$ up to conjugation, with v_1, v_2, v_3 as previously. Up to conjugation and complex conjugation, it is determined by $\mathrm{tr}(I_i I_j I_k I_j)$, with i, j, k pairwise distinct.

Furthermore, θ verifies

$$\cos(\theta) < \frac{-1 + \cos\left(\frac{\pi}{p}\right)^2 + \cos\left(\frac{\pi}{q}\right)^2 + \cos\left(\frac{\pi}{r}\right)^2}{2\cos\frac{\pi}{p}\cos\frac{\pi}{q}\cos\frac{\pi}{r}} \tag{22}$$

and conversely, this condition suffices to define a representation with that value of θ.

The parameter θ can be taken in $[0,\pi/2]$ since the complex conjugation exchanges θ and $2\pi - \theta$. The possible values of $\mathrm{tr}(I_i I_j I_k I_j)$ are constrained by the preceding condition. For example, when $(p,q,r) = (3,3,r)$, we have

$$\cos\theta < \frac{-\frac{1}{2} + \cos\left(\frac{\pi}{r}\right)^2}{\frac{1}{2}\cos\frac{\pi}{r}} = \frac{-1 + 2\cos\left(\frac{\pi}{r}\right)^2}{\cos\frac{\pi}{r}}. \tag{23}$$

If we take a look at the trace of $I_i I_j I_k I_j$, its maximum is given for $\cos\theta$ minimal (that is to say -1) and its minimum by the maximum of $\cos\theta$. We have the inequalities:

$$\mathrm{tr}(I_i I_j I_k I_j) \le 4\cos\frac{\pi}{r}\left(\cos\frac{\pi}{r} + 1\right), \tag{24}$$

$$\mathrm{tr}(I_i I_j I_k I_j) > 4\cos\left(\frac{\pi}{r}\right)^2 - 4\cos\frac{\pi}{r}\left(\frac{-1 + 2\cos\left(\frac{\pi}{r}\right)^2}{\cos\frac{\pi}{r}}\right) = 4\left(1 - \cos\left(\frac{\pi}{r}\right)^2\right) > 0. \tag{25}$$

This computation shows that the range of the values of $\mathrm{tr}(I_i I_j I_k I_j)$ is included in $\mathbf{R}_+$ and therefore, the range for which $\Delta(p,q,r;\theta)$ is discrete is of the form $[3,m]$, with m the maximum stated before. The value 3 is indeed reachable: the minimum value for r is 4, since we have to verify $\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}<\pi$, and the maximum value of $4(1-\cos(\frac{\pi}{r})^2)$ is indeed reached when r is minimal. This value for $r=4$ is 1.

One can compute the angular invariant required to have $I_i I_j I_k I_j$ unipotent. It is given by:

$$\cos\theta = \cos\frac{\pi}{r} - \frac{3}{4\cos\frac{\pi}{r}}. \tag{26}$$

When one or several of p, q and r are non finite, we can get similar results by replacing the undefined c_{ij} with $\cosh(l_{ij}/2)$, where l_{ij} is the distance between the two complex hyperbolic geodesics H_i and H_j. See [Pra05]. In particular, it is still true that $\cos\theta$ is determined by $\mathrm{tr}(I_i I_j I_k I_j)$.

3 Experimental approach

In this section, we explain how we approximate the limit sets of the representations appearing in the census [Gör].

We also propose a comparative experiment by simulating the limit sets associated to the $(3,3,n)$ triangle groups. Since, up to conjugation and complex conjugation, there is a unique $\Delta(3,3,n;\theta_\infty)$ with $I_3 I_2 I_1 I_2$ unipotent, the limit set is itself unique up to translation and complex conjugation.

It will allow us to propose visual clues in order to distinguish the different $(3,3,n)$ triangle group fractals when $I_3 I_2 I_1 I_2$ is unipotent.

The source code of the simulations and most of their results are available on the author's webpage [Aleb]. The code has been made open-source [Alea].

3.1 Computing limit sets

Lemma 3.1. *Let $\Gamma \subset \mathrm{PU}(n,1)$ be a subgroup. Let Γ_L denote the subset of the loxodromic elements. Suppose $\Gamma_L \neq \emptyset$. Then the closure of the accumulation points for the iteration dynamic:*

$$\overline{\{x \mid \exists g \in \Gamma_L,\ \lim g^n x_0 = x\}} \text{ for any } x_0 \in \mathbf{H}^2_{\mathbf{C}}, \tag{27}$$

is equal to the full limit set $L(\Gamma)$.

Proof. Note that if g is loxodromic then after any conjugation, $\gamma g \gamma^{-1}$ is again loxodromic.

The set $A = \overline{\{x \mid \exists g \in \Gamma_L,\ \lim g^n x_0 = x\}} \subset \partial\mathbf{H}^2_{\mathbf{C}}$ is compact and has at least two points. By definition, $A \subset L(\Gamma)$. So we only need to show that A is Γ-invariant, implying $L(\Gamma) \subset A$.

For any $\gamma \in \Gamma$, we have $\gamma(\lim g^n x_0) = \lim(\gamma g \gamma^{-1})^n(\gamma x_0)$. But $\gamma g \gamma^{-1}$ is loxodromic and its only attractive fixed point is the limit point for the orbits from both x_0 and γx_0. Therefore $\lim(\gamma g \gamma^{-1})^n(\gamma x_0) = \lim(\gamma g \gamma^{-1})^n x_0$ which shows the Γ-invariance. It implies $L(\Gamma) \subset A$ by minimality of $L(\Gamma)$. □

This lemma suggests a strategy to approximate $L(\Gamma)$: compute the attractive limit points of the loxodromic elements of Γ. However, this strategy requires to compute a very large number of elements $g \in \Gamma$. This can be done by generating words of length n. If Γ is described by two generators, then there are approximately 3^n words of length n.

In practice, and this is particularly true with complex hyperbolic triangle groups, it is hard to get *different* points from such a computation. One can often see large concentrations of points in tiny boxes and even many copies of the same point. This is partly due to unknown relations between words, even at small length words.

Instead of only computing words and getting their attractive limits, we used a second strategy in complement. When enough points are acquired, one can apply words on them (loxodromic or not) to get a better picture of the limit set. This method is much more efficient for it rarely makes redundant images. When nice symmetries are known (and for example, with complex hyperbolic triangles one knows the reflections I_1, I_2 and I_3), this allows a much better result.

In practice, we first compute the attractive points of n_1-length words, then apply given symmetries on the set obtained, then apply n_2-length words on them, and apply again symmetries.

We show the different steps for two examples: one fractal limit set and one dense limit set from representations of the fundamental group of m004 (the figure-eight knot's complement). See figures 3 and 4 in the appendix.

Once this numerical approximation is done, we find the one giving fractals by looking if it is neither dense (high density of points) nor a smooth circle (easy to observe). The fractals we found are listed in the figures 1 and 2.

3.2 Complex hyperbolic $(3,3,n)$ triangle groups

To compute the limit sets associated to $\mathrm{PU}(2,1)$-representations of $(3,3,n)$ triangle groups, we used our previous parametrization of the complex reflections I_1, I_2, and I_3. We picked the angular invariant for which $I_3 I_2 I_1 I_2$ is unipotent.

For $n \in \{4,5,6,7\}$ we show three projections of the limit set and an additional diagram proposing a visual clue to recognize the limit set, see figures 5, 6, 7, and 8 in the appendix. This visual clue consists in looking for a pair of symmetric spikes and inspect the middle. We count the largest outer holes. When $n = 4$ there is none, when $n = 5$ there is one, when $n = 6$ there are two, when $n = 7$ there are three, etc.

4 Morphisms and redundancy

From the census of the boundary unipotent representations in [FKR15] and stored in SnapPy [Gör], we numerically approximated all their limit sets. After a visual inspection, we kept the representations that gave fractals (see [Aleb] for the numerical results and the figures 1 and 2).

Those representations come in pairs by complex conjugation of the coefficients. In this section, we study many of them by a classification into $(3,3,n)$ triangle groups. 19 pairs will be directly classified and 4 more will be constructed. The classification will be systematic, following a method that we will describe.

Notes on the selection It is already known that m004-1 and m004-3 are related by the composition of a figure-eight knot's symmetry (see [DF15]). Therefore we only classify one of the two with the systematic procedure.

A representation of m038 presents the characteristics of a $(3,4,4)$ complex hyperbolic triangle group. Indeed, m038 has such a representation according to a preprint of Ma and Xie [MX20] that the author has been able to consult. A representation of m137 presents the characteristics of a $(3,4,5)$ complex hyperbolic triangle group.

We summarize the mathematical result in the following theorem. It is fully independent from the numerical computations. The method of the proof will be explained later.

Recall that $\Delta_2(3,3,n;\theta_\infty)$ denotes the even-length subgroup of $\Delta(3,3,n;\theta_\infty)$, where the angular invariant θ_∞ is chosen so that $I_3I_2I_1I_2$ is unipotent.

Theorem 4.1. *Consider the following table. Each row consists of manifolds having a boundary unipotent representation with the group $\Delta_2(3,3,n;\theta_\infty)$ denoted in the first column for image (up to conjugation in $\mathrm{PU}(2,1)$ and complex conjugation).*

The manifolds in the column "Uniformization" are (spherical CR) uniformized by their representation and the others in the column "Redundant" are not.

Group	Uniformization	Redundant
$\Delta_2(3,3,4;\theta_\infty)$	m004	m022, m029, m034, m053, m081, m117
$\Delta_2(3,3,5;\theta_\infty)$	m009	m015, m035, m142, m146
$\Delta_2(3,3,6;\theta_\infty)$	m023	
$\Delta_2(3,3,7;\theta_\infty)$	m039	m032, m045
$\Delta_2(3,3,8;\theta_\infty)$	s000	m053
$\Delta_2(3,3,\infty;\theta_\infty)$	m129	m203

The manifolds m039 and s000 do not appear in the census of [FKR15] for there are respectively described by five and six tetrahedra. But the result still applies and we will construct their triangular representations with boundary unipotent holonomy from their fundamental groups. We found those manifolds by applying the theorem of Acosta (see below).

The manifold m053 will be treated differently than the other members of the census. The initial description in SnapPy of its two triangular representations m053-1 and m053-7 are quite hard to use. We proceed the computation of another description (with the help of the method `Manifold.randomize()` of SnapPy).

Note that for each row, at most one representation is a uniformization of the corresponding manifold. Deraux [Der15] encountered the same phenomenon with the manifolds m009 and m015. In fact, we identify the uniformizations with the following result of Acosta.

Theorem ([Aco19]). *Let $4 \le n \le \infty$. The manifold at infinity of $\mathbf{H}^2_{\mathbf{C}}/\Delta_2(3,3,n;\theta_\infty)$ is the Dehn filling with slope $(1,n-3)$ of any cusp of the Whitehead link complement.*

With SnapPy, it is possible to compute Dehn surgeries on the Whitehead link complement. Note that m129 is the Whitehead link complement in our table. One has to be careful: the marking of the peripheral holonomy is not unique, with SnapPy one needs to call the manifold 5^2_1 and fill a cusp with the meridian equal to $n-3$ and the longitude equal to 1.

This procedure gives the selection of the uniformized manifold along each row. Note that we do have a true CR uniformization since Acosta did construct the CR structures by CR surgery. (It is not only a topological result.)

The first uniformization of m129 (the Whitehead link complement) was shown by Schwartz [Sch01], but the present uniformization by a $(3,3,\infty)$ triangle group with boundary unipotent holonomy was studied by Parker and Will [PW17a].

We now explain how we prove the rest of theorem 4.1. We start with the following fundamental result about $\mathrm{PU}(2,1)$ subgroups generated by two elements. It is the restricted case of a more general result in $\mathrm{SL}_3(\mathbf{C})$.

Proposition 4.2 ([Law07; Wil09])**.** *If* Γ *is a group generated by two elements a and b, then any irreducible representation* $\xi\colon \Gamma \to \mathrm{PU}(2,1)$ *is fully determined up to conjugation in* $\mathrm{PU}(2,1)$ *by* $\mathrm{tr}(\xi(a)), \mathrm{tr}(\xi(b)), \mathrm{tr}(\xi(ab)), \mathrm{tr}(\xi(ab^{-1}))$ *and* $\mathrm{tr}(\xi([a,b]))$.

In fact, in $\mathrm{PU}(2,1)$, $\mathrm{tr}(\xi([a,b]))$ is a root of a second degree polynomial with real coefficients and the two roots are either equal or complex conjugates.

With

$$\Lambda_2(3,3,n) = \left\langle x, y \,\middle|\, x^3 = y^3 = (xy)^n = e \right\rangle, \tag{28}$$

recall that we have a surjective morphism $\Lambda_2(3,3,n) \to \Delta_2(3,3,n;\theta_\infty)$ given by $(x,y) \mapsto (I_1 I_2, I_2 I_3)$. The traces of the images of x, y, xy and xy^{-1} are all real and prescribed by the data $(3,3,n)$. (When $n = \infty$, we send xy to a unipotent transformation.) Therefore, by doing a complex conjugation on the representation, we only conjugate the trace of the image of $[x,y]$.

Corollary 4.3. *Let* $\pi_1(M)$ *be a fundamental group generated by two elements a and b. Let* $\xi\colon \pi_1(M) \to \mathrm{PU}(2,1)$ *be a representation. Let* $\rho\colon \pi_1(M) \to \Lambda_2(3,3,n)$ *be a surjective morphism and* $\gamma\colon \pi_1(M) \to \Delta_2(3,3,n;\theta_\infty)$ *be the composition of* ρ *with* $\Lambda_2(3,3,n) \to \Delta_2(3,3,n;\theta_\infty)$.

Let $w_x, w_y, w_{xy}, w_{xy^{-1}} \in \pi_1(M)$ *be such that* $\rho(w_x) = x, \rho(w_y) = y$ *and* $w_{xy} = w_x w_y$, $w_{xy^{-1}} = w_x w_y^{-1}$. *We make the following hypotheses:*

1. *The traces* $\mathrm{tr}\circ\xi$ *and* $\mathrm{tr}\circ\gamma$ *coincide on the words* w_x, w_y, w_{xy} *and* $w_{xy^{-1}}$.
2. *There exists* $w_a, w_b \in \langle w_x, w_y\rangle \subset \pi_1(M)$ *such that* $\xi(w_a) = \xi(a)$, $\xi(w_b) = \xi(b)$ *and* $\gamma(w_a) = \gamma(a)$, $\gamma(w_b) = \gamma(b)$.

Then γ *and* ξ *are equal up to conjugation and complex conjugation.*

Proof. Given the notations, the first assumption shows that ξ and γ are equal up to complex conjugation and conjugation on $\langle w_x, w_y\rangle \subset \pi_1(M)$.

Up to the conjugations, assume that $\xi = \gamma$ on $\langle w_x, w_y\rangle$. Then by the second hypothesis $\xi(a) = \xi(w_a) = \gamma(w_a) = \gamma(a)$ and $\xi(b) = \xi(w_b) = \gamma(w_b) = \gamma(b)$. Therefore, $\xi = \gamma$ on $\pi_1(M)$. $\square$

Recall that the trace of a transformation w determines its type. If a transformation w is elliptic with order k, then $\mathrm{tr}(w) = 4\cos(\pi/k)^2 - 1$. The transformation w is unipotent if, and only if, $\mathrm{tr}(w) = 3$ and $w \neq \mathrm{id}$.

The method This is how we will identify all our boundary unipotent representations. An application of this method is detailed in the next section.

1. We describe the relation of $\pi_1(M)$ as consequence of several simpler relations "of triangle type". Those simpler relations are verified by the representation in the census (but not by $\pi_1(M)$).

   For example, $a^2ba^{-1}b^{-2}a^{-1}ba$ is implied by a^4, b^3 and $(a^{-1}b)^3$. It is the most challenging step and also the most important. It is in general very difficult to find such simpler relations.

   This step is inscribed in the first table. The "id" column allows us to recover the representation from SnapPy. Relations can be checked with the SageMath code in the appendix.

2. We construct the surjective morphism ρ and recover the values of the various words w_x, w_y, w_{xy} and $w_{xy^{-1}}$. We use the previous relations "of triangle type" to check that we have a morphism. This step is inscribed in the second and third tables. It is a completely autonomous step from all the computations. It only involves $\pi_1(M)$ and by hand computations.

3. We find values of w_a and w_b and it can be checked by hand that they verify the required equalities for both ξ and γ. Those words are inscribed in the third table.

4. We verify that the traces of the four elements match with the evidence of the first step and by exact computations, *e.g.* for $w_{xy^{-1}}$ (see the SageMath code in the appendix for this).

Those four steps prove the identification we are looking for. But we add a last table with the description of the peripheral holonomy. The reason is that one can only consider the second step (doable by hand) and with the last table show that the morphism constructed has boundary unipotent holonomy, independently from the census. (One should play with various the conjugations such as $y(xy^{-1})y^{-1} = yxy$ and $y^{-1}(xy^{-1})y = y^{-1}x$ in $\Lambda_2(3,3,n)$.)

For the construction of this last table, we only use the relations described in the first step and implied by the description of the morphism.

Proof of theorem 4.1. Let M be a manifold in the table and $\Delta_2(3,3,n;\theta_\infty)$ an assigned group. By the second step, we construct a morphism $\rho\colon \pi_1(M) \to \Lambda_2(3,3,n)$. This morphism gives a representation ξ by the composition $\pi_1(M) \to \Lambda_2(3,3,n) \to \Delta_2(3,3,n;\theta_\infty)$. Finally, ξ has boundary unipotent holonomy by the last table. $\square$

On the notations We keep the notations of the corollary and write A, B, X, Y instead of $a^{-1}, b^{-1}, x^{-1}, y^{-1}$ respectively.

4.1 $\Lambda_2(3,3,4)$ – m004, m022, m029, m034, m081 and m117

We survey the method for m004-1. We denote this representation by $\xi\colon \pi_1(\text{m004}) \to \mathrm{PU}(2,1)$. By SnapPy, the fundamental group of m004 is presented by the generators a and b and the relation a^2bAB^2Aba.

In the census, ξ is a representation verifying (by exact computation) the relations $\xi(a)^4 = e$, $\xi(b)^3 = e$ and $\xi(Ab)^3 = e$. Those three relations alone imply the one of the fundamental group: $(a^2)bA(B^2)Aba = (AA)bA(b)Aba = A(Ab)^3a = e$.

Those first datas are reported in the first table. They suggest that we should seek for a $(3,3,4)$ triangle group representation.

We define a new morphism $\rho\colon \pi_1(\text{m004}) \to \Lambda_2(3,3,4)$ by $\rho(a) = xy$ and $\rho(b) = y$. To check that this is indeed a morphism, it suffices to check the relations $a^4, b^3, (Ab)^3$. And that is the case: in $\Lambda_2(3,3,4)$, xy has order 4, y has order 3 and Ab is sent on Yxy that has order 3 since x has order 3. This is given in the second table.

With the data of the third table, we show that we have constructed a *surjective* morphism $\rho\colon \pi_1(\text{m004}) \to \Lambda_2(3,3,4)$. Since $\Lambda_2(3,3,4)$ is generated by x and y, it suffices to give antecedents to both. The word $w_x = aB$ in $\pi_1(\text{m004})$ is sent to $xyY = x$ and the word $w_y = b$ is sent to y.

Now we prove that γ (which is the composition of ρ with $\Lambda_2(3,3,4) \to \Delta_2(3,3,4;\theta_\infty)$) and ξ have equal images up to conjugation and complex conjugation. According to corollary 4.3, it suffices to find the words $w_x, w_y, w_{xy}, w_{xy^{-1}}, w_a$ and w_b. This is all given in the third table and verifications are straightforward.

The traces of w_x, w_y, w_{xy} will coincide since they verify the triangular relations in both representation γ and ξ. With the word $w_{xy^{-1}}$ we check by formal computation that the trace under ξ is indeed 3.

If one does not want to involve the representation ξ of the census, but only construct the morphism ρ *a priori*, the last table allows us to check that the boundary is unipotent under ρ composed with the usual map $\Lambda_2(3,3,4) \to \Delta_2(3,3,4;\theta_\infty)$. Here, the boundary is described by the two words ab and $aBAbABab$. Under ρ, we have $\rho(aBAbABab) = \rho(ab)^3$ and $\rho(ab) = xyy = xY$ that is indeed mapped to a unipotent element of $\mathrm{PU}(2,1)$.

	id	π_1 relation	Relations
m004-1	[0,0]	a^2bAB^2Aba	a^4, b^3, $(Ab)^3$
m022-1	[0,0]	ab^5abA^2b	a^3, b^4, $(ab)^3$
m029-1	[0,0]	$aBab^3A^2b^3$	a^3, b^4, $(aB)^3$
m034-1	[0,0]	$a^3b^2ABAb^2$	a^4, b^3, $(AB)^3$
m081-1	[0,0]	ab^3aBa^4B	a^3, b^4, $(aB)^3$
m117-1	[0,0]	$a^2b^2a^2b^2ABAb^2$	a^3, b^3 $(AB)^4$

	a	b
m004	xy	y
m022	y	$(xy)^{-1}$
m029	y	xy
m034	$(xy)^{-1}$	x
m081	x^{-1}	$(xy)^{-1}$
m117	x^{-1}	y^{-1}

	w_x	w_y	w_{xy}	$w_{xy^{-1}}$	w_a	w_b
m004	aB	b	a	aB^2	w_{xy}	w_y
m022	BA	a	B	BA^2	w_y	w_{xy}^{-1}
m029	bA	a	b	bA^2	w_y	w_{xy}
m034	b	BA	A	bab	w_{xy}^{-1}	w_x
m081	A	aB	B	Ba^2	w_x^{-1}	w_{xy}^{-1}
m117	A	B	AB	Ab	w_x^{-1}	w_y^{-1}

	Peripheral	holonomy
m004	ab	$aBAbABab \equiv (ab)^3$
m022	Ba	$A^2babA \equiv Ba$
m029	ab^2	$bA^3b^3 \equiv e$
m034	$b^2a^2 \equiv BA^2$	$A^3B^3A \equiv e$
m081	$b^2a \equiv B^2a$	$Ba^3Ba \equiv B^2a$
m117	bA	$BA^3BA \equiv bA$

4.2 $\Lambda_2(3,3,5)$ – m009, m015, m035, m142 and m146

	id	π_1 relation	Relations
m009-1	[0,1]	a^2bABa^2BAb	a^5 , $(a^2B)^3$, $(a^2b)^3$
m015-2	[0,1]	$ab^2A^2b^2aB^3$	a^3, b^5, $(abb)^3$
m035-5	[0,3]	$ab^3A^2b^3aB^2$	a^3, b^5, $(aBB)^3$
m142-1	[0,0]	$ab^2aBab^2aBa^4B$	a^3, b^3, $(aB)^5$
m146-3	[0,3]	$a^2b^2a^3b^2a^2BAB$	a^3, b^5, $(AB)^3$

	a	b
m009	xy	$(YX)^2Y$
m015	Y	$(YX)^2$
m035	y	$(YX)^2$
m142	YXy	y
m146	y	YX

	w_x	w_y	w_{xy}	$w_{xy^{-1}}$	w_a	w_b
m009	a^3b	BA^2	a	a^3ba^2b	w_{xy}	$w_{xy}^{-2}w_y^{-1}$
m015	b^2a	A	b^2	b^2A	w_y^{-1}	w_{xy}^{-2}
m035	b^2A	a	b^2	b^2a	w_y	w_{xy}^{-2}
m142	bAB	b	bA	bAB^2	$w_{xy}^{-1}w_y$	w_y
m146	BA	a	B	BA^2	w_y	w_{xy}^{-1}

	Peripheral	holonomy
m009	ab	$ABa^3BAb \equiv (ab)^2$
m015	bA	$ab^2A^3b^2 \equiv (bA)^{-1}$
m035	ab	$ab^3A^3b^3 \equiv ab$
m142	BA	$bA^3bA \equiv BA$
m146	$ba^2 \equiv bA$	$BABAB^2 \equiv aB$

4.3 $\Lambda_2(3,3,6)$ – m023

	id	π_1 relation	Relations
m023-1	[0,0]	$aBAb^2ABab^3$	b^6, $(Abb)^3$, $(aB^3)^3$

	a	b
m023	$(YX)^2Y^2X$	xy

	w_x	w_y	w_{xy}	$w_{xy^{-1}}$	w_a	w_b
m023	b^3ab	BAB^2	b	b^3ab^3ab	$w_{xy}^{-3}w_xw_{xy}^{-1}$	w_{xy}

	Peripheral	holonomy
m023	bab	$b^2aBABab^2 \equiv (BAB)^2$

4.4 $\Lambda_2(3,3,7)$ – m032 and m045

	id	π_1 relation	Relations
m032-7	[0,2]	$a^2B^2Ab^5AB^2$	a^3, b^7, $(AB^2)^3$
m045-8	[0,4]	$a^3b^2a^3BA^4B$	a^7, b^3, $(a^3B)^3$

	a	b
m032	x	$(xy)^3$
m045	$(xy)^2$	X

	w_x	w_y	w_{xy}	$w_{xy^{-1}}$	w_a	w_b
m032	a	AB^2	B^2	ab^2a	w_x	w_{xy}^3
m045	B	bA^3	A^3	Ba^3B	w_{xy}^2	w_x^{-1}

	Peripheral	holonomy
m032	B^3a	$b^2A^3b^2a \equiv B^3a$
m045	AB	$bA^3B^3A^3 \equiv ba$

4.5 $\Lambda_2(3,3,\infty)$ – m129 and m203

With the complex hyperbolic triangle group $\Delta_2(3,3,\infty;\theta_\infty)$ we ask I_1I_3 to be unipotent: we send xy to a unipotent transformation.

	id	π_1 relation	Relations
m129-1	[0,0]	$a^3B^2abA^3b^2AB$	a^3, b^3
m203-1	[0,0]	$a^3b^2a^2BA^3B^2A^2b$	a^3, b^3

	a	b
m129	xyx^{-1}	x
m203	xyx^{-1}	x

	w_x	w_y	w_{xy}	$w_{xy^{-1}}$	w_a	w_b
m129	b	Bab	ab	Ab	$w_{xy}w_x^{-1}$	w_x
m203	b	Bab	ab	Ab	$w_{xy}w_x^{-1}$	w_x

	Peripheral	holonomy
m129	$A^2b \equiv ab$	$A^3b^2A \equiv BA$
	Ab	$bA^3ba \equiv Ba$
m203	$a^2b \equiv Ab$	$B^2A^3B \equiv e$
	ab	$BA^3B^2A^3 \equiv e$

4.6 m039, s000 and m053

In this final section, we construct the remaining representations with boundary unipotent holonomy. We were not able to identify them with representations from the census for the following reasons.

For m039 and s000 this comes from the fact that they are described by respectively five and six tetrahedra and therefore are not part of the census. For m053, the initial description provided is too complicated to be used out of the box. But with an other description, one can still give triangular representations and the identification with m053-1 and m053-7 follows from the visual clues.

For those representations, we follow the same method to construct the representations, with the exception of the experimental verification of the traces and relations that are not necessary here.

	π_1 relation	Relations
m039	a^6BAb^2AB	a^7, b^3, $(AB)^3$
m053	$a^3b^2a^3BA^5B$	a^4, b^3, $(AB)^3$
		a^8, b^3, $(a^3B)^3$
s000	a^7BAb^2AB	a^8, b^3, $(AB)^3$

	Triangle	a	b
m039	$\Lambda_2(3,3,7)$	YX	y
m053	$\Lambda_2(3,3,4)$	xy	X
m053	$\Lambda_2(3,3,8)$	$(xy)^3$	x
s000	$\Lambda_2(3,3,8)$	YX	y

	Triangle	w_x	w_y	w_{xy}	$w_{xy^{-1}}$
m039	$\Lambda_2(3,3,7)$	AB	b	A	bA
m053	$\Lambda_2(3,3,4)$	B	ba	a	BAB
m053	$\Lambda_2(3,3,8)$	b	Ba^3	a^3	bA^3b
s000	$\Lambda_2(3,3,8)$	AB	b	A	Ab

	Peripheral	holonomy
m039	$A^5b \equiv a^2b$	$aB^3ab \equiv a^2b$
m053	BA^3	$B^2A^3bA^3B$
with $\Lambda_2(3,3,4)$	$\equiv Ba$	$\equiv Ab$
with $\Lambda_2(3,3,8)$	$\equiv BA^3$	$\equiv a^3b$
s000	bA	$BabaB^2 \equiv bA$

5 More diverse examples

To study more examples, we note the following. Let M be a hyperbolic manifold with one cusp. If a representation $\pi_1(M) \to \mathrm{PU}(2,1)$ has for boundary a parabolic subgroup *of rank one* (generated by a single element), then there exists a relation between the two generators of its peripheral holonomy. This relation can be used to factorize the morphism through a Dehn filling.

Now, a manifold on which we executed a Dehn filling will have a different presentation. Often, this presentation enables to see more rapidly the triangle group involved if we are in the case of theorem 4.1.

5.1 A homological obstruction

The existence of a *surjective* morphism (as in theorem 4.1) is submitted to an easy criterium.

Lemma 5.1. *If there exists a surjective morphism $\rho\colon \pi_1(M) \to \Lambda_2(p,q,r)$ then there exists a surjective morphism $\widetilde{\rho}\colon H_1(M;\mathbf{Z}) \to \Lambda_2(p,q,r)^{\mathrm{ab}}$, where we took the abelianizations.* □

Among the representations not covered in theorem 4.1, there are three that have a fractal limit set corresponding to a $\Delta_2(3,3,n;\theta_\infty)$.

The representation m045-1 shows the sign of a $\Delta_2(3,3,4;\theta_\infty)$ group as image. See its approximated limit set (figure 11) and compare with the visual clues (figure 5) in the appendix.

The representations m035-1 and m130-1 both resemble to $\Delta_2(3,3,\infty;\theta_\infty)$. Compare figures 9 and 10. See also [Aleb] for the numerical approximations in a three-dimensional display.

Proposition 5.2. *The boundary unipotent representations m045-1, m035-1 and m130-1 do not verify theorem 4.1. That is to say, those representations are not surjective morphisms of the form* $\rho\colon \pi_1(M) \to \Lambda_2(3,3,n)$.

In the case of $\Lambda_2(3,3,n)$, the abelianization of this abstract group is $\mathbf{Z}/3\mathbf{Z}$ or $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ depending on the class of n modulo 3. For $\Lambda_2(3,3,4)$ we get $\mathbf{Z}/3\mathbf{Z}$, and for $\Lambda_2(3,3,\infty)$ we get $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$.

Proof of proposition 5.2. The representation m045-1 (with id [0,0]) has boundary unipotent holonomy of rank one. Therefore, with a relation between the two peripheral curves.

This relation gives a factorization by the corresponding Dehn filling. In fact, in the terms of SnapPy and the data of m045-1 in [Gör], we get the Dehn filling m045(5, 1). This can be verified with exact computations with the code in the appendix.

Computations of SnapPy show $H_1(\mathrm{m045}(5,1)) = \mathbf{Z}/14\mathbf{Z}$. But 14 and 3 are relatively prime and therefore there exists no surjective morphism from $H_1(\mathrm{m045}(5,1))$ to $\mathbf{Z}/3\mathbf{Z}$, giving an impossibility with the previous lemma.

Similarly, the representations m035-1 (of id [0,0]) and m130-1 (of id [0,1]) factorize through the Dehn fillings m035(3, −1) and m130(2, −1) respectively.

Now, $H_1(\mathrm{m035}(3,-1)) = \mathbf{Z}/20\mathbf{Z}$ and $H_1(\mathrm{m130}(2,-1)) = \mathbf{Z}/16\mathbf{Z}$. Thus there can't be any surjective morphism to $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$, raising an impossibility with the lemma. □

The fact that m045-1 has a limit set so similar to the one of $\Delta_2(3,3,4;\theta_\infty)$ suggests the existence of a close relationship. For instance, there might exist a morphism with finite index image in $\Delta_2(3,3,4;\theta_\infty)$. But this remains an open question.

5.2 A Lagrangian triangle group

As we will see (proposition 5.4), there exists at least one representation with boundary unipotent holonomy,

$$\xi\colon \pi_1(M) \to \Lambda_2(p,q,r) \to \mathrm{PU}(2,1), \tag{29}$$

surjective on $\Lambda_2(p,q,r)$ but *that cannot be factorized* by

$$\pi_1(M) \to \Lambda_2(p,q,r) \to \Delta_2(p,q,r;\theta) \subset \mathrm{PU}(2,1). \tag{30}$$

Lemma 5.3. *For any* $4 \le n < \infty$*, let* $\eta\colon \Lambda_2(3,3,n) \to \mathrm{PU}(2,1)$ *be a representation with* $\eta([x,y])$ *unipotent. Then* $\mathrm{tr}(\eta(xy^{-1}))$ *is not real.*

Proof. We use the exact parametrization of Guilloux and Will [GW19]. The formal computations that we will described have too many terms to be reproduced here, but they can be recovered in the code of the author [Alea] or directly reproduced with the description of [GW19].

Recall with proposition 4.2 that the discriminant Δ of the equation for which $\mathrm{tr}(\eta([x,y]))$ is solution is always negative or null. Since we want $\mathrm{tr}(\eta([x,y])) = 3$, we must have $\Delta = 0$. This equation can be solved by exact computations and is only a function of $\mathrm{tr}(\eta(xy^{-1}))$ since $\mathrm{tr}(\eta(x)), \mathrm{tr}(\eta(y))$ and $\mathrm{tr}(\eta(xy))$ are determined by their elliptical order.

With the resolution of $\Delta = 0$ in terms of the only free variable $\mathrm{tr}(\eta(xy^{-1}))$, one can compute $\mathrm{tr}(\eta([x,y]))$ explicitly and solve the equation $\mathrm{tr}(\eta([x,y])) = 3$ in terms of

$\operatorname{tr}(\eta(xy^{-1}))$. It is again a quadratic equation. There are in general two solutions and are complex conjugates. We take the solution with positive imaginary part.

Numerical approximations show that for any $n < \infty$, the imaginary part of $\operatorname{tr}(\eta([x,y]))$ is always non zero. For example, when $n = 4$, $\operatorname{tr}(\eta(xy^{-1})) \simeq 2.820 + \boldsymbol{i} \cdot 0.222$. $\square$

The representation m023-7 (with id [0,4]) of the census has a limit set different from the usual triangular subgroups, see figure 12. The boundary holonomy has rank one, and therefore m023-7 factorizes through a Dehn filling. In this case, it is m023(2, −1).

Proposition 5.4. *There exists a surjective morphism*

$$\rho\colon \pi_1(\mathrm{m023}) \to \Lambda_2(3,3,4), \tag{31}$$

and a composition $\pi_1(\mathrm{m023}) \to \Lambda_2(3,3,4) \to \mathrm{PU}(2,1)$ *having boundary unipotent holonomy. The boundary unipotent representation in* $\mathrm{PU}(2,1)$ *is not induced by any representation* $\Lambda_2(3,3,4) \to \Delta_2(3,3,4;\theta)$.

Note that any inclusion $\Lambda_2(p,q,r) \to \Delta_2(p,q,r;\theta)$ sends xy^{-1} to a matrix with real trace in $\mathrm{PU}(2,1)$. Indeed, when described in terms of I_1, I_2 and I_3, we have $xy^{-1} \mapsto I_1 I_2 I_3 I_2 = I_1(I_2 I_3 I_2)$ and this is the product of two reflections.

Proof. One of the presentations of $\pi_1(\mathrm{m023}(2,-1))$ given by SnapPy (using also `M.randomize()`) is

$$\pi_1(\mathrm{m023}(2,-1)) = \{a, b \mid a^4 b^3 = aBab^2ab^2 = e\} \tag{32}$$

with the peripheral curves given by $aBAb$ and $aBAbaBAb$. Now, we define

$$\rho(a) = xy,\ \rho(b) = x \tag{33}$$

and one can verify the relations of the fundamental group but also $(aBAb)^2 = aBAbaBAb$.

The boundary holonomy of this representation is prescribed by $aBAb$ and this word has for image $xyXYXx = xyXY = [x,y]$. We conclude with the previous lemma. $\square$

Now we can compare the fractal of m023-7 with the one of $\Lambda_2(3,3,4) \to \mathrm{PU}(2,1)$ sending $[x,y]$ to a unipotent element as in the lemma (figure 13). Visually, they match indeed.

Denote $\rho\colon \pi_1(\mathrm{m023}) \to \Lambda_2(3,3,4)$ the surjective morphism from the proposition. Denote $\xi\colon \pi_1(\mathrm{m023}) \to \mathrm{PU}(2,1)$ its composition with $\eta\colon \Lambda_2(3,3,4) \to \mathrm{PU}(2,1)$ sending $[x,y]$ to a unipotent element.

A consequence of the fact that $\operatorname{tr}(\eta([x,y])) = 3$ is that $\xi(\pi_1(\mathrm{m023})) = \langle \eta(x), \eta(y) \rangle$ is $\mathbf{R}$-decomposable: it is generated by three antiholomorphic reflections. This comes from a theorem of Paupert and Will:

Theorem. *[PW17b] Let* $A, B \in \mathrm{PU}(2,1)$ *be two isometries not fixing a common point in* $\mathbf{H}^2_{\mathbf{C}}$. *Then the pair* A, B *is* $\mathbf{R}$*-decomposable if and only if the commutator* $[A,B]$ *has a fixed point in* $\mathbf{H}^2_{\mathbf{C}}$ *whose associated eigenvalue is real and positive.*

Now, since η is generated by three antiholomorphic reflections, we are in presence of a Lagrangian triangular group [Wil07; FK00]: the three antiholomorphic reflections preserve a real plane inside $\mathbf{H}^2_{\mathbf{C}}$, also called *Lagrangian* plane in complex hyperbolic geometry.

6 Appendix: figures and code

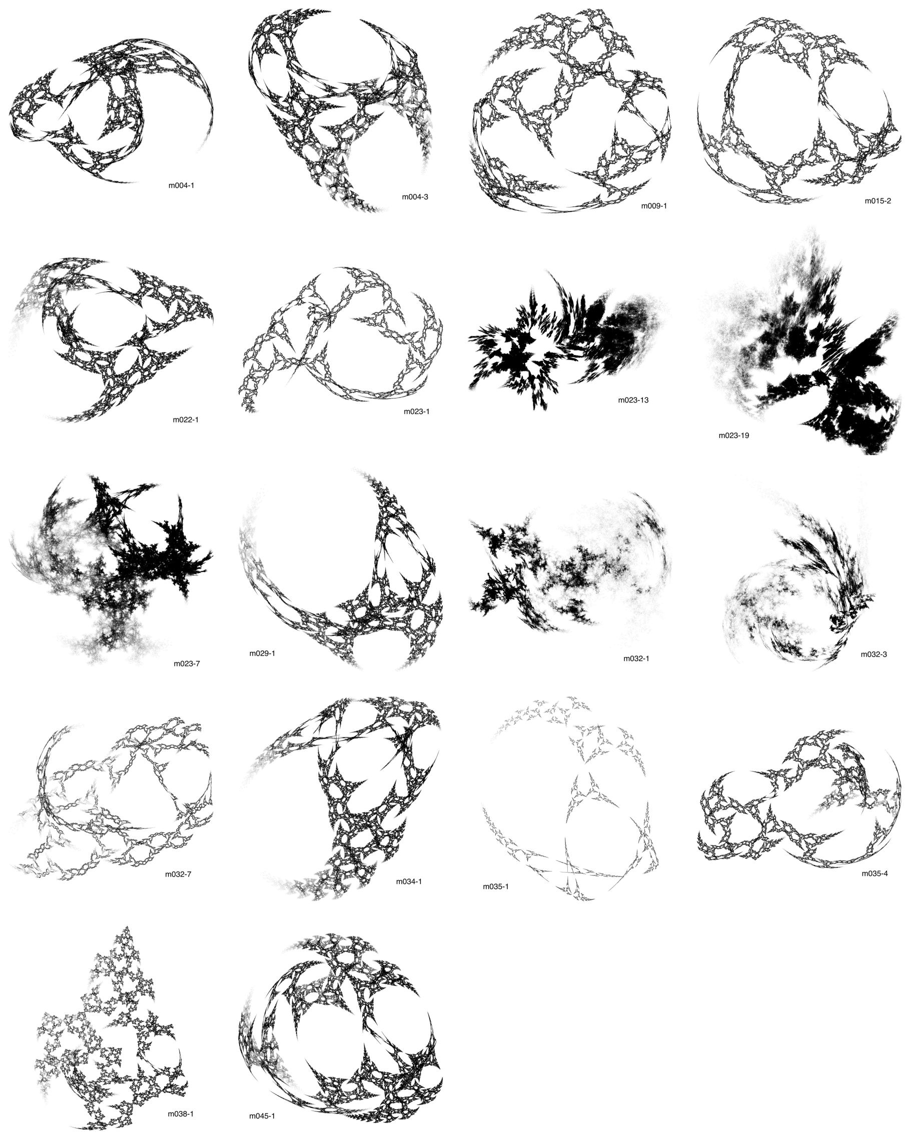

Figure 1: The fractal limit sets found in the census, part 1.

```
import SnapPy
import numpy

# Data
s = 'm009'
i,j = 0,1 # "id" in the table
#

M = SnapPy.Manifold(s)
G = M.fundamental_group()
```

```
print(G,G.peripheral_curves())

P = M.ptolemy_variety(3,'all').retrieve_solutions(prefer_rur=True)
S = [[component
      for component in per_obstruction
      if component.dimension == 0]
     for per_obstruction in P]
K = S[i][j]

def f(x): # evaluates words
    mat_x = K.evaluate_word(x,G)
    return [[z.lift() for z in y] for y in mat_x]

print(f('aaabaab'))
```

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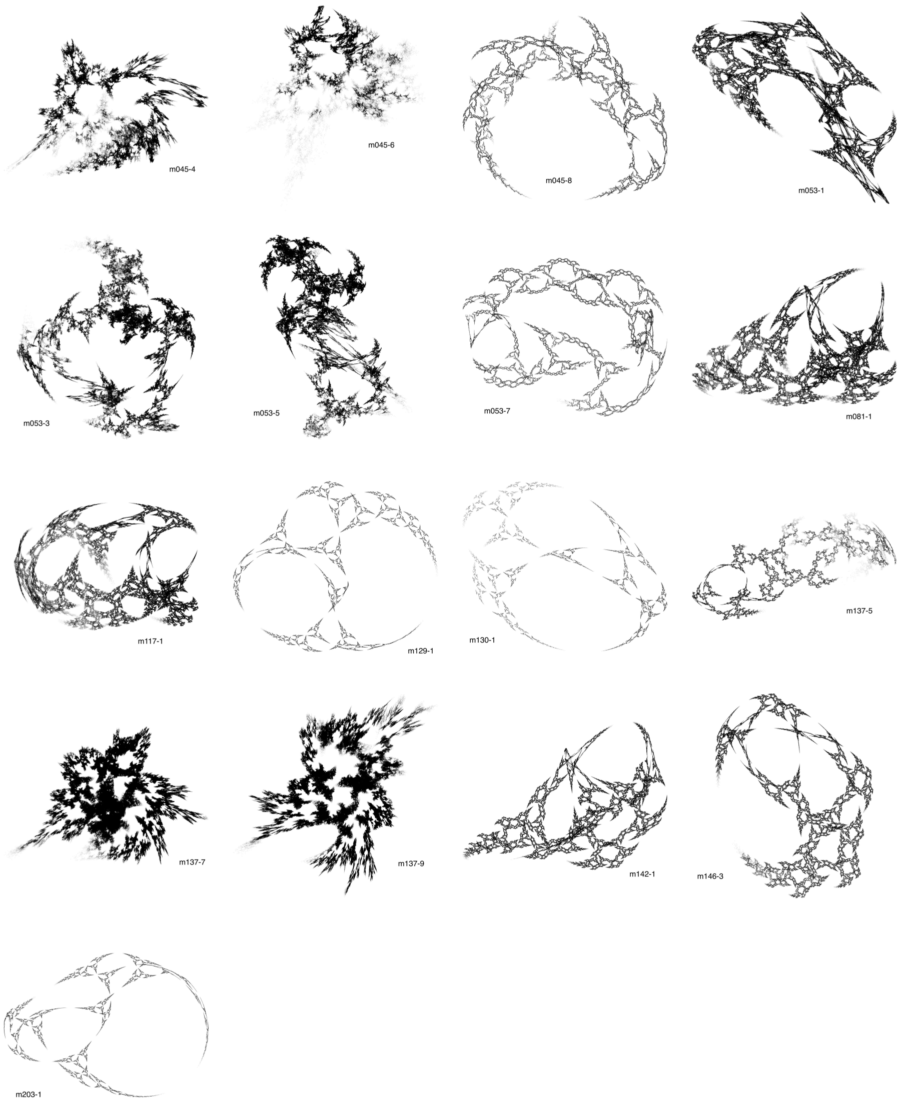

Figure 2: The fractal limit sets found in the census, part 2.

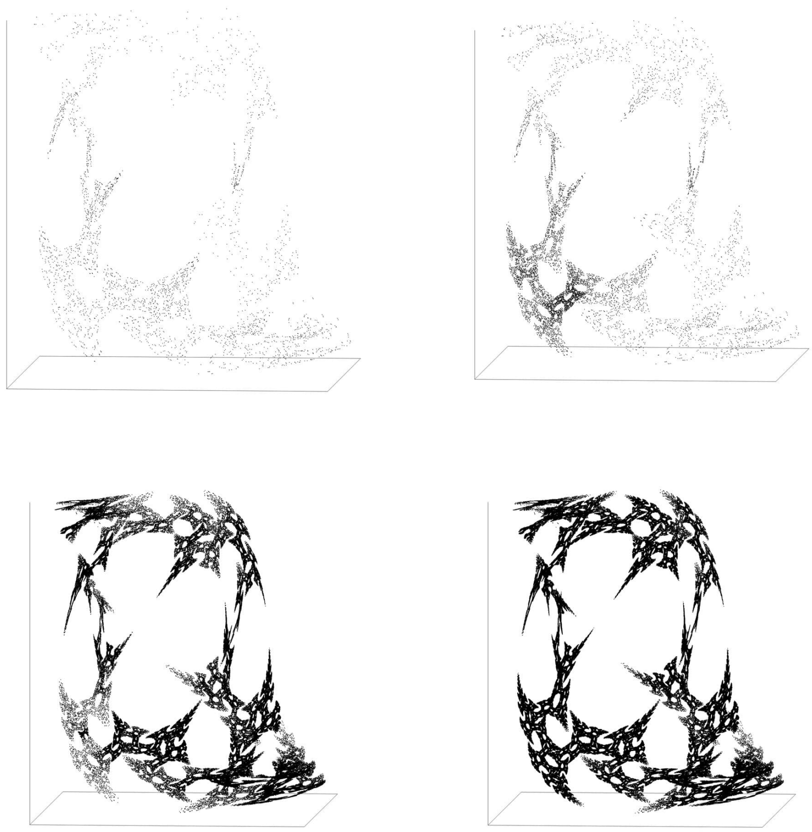

Figure 3: Limit set of m004-1, with steps illustrated.

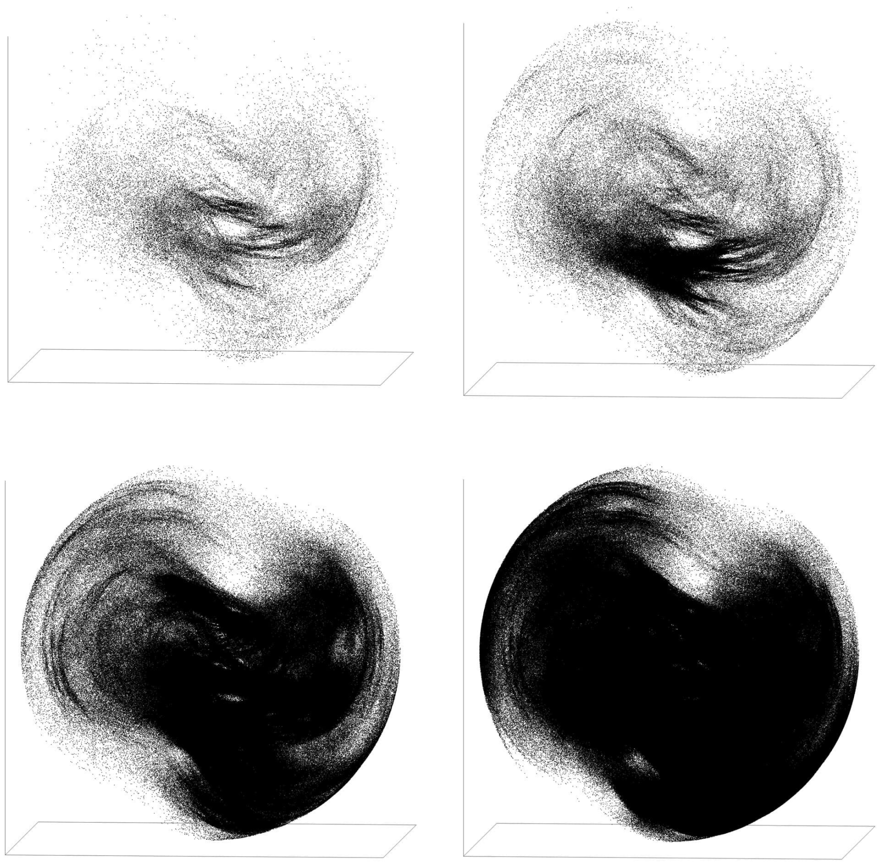

Figure 4: Limit set of m004-5, with steps illustrated.

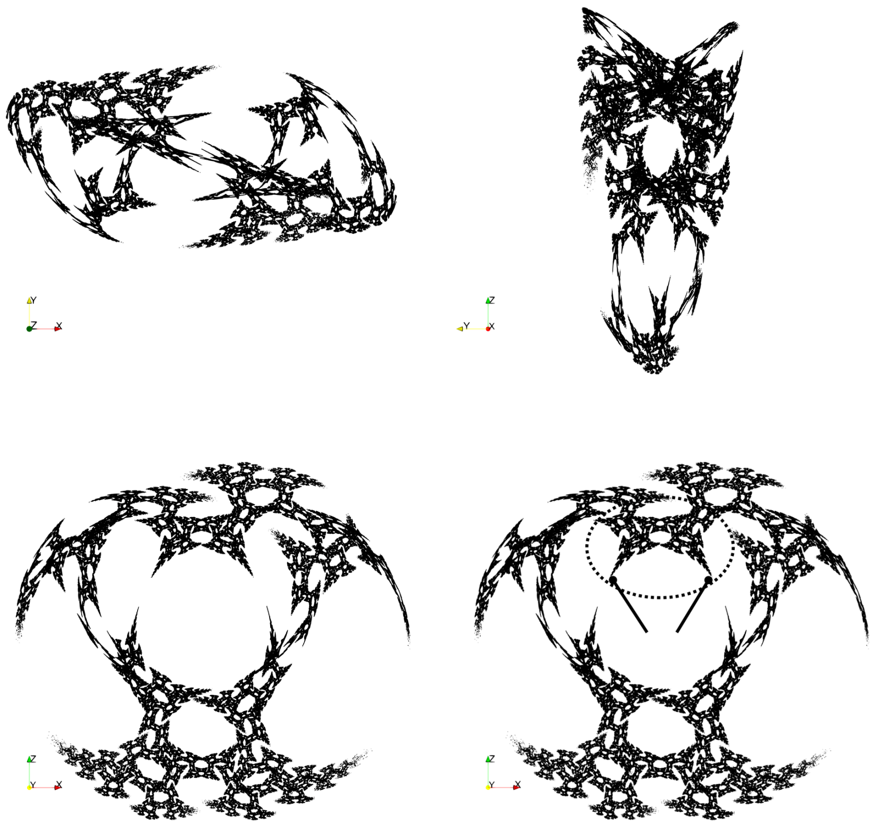

Figure 5: Limit set of $\Delta(3,3,4;\theta_\infty)$.

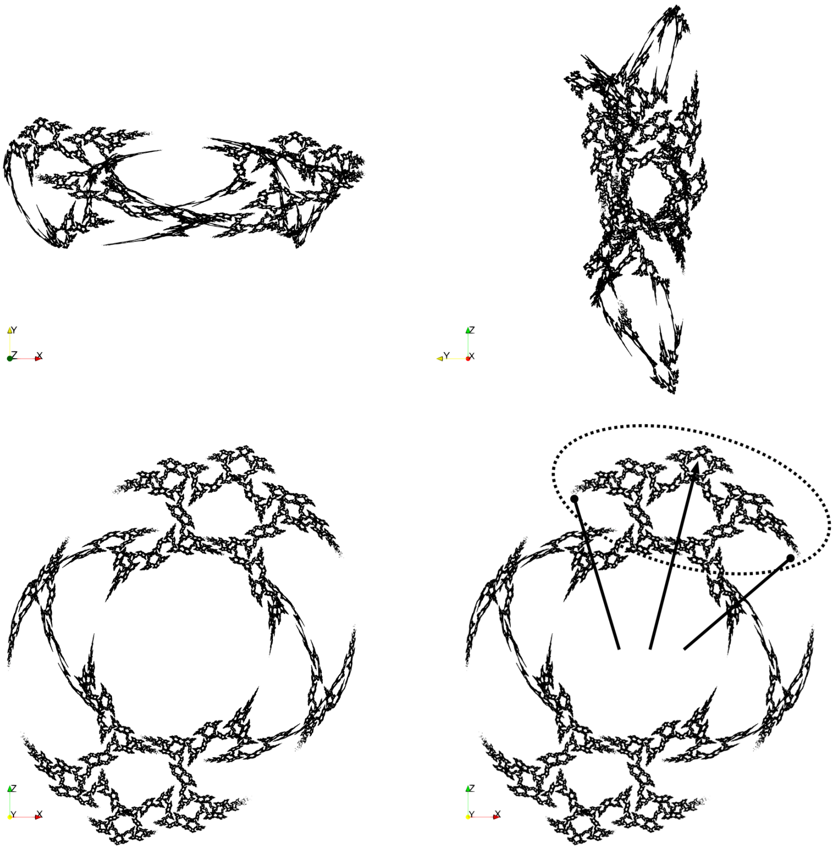

Figure 6: Limit set of $\Delta(3,3,5;\theta_\infty)$.

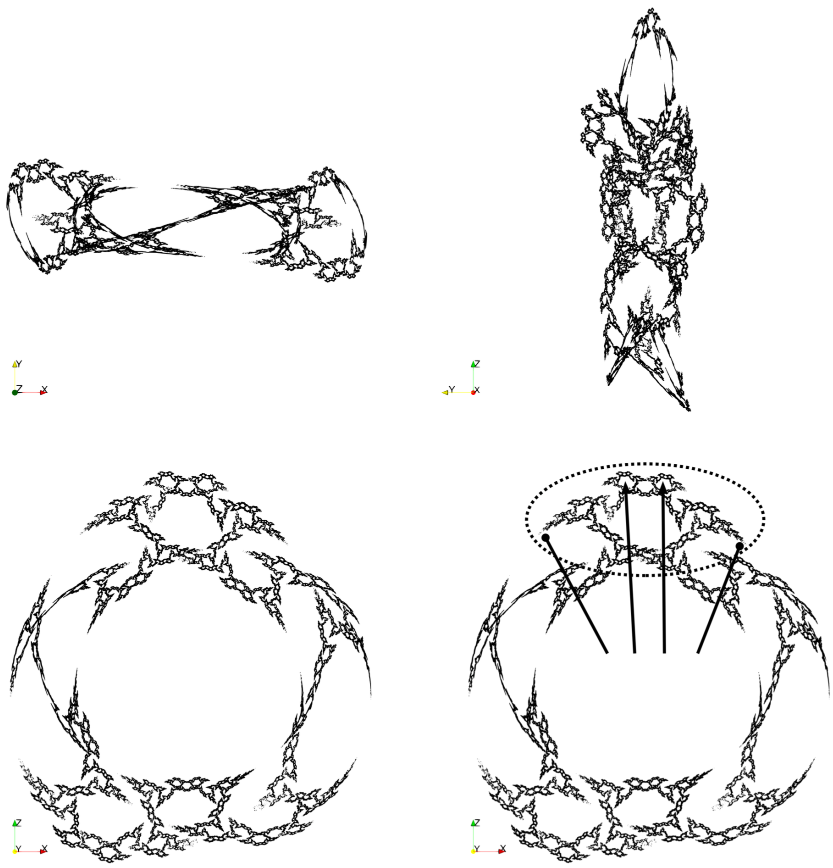

Figure 7: Limit set of $\Delta(3,3,6;\theta_\infty)$.

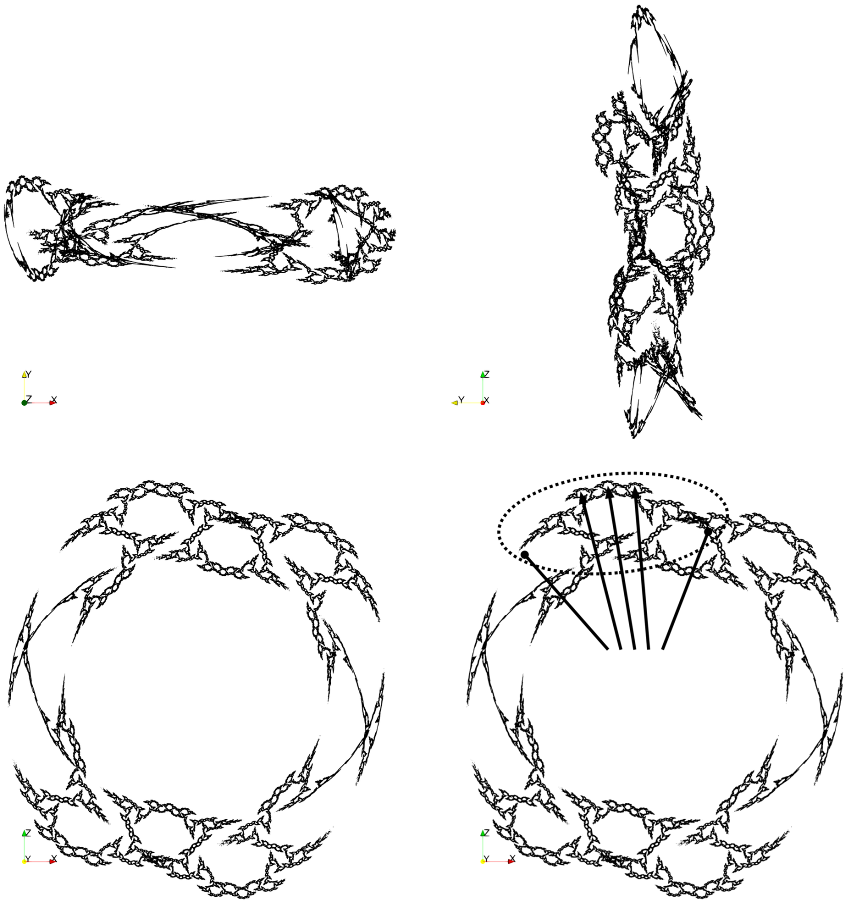

Figure 8: Limit set of $\Delta(3,3,7;\theta_\infty)$.

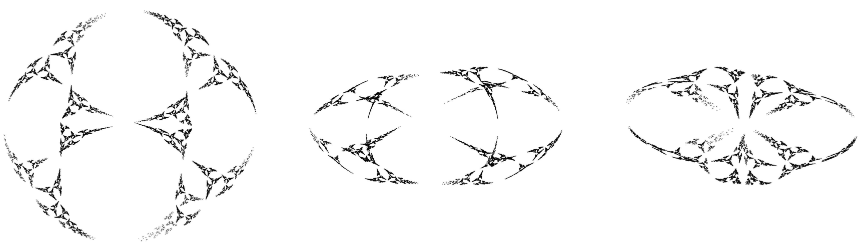

Figure 9: Limit set of $\Delta(3,3,\infty;\theta_\infty)$.

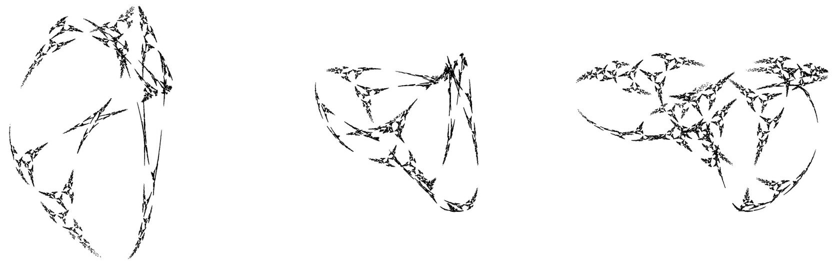

Figure 10: Limit set of m035-1.

Figure 11: Limit set of m045-1.

Figure 12: Limit set of m023-7.

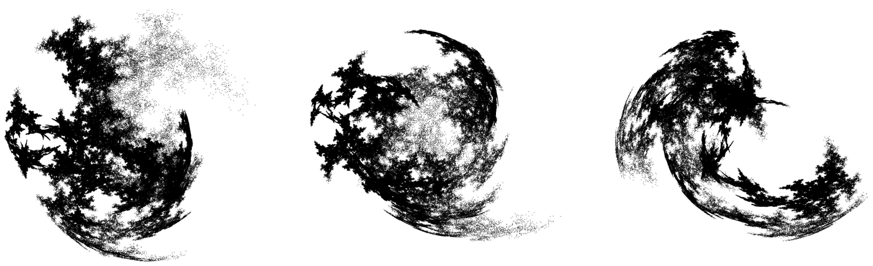

Figure 13: Limit set of the triangle $\Lambda_2(3,3,4) \to \mathrm{PU}(2,1)$ with $[x, y]$ unipotent.