ON THE SPECIAL IDENTITIES OF GELFAND–DORFMAN ALGEBRAS

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ABSTRACT. A Gelfand–Dorfman algebra (GD-algebra) is said to be special if it can be embedded into a differential Poisson algebra. In this paper, we prove that the class of all special GD-algebras is closed with respect to homomorphisms and thus forms a variety. We describe a technique for finding potentially all special identities of GD-algebras and derive two known special identities of degree 4 in this way. By computing the Gröbner basis for the corresponding shuffle operad, we show that these two identities imply all special ones up to degree 5.

INTRODUCTION

A linear space V with two bilinear operations \circ and $[\cdot, \cdot]$ is called a *Gelfand–Dorfman algebra* (or simply GD-algebra) if (V, \circ) is a Novikov algebra, $(V, [\cdot, \cdot])$ is a Lie algebra, and the following additional identity holds:

$$b \circ [a, c] = [a, b \circ c] - [c, b \circ a] + [b, a] \circ c - [b, c] \circ a.$$
(1)

Recall that the variety of Novikov algebras is defined by the following identities:

$$(a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c), \tag{2}$$

$$(a \circ b) \circ c = (a \circ c) \circ b. \tag{3}$$

The axioms above emerged in the paper [7] as a tool for constructing Hamiltonian operators in formal calculus of variations. Later it was shown [19] that GD-algebras are in one-to-one correspondence with quadratic Lie conformal algebras playing an important role in the theory of vertex operators. The class of GD-algebras is governed by a binary quadratic operad ([9], see also [11] for the definition) denoted GD. As it was shown in [10], the Koszul dual operad GD[!] corresponds to the class of differential Novikov–Poisson algebras introduced in [4]. The latter algebras play an important role in the combinatorics of derived operations on non-associative algebras [10].

In the present paper, we study special GD-algebras, i.e., those embeddable into Poisson algebras with a derivation. We prove in section 1 that the class of special GD-algebras is closed under homomorphic images and thus forms a variety. Nonspecial GD-algebras exist: the examples were found implicitly in [10] and explicitly

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in [17]. Note that all these examples are of dimension three. We apply the Gröbner–Shirshov bases technique for Poisson algebras to prove that all 2-dimensional GD-algebras are special.

In section 2, we give a technical method to find all special identities of GDalgebras and explicitly find all special identities of GD-algebras up to degree 5. In section 3, we prove that all special identities up to degree 5 are consequences of two independent special identities of degree 4 found in [10]. For that purpose, we convert the symmetric operad GD with special identities of degree 4 into a shuffle operad [11] and compute the first five components of its Gröbner basis [13] by means of the computer algebra package [14]. In the same way, we calculate the dimensions of GD operad up to degree 6.

1. Special GD-Algerbas

Suppose $(P, \cdot, \{\cdot, \cdot\})$ is a Poisson algebra with a derivation d. Hereinafter, we will denote d(x) by x' for simplicity. Hence, (P, \cdot) is an associative and commutative algebra, $(P, \{\cdot, \cdot\})$ is a Lie algebra and the *Leibniz identity* holds:

$$\{x, yz\} = y\{x, z\} + z\{x, y\}, \quad x, y, z \in P.$$

The linear map $d: P \to P$ acts as a derivation relative to both operations:

$$(xy)' = xy' + x'y, \quad \{x, y\}' = \{x', y\} + \{x, y'\}, \quad x, y \in P.$$

Then the same space P equipped with operations

$$x \circ y = xy', \quad [x, y] = \{x, y\}$$

is a GD-algebra denoted $P^{(d)}$ [20]. A GD-algebra V is said to be *special* if it can be embedded into a GD-algebra $P^{(d)}$ for an appropriate differential Poisson algebra P.

It is clear that the class of special GD-algebras is closed with respect to subalgebras and Cartesian products, so the class of all homomorphic images of special GDalgebras is a variety denoted SGD.

The relation between differential Poisson algebras and GD-algebras described above is similar to the well-known relation between associative and Jordan algebras [1] (see also [8, Chapter 3]). However, in contrast to the Jordan algebra case, the class of special GD-algebras turns to be closed with respect to homomorphisms.

Theorem 1. Let V be a special GD-algebra. Then every homomorphic image of V is special.

Proof. As V is a special GD-algebra, it is a homomorphic image of the free special GD-algebra $\text{SGD}\langle X \rangle$ generated by an appropriate set X. Therefore, it is enough to show that for every set X all homomorphic images of $\text{SGD}\langle X \rangle$ are special.

Let us recall the structure of $\text{SGD}\langle X \rangle$ [10]. Consider the free differential Poisson algebra PoisDer $\langle X, d \rangle$ generated by the set X. The operation set of this system consists of a commutative multiplication, Poisson bracket $\{\cdot, \cdot\}$, and a unary operation d which acts as a derivation with respect to both binary operations. As above,

we will write u' for d(u). Define the *weight* of a monomial from PoisDer $\langle X, d \rangle$ by induction as follows:

$$\operatorname{wt}(x) = -1, \quad x \in X; \quad \operatorname{wt}(u') = \operatorname{wt}(u) + 1;$$

 $\operatorname{wt}(\{u, v\}) = \operatorname{wt}(u) + \operatorname{wt}(v) + 1; \quad \operatorname{wt}(uv) = \operatorname{wt}(u) + \operatorname{wt}(v).$

For a weight-homogeneous polynomial $f \in \text{PoisDer}\langle X, d \rangle$ we denote by wt(f) the weight of its monomials.

As shown in [10], SGD $\langle X \rangle$ is isomorphic to the subspace of PoisDer $\langle X, d \rangle^{(d)}$ spanned by all monomials of weight -1.

Suppose I is an ideal of the GD-algebra $\mathrm{SGD}\langle X \rangle$, the latter is immersed into $\mathrm{PoisDer}\langle X, d \rangle$. Consider the ideal \hat{I} of $\mathrm{PoisDer}\langle X, d \rangle$ generated by I. It is enough to show $\hat{I} \cap \mathrm{SGD}\langle X \rangle = I$. If the latter holds, then $\mathrm{PoisDer}\langle X, d \rangle / \hat{I}$ is a differential Poisson envelope of the GD-algebra $V = \mathrm{SGD}\langle X \rangle / I$.

An arbitrary element in $f \in \hat{I}$ may be obtained as

$$f = \sum_{i} F_i(X, t)|_{t=u_i},$$

where $F_i(X,t) \in \text{PoisDer}\langle X \cup \{t\}, d\rangle$, $u_i \in I \subset \text{SGD}\langle X \rangle \subset \text{PoisDer}\langle X, d\rangle$. Note that $\text{wt}(f) = \text{wt}(u_i) = -1$, so we may assume $\text{wt}(F_i) = -1$ for all *i* (otherwise, it is enough to choose the homogeneous components of weight -1 for each F_i). Therefore,

$$F_i(X,t) \in \mathrm{SGD}\langle X \cup \{t\}\rangle,\$$

and $F_i(X, u_i) \in I$. Hence, $f \in I$.

Corollary 1. The class of special Gelfand–Dorfman algebras forms a variety.

Remark 1. Novikov algebras are particular cases of GD-algebras (with [x, y] = 0). It was shown in [15] that the free Novikov algebra embeds into the free differential commutative algebra. Hence, restricting the proof of Theorem 1 to Novikov algebras we obtain another proof of speciality of all Novikov algebras [3].

The examples of non-special GD-algebras were constructed in [10], [17]. It is worth mentioning that all these examples are of dimension 3. In the next statement we show that dimension 3 is a minimal one: all 2-dimensional GD-algebras are special.

Theorem 2. Let V be a 2-dimensional GD-algebra. Then V is special.

Proof. Let $X = \{u, v\}$ be a basis of a 2-dimensional GD-algebra V. If [u, v] = 0 then V is a pure Novikov algebra and by [3] V embeds into the commutative differential algebra ComDer $\langle X \rangle / (xy' - x \circ y)$ which may be considered as a Poisson one with respect to the trivial bracket.

If V is not abelian as a Lie algebra then we may assume [u, v] = v. It is straightforward do deduce from (1), (2), and (3) that the multiplication table for \circ on V

has the following form:

$$u \circ u = \alpha u + \delta v,$$

$$v \circ v = 0,$$

$$u \circ v = \gamma v,$$

$$v \circ u = \alpha v,$$

for $\alpha, \gamma, \delta \in \mathbb{k}$. It is enough to show that V can be embedded into a differential Poisson algebra. We consider three cases: (1) $\alpha \neq \gamma$, (2) $\alpha = \gamma \neq 0$, (3) $\alpha = \gamma = 0$. Case 1: $\alpha \neq \gamma$. Note that $u \circ v - v \circ u = (\gamma - \alpha)v = (\gamma - \alpha)[u, v]$, i.e.,

$$[u,v] = rac{1}{\gamma-lpha}(u\circ v - v\circ u).$$

Therefore, the structure of a GD-algebra on V is almost completely similar to the commutator GD-algebra on a Novikov algebra considered in [17]. The only difference is the multiplicative constant $\frac{1}{\gamma-\alpha}$ which does not affect the construction of a differential Poisson envelope. Namely, consider the free differential commutative algebra $F = \text{ComDer}\langle\{u, v\}, d\rangle = \text{Com}\langle u, v, u', v', u'', v'', \dots \rangle$, and define a bracket $\{\cdot, \cdot\}$ on F as follows. On the generators, let

$$\{u^{(m)}, v^{(n)}\} = \frac{1}{\gamma - \alpha} ((n-1)u^{(m+1)}v^{(n)} - (m-1)u^{(m)}v^{(n+1)}), \quad n, m \ge 0.$$
(4)

Then apply the Leibniz rule to expand the bracket on the entire F. The bracket obtained is proportional to the one considered in [17]. Therefore, the operation (4) satisfies the Jacobi identity, and the operation d is a derivation with respect to this bracket. Consider the ideal I_V of the free differential commutative algebra Fgenerated by $xy' - x \circ y$, $x, y \in \{u, v\}$. The ideal I_V is invariant under all operations $\{f, \cdot\}, f \in F$. Hence, the universal differential commutative envelope F/I_V of the Novikov algebra V is also a differential Poisson envelope of V as of a GD-algebra.

Case 2: $\alpha = \gamma \neq 0$. An obvious change of variables $\frac{1}{\alpha}u - \frac{\delta}{\alpha^2}v \rightarrow u \ (v \rightarrow v)$ leads to the following multiplication table in V:

$$[u,v] = \frac{1}{\alpha}v, \quad u \circ u = u, \quad u \circ v = v, \quad v \circ u = v, \quad v \circ v = 0.$$

$$(5)$$

Consider the polynomial algebra k[x, e] equipped with two commuting derivations

$$d_1 = \frac{\partial}{\partial x}, \quad d_2 = \frac{e}{\alpha} \frac{\partial}{\partial e}$$

Then $\{f, g\} = d_1(f)d_2(g) - d_2(f)d_1(g), f, g \in \mathbb{k}[x, e]$, is a Poisson bracket on $\mathbb{k}[x, e]$, and d_1 is a derivation on the Poisson algebra obtained. In particular,

$$\{x, e\} = \frac{1}{\alpha}e.$$
 (6)

Denote by A the quotient of $\Bbbk[x, e]$ modulo the ideal I generated by e^2 . As $d_1(I), d_2(I) \subseteq I$, the commutative algebra A is a differential Poisson algebra generated by x and e

with a bracket defined via (6) and with a derivation

$$d(x) = 1, \quad d(e) = 0$$

Now, the initial GD-algebra V embeds into $A^{(d)}$ by

$$u \mapsto x, \quad v \mapsto ex.$$

Indeed, it is straightforward to check that this map preserves the multiplication table (5).

Case 3: $\alpha = \gamma = 0$. If $\delta = 0$ then V is a pure Lie algebra and thus embeds into the associated graded Poisson algebra $P(V) = \operatorname{gr} U(V)$ of the universal associative envelope of V with zero derivation.

If $\delta \neq 0$ then, up to a change of variables, we may assume

$$[u, v] = v, \quad u \circ u = v, \quad v \circ v = u \circ v = v \circ u = 0.$$

Consider the polynomial algebra $F = \mathbb{k}[u, v, u', v']$ in four formal variables. Define a skew-symmetric bracket $\{\cdot, \cdot\}$ on F by the Leibniz rule starting from

$$\{u, v\} = v, \quad \{u, u'\} = u', \quad \{u, v'\} = 2v', \{v, u'\} = v', \quad \{v, v'\} = \{u', v'\} = 0.$$
 (7)

This is a Poisson bracket since it is straightforward to check the Jacobi identity on the generators.

Let I be the ideal of F generated by

$$uu' - v, uv', vu', vv', vv, u'u' - v', u'v', v'v'.$$

These polynomials form a Gröbner basis in $F = \Bbbk[u, v, u', v']$, so the quotient A = F/I contains V as a subspace. The ideal I is closed under applying the bracket $\{x, \cdot\}$ for x = u, v, u', v'. Hence, A = F/I is a Poisson algebra.

The ideal I is closed under the derivation d of F (as of commutative algebra) defined by

$$d(u) = u', \quad d(v) = v', \quad d(u') = 0, \quad d(v') = 0.$$

Therefore, d induces a derivation on A.

It remains to check that d is a derivation relative to the Poisson bracket on A, i.e., $d\{f,g\} = \{d(f),g\} + \{f,d(g)\}$ for $f,g \in A$. Note that the linear basis of A consists of reduced monomials relative to the Gröbner basis (7): $1, u', v', u^n, u^m v$, for $n, m \geq 0$. By definition,

$$d(u^{n}) = \begin{cases} 0, & n = 0, \\ u', & n = 1, \\ nu^{n-2}v, & n \ge 2, \end{cases} \quad d(u^{m}v) = \begin{cases} v', & m = 0, \\ 0, & m \ge 1. \end{cases}$$

If f = u, g = v then

$$\{d(f),g\} + \{f,d(g)\} = \{u',v\} + \{u,v'\} = -v' + 2v' = v' = d\{f,g\}.$$

If $f = u^n$, n > 1, g = v then

$$\{d(f),g\} + \{f,d(g)\} = \{nu^{n-2}v,v\} + \{u^nv',v'\}$$

= $n(n-2)u^{n-3}vv + 2nu^{n-1}v'v' = 0 = d\{f,g\}.$

For the remaining pairs of basic elements f, g, the derivation property for d may be checked in a similar way.

Hence, A is a differential Poisson algebra. It follows from the definition of d and A that V as a GD-algebra embeds into $A^{(d)}$.

2. Special identities of GD-algebras

Since both classes GD and SGD are varieties and thus defined by some sets of identities, the interesting problem is to determine a list of those identities that hold in SGD but do not hold in GD. As in the case of Jordan algebras, let us call such identities *special*.

In [10], the following description of SGD was proposed: the operad of special GD-algebras is a sub-operad in the Manin white product GD[!] •Pois. Here Pois is the operad of Poisson algebras and GD[!] is Koszul dual to GD. However, finding special identities in this way is technically hard since the entire operad GD[!] •Pois is pretty large.

In this section, we develop another approach which allows us to show that the two identities found in [10] exhaust all special identities of degree 4 and find the complete list of special identities of degree 5. In the next section, we apply Gröbner basis technique for operads to show that the identities of degree 5 found in this section follow from the identities of degree 4.

Let $X = \{x_1, x_2, ...\}$ be a countable set, and let $GD\langle X \rangle$ be the free GD-algebra generated by X. Suppose B is a linear basis of $GD\langle X \rangle$, $X \subset B$, equipped with a linear order \leq . Note that finding the explicit form of B, or at least of the multilinear part of B, is a separate interesting problem. Let us construct $U = \text{PoisDer}\langle B, d \rangle / I$, where I is the differential ideal generated by

$$ad(b) - a \circ b, \quad a, b \in B,$$

 $\{a, b\} - [a, b], \quad a, b \in B, \ a > b.$

The algebra U constructed is the universal Poisson differential envelope of $\mathrm{GD}\langle X \rangle$. The kernel of the natural homomorphism $\tau : \mathrm{GD}\langle X \rangle \to U$, $\tau(b) = b + I$, is exactly the set of special identities of GD-algebras.

Denote $\dot{B}^{(\omega)} = \{b^{(n)} \mid b \in B, n \ge 0\}$ and expand the order \le to $B^{(\omega)}$ by the following rule:

$$b^{(n)} < a^{(m)} \iff (b, -n) < (a, -m)$$
 lexicographically. (8)

The ordering is motivated by the Shirshov's argument [18] concerning the standard bracketing on Lyndon–Shirshov words.

Let us identify $\operatorname{PoisDer}\langle B, d \rangle$ with $\operatorname{Pois}\langle B^{(\omega)} \rangle$ assuming $d^n(b) = b^{(n)}$, then I coincides with the ideal in $\operatorname{Pois}\langle B^{(\omega)} \rangle$ generated by

$$ab^{(n)} - (a \circ b)^{(n-1)} + \sum_{i=1}^{n-1} \binom{n-1}{i} a^{(i)} b^{(n-i)}, \ n \ge 1,$$
(9)

$$\{a, b^{(n)}\} - [a, b]^{(n)} + \sum_{i=1}^{n} \binom{n}{i} \{a^{(i)}, b^{(n-i)}\}, \ a > b, \ n \ge 0.$$
(10)

In order to find the kernel of τ it is enough to calculate the intersection of I with $\mathbb{k}B^{(0)}$. The latter can be done if we know the Gröbner–Shirshov basis (GSB) of I in the free Poisson algebra Pois $\langle B^{(\omega)} \rangle$ [5].

Let us proceed as follows. Consider the Lie algebra $L = \text{Lie}\langle B^{(\omega)} \rangle / J$, where J is the ideal generated by (10). Then $\text{Pois}\langle B^{(\omega)} \rangle / I$ is the same as the quotient of the symmetric algebra S(L) modulo the ideal generated by all elements of the form

$$\{a_1^{(n_1)}, \{a_2^{(n_2)}, \dots, \{a_k^{(n_k)}, f\} \dots\}\},$$
(11)

for all f in (9), $a_i \in B$, $n_i \ge 0$. Thus it is enough to calculate the intersection of L with the Gröbner basis of an ideal in S(L) generated by relations (11). Let us first find the Gröbner–Shirshov basis of the Lie algebra L in a slightly more general context.

Lemma 1. Let \mathfrak{g} be a Lie algebra with a linearly ordered basis B. Then (10) is a Gröbner-Shirshov basis in $\operatorname{Lie}\langle B^{(\omega)}\rangle$ relative to the deg-lex ordering based on (8).

Remark 2. Given a Lie algebra \mathfrak{g} with a linear basis B as above, the Lie algebra generated by $B^{(\omega)}$ with defining relations (10) is the universal differential envelope of \mathfrak{g} .

Proof. One may simply check that all compositions of intersection of the relations (10) are trivial in the sense of [18] (see also [2]). Note that a specific technique for calculating Gröbner–Shirshov bases in differential Lie algebras was proposed in [6], but we have a different ordering.

A more conceptual way is based on the following observation. Denote by L the quotient of $\text{Lie}\langle B^{(\omega)}\rangle$ by the Lie ideal generated by (10). The multiplication table on \mathfrak{g} corresponds to n = 0. Add a new letter t to the alphabet B (assuming t < B) and construct

$$U = \operatorname{Lie}\langle B, t \mid \{a, b\} - [a, b], \ a > b \rangle.$$

Since the multiplication table is always a Gröbner–Shirshov basis, the linear basis of U consists of all Lyndon–Shirshov words [u] in $B \cup \{t\}$ such that their associative images u do not contain subwords ab for a > b. Such a word u is either equal to t or may be written as

$$u = u_1 a_1 t^{k_1} \dots u_2 a_2 t^{k_2} \dots u_m a_m t^{k_m},$$

for $u_i \in B^*$, $a_i \in B$, $k_i \ge 1$. As shown in [18], the standard bracketing [u] on u is constructed in the same way as on

$$u_1 a_1^{(k_1)} \dots u_2 a_2^{(k_2)} \dots u_m a_m^{(k_m)},$$

where $a^{(k)} = \{\{\ldots, \{a, t\}, \ldots, t\}, t\}$ and the order is defined by (8). Hence, the linear basis of U consists of t and of all those Lyndon–Shirshov words that are reduced modulo (10). Therefore, the latter relations form a Gröbner–Shirshov basis in $\text{Lie}\langle B^{(\omega)}\rangle$, and, in particular, $U \simeq \Bbbk t \ltimes L$, where $\{f, t\} = f'$ for $f \in L$. \Box

Corollary 2. The linear basis of L consists of all nonassociative Lyndon–Shirshov words of the form

$$[x_{11}\dots x_{1l_1}a_1^{(k_1)}\dots x_{m1}\dots x_{ml_m}a_m^{(k_m)}], \quad k_i \ge 1, \ x_{i1} \le \dots \le x_{il_i} \le a_i$$

In order to find the intersection of L with the ideal in S(L) generated by (11) let us define a rewriting system in $S(\text{Lie}\langle B^{(\omega)}\rangle)$ based on the relations (10) with principal parts $\{a, b^{(n)}\}, a > b$, and on the relations (11) by choosing the principal parts as

$$aLS(\{a_1^{(n_1)}, \{a_2^{(n_2)}, \dots, \{a_k^{(n_k)}, b^{(n)}\} \dots\}\}), \quad n_i \ge 0, \ n \ge 1,$$

where LS(u) stands for the principal Lyndon–Shirshov word in the expansion of $u \in \text{Lie}\langle B^{(\omega)} \rangle$.

For example, if $f = ac' - a \circ c$, b > c, then the relation $\{b, f\} = \{b, ac'\} - \{b, a \circ c\}$ from (11) gives rise to the following rewriting rule:

$$a\{b,c'\} \to [a,b] \circ c + [b,a \circ c].$$

Similarly, $\{c, ab' - a \circ b\}$ gives us

$$a\{c, b'\} \to [a, c] \circ b + [c, a \circ b].$$

On the other hand, we have a rule

$$\{b, c'\} \to \{c, b'\} + [b, c]'.$$

from (10).

Here we have an ambiguity in $a\{b, c'\}$: there are two ways how to rewrite it. This particular critical pair is convergent by (1):

$$a\{b,c'\} \to a\{c,b'\} + a[b,c]' \to [a,c] \circ b + [c,a \circ b] + a \circ [b,c] = [a,b] \circ c + [b,a \circ c].$$

Denote by $\mathcal{G}(B)$ the oriented graph with vertices $S(\text{Lie}\langle B^{(\omega)}\rangle)$ and edges defined by the rewriting rules based on (10) and (11). These rules preserve the weight (wt) of differential polynomials in $B^{(\omega)}$ as well as the degree in X. The set $\Bbbk B$ is homogeneous of weight -1. Hence, $\mathcal{G}(B)$ splits into connected components $\mathcal{G}_{n,w}$ that contain vertices of degree n in X of weight w. Denote $\mathcal{G}_{n,-1}$ by $\mathcal{G}_n(B)$. This graph has no infinite chains (i.e., it is a rewriting system): as shown in [10], these rules applied to a differential monomial of weight -1 rewrite it to an element of $\Bbbk B$ in a finite number of steps. This observation allows us not to define any ordering on commutative monomials in $S(\text{Lie}\langle B^{(\omega)}\rangle)$. In order to find the special identities of degree n (with respect to X) it is enough to find the relations in $\Bbbk B$ that make the rewriting system $\mathcal{G}_n(B)$ confluent.

Given a fixed integer $n \geq 3$, it is enough to consider all multilinear differential monomials of weight -1 and of degree n in X, and expand all critical pairs in the rewriting system $\mathcal{G}_n(B)$. Note that all such pairs without brackets (i.e., those from $\operatorname{Com}\langle B^{(\omega)}\rangle$) are confluent by [3] and we do not need to consider them. Similarly, Lemma 1 shows that all "pure Lie" critical pairs $f \leftarrow u \rightarrow g$, $u \in \operatorname{Lie}\langle B^{(\omega)}\rangle$, are trivial.

For n = 3, the remaining critical pairs correspond to the ambiguities of the form $a\{b, c'\}, b > c$. As shown above, these critical pairs are convergent.

For n = 4, there are five potential ambiguities:

 $\begin{array}{ll} (A1) \ a\{b,\{c,d'\}\}, \ c>d;\\ (A2) \ a\{\{b,c'\},d\}, \ c\leq d, \ b>c;\\ (A3) \ ab'\{c,d'\}, \ c\geq d;\\ (A4) \ ab\{c',d'\}, \ c>d;\\ (A5) \ ab\{c,d''\}, \ c\geq d. \end{array}$

In the case (A1), one may apply either $\{c, d'\} \rightarrow \{c', d\} + [c, d]'$ or the rule coming from $\{b, \{c, ad' - a \circ d\}\}$ in (11):

$$a\{b, \{c, d'\}\} \to f_1 = a\{b, \{c', d\}\} + a\{b, [c, d]'\}$$

or

$$a\{b, \{c, d'\}\} \to f_2 = \{a, b\}\{c, d'\} + \{a, c\}\{b, d'\} - [b, [c, a]]d' + \{b, \{c, a \circ d\}\}.$$

Then apply the rewriting rules coming from (11) to get

$$f_2 \to [c, [a, b] \circ d] - [c, [a, b]] \circ d + [b, [a, c] \circ d] + [b, [c, a \circ d]]$$

and, similarly,

 $f_1 \rightarrow -[d, [a, b] \circ c] + [d, [a, b]] \circ c - [b, [a, d] \circ c] - [b, [d, a \circ c]] + [b, a \circ [c, d]] - [b, a] \circ [c, d]$ One may see that $f_1 = f_2$ due to (1), namely, $f_2 - f_1$ is zero modulo the Gel'fand-Dorfman relations at (a, c, d) and (c, [a, b], d).

The ambiguity (A2) also gives rise to a convergent critical pair modulo (1), as one may check in a similar way.

In the case (A3), we have three possible rewriting rules:

$$\{c, d'\} \rightarrow \{c', d\} + [c, d]', \quad \text{if } c > d, \\ a\{c, d'\} \rightarrow [c, a \circ d] + [a, c] \circ d, \\ ab' \rightarrow a \circ b.$$

The first one, combined with either of other rules, leads to a convergent critical pair due to (1). Consider the pair coming from the last two rules:

$$ab'\{c,d'\} \to [c,a\circ d] \circ b + ([a,c]\circ d) \circ b,$$
$$ab'\{c,d'\} \to (a\circ b)\{c,d'\} \to [c,(a\circ b)\circ d] - [c,a\circ b]\circ d.$$

The relation obtained

$$[c, a \circ d] \circ b + ([a, c] \circ d) \circ b = [c, (a \circ b) \circ d] - [c, a \circ b] \circ d$$

$$(12)$$

is one of the special identities found in [10].

For (A4), it is enough to consider the following rules:

$$\begin{aligned} a\{c',d'\} &\to \{a,c'\}d' + \{c',a\circ d\}, \\ b\{c',d'\} &\to \{b,c'\}d' + \{c',b\circ d\}, \\ b\{c',d'\} &\to -\{b,d'\}c' - \{d',b\circ c\} \end{aligned}$$

For example, the second rule allows us to rewrite $ab\{c', d'\}$ as

$$\{b, (a \circ d) \circ c\} - \{b, a \circ d\} \circ c + \{a \circ c, b \circ d\} - \{a, b \circ d\} \circ c.$$

Now use (12) to replace the first term with $[b, (a \circ c)] \circ d - ([b, a] \circ c) \circ d + [b, a \circ d] \circ c$. As a result, we obtain

$$ab\{c',d'\} \to [b,a\circ c] \circ d - ([b,a]\circ c) \circ d + [a\circ c,b\circ d] - [a,b\circ d] \circ c \tag{13}$$

The convergence of the corresponding three critical pairs starting at $ab\{c', d'\}$ is equivalent to the conditions that the right-hand side of (13) is symmetric with respect to the permutation (a, b) and skew-symmetric by (c, d).

Either of these two conditions (due to (3) and anti-commutativity of $[\cdot, \cdot]$) gives rise to the following relation:

$$2([a,b] \circ c) \circ d = [b \circ c, a \circ d] - [a \circ c, b \circ d] + ([a,b \circ c] - [b,a \circ c]) \circ d + ([a,b \circ d] - [b,a \circ d]) \circ c.$$
(14)

This is another special identity found in [10].

In the case (A5), we have two types of critical pairs: the first one coming from the rules based on $\{c, bd'' - (b \circ d)' + b'd'\}$ and $\{c, ad'' - (a \circ d)' + a'd'\}$, the second one (for c > d) coming from either of the rules above and $\{c, d''\} + 2\{c', d'\} + \{c'', d\} - [c, d]''$ in (10).

For example,

$$ab\{c, d''\} \to f_1 = a\{c, (b \circ d)'\} - a\{c, b'\}d' - ab'\{c, d'\} - a[c, b]d''.$$

The polynomial f_1 rewrites in $\mathcal{G}_4(B)$ as follows:

$$f_1 \to ([c, a], b, d) - [c, (a, b, d)] + (a, [c, b], d).$$

Here (x, y, z) stands for $(x \circ y) \circ z - x \circ (y \circ z)$. The expression obtained is symmetric with respect to (a, b) by (2) which means the convergence of the first type critical pair.

The second type critical pairs based on (A5) appear when we rewrite

$$ab[c, d''] \to f_2 = ab\{d, c''\} - 2ab\{c', d'\} + ab[c, d]''.$$

We already know how to rewrite all terms in the right-hand side to get an element in $\mathbb{k}B$. Comparing the result with what is obtained from f_1 we obtain a relation which

is a corollary of (1), (12), and (14). (Note that (1) and (12) allow us to rewrite both summands in [c, (a, b, d)] via shorter terms, like those in (14).)

In [10] it was proved that (12) and (14) are independent identities on a GD-algebra. Now we may state the following

Proposition 1. All special identities of GD-algebras of degree ≤ 4 are consequences of (12) and (14).

In a similar way, we may find the complete list of special identities of degree 5. The list of ambiguities in $\mathcal{G}_5(B)$ is relatively long since we have to consider all Poisson differential monomials of degree 5 and weight -1:

$$[a, [b, [c, d']]]e, [a, [b, c'']]de, [a, [b', c']]de, [a', [b, c']]de, [a, [b, c']]de', [a, b'][c, d']e, \\ [a, b^{(3)}]cde, [a', b'']cde, [a, b'']cde', [a', b']cde', [a, b']cde'', [a, b']cd'e'.$$

As a result of the same computations as for n = 4, we obtain three special identities of degree 5:

$$\begin{aligned} [d \circ a, [b, e \circ c]] &= [e \circ a, [b, d \circ c]] + [d, [b, e \circ c] \circ a] - [d, [b, e] \circ a] \circ c + ([d, [b, e]] \circ c) \circ a \\ &- [d, [b, e \circ c]] \circ a - [e \circ a, [b, d] \circ c] - [e, [b, d \circ c]] \circ a + [e, [b, d] \circ c] \circ a + [d \circ a, [b, e] \circ c] \\ &+ [d, [b, e \circ c]] \circ a - [d, [b, e] \circ c] \circ a - [e, [b, d \circ c] \circ a] + [e, [b, d] \circ a] \circ c - ([e, [b, d]] \circ c) \circ a \\ &+ [e, [b, d \circ c]] \circ a, (15) \end{aligned}$$

$$[c, [a, e \circ b] \circ d] = [a, [c, e \circ d] \circ b] - [a, [c, e \circ d]] \circ b - [a, [c, e] \circ b] \circ d + ([a, [c, e]] \circ d) \circ b + [c, [a, e] \circ d] \circ b + [c, [a, e \circ b]] \circ d - ([c, [a, e]] \circ d) \circ b,$$
(16)

and

$$[a, d \circ b] \circ (c \circ e) = [a, c \circ b] \circ (d \circ e) + ([a, d \circ b] \circ c) \circ e + ([a, d] \circ (c \circ e)) \circ b - (([a, d] \circ c) \circ e) \circ b - ([a, c \circ b] \circ d) \circ e - ([a, c] \circ (d \circ e)) \circ b + (([a, c] \circ d) \circ e) \circ b.$$
(17)

Other critical pairs are convergent modulo (12), (14), (15), (16), and (17). We do not state the details here since in the next section we show that in fact all special identities of degree ≤ 5 are corollaries of (12) and (14).

3. On the Gröbner basis of the Gelfand–Dorfman operad with special identities

Let us recall the basic definitions related with operads following [12, Chapter 5]. A (symmetric) operad \mathcal{P} in the category $\operatorname{Vec}_{\Bbbk}$ of linear spaces over a field \Bbbk is a collection of spaces $\mathcal{P}(n), n \geq 1$, equipped with linear composition maps

$$\gamma_{n_1,\dots,n_m}^m:\mathcal{P}(m)\otimes\mathcal{P}(n_1)\otimes\ldots\otimes\mathcal{P}(n_m)\to\mathcal{P}(n_1+\dots+n_m)$$

for all integers $m, n_1, \ldots, n_m \geq 1$, each $\mathcal{P}(n)$ is a module over the symmetric group S_n . These data have to satisfy the following conditions: the composition is associative and equivariant relative to the action of S_n ; the space $\mathcal{P}(1)$ contains an element 1 which acts as an identity relative to the composition.

Every operad may be considered as an image of an appropriate free operad, i.e., a quotient modulo an operad ideal.

Namely, for every graded space $V = \bigoplus_{n \ge 1} V(n)$ there exists a uniquely defined free operad $\mathcal{F}(V)$ generated by V. An operad ideal of $\mathcal{F}(V)$ may be presented as a minimal one that contains a given series of elements from $\bigcup_{n \ge 1} \mathcal{F}(V)(n)$. Therefore,

an operad may be defined by generators and relations.

For example, the operad Lie governing the variety of Lie algebras is generated by $V_1 = V_1(2)$, where dim $V_1(2) = 1$, this is a skew-symmetric S_2 -module. The free operad $\mathcal{F}(V_1)$ is exactly the operad of anti-commutative algebras. The set of defining relations of Lie consists of the Jacobi identity:

$$\gamma_{2,1}^2(\mu,\mu,1) + \gamma_{2,1}^2(\mu,\mu,1)^{(123)} + \gamma_{2,1}^2(\mu,\mu,1)^{(132)}$$

The operad Nov of the variety of Novikov algebras is generated by $V_2 = V_2(2)$, dim $V_2 = 2$, this is the regular S_2 -module. Namely, a basis of $V_2(2)$ consists of ν , $\nu^{(12)}$. The free operad $\mathcal{F}(V_2)$ is exactly the magmatic one. The defining relations of Nov include left symmetry and right commutativity:

$$\gamma_{2,1}^{2}(\nu,\nu,1) - \gamma_{2,1}^{2}(\nu,\nu,1)^{(12)} - \gamma_{1,2}^{2}(\nu,1,\nu) + \gamma_{1,2}^{2}(\nu,1,\nu)^{(12)}, \gamma_{1,2}^{2}(\nu,1,\nu) - \gamma_{1,2}^{2}(\nu,1,\nu)^{(23)}.$$

There is an intermediate notion between nonsymmetric and symmetric operads, known as a *shuffle operad*. By definition, a shuffle operad \mathcal{V} is a collection of spaces $\mathcal{V}(n), n \geq 1$, equipped with a collection of compositions

$$\gamma_{\pi}: \mathcal{V}(m) \otimes \mathcal{V}(n_1) \otimes \ldots \otimes \mathcal{V}(n_m) \to \mathcal{V}(n_1 + \ldots + n_m)$$

where π is a shuffle partition of the set $\{1, \ldots, n\}$, $n = m_1 + \cdots + m_n$, into m disjoint subsets I_j , $j = 1, \ldots, m$, such that $\min I_1 < \min I_2 < \cdots < \min I_m$. Thus, a shuffle operad has no symmetric module structure, but its composition structure still carries some information about the order of arguments.

Shuffle operads provide a convenient framework for the computation of Gröbner bases and normal forms in an operad defined by generators and relations [13]. There is a forgetful functor $\mathcal{P} \mapsto \mathcal{P}^f$ that turns a symmetric operad \mathcal{P} into a shuffle one [11, Section 5.3] such that the normal form of elements in $\mathcal{P}^f(n)$ allows us to recover the normal form in $\mathcal{P}(n)$. In particular, we may find the dimensions of $\mathcal{P}(n)$ in this way.

Following [11], we may convert the symmetric operad GD into a shuffle operad GD^{f} , a homomorphic image of the shuffle tree operad $\mathcal{T}_{\text{III}}(\mathcal{X})$ for an appropriate language \mathcal{X} .

Let us replace the operations ν and μ with the set of three operations $\mathcal{X} = \{x, y, z\}$, where x and y represent ν and $\nu^{(12)}$, z represents μ . We will use the natural notation $x(1 \ 2) = \nu(x_1, x_2), \ y(1 \ 2) = \nu(x_2, x_1)$, etc. To convert the defining relations of the operad GD into elements of $\mathcal{T}_{\text{III}}(\mathcal{X})$ we may use the equivariance property of the composition (see [11] for details). As a result, (2) and (3) turn into the following relations for x and y:

$$\begin{aligned} x(x(1\ 2)\ 3) &- x(1\ x(2\ 3)) - x(y(1\ 2)\ 3) + y(x(1\ 3)\ 2), \\ x(x(1\ 3)\ 2) &- x(1\ y(2\ 3)) - x(y(1\ 3)\ 2) + y(x(1\ 2)\ 3), \\ y(1\ x(2\ 3)) &- y(y(1\ 3)\ 2) - y(1\ y(2\ 3)) + y(y(1\ 2)\ 3), \\ x(x(1\ 2)\ 3) &- x(x(1\ 3)\ 2), \\ x(y(1\ 2)\ 3) &- y(1\ x(2\ 3)), \\ x(y(1\ 3)\ 2) &- y(1\ y(2\ 3)). \end{aligned}$$

The Jacobi identity may be expressed in terms of z as

$$z(z(1\ 2)\ 3) - z(1\ z(2\ 3)) - z(z(1\ 3)\ 2).$$

Similarly, converting (1) (namely, all relations in the symmetric group orbit of the defining relation), we obtain

$$z(1 x(2 3)) + z(y(1 2) 3) - x(z(1 2) 3) - y(1 z(2 3)) - y(z(1 3) 2),$$

-z(x(1 3) 2) + z(x(1 2) 3) + x(z(1 2) 3) - x(z(1 3) 2) - x(1 z(2 3)),
-y(z(1 2) 3) + z(1 y(2 3)) + z(y(1 3) 2) - x(z(1 3) 2) + y(1 z(2 3)).

The elements (*shuffle tree polynomials* [11]) obtained generate the ideal of defining relations for the operad GD^f , a quotient of $\mathcal{T}_{\text{III}}(\mathcal{X})$.

Calculating the Gröbner base of GD^f by means of the package [14], we get the following result for dim $\text{GD}(n) = \dim \text{GD}^f(n)$.

The first five terms of the sequence coincide with the number of certain planar graphs (see OEIS A322137, A291842). However, the sixth one is different. Finding a linear basis of the free GD-algebra, or at least the sequence GD(n) is an interesting open problem. Note that the operad GD is not Koszul since so is Nov [16].

By Corollary 1, the class SGD is defined by identities. There should exist identities separating SGD from GD, i.e., independent special identities of GD-algebras. Let us consider the class of GD-algebras with additional identities (12) and (14). Denote this class by wSGD (weak special Gelfand–Dorfman algebra).

As above, we may convert the defining relations of wSGD into shuffle tree polynomials by adding the orbits of relations (12) and (14) to GD^{f} . Namely, (12) and

(14) give rise to the following elements of $\mathcal{T}_{\text{III}}(\mathcal{X})$:

$$\begin{split} & z(1\;x(x(2\;3)\;4)) - x(z(1\;x(2\;3))\;4) - x(z(1\;x(2\;4))\;3) + x(x(z(1\;2)\;3)\;4), \\ & z(1\;x(y(2\;3)\;4)) - x(z(1\;y(2\;3))\;4) - x(z(1\;x(3\;4))\;2) + x(x(z(1\;3)\;2)\;4), \\ & z(1\;y(2\;y(3\;4))) - x(z(1\;y(3\;4))\;2) - x(z(1\;y(2\;4))\;3) + x(x(z(1\;4)\;2)\;3), \\ & -z(x(x(1\;3)\;4)\;2) + x(z(x(1\;3)\;2)\;4) + x(z(x(1\;4)\;2)\;3) - x(x(z(1\;2)\;3)\;4), \\ & -z(x(y(1\;3)\;4)\;2) + x(z(y(1\;3)\;2)\;4) - y(1\;z(2\;x(3\;4)))) + x(y(1\;z(2\;3))\;4), \\ & -z(x(y(1\;4)\;3)\;2) + x(z(y(1\;4)\;2)\;3) - y(1\;z(2\;y(3\;4)))) + x(y(1\;z(2\;4))\;3), \\ & -z(x(x(1\;2)\;4)\;3) + x(z(x(1\;2)\;3)\;4) + x(z(x(1\;4)\;3)\;2) - x(x(z(1\;3)\;2)\;4), \\ & -z(x(y(1\;2)\;4)\;3) + x(z(y(1\;2)\;3)\;4) + y(1\;z(x(2\;4)\;3)) - y(1\;x(z(2\;3)\;4)), \\ & -z(x(y(1\;4)\;2)\;3) + x(z(y(1\;4)\;3)\;2) + y(1\;z(y(2\;4)\;3)) + y(1\;y(2\;z(3\;4))), \\ & -z(x(y(1\;2)\;3)\;4) + x(z(x(1\;2)\;4)\;3) + x(z(x(1\;3)\;4)\;2) - x(x(z(1\;4)\;2)\;3), \\ & -z(x(y(1\;2)\;3)\;4) + x(z(y(1\;2)\;4)\;3) + y(1\;z(x(2\;3)\;4)) - x(y(1\;z(2\;4))\;3), \\ & -z(x(y(1\;2)\;3)\;4) + x(z(y(1\;2)\;4)\;3) + y(1\;z(x(2\;3)\;4)) - x(y(1\;z(2\;4))\;3), \\ & -z(x(y(1\;3)\;2)\;4) + x(z(y(1\;3)\;4)\;2) + y(1\;z(y(2\;3)\;4)) - x(y(1\;z(2\;4))\;3), \\ & -z(x(y(1\;3)\;2)\;4) + x(z(y(1\;3)\;4)\;2) + y(1\;z(y(2\;3)\;4)) - x(y(1\;z(2\;4))\;3), \\ & -z(x(y(1\;3)\;2)\;4) + x(z(y(1\;3)\;4)\;2) + y(1\;z(y(2\;3)\;4)) - x(y(1\;z(3\;4))\;2) \end{split}$$

and

$$z(x(1\ 2)\ x(3\ 4)) - x(z(x(1\ 2)\ 3)\ 4) - x(z(1\ x(3\ 4))\ 2) + 2x(x(z(1\ 3)\ 2)\ 4) + z(x(1\ 4)\ y(2\ 3)) - x(z(1\ y(2\ 3))\ 4) - x(z(x(1\ 4)\ 3)\ 2),$$

$$\begin{split} z(x(1\ 3)\ x(2\ 4)) &- x(z(1\ x(2\ 4))\ 3) - x(z(x(1\ 3)\ 2)\ 4) + 2x(x(z(1\ 2)\ 3)\ 4) \\ &+ z(x(1\ 4)\ x(2\ 3)) - x(z(1\ x(2\ 3))\ 4) - x(z(x(1\ 4)\ 2)\ 3), \end{split}$$

 $\begin{aligned} z(y(1\ 2)\ y(3\ 4)) - y(1\ z(2\ y(3\ 4))) - x(z(y(1\ 2)\ 4)\ 3) + 2y(1\ x(z(2\ 4)\ 3)) \\ &- z(y(1\ 4)\ x(2\ 3)) + x(z(y(1\ 4)\ 2)\ 3) - y(1\ z(x(2\ 3)\ 4)), \end{aligned}$

$$\begin{split} z(y(1\ 2)\ x(3\ 4)) - y(1\ z(2\ x(3\ 4))) - x(z(y(1\ 2)\ 3)\ 4) + 2y(1\ x(z(2\ 3)\ 4)) \\ &- z(y(1\ 3)\ x(2\ 4)) + x(z(y(1\ 3)\ 2)\ 4) - y(1\ z(x(2\ 4)\ 3)), \end{split}$$

$$\begin{aligned} z(y(1\ 3)\ y(2\ 4)) + y(1\ z(y(2\ 4)\ 3)) - x(z(y(1\ 3)\ 4)\ 2) + 2y(1\ y(2\ z(3\ 4)))) \\ &- z(y(1\ 4)\ y(2\ 3)) - y(1\ z(y(2\ 3)\ 4)) + x(z(y(1\ 4)\ 3)\ 2), \end{aligned}$$

$$z(x(1\ 2)\ y(3\ 4)) - x(z(1\ y(3\ 4))\ 2) - x(z(x(1\ 2)\ 4)\ 3) + 2x(x(z(1\ 4)\ 2)\ 3) + z(x(1\ 3)\ y(2\ 4)) - x(z(x(1\ 3)\ 4)\ 2) - x(z(1\ y(2\ 4))\ 3).$$

Calculating the Gröbner base of the operad wSGD^f by means of the package [14], we get the following result:

The dimensions of SGD(n) were computed in [10, Section 4]. For $n \leq 5$, we have $\dim SGD(n) = \dim wSGD(n)$. Hence, all special identities of degree ≤ 5 are corollaries of (12) and (14). We obtain the following result:

Corollary 3. The relations (15), (16) and (17) are consequences of (12) and (14).

It remains an open problem whether (12) and (14) exhaust all independent special identities of GD-algebras.

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