Formal Explanations as Logical Derivations

Francesco A Genco

Formal Explanations as Logical Derivations

Francesco A. Genco

IHPST – UMR 8590
Université Paris 1 Panthéon-Sorbonne and CNRS
13, Rue du Four, 75006 Paris, France
frgenco@gmail.com

ARTICLE HISTORY
Compiled February 17, 2022

ABSTRACT
According to a longstanding philosophical tradition dating back to Aristotle's *Posterior Analytics* and carried on in Bolzano's *Theory of Science*, certain proofs do not only certify the truth of their conclusion but also explain the reasons why their conclusion holds. In contemporary philosophy, the explanatory relation of *grounding* has taken the stage and much effort is being devoted to logically characterise it, especially by proof-theoretical means. Nevertheless, no thorough investigation of the resulting notion of formal explanation exists. In the present work, we show that these formal explanations can indeed be seen as logical derivations of a particular kind and we conduct a formal study of the interactions between grounding rules and logical rules, formal explanations and logical derivations. In order to do so, we define a minimal calculus that captures both grounding and logical derivability, we show by a normalisation procedure that grounding rules are proof-theoretically balanced with respect to logical elimination rules, and that the obtained normal proofs are analytic. The introduced calculus enables us, moreover, to combine logical derivations and formal explanations, to distinguish the explanatory parts of derivations from their non-explanatory parts, and to compose formal explanations in order to construct chains of consecutive grounding steps, thus formalising also a notion of mediate grounding.

KEYWORDS
Grounding; formal explanation; classical logic; proof theory; normalisation

1. Introduction

The act of proving a sentence is usually associated with the question whether the sentence is true or false. The existence of a proof is supposed to guarantee that the sentence is true. Nevertheless, some proofs are more informative than others. It happens sometimes that a proof stands out among the other proofs of a sentence because it does not only certify the truth of the sentence, but it also displays in the clearest way the reasons of its truth. In other words, such a proof *explains why* the sentence is true. The idea that certain proofs can be considered as rigorous explanations of the reason why a truth holds goes far back in the history of philosophy. A longstanding tradition of investigations on this notion of explanation has its origins in Aristotle's *Posterior Analytics* (Barnes, 1984, Post. An. I, 2–8) and has been carried on, for

Funded by the IBS project (ANR-18-CE27-0012-01).

instance, by Bolzano (2014) in his *Theory of Science*. According to this tradition, rigorous explanations are proofs of a particular kind and thus two kinds of proof can be distinguished: proofs that just guarantee that a statement is true—usually called *proofs-that*—and proofs that also provide the reason why the statement is true—usually called *proofs-why*.

In contemporary philosophy, an explanatory relation which can be traced back to Bolzano's notion of *Abfolge* (Bolzano, 2014, §162, §168, §198–221) has taken the stage and is receiving considerable attention in several fields of philosophy: the grounding relation. Grounding is usually introduced as an objective and explanatory relation that connects two relata—the ground and the consequence—if the first determines or explains the second.[1] In other terms, we can say that the consequence holds in virtue of the ground. Logical systems of various kinds have been employed for the characterisation and study of the general features of grounding, see for instance the works by Fine (2012a, 2012b) and Korbmacher (2018a, 2018b). Moreover, much work is being devoted to characterise the relation that holds between a logically complex formula F and the formulae in virtue of which F holds.[2] This relation is usually called *logical grounding*. In order to characterise and study the relation of logical grounding, several proof systems have been developed, as for instance in the works by Schnieder (2011), Correia (2014) and Poggiolesi (2018). While the methods of proof-theory are not yet considered standard tools for the investigation of grounding, promising endeavours of proof-theoretical analysis of grounding made their appearance—as the works by Rumberg (2013), Poggiolesi (2018), and Prawitz (2019)[3] witness—and constitute very interesting instruments for the philosopher willing to formally frame the notion of grounding and precisely formulate related problems.

While much has been done in order to define satisfactory logical grounding rules, no thorough study of the resulting notion of formal explanation and of the relationship between grounding rules and logical rules exists. Some even contest the legitimacy of considering logical grounding as different from logical consequence, see for instance the arguments presented by McSweeney (2020). We will show in this paper that there is a sensible notion of grounding based on Bolzano's *Abfolge*[4] which can be clearly and formally distinguished from logical entailment, but still induces a notion of formal explanation which determines a meaningful subclass of the class of logical derivations. We will show that this grounding relation can indeed be considered a derivability relation of a particular kind—as Bolzano argued—and we will present a thorough study of this relation by proof-theoretical means, focusing in particular on the usage of grounding rules as introduction rules for the classical connectives and on their interaction with logical elimination rules.

In particular, we will consider the notion of *complete logical grounding* introduced by Poggiolesi (2016) and based on Bolzano's analysis of *Abfolge*, and we will study the resulting notion of formal explanation, that is, of derivation constructed by only employing the grounding rules that characterise this notion of logical grounding. Our

[1] See Betti (2010); Correia (2014); Correia and Schnieder (2012); Fine (2012a); Korbmacher (2018b); McSweeney (2020); Poggiolesi (2016); Rosen (2010); Rumberg (2013); Schnieder (2011); Sider (2018).

[2] See Correia (2014); Correia and Schnieder (2012); Fine (2012a); Poggiolesi (2016, 2018); Schnieder (2011).

[3] Notice that Prawitz also developed a different notion of grounding, see for instance Prawitz (2015), in order to explain why certain inferences enable us obtain grounds for accepting their conclusion from the grounds that we might have for accepting their premisses. Even though this notion could entertain interesting connections with that of logical grounding, it certainly cannot be identified with it; and the investigation of the relationship between the two lies outside the scope of this work.

[4] This kind of grounding is sometimes referred to as *complete grounding*, for instance by Poggiolesi (2016), as opposed to *full grounding* as defined by Fine (2012a).

notion of formal explanation will hence comply with the requirements by Bolzano on *Abfolge* which have been formalised by Poggiolesi (2016). This notion of formal explanation will coincide with a particular kind of logical derivation in which the truth of the premisses of each rule application occurring in it—that is, the application of a logical grounding rule—determines the truth of each part of the conclusion. The premisses of each step of a formal explanation, therefore, explain its conclusion in the sense that they provide a complete account of the truth of the latter. The rules that enable us to construct formal explanations of this kind are very particular rules: they look rather different from traditional logical rules, and, in most cases, require more information to be applied than the corresponding logical introduction rules. Nevertheless, a closer inspection reveals that the role that these logical grounding rules play is not so different from the one played by the traditional logical introduction rules. We endeavour, hence, in the task of formally showing that these rules for complete logical grounding can indeed be used instead of logical introduction rules; which means that, in order to define a complete reasoning system for classical logic in which we can also construct formal explanations, there is no need to add grounding rules to a complete calculus for classical logic, because grounding rules already suitably play the role of introduction rules. This result, in turn, corroborates the idea that a formal explanation is a logical derivation of a particular kind, and does it in a stronger sense than usual. Indeed by most logical grounding calculi we can show that, even though one needs to use special rules to construct formal explanations, the resulting explanations are also sound logical derivations. By the calculus that we will present, on the other hand, we can also show that formal explanations are just some of the logical derivations that we anyway need to be able to construct in order to show that the calculus is complete with respect to classical logic. In other terms, formal explanations are simply constructed in our calculus by some of the rules that we anyway need to use for regular logical reasoning. In set-theoretical terms this means that we do not need to consider a complete set of logical derivations **L**—those constructed by a traditional complete set of logical rules—and add to it an additional set **E** of formal explanations which are also sound logical derivations, as is usually done; but we can directly construct a complete set of logical derivations **D** *inside* which we can find a subset **S** that contains all our formal explanations. And the formal explanations in **S** are simply constructed by some of the rules that we anyway need to use for constructing **D**. Determining the set of formal explanations in this way—that is, as a non-redundant subset **S** of **D**—confirms that formal explanations are a particular kind of logical derivations in a strong sense: formal explanations are some of the logical derivations that we anyway need if we want a complete set of logical derivations. The fact that our complete set of logical derivations **D** is not *ad hoc* and unnatural is confirmed by the normalisation and analyticity results, which guarantee us that the calculus employed to construct it is formed by a proof-theoretically balanced pair of sets of rules, one of introduction rules—with a non-dispensable subset of grounding rules—and one of elimination rules.

In order to formally develop this analysis, we will present the calculus Gr, which is at the same time a grounding calculus and a complete calculus for classical logic. The calculus Gr is based on a set of logical rules to eliminate connectives and a set of grounding rules to introduce them. The first difference between grounding rules and logical rules appears already: while logical rules can both introduce and eliminate connectives, grounding rules can only be introductory since the complexity of a ground must be smaller than the complexity of its consequence. This calculus will enable us to study the direct interplay between grounding rules and logical elimination rules. We will, first, study the calculus from a ground-theoretical perspective and, in particular, prove that it is sound and complete with respect to the formalisation by Poggiolesi (2016) of Bolzano's notion of grounding. Secondly, we will study it from a logical perspective, and prove that it is sound and complete with respect to classical logic. Thirdly, we will conduct a proof-theoretical analysis by defining a normalisation procedure, by proving that the procedure is terminating, and by showing that it yields normal derivations which are analytic. As we will argue in Section 4, the normalisation result guarantees that grounding rules can be suitably seen as introduction rules, in the sense that the information required to apply grounding rules in order to introduce a connective does not exceed the information obtained by eliminating the connective. The analyticity result guarantees, in turn, that the normalisation procedure is satisfactorily defined. An exemplification of the expressive possibilities offered by Gr is also presented. We will show, in particular, that by Gr we can construct grounding derivations, logical derivations and derivations combining logical and grounding steps. For any derivation constructed in the calculus, we can immediately determine which of its parts are explanatory and which are purely logical. The calculus Gr also constitutes an improvement with respect to the calculus presented by Poggiolesi (2018) in that it enables us to construct chains of formal explanations and thus to formalise the notion of mediate grounding.

The article is structured as follows. In Section 2 we present some basic definitions, introduce the grounding calculus Gr and discuss the function and meaning of its rules. In Section 3.1 we prove that the notion of grounding induced by Gr complies with the formalisation of Bolzano's *Abfolge* proposed by Poggiolesi (2016). In Section 3.2 we prove that the calculus Gr is sound and complete with respect to classical logic. In Section 4 we prove a normalisation result for Gr and in Section 4.1 we show that normal form Gr derivations satisfy the subformula property and thus are analytic. In Section 4.2 we exemplify and discuss the fact that Gr enables us to construct, combine and distinguish formal explanations and purely logical derivations, and that a Gr derivation can contain chains of explanations and thus formalise mediate grounding as well. Finally, in Section 5 we conclude the article by some general considerations and by pointing at some unsolved problems that may prove of interest for future work.

2. A calculus for grounding: Gr

In this section, we define the calculus Gr which formalises the notion of complete logical grounding introduced by Poggiolesi (2016).

In order to provide the reader with some informal intuitions that should be useful to have a better grasp on the technical details that will follow, we first show what kind of grounding statements Gr is supposed to let us derive and how.

A complete logical grounding statement, see Definition 3.5, will be expressed in our logical language by formulae of the form $(A \triangleright B)$, $(C, D \triangleright F)$ or $(G \mid H \triangleright L)$. The first two formulae respectively express that A is the complete logical ground of B and that C and D form the complete logical ground of F. The third formula expresses that G is the complete logical ground of L under the condition H. From a proof-theoretical perspective, formulae of these forms respectively correspond to grounding rule applications of the following forms:[5]

$$\frac{A}{B} \qquad \frac{C \quad D}{F} \qquad \frac{G \quad | \, H}{L}$$

The rules for introducing the grounding operator $\triangleright$, presented in Table 4, will enable us to derive the corresponding grounding statement from the conclusion of the relative grounding rule application.

Notice that the vertical bar | appearing in the third rule schema is not part of the logical language and does not play any active role in the construction of derivations: all premisses of any grounding rule must be derived before the rule can be applied. This occurrence of the symbol | exclusively indicates that the premiss to the right of | constitutes the condition required to be able to consider the premiss to the left of | as the complete logical ground of the conclusion. This information is only used when introducing the grounding operator $\triangleright$ to conclude the correct grounding statement from the rule application.

Now that a few basic intuitions about the interpretation of grounding statements and about the use of grounding rules have been presented, let us briefly discuss the main innovations and differences with respect to the calculus presented by Poggiolesi (2018). Afterwards, we will be able to introduce in a straightforward manner the machinery required to make these intuitions precise.

2.1. Innovations and differences

Gr presents several notational innovations with respect to the grounding calculus presented in Poggiolesi (2018). These will be essential to prove the normalisation result. The main differences concern the handling of ground-theoretically equivalent formulae, the notation for converse formulae and the definition of grounding rules for negated formulae. We will now discuss these three issues, briefly presenting the way they are dealt with by Poggiolesi (2018) and the innovations concerning them that will be adopted in Gr in order to define a calculus that admits a relatively simple normalisation procedure.

Let us begin with ground-theoretic equivalences. While logical grounding does not comply with logical equivalence—which means that there are formulae which are logically equivalent but cannot be interchangeably used in a grounding statement—there are stricter forms of equivalence with which grounding complies. These are often called *ground-theoretic equivalences*, see e.g. Correia (2010) and Poggiolesi (2016, 2018). Ground-theoretically equivalent formulae might be different from a purely syntactical point of view but should be regarded as identical with respect to the grounding relation. For instance, if $A \wedge B$ is the ground of C, so is $B \wedge A$; but this does not mean that C has two grounds, it just means that $A \wedge B$ and $B \wedge A$ represents the same ground. The technical notion that we employ here to handle ground-theoretic equivalence has been introduced by Poggiolesi (2016, 2018) and is that of a-c equivalence, see Definition 3.2. While in the calculus presented by Poggiolesi (2018) the notion of a-c equivalence appears in the side conditions on grounding rules, though, the calculus Gr includes rules to explicitly transform a formula into any a-c equivalent formula inside a derivation: the $\alpha\kappa$ rules, see Section 2.3. By using $\alpha\kappa$ rules in combination with the $\triangleright$ introduction rule—the first rule presented in Table 4—we introduce the possibility of

[5]The grounding rules for negation might have more complex structures, and they will be discussed in due time.

interchangeably using a-c equivalent grounds in a grounding statement. In particular, if we show that A is a-c equivalent to A' and there is a grounding rule with premiss A' and conclusion B, we can infer that A is the ground of B, and not only that A' is. This does not mean that we can derive more than one ground for each truth, since a-c equivalent formulae are supposed to be different ways to refer to the same truth. As a consequence, one grounding step according to the calculus presented by Poggiolesi (2018) corresponds in Gr to one grounding rule application immediately preceded by some $\alpha\kappa$ rule applications, as specified in Table 4 for the $\triangleright$ introduction rule.

The second main innovation of Gr concerns the handling of converse formulae. Let us first intuitively introduce the role of converse formulae with respect to the notion of grounding that we adopt and then discuss how they are handled in Poggiolesi (2018) and here. Sometimes, the logical ground of a formula F must contain a formula equivalent to the negation of a subformula of F. For instance, the logical ground of $p \to q$ might, in certain cases, also contain the formula $\neg p$. Nevertheless, as formalised by Definition 3.5, according to the notion of logical grounding that we adopt here, the logical ground of a formula F must be simpler than F itself. Technically, we measure the relative complexity of two formulae by employing g-complexity, see Definition 3.4, and according to g-complexity the negation $\neg S$ of a subformula S of F might not be simpler than F. Hence, in order to construct the ground of F, we never directly employ the negation $\neg S$ of S, but we employ its converse S^*—where the converse S^* of S is defined as the formula equivalent to $\neg S$ which is as g-complex as S itself, see Definition 3.1. Intuitively, S^* is obtained by adding one negation to S if this does not increase its g-complexity, or by removing one negation from S otherwise. While converses in Poggiolesi (2018) are referred to by the meta-notation $(\ )^*$ that directly indicates the converse formula of its argument, in the present work we employ the structural connective $(\ )^\perp$ for which explicit rules are defined. Notice that $F^\perp$—as opposed to F^*—is not a formula in the language of the logic, but only an expression in the language of the calculus Gr. The essential proof-theoretical difference is that, while the application of certain rules of the calculus presented by Poggiolesi (2018) requires us to check whether a formula is the converse of another one, Gr internalises this check by rules that enable us to introduce the superscript $(\ )^\perp$ in compliance with the definition of converse.

The third main innovation of Gr concerns the rules for grounding negations. While the Gr rules for introducing single negations follow the structure of the relative rule presented by Poggiolesi (2018), these rules do not require any side condition since all restrictions are handled in a local and purely syntactic fashion by exploiting the constrained interplay of the rules presented in the last two lines of Table 2. The technical details concerning the use of these rules will be discussed in Section 2.4.

2.2. The logical language and the language of the calculus **Gr**

In this section, we formally define the logical language that we will use for Gr and fix some notational conventions.

Definition 2.1 (Language $\mathcal{L}$)**.** The language $\mathcal{L}$ is defined by the following grammar:

$$\begin{aligned} \varphi &::= \xi \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid (\varphi \triangleright \varphi) \mid (\varphi, \varphi \triangleright \varphi) \mid (\varphi \mid \varphi \triangleright \varphi) \\ \xi &::= p \mid q \mid r \mid \ldots \end{aligned}$$

where $p, q, r, \ldots$ are all propositional variables of the language.

Unless stated otherwise, when we talk about formulae, we assume that we are talking about formulae of $\mathcal{L}$. The logical constants $\bot, \neg, \wedge, \vee, \rightarrow$ are the standard ones for falsity, negation, conjunction, disjunction and implication, and require no explanation. In addition to these constants, we introduce the logical constant $\triangleright$ in order to formulate grounding statements. By a formula of the form $A \triangleright B$ we express the fact that A is the ground of B, by a formula of the form $A, B \triangleright C$ the fact that A and B together form the ground of C, and by a formula of the form $A \mid B \triangleright C$ the fact that A is the ground of C under the condition that B is true.

Notation. We denote by $\neg^n$ a sequence $\neg \ldots \neg$ containing n consecutive occurrences of the symbol $\neg$. For instance $\neg^3 p$ will denote the formula $\neg\neg\neg p$.

In order to obtain an elegant normalisation proof, we avoid the usage in Gr of rules with side conditions. To do so, we introduce some notational devices. These devices will be essential, for instance, in the definition of the grounding rules for negations and converse formulae.

Definition 2.2 (Expressions)**.** For any language $\mathcal{L}_x$, expressions in the language $\mathcal{L}_x$ are defined by the following grammar:

$$\begin{aligned} \eta &::= \mu \mid \{\mu\} \mid [\mu] \mid \{\{\mu\}\} \mid [[\bot]] \\ \mu &::= \nu \mid \nu^{\bot} \\ \nu &::= A \mid B \mid C \mid \ldots \end{aligned}$$

where the terminal elements are $[[\bot]]$ and all formulae $A, B, C \ldots$ of the language $\mathcal{L}_x$.

Curly and square brackets will be used in the calculus to restrict the form of certain parts of derivations. In particular, they will be used to guarantee that only certain elimination rules are used for deriving the minor premisses of an application of the grounding rules for negation, see Table 2. The superscript $(\ )^{\bot}$, on the other hand, will be used to represent the converse of a formula, see Definitions 2.7 and 3.1. Just like curly and square brackets, the superscript $(\ )^{\bot}$ is a structural connective of the calculus: it is not in the language of the logic, but only appears inside derivations.

We conclude the preliminary definitions by introducing two simple adaptations of the usual notation for contexts. This notation is used to be able to modify a part of an expression without bothering about the rest of the expression. For instance, the formula $(p \wedge (q \vee r))^{\bot}$ can be represented as $\mathcal{C}[q \vee r]$ where the context $\mathcal{C}[\ ]$ indicates the part of the formula that we do not intend to modify for the moment: $(p \wedge (\ ))^{\bot}$. So, if a rule enables us to transform an expression of the form $\mathcal{C}[A \vee B]$ into one of the form $\mathcal{C}[B \vee A]$, then, by applying it to $\mathcal{C}[q \vee r]$ we can obtain $\mathcal{C}[r \vee q]$, which means—since $\mathcal{C}[\ ] = (p \wedge (\ ))^{\bot}$—that if we apply it to $(p \wedge (q \vee r))^{\bot}$ we obtain $(p \wedge (r \vee q))^{\bot}$.

Definition 2.3 (Expression Context)**.** An expression context $\mathcal{C}[x]$ is an expression containing a distinguished propositional atom x. For any formula A, by the notation $\mathcal{C}[A]$, we denote the expression obtained by replacing x with A in $\mathcal{C}[x]$.

We also define a context notation for contexts composed of parentheses only. By $\mathcal{P}[E]$, we indicate an expression of one of the following forms: $E, [E], \{E\}$ or $\{\{E\}\}$.

Definition 2.4 (Parenthesis Context)**.** A parenthesis context $\mathcal{P}[x]$ is an expression of the form x, $[x]$, $\{x\}$ or $\{\{x\}\}$ where x is a distinguished propositional atom. For any formula A, by the notation $\mathcal{P}[A]$, respectively $\mathcal{P}[A^{\bot}]$, we denote the expression obtained by replacing x with A, respectively $A^{\perp}$, in $\mathcal{P}[x]$.

We present now the rules of the calculus Gr in separate groups so that we can orderly explain their features and functions.

2.3. Rules for a-c equivalent and converse formulae

As argued for instance by Correia (2010) and Poggiolesi (2016, 2018), grounding is not supposed to distinguish between two formulae if, by applying commutativity or associativity of $\wedge$ or $\vee$ to some subformula of one of the two formulae, we can obtain the other. As already mentioned in Section 2.1, in Gr we implement this by the $\alpha\kappa$ rules presented in Table 1. The $\alpha\kappa$ rules enable us to explicitly transform a formula into any ground-theoretically equivalent formula. By using $\alpha\kappa$ rules in combination with the $\triangleright$ introduction rule—the first rule presented in Table 4—we can interchangeably use ground-theoretically equivalent formulae in a grounding statement. In particular, if we show that A is ground-theoretically equivalent to A' and there is a grounding rule with premiss A' and conclusion B, we cannot only infer that A' is the ground of B, but also that A is.

In Gr also converse formulae are handled explicitly. In particular, the expression $A^{\perp}$ represents the converse of A in the language of the calculus. Notice that $A^{\perp}$ is not a formula of the logical language and that $(\ )^{\perp}$ is a structural element of Gr which is only used inside derivations and for the schematic representation of rules. Technically, we can introduce $(\ )^{\perp}$ by the relative rules in Table 1 or by assuming a hypothesis of the form $F^{\perp}$, as detailed in Definition 2.6. The elimination rules for $(\ )^{\perp}$, on the other hand, are in Table 2 and Table 3. These rules have the same form as negation elimination rules since a pair of converse formulae can always be written as F and $\neg F$ for some formula F, and hence they always entail $\perp$.

$$\frac{\mathcal{C}[A \wedge B]}{\mathcal{C}[B \wedge A]}\ \alpha\kappa \qquad \frac{\mathcal{C}[A \wedge (B \wedge C)]}{\mathcal{C}[(A \wedge B) \wedge C]}\ \alpha\kappa \qquad \frac{\mathcal{C}[A \vee B]}{\mathcal{C}[B \vee A]}\ \alpha\kappa \qquad \frac{\mathcal{C}[A \vee (B \vee C)]}{\mathcal{C}[(A \vee B) \vee C]}\ \alpha\kappa$$

$$\frac{\mathcal{P}[\neg\neg^{2n} A]}{\mathcal{P}[(\neg^{2n} A)^{\perp}]} \qquad \frac{\mathcal{P}[\neg^{2n} A]}{\mathcal{P}[(\neg\neg^{2n} A)^{\perp}]}$$

where $\mathcal{C}$ is an expression context and $\mathcal{P}$ is a parethesis context

Table 1. Equivalence and Converse Rules

Notice that $\alpha\kappa$ rules can be applied at any depth inside any expression, while converse rules can only be applied inside curly or square brackets. Introducing the superscript $(\ )^{\perp}$ inside a formula—as in $A^{\perp} \wedge B$—would indeed generate an ill-formed sequence of symbols, which is neither a formula nor an expression of the calculus.

It is easy to show that that the full associativity of $\wedge$ and $\vee$ is captured by the $\alpha\kappa$ rules.

Proposition 2.5. *The rules* $\dfrac{\mathcal{C}[(A \star B) \star C]}{\mathcal{C}[A \star (B \star C)]}\ \alpha\kappa$ *for* $\star \in \{\wedge, \vee\}$ *are derivable by using applications of rules in Table 1 for the same context* $\mathcal{C}[x]$*.*

2.4. Grounding rules

We present the grounding rules of the calculus. We distinguish the application of these rules from the applications of other rules by a double inference line. This distinction will be important later when we will discuss the possibility of separating explanatory and non-explanatory parts of a derivation.

As discussed at the beginning of Section 2 and as we will detail in Section 2.6, the grounding rules presented in the first two lines of Table 2 directly correspond to grounding statements of the form $(A \rhd B)$, $(C, D \rhd F)$ or $(G \mid H \rhd L)$—depending on the number of premisses of the rule and on the presence, or absence, of the symbol $\mid$ between them—and will be used in combination with the introduction rules for the grounding operator $\rhd$ to derive the corresponding grounding statements. The symbol $\mid$ occurring between the premisses of rules of the form $\dfrac{G \quad \mid H}{L}$ is not part of the logical language but only a device of the calculus. This symbol is used to distinguish between the premiss of a grounding rule that constitutes the ground of L from the premiss that only acts as condition to the ground. More precisely, the premiss G is the ground of L because it occurs to the left of $\mid$, while the premiss H acts as condition to the ground G of L because it occurs to the right of $\mid$. This distinction is also employed to correctly apply the introduction rules for the grounding operator $\rhd$, as shown in Table 4. A rule application of the form displayed above, for instance, would enable us to introduce $\rhd$ in a grounding statement of the form $(G \mid H \rhd L)$.

While the rule for grounding double negations is rather simple, the rules for grounding single negations—displayed in the third line of Table 2—are rather complex and require some explanations. These rules are based on the idea that two formulae A and B form the complete logical ground of a formula $\neg C$ if, and only if, A and B directly yield a contradiction in combination with the immediate subformulae of C. In order to guarantee that the premisses $[[\bot]]$ of a grounding rule for negations are actually obtained through direct contradictions involving A, B and the immediate subformulae of C, we employ square, curly and double curly brackets to limit the rules that we can use in the derivations of $[[\bot]]$.[6] For instance, if we have an expression $\{F\}$ where F is a formula, we can only eliminate its outermost connective by one of the rules in Table 3 that has a premiss between curly brackets. The result of this elimination will yield a formula between double curly brackets, and, again, only certain rules can be applied to such a formula. These constraints enable us to be certain of the fact that if we obtained $[[\bot]]$ from the hypotheses between curly and square brackets generated by a grounding rule for negation, $[[\bot]]$ was obtained through direct contradictions involving $[A], [B]$ and the immediate subformulae of $\{C\}$.

For instance, $\neg p$ and $\neg q$ form the complete logical ground of $\neg(p \wedge q)$ because $\neg p$ directly contradicts p and $\neg q$ directly contradicts q. The series of rule applications that correspond to this argument is the following:

$$\dfrac{\neg p \quad \neg q \quad \dfrac{\dfrac{[\neg p]^1}{[p^\bot]}\alpha\kappa \quad \dfrac{\{p \wedge q\}^1}{\{\{p\}\}}}{[[\bot]]} \quad \dfrac{\dfrac{[\neg q]^1}{[q^\bot]}\alpha\kappa \quad \dfrac{\{p \wedge q\}^1}{\{\{q\}\}}}{[[\bot]]}}{\neg(p \wedge q)}\,1$$

where the conjunction elimination and converse elimination steps are forced. Indeed, if

[6] Notice that square brackets are not used here to discharge hypotheses, superscripts are used for that.

we have a conjunction $\{\{p \wedge q\}\}$ inside square brackets, the only rule that we can apply to it—apart from $\alpha\kappa$ and converse rules—is the one that eliminates the conjunction. Similarly, the only rule that we can apply to the expression $[q^\perp]$—apart from $\alpha\kappa$ and converse rules—is the one that concludes $[[\perp]]$ from the premisses $[q^\perp]$ and $\{\{q\}\}$.

The introduction rules for single negations are three because, depending on the outermost connective of C, we might need to construct derivations of $[[\perp]]$ with different structures. For instance, if $A = \neg p$, $B = \neg q$ and $C = p \vee q$ we have

$$\dfrac{\neg p \quad \neg q \quad \dfrac{\{p \vee q\}^1 \quad \dfrac{\dfrac{[\neg p]^1}{[p^\perp]}\,\alpha\kappa \quad \{\{p\}\}^2}{[[\perp]]} \quad \dfrac{\dfrac{[\neg q]^1}{[q^\perp]}\,\alpha\kappa \quad \{\{q\}\}^2}{[[\perp]]}}{[[\perp]]}\,2}{\neg(p \vee q)}\,1$$

And if $A = \neg p$, $B = q$ and $C = p \wedge q$ we have

$$\dfrac{\neg p \quad \dfrac{\dfrac{[\neg p]^1}{[p^\perp]}\,\alpha\kappa \quad \dfrac{\{p \wedge q\}^1}{\{\{p\}\}}}{[[\perp]]} \quad \dfrac{[q^\perp]^1 \quad \dfrac{\{p \wedge q\}^1}{\{\{q\}\}}}{[[\perp]]} \quad \mid q}{\neg(p \wedge q)}\,1$$

$$\dfrac{A \quad B}{A \wedge B} \qquad \dfrac{A \quad B}{A \vee B} \qquad \dfrac{A \quad \mid B^\perp}{A \vee B} \qquad \dfrac{B \quad \mid A^\perp}{A \vee B}$$

$$\dfrac{B \quad \mid A}{A \to B} \qquad \dfrac{A^\perp \quad B}{A \to B} \qquad \dfrac{A^\perp \quad \mid B^\perp}{A \to B} \qquad \dfrac{A}{\neg\neg A}$$

$$\dfrac{A \quad B \quad \begin{matrix}[A]^n\ [B]^n\ \{C\}^n \\ \vdots \\ [[\perp]]\end{matrix}}{\neg C}\,n \qquad \dfrac{A \quad B \quad \begin{matrix}[A]^n\ \{C\}^n_1 \\ \vdots \\ [[\perp]]\end{matrix} \quad \begin{matrix}[B]^n\ \{C\}^n_2 \\ \vdots \\ [[\perp]]\end{matrix}}{\neg C}\,n \qquad \dfrac{A \quad \begin{matrix}[A]^n\ \{C\}^n_1 \\ \vdots \\ [[\perp]]\end{matrix} \quad \begin{matrix}[B^\perp]^n\ \{C\}^n_2 \\ \vdots \\ [[\perp]]\end{matrix} \quad \mid B}{\neg C}\,n$$

where the hypotheses $A, B, B^\perp, C$ appear exactly once in the derivations of $[[\perp]]$

$$\dfrac{\{A \wedge B\}_1}{\{\{A\}\}} \qquad \dfrac{\{A \wedge B\}_2}{\{\{B\}\}} \qquad \dfrac{\{A \vee B\} \quad \begin{matrix}\{\{A\}\}^n \\ \vdots \\ [[\perp]]\end{matrix} \quad \begin{matrix}\{\{B\}\}^n \\ \vdots \\ [[\perp]]\end{matrix}}{[[\perp]]}\,n \qquad \dfrac{\{A \to B\} \quad [A]}{\{\{B\}\}} \qquad \dfrac{[A^\perp] \quad \{\{A\}\}}{[[\perp]]}$$

where $n \in \mathbb{N}$

Table 2. Grounding Rules and Auxiliary Rules for Grounding Negations

2.5. Logical rules

We present here the logical rules for the connectives $\bot, \neg, \wedge, \vee$ and $\to$. These comprise the traditional elimination rules for these connectives, the double negation elimination rule and the negation introduction rule.

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A \vee B \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n \qquad \frac{A \to B \quad A}{B}$$

$$\frac{\begin{matrix} A^n \\ \vdots \\ \bot \end{matrix}}{\neg A}\, n \qquad \frac{\neg A \quad A}{\bot} \qquad \frac{A^{\bot} \quad A}{\bot} \qquad \frac{\bot}{P} \qquad \frac{\neg\neg D}{D}$$

where $n \in \mathbb{N}$, P is a propositional variable, and D is not a negated formula

Table 3. Logical Rules

The negation introduction rule is the only introduction rule in this table. While the idea behind the definition of the calculus Gr is to replace logical introduction rules by grounding rules, this does not work for negation. Indeed, the restrictions on the grounding rules for negation make them wanting as logical introduction rules. An easy solution to this problem is to include in the calculus an unrestricted version of the grounding rules for negation: the traditional negation introduction rule.

Definition 2.6 (Calculus Gr)**.** The grounding calculus Gr consists of the rules of Tables 1, 2 and 3. A Gr derivation is built by starting from hypotheses of the form A, $\{A\}$, $\{\{A\}\}$, $[A]$ or $[A^{\bot}]$ and by applying the rules of Gr.

Notice that expressions of the form $A^{\bot}$ can only be used inside hypotheses of the form $[A^{\bot}]$.

2.6. Rules for $\rhd$

In order to prove that the calculus Gr actually defines a calculus for logical grounding, we also introduce a grounding operator $\rhd$ which internalises the grounding relation defined by the Gr rules. We will not investigate the properties of this operator here, but only show that $\rhd$ defines a grounding relation which corresponds to that of Poggiolesi (2016, 2018), see Section 3.1.

The connective $\rhd$ will appear in expressions of the form or $A_1, \dots, A_n \rhd C$ which express the fact that $A_1, \dots, A_n$ form the ground of C, or of the form $A_1, \dots, A_n \mid B \rhd C$ where $A_1, \dots, A_n$ form the ground of C under the condition B. Intuitively, $A_1, \dots, A_n \mid B \rhd C$ can only be derived immediately after a rule application of the form

$$\frac{A_1 \quad \dots \quad A_n \quad \mid B}{C}$$

Obivously, if no condition B occurs in the rule application, then we derive the formula $A_1, \ldots, A_n \rhd C$. Nevertheless, some technical issue must be addressed: the subformulae $A_1, \ldots, A_n, B$ and C of the formula $A_1, \ldots, A_n \mid B \rhd C$ must obviously be formulae themselves, but among the premisses $A_1, \ldots, A_n, B$ of a grounding rule, there might be expressions of the form $D^\perp$, which are not formulae. To overcome this difficulty, we define a simple function f that maps expressions of the form $D^\perp$ into their interpretation as formulae. By using f in $\rhd$ introduction rules, we will be able to introduce $\rhd$ without adding side conditions to the rules of the calculus. Intuitively, $f(D^\perp)$ is the formula from which we can infer $D^\perp$ by one of the rules for introducing $(\ )^\perp$. Or, in other terms, $f(D^\perp)$ is $\neg D$ if D has an even prefix of negations; otherwise, $D = \neg E$, for some E, and $D^\perp = E$.

Since no ambiguity arises, for the sake of simplicity, we will exploit f also to map expressions of the form $\{A\}$, $\{\{A\}\}$, $[A]$, and $[[A]]$ into the formula A.

Here is the formal definition of f.

Definition 2.7 (Formula Form)**.** For any expression E and formula A,

- $f(\{E\}) = f(\{\{E\}\}) = f([E]) = f([[E]]) = f(E)$,
- $f(A) = A$,
- if $A = \neg^{2n} B$, then $f(A^\perp) = \neg A$,
- if $A = \neg\neg^{2n} B$, then $f(A^\perp) = \neg^{2n} B$.

For any set of expressions Γ, we denote by $f(\Gamma)$ the set of expressions $\{f(E) : E \in \Gamma\}$.

The first clause of Definition 2.7 makes f behave as the identity on formulae. The second clause makes f ignore parentheses. The third and the fourth clauses simply define the formula interpretation of $A^\perp$ as the converse formula of A.

It is easy to show that the value of $f(A^\perp)$ is exactly the formula from which we can infer $A^\perp$ by one of the rules for introducing $(\ )^\perp$. This will be used in the proof of Theorem 3.18 to show the completeness of Gr with respect to classical logic.

Proposition 2.8. *For any A, the following is a* $\mathrm{Gr}^\rhd$ *derivation:*

$$\frac{f(A^\perp)}{A^\perp}$$

Proof. See the appendix. □

A second technical issue concerns $\alpha\kappa$ rules. These rules have been introduced, as explained is section 2.3, in order to account for the fact that certain formulae should be regarded as identical with respect to grounding. We must reflect this also in the $\rhd$ introduction rules. In order to do so, we enable to infer a formula $A_1, \ldots, A_n \mid C \rhd B$ from a rule application

$$\frac{A'_1 \quad \ldots \quad A'_n \quad \mid C'}{B}$$

whenever $A'_1, \ldots, A'_n$ and C' have been obtained only using $\alpha\kappa$ rules from $A_1, \ldots, A_n$ and C, respectively. The same holds for the formula $A_1, \ldots, A_n \rhd B$ when the condition C' does not appear in the rule application.

The rules for $\triangleright$ are presented in Table 4.[7]

$$\text{If} \quad \cfrac{\begin{matrix} A_1 & & A_n & & C \\ \alpha\kappa_1 & & \alpha\kappa_n & & \alpha\kappa_{n+1} \\ A'_1 & \dots & A'_n & \mid & C' \end{matrix}}{\overline{B}} \quad \text{is a derivation}$$

in which $\alpha\kappa_1, \dots, \alpha\kappa_{n+1}$ contain zero or more $\alpha\kappa$ rule applications,

$$\text{then} \quad \cfrac{\cfrac{\begin{matrix} A_1 & & A_n & & C \\ \alpha\kappa_1 & & \alpha\kappa_n & & \alpha\kappa_{n+1} \\ A'_1 & \dots & A'_n & \mid & C' \end{matrix}}{\overline{B}}}{f(A_1), \dots, f(A_n) \mid f(C) \triangleright B} \quad \text{is a derivation.}$$

$$\frac{A_1, \dots, A_n \mid C \triangleright B}{B} \qquad \frac{A_1, \dots, A_n \mid C \triangleright B}{A_i} \qquad \frac{A_1, \dots, A_n \mid C \triangleright B}{C}$$

$$\text{for } i \in \{1, \dots, n\}$$

$$\text{If there is no rule application} \quad \cfrac{A'_1 \quad \dots \quad A'_n \quad \mid C'}{\overline{B}}$$

—with possibly some more premisses $[[\bot]]$—

such that for $i \in \{1, \dots, n\}$ A'_i, resp. C', is a-c equivalent to A_i, resp. C,

$$\text{then} \quad \frac{f(A_1), \dots, f(A_n) \mid f(C) \triangleright B}{\bot} \quad \text{is a derivation.}$$

Table 4. Rules for $\triangleright$

We explain now the $\triangleright$ elimination rules. The first three of these rules are

$$\frac{A_1, \dots, A_n \mid C \triangleright B}{B} \qquad \frac{A_1, \dots, A_n \mid C \triangleright B}{A_i} \qquad \frac{A_1, \dots, A_n \mid C \triangleright B}{C}$$

$$\text{for } i \in \{1, \dots, n\}$$

and enable us to infer, from a grounding statement, each component of the ground, the condition and the explained formula. These rules reflect what is called the *factivity* of grounding, that is, if A grounds C under the condition B, then A, B and C are true. Since grounding rules do not discharge any hypothesis, this is perfectly coherent with the structure of grounding derivations. For instance, if we have a grounding rule application

$$\cfrac{A \quad \mid B}{\overline{C}}$$

then A and B can be hypotheses, derivable from hypotheses, or proven valid. In any of these cases, it is sound to derive them from the formula $A \mid f(B) \triangleright C$.

[7]The elimination rules for $\triangleright$ presented here differ with respect to those presented by Poggiolesi (2018), but it is easy to see that by our rules we can derive all elimination rules of Poggiolesi (2018). In addition, by our elimination rules it is possible to derive from a grounding statement all elements of the ground, the ground condition and the consequence. This reflects the factivity of grounding.

The fourth elimination rule for $\triangleright$ is

$$\frac{f(A_1), \ldots, f(A_n) \mid f(C) \triangleright B}{\perp}$$

and it can only be applied if there is no rule application

$$\frac{A_1' \quad \ldots \quad A_n' \quad \mid C'}{B}$$

with possibly some more premisses of the form $[[\perp]]$ such that A_i', respectively C', is a-c equivalent to A_i, respectively C, for $i \in \{1, \ldots, n\}$. This rule enables us to use the calculus in order to characterise also in a negative way the notion of logical grounding that we adopt—that is, the notion of *complete logical grounding* defined in Poggiolesi (2016). The rule is required, in particular, to derive the negation of false grounding statements. For instance, we would like to be able to derive that the atom P is not the ground of the atom Q, or that the atom R is not the ground of $\neg R$, and we can do it by this rule as follows:

$$\dfrac{\dfrac{P \triangleright Q^1}{\perp}}{\neg(P \triangleright Q)}\,1 \qquad\qquad \dfrac{\dfrac{R \triangleright \neg R^1}{\perp}}{\neg(R \triangleright \neg R)}\,1$$

since it is clear by simple inspection of the grounding rules of Gr that there is no way in Gr to construct derivations of the following two forms:

$$\frac{P}{Q} \qquad\qquad \frac{R}{\neg R}$$

We define the calculus $\mathrm{Gr}^{\triangleright}$ by extending the calculus Gr with the rules presented in Table 4 for the connective $\triangleright$.

Definition 2.9 (Calculus $\mathrm{Gr}^{\triangleright}$)**.** The grounding calculus $\mathrm{Gr}^{\triangleright}$ consists of the rules presented in Tables 1, 2, 3, and 4. A $\mathrm{Gr}^{\triangleright}$ derivation is built by starting from hypotheses of the form A, $\{A\}$, $\{\{A\}\}$, $[A]$ or $[A^{\perp}]$—expressions of the form $A^{\perp}$ cannot be used as hypotheses—and by applying the rules of $\mathrm{Gr}^{\triangleright}$.

3. Grounding and classical logic: two soundness and completeness results

In this section we first prove that $\mathrm{Gr}^{\triangleright}$ is a calculus for complete and immediate grounding and then we present a soundness and completeness result for Gr with respect to classical logic. Thus we show that our grounding calculus can be also considered as a calculus for classical logic, if we employ grounding rules as introduction rules.

3.1. Complete and immediate grounding

We show now that $\mathrm{Gr}^{\triangleright}$ captures the notion of complete and immediate grounding defined by Poggiolesi (2016). We start by recalling the essential definitions employed in these two works.

The following two definitions correspond to Definitions 3.6 and 3.7 used by Poggiolesi (2018) to fix the notion of complete and immediate grounding and to show soundness and completeness for the grounding part of the calculus PGr. We will use them here to show that the grounding relation induced by the grounding rules of Gr is the same complete and immediate grounding relation defined by Poggiolesi (2016).

Definition 3.1 (Definition 3.2 in Poggiolesi (2018)). Let $D \in \mathcal{L}_{\text{CL}}$. The converse D^* of D is

- $\neg^{2n}B$ if $D = \neg\neg^{2n}B$ and the outermost connective of B is not a negation,
- $\neg\neg^{2n}B$ if $D = \neg^{2n}B$ and the outermost connective of B is not a negation.

Coherently with the rules for $(\ )^{\perp}$ in Table 1, to obtain the converse of a formula D, we add a negation to D, if D has a prefix which consists of an even number of negations; we remove one negation from D, if D has a prefix which consists of an odd number of negations.

Definition 3.2 (Definition 3.4 in Poggiolesi (2018)). For any formulae $A, B \in \mathcal{L}_{\text{CL}}$, A is a-c equivalent to B if one of the following holds:

- $A = B$,
- $E \star F$ for $\star \in \{\wedge, \vee\}$ is a subformula of A the formula obtained by substituting $F \star E$ for $E \star F$ in A is a-c equivalent to B,
- $(E \star F) \star G$ for $\star \in \{\wedge, \vee\}$ is a subformula of A and the formula obtained by substituting $E \star (F \star G)$ for $(E \star F) \star G$ in A is a–c equivalent to B.

Notice that this definition corresponds to the relation between formulae obtained by applying $\alpha\kappa$ rules in Table 1, also considering Proposition 2.5.

The following definition introduces the the $\cong$ relation which combines a-c equivalence and the converse relation.

Definition 3.3 (Definition 3.5 in Poggiolesi (2018)). For any $A, B \in \mathcal{L}_{\text{CL}}$ $A \cong B$ if either A is a-c equivalent to B, or A is a-c equivalent to B^*.

We recall now the complexity measure for complete grounding.

Definition 3.4 (Definition 3.6 in Poggiolesi (2018)). For any multiset of formulas $\Gamma \subseteq \mathcal{L}_{\text{CL}}$ and formula $C \in \mathcal{L}_{\text{CL}}$, we say that Γ is completely and immediately less g-complex than C if one of the following holds:

- $C \cong \neg\neg B$ and either $\Gamma = \{B\}$ or $\Gamma = \{B^*\}$
- $C \cong B \star D$, for $\star \in \{\wedge, \vee, \rightarrow\}$, and one of the following holds:
  $\Gamma = \{B, D\} \mid \Gamma = \{B, D^*\} \mid \Gamma = \{B^*, D\} \mid \Gamma = \{B^*, D^*\}$

Finally, we recall the definition of complete and immediate grounding.

Definition 3.5 (Definition 3.7 in Poggiolesi (2018)). For any consistent multiset of formulas $C \cup \Gamma$ such that $C \cup \Gamma \subseteq \mathcal{L}_{\text{CL}}$, we say that, under the robust condition C (that may be empty), Γ completely and immediately grounds A if all the following hold:

- A is derivable in a calculus for classical logic from the hypotheses Γ
- $\neg A$ is derivable in a calculus for classical logic from the hypotheses $\neg\Gamma, C$
- $C \cup \Gamma$ is completely and immediately less g-complex than A, see Def. 3.4.

where $\neg\Gamma = \{\neg B : B \in \Gamma\}$.

We define now the language $\mathcal{L}_1$, which represents the fragment of $\mathcal{L}$ containing those formulae in which there is no nesting of occurrences of the connective $\rhd$. This fragment contains exactly those $\rhd$-statements that correspond to the grounding statements considered in PGr.

Definition 3.6 (Language $\mathcal{L}_1$)**.** The language $\mathcal{L}_1$ is defined by the following grammar:

$$\begin{aligned}
\varphi &::= \psi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid (\psi, \ldots, \psi \mid \psi \rhd \psi) \\
\psi &::= \xi \mid \bot \mid \neg\psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi \to \psi \\
\xi &::= p \mid q \mid r \mid \ldots
\end{aligned}$$

where $p, q, r, \ldots$ are all propositional variables of the language and $\psi, \ldots, \psi$ is a list of non-terminal symbols ψ separated by commata.

Before proving the two theorems that guarantee that $\mathrm{Gr}^{\rhd}$ captures exactly the considered notion of grounding, we state a simple lemma about the negation of the formula interpretation $f(A^{\perp})$ of expressions of the form $A^{\perp}$.

Lemma 3.7. *For any formula A, there is a derivation of A in* NC *from the hypothesis $\neg f(A^{\perp})$.*

Proof. See the appendix. $\square$

Theorem 3.8 (Ground Soundness)**.** *For any consistent set of formulae $\{G_1, \ldots, G_n, D, C\} \subseteq \mathcal{L}_{CL}$ if we can derive*

$$G_1, \ldots, G_n \mid C \rhd D$$

in $\mathrm{Gr}^{\rhd}$ *from the hypotheses $G_1, \ldots, G_n, D$, then $\{G_1, \ldots, G_n\}$ completely and immediately grounds D under the, possibly empty, robust condition C according to Definition 3.5.*

Proof. Since $G_1, \ldots, G_n, D, C \in \mathcal{L}_{\mathrm{CL}}$, the connective $\rhd$ is introduced immediately below a grounding rule application—see Table 4. We reason on the grounding rule which is applied immediately above the introduction of $\rhd$ and on the derivations $\alpha\kappa_1 \ldots, \alpha\kappa_n$ of its premises: $\dfrac{\begin{matrix} G_1 \\ \alpha\kappa_1 \\ G_1' \end{matrix} \quad \ldots \quad \begin{matrix} G_n \\ \alpha\kappa_n \\ G_n' \end{matrix} \quad \mid \quad \begin{matrix} C \\ \alpha\kappa_{n+1} \\ C' \end{matrix}}{D}$. In the following we denote by A' the premiss of the grounding rule which is derived by $\alpha\kappa$ rules from the ground, or robust condition, A.

We present a few interesting cases, see the appendix for an unabridged version of the proof.

- $\dfrac{A' \quad \mid (B')^{\perp}}{A' \vee B'}$ First, we can derive the conclusion of this inference from the hypothesis A by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and disjunction introduction.

  Second, we can derive $\neg(A' \vee B')$ from $\neg A$ and the robust condition $f(B^{\perp})$ since we can derive $\neg A'$ and $f(B'^{\perp})$ from these two formulae by $\alpha\kappa$ rules and then, by Lemma 3.15, construct the following derivation:

$$\dfrac{A' \vee B'^1 \qquad \dfrac{\neg A' \quad A'^2}{\bot} \qquad \dfrac{f(B'^{\perp}) \quad B'^2}{\bot}}{\dfrac{\bot}{\neg(A' \vee B')}\, 1}\, 2$$

Finally, $\{A, f(B^{\perp})\}$ is less g-complex than $A' \vee B'$ because $A' \vee B' \cong A \vee B$, since they are a-c equivalent, and $\{A, f(B^{\perp})\} = \{A, B^*\}$.

- $$\dfrac{A' \quad B' \quad \begin{matrix}[A']^n\ [B']^n\ \{C'\}^n \\ \vdots \\ [[\bot]]\end{matrix}}{\neg C'}\, n$$ First of all, we notice that the hypotheses $[A']$, $[B']$ and $\{C'\}$ are between parentheses. Now, the only rules that can be applied to formulae between parentheses are $\alpha\kappa$ rules, converse rules and the following five rules:

$$\dfrac{\{A \wedge B\}}{\{\{A\}\}} \qquad \dfrac{\{A \wedge B\}}{\{\{B\}\}} \qquad \dfrac{\{A \vee B\} \quad \begin{matrix}\{\{A\}\}^n \\ \vdots \\ [[\bot]]\end{matrix} \quad \begin{matrix}\{\{B\}\}^n \\ \vdots \\ [[\bot]]\end{matrix}}{[[\bot]]}\, n \qquad \dfrac{\{A \to B\} \quad [A]}{\{\{B\}\}} \qquad \dfrac{[A^{\perp}] \quad \{\{A\}\}}{[[\bot]]}$$

Let us call the first two rules displayed here *bracketed conjunction eliminations*, the third *bracketed disjunction elimination*, the fourth *bracketed implication elimination*, and the fifth *bracketed converse elimination*. We argue, considering the restrictions on the applicability of these five rules and considering that the hypotheses $[A']$, $[B']$ and $\{C'\}$ must appear exactly once in the derivation of $[[\bot]]$, that C' must be either of the form $C_1 \vee C_2$ or of the form $C_1 \to C_2$. Indeed, $[[\bot]]$ can only be obtained by bracketed eliminations applied to $\{C'\}$. Moreover, only bracketed implication and bracketed disjunction elimination enable us to use both hypotheses $[A']$ and $[B']$ exactly once, as required by the grounding rule for negation. Indeed, bracketed disjunction elimination enables us to obtain $[[\bot]]$ twice—once for each disjunct, once in combination with $[A']$ and once in combination with $[B']$; and bracketed implication elimination enables us to use one among $[A']$ and $[B']$ to eliminate the implication and the other one to obtain $[[\bot]]$ in combination with the consequent of the implication. If we used bracketed conjunction elimination, on the other hand, we would obtain only one formula from $\{C'\}$ and we would not be able to use it in combination with both $[A']$ and $[B']$ to obtain $[[\bot]]$. If we consider moreover that the $\alpha\kappa$ rules do not change the main connective of their premiss and that, if we applied a converse introduction rule to $\{C'\}$, we would obtain an expression that cannot be used as premiss of any rule—since there are no rules that act on expressions of the form $\{F^{\perp}\}$—we can conclude that C' must be either of the form $C_1 \vee C_2$ or of the form $C_1 \to C_2$. We reason then by cases on the form of C'.

If $C' = C_1 \vee C_2$, without loss of generality, we have that A' and $C_1^{\perp}$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules; and B' and $C_2^{\perp}$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules. Therefore, we need to show that we can derive $\neg(C_1 \vee C_2)$ from the hypotheses A and B. We know that we can derive $f(A')$ and $f(C_1^{\perp})$, and $f(B')$ and $f(C_2^{\perp})$ from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following NC derivation:

$$\dfrac{\dfrac{C_1 \vee C_2 \,^1 \qquad \dfrac{f(C_1^{\perp}) \quad C_1\,^2}{\perp} \qquad \dfrac{f(C_2^{\perp}) \quad C_2\,^2}{\perp}}{\perp}\,2}{\neg(C_1 \vee C_2)}\,1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

If $C' = C_1 \to C_2$, without loss of generality, A' and C_1 can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules; and B' and $C_2^{\perp}$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules. Therefore, we need to show that we can derive $\neg(C_1 \to C_2)$ from the hypotheses A and B. We know that we can derive A' and C_1, and $f(B')$ and $f(C_2^{\perp})$ from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following NC derivation:

$$\dfrac{\dfrac{f(C_2^{\perp}) \qquad \dfrac{C_1 \to C_2\,^1 \quad C_1}{C_2}}{\perp}}{\neg(C_1 \vee C_2)}\,1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Second, we need to show that if $C' = C_1 \vee C_2$ we can derive $\neg\neg(C_1 \vee C_2)$ from the hypotheses $\neg A$ and $\neg B$, and if $C' = C_1 \to C_2$ we can derive $\neg\neg(C_1 \to C_2)$ from the hypotheses $\neg A$ and $\neg B$. In the first case, we can derive $\neg f(A') = \neg f(C_1^{\perp})$ and $\neg f(B') = \neg f(C_2^{\perp})$ from $\neg A$ and $\neg B$ by $\alpha\kappa$ rules. Hence, by Lemma 3.7, we can construct the following derivations:

$$\dfrac{\dfrac{\neg(C_1 \vee C_2)^1 \qquad \dfrac{\begin{array}{c}\neg f(C_1^{\perp})\\ \vdots\\ C_1\end{array}}{(C_1 \vee C_2)}}{\perp}\,2}{\neg\neg(C_1 \vee C_2)}\,1 \quad \text{and} \quad \dfrac{\dfrac{\neg(C_1 \vee C_2)^1 \qquad \dfrac{\begin{array}{c}\neg f(C_2^{\perp})\\ \vdots\\ C_2\end{array}}{(C_1 \vee C_2)}}{\perp}\,2}{\neg\neg(C_1 \vee C_2)}\,1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications. In the second case, we can derive $\neg A' = \neg C_1$ and $\neg f(B') = \neg f(C_2^{\perp})$ from $\neg A$ and $\neg B$ by $\alpha\kappa$ rules. Hence, by Lemma 3.7, we can construct the following derivations:

$$\dfrac{\dfrac{\neg(C_1 \to C_2)^1 \qquad \dfrac{\dfrac{\dfrac{\neg C_1 \quad C_1\,^2}{\perp}}{C_2}}{C_1 \to C_2}\,2}{\perp}}{\neg\neg(C_1 \to C_2)}\,1 \quad \text{and} \quad \dfrac{\dfrac{\neg(C_1 \to C_2)^1 \qquad \dfrac{\begin{array}{c}\neg f(C_2^{\perp})\\ \vdots\\ C_2\end{array}}{(C_1 \to C_2)}}{\perp}\,2}{\neg\neg(C_1 \to C_2)}\,1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Finally, we show that $\{f(A), f(B)\}$ is less g-complex than $\neg C'$. Now, if $C' = C_1 \vee C_2$. we have that $A' = C_1^\perp$ and $B' = C_2^\perp$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there are formulae D and E such that

- $A = D^\perp$ and $B = E^\perp$, and thus $D^* = f(A)$ and $E^* = f(B)$,
- D is a-c equivalent to C_1 and E is a-c equivalent to C_2.

Therefore, $\neg C' = \neg(C_1 \vee C_2) \cong (D \vee E)$ and $\{f(A), f(B)\} = \{D^*, E^*\}$. In conclusion, $\{f(A), f(B)\}$ is less g-complex than $\neg C'$. If, on the other hand, $C' = C_1 \to C_2$. we have that $A' = C_1$ and $B' = C_2^\perp$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there is a formula E such that

- $B = E^\perp$, and thus $E^* = f(B)$,
- E is a-c equivalent to C_2.

Therefore, $\neg C' = \neg(C_1 \to C_2) \cong (A \to E)$ and $\{A, f(B)\} = \{A, E^*\}$. In conclusion, also in this case, $\{A, f(B)\}$ is less g-complex than $\neg C'$.

□

Theorem 3.9 (Ground Completeness)**.** *For any $A, C \in \mathcal{L}_{CL}$ and $\Gamma \subseteq \mathcal{L}_{CL}$, if Γ completely and immediately grounds A under the, possibly empty, robust condition C according to Definition 3.5, then we can derive $\Gamma \mid C \rhd A$ in* $\mathrm{Gr}^{\rhd}$.

Proof. The proof proceeds as follows. For any formula $A \in \mathcal{L}_{\mathrm{CL}}$, we select each multiset $\Gamma \subseteq \mathcal{L}_{\mathrm{CL}}$ such that

- Γ is completely and immediately less g-complex than A according to Definition 3.4.
- Γ is consistent
- For some formula $C \in \mathcal{L}_{\mathrm{CL}}$,
  - There is a derivation of A in NC from a non-empty multiset $\Gamma \setminus \{C\}$.
  - There is a derivation of $\neg A$ in NC from the multiset $\neg(\Gamma \setminus \{C\}) \cup (\Gamma \cap \{C\})$.

where $\neg\Delta = \{\neg B : B \in \Delta\}$ and $\Gamma \setminus \Delta$ is the multiset of elements of Γ that do not occur in Δ.

For each selected Γ, if some $C \notin \Gamma$ satisfies the conditions above, we show that $\Gamma \rhd A$ is derivable in $\mathrm{Gr}^{\rhd}$ from the hypotheses Γ; if some $C \in \Gamma$ satisfies the conditions above, we show that $\Gamma \mid C \rhd A$ is derivable in $\mathrm{Gr}^{\rhd}$ from the hypotheses Γ, C.

Notice that if $C \notin \Gamma$, the choice of C is completely irrelevant.

According to Definition 3.4, if Γ exists, either (i) $A \cong \neg\neg B$ or (ii) $A \cong (B \star C)$, for $\star \in \{\wedge, \vee, \to\}$.

If (i) is the case, A could either be a-c equivalent to $\neg\neg B$ or to its converse $(\neg\neg B)^*$. In this case, Γ is either $\{B\}$ or $\{B^*\}$. It is easy to see that from B we can derive $\neg\neg B$ but not $(\neg\neg B)^*$, and from B^* we can derive $(\neg\neg B)^*$ but not $\neg\neg B$. Moreover, from $\neg B$ we can derive $\neg\neg\neg B$, and from $\neg(B^*)$ we can derive $\neg((\neg\neg B)^*)$.

Accordingly:

$$\dfrac{\dfrac{B}{\neg\neg B}}{B \rhd \neg\neg B} \qquad \dfrac{\dfrac{B^*}{(\neg\neg B)^*}}{B^* \rhd (\neg\neg B)^*}$$

are $\mathrm{Gr}^{\rhd}$ derivation.

If (ii) is the case, A could either be a-c equivalent to $(B \star C)$ or to its converse $\neg(B \star C)$. In this case, Γ is one of the following: $\{B, C\}$, $\{B, C^*\}$, $\{B^*, C\}$, $\{B^*, C^*\}$.

As for derivability, we only have the following combinations:

from $\{B, C\}$ we derive $B \wedge C$	from $\{B, C\}$ we derive $B \vee C$
from $\{B\}$ we derive $B \vee C$	from $\{C\}$ we derive $B \vee C$
from $\{B, C^*\}$ we derive $B \vee C$	from $\{B^*, C\}$ we derive $B \vee C$
from $\{B, C\}$ we derive $B \to C$	from $\{C\}$ we derive $B \to C$
from $\{B^*\}$ we derive $B \to C$	from $\{B^*, C\}$ we derive $B \to C$
from $\{B^*, C^*\}$ we derive $B \to C$	from $\{B^*\}$ we derive $(B \wedge C)^*$
from $\{C^*\}$ we derive $(B \wedge C)^*$	from $\{B^*, C\}$ we derive $(B \wedge C)^*$
from $\{B, C^*\}$ we derive $(B \wedge C)^*$	from $\{B^*, C^*\}$ we derive $(B \wedge C)^*$
from $\{B^*, C^*\}$ we derive $(B \vee C)^*$	from $\{B, C^*\}$ we derive $(B \to C)^*$

It is easy to construct these combinations from the truth tables of the connectives. Each of these combinations can be constructed as pairs of the form $(\Gamma \setminus \{E\}, A)$. We select now those that enable us to derive $\neg A$ from $\neg(\Gamma \setminus \{E\}) \cup (\Gamma \cap \{E\})$. We obtain the following triples $(\Gamma \setminus \{E\}, E, A)$ in which we denote by $-$ the absence of the formula E:

$(\{B, C\}, -, B \wedge C)$	$(\{B, C\}, -, B \vee C)$	$(\{B\}, C^*, B \vee C)$
$(\{C\}, B^*, B \vee C)$	$(\{C\}, B, B \to C)$	$(\{B^*, C\}, -, B \to C)$
$(\{B^*\}, C^*, B \to C)$	$(\{B^*\}, C, (B \wedge C)^*)$	$(\{C^*\}, B, (B \wedge C)^*)$
$(\{B^*, C^*\}, -, (B \wedge C)^*)$	$(\{B^*, C^*\}, (B \vee C)^*)$	$(\{B, C^*\}, (B \to C)^*)$

It is easy to see that the combinations involving $B \wedge C, B \vee C$ and $B \to C$ exactly correspond to the grounding rules in Table 2 for grounding formulae of the relevant form. For the triple $(\{B^*\}, C, (B \wedge C)^*)$, the grounding derivation is the following:

$$\dfrac{B^* \qquad \dfrac{\dfrac{[B^*]^1}{[B^\perp]} \quad \dfrac{\{B \wedge C\}^1}{\{\{B\}\}}}{[[\perp]]} \qquad \dfrac{[C^\perp]^1 \quad \dfrac{\{B \wedge C\}^1}{\{\{C\}\}}}{[[\perp]]} \qquad \mid C}{\neg(B \wedge C)}\ 1$$

For the triple $(\{C^*\}, B, (B \wedge C)^*)$, the grounding derivation is the following::

$$\dfrac{C^* \qquad \dfrac{[B^\perp]^1 \quad \dfrac{\{B \wedge C\}^1}{\{\{B\}\}}}{[[\perp]]} \qquad \dfrac{\dfrac{[C^*]^1}{[C^\perp]} \quad \dfrac{\{B \wedge C\}^1}{\{\{C\}\}}}{[[\perp]]} \qquad \mid B}{\neg(B \wedge C)}\ 1$$

For the triple $(\{B^*, C^*\}, -, (B \wedge C)^*)$, the grounding derivation is the following:

$$\dfrac{B^* \quad C^* \qquad \dfrac{\dfrac{[B^*]^1}{[B^\perp]} \quad \dfrac{\{B \wedge C\}^1}{\{\{B\}\}}}{[[\perp]]} \qquad \dfrac{\dfrac{[C^*]^1}{[C^\perp]} \quad \dfrac{\{B \wedge C\}^1}{\{\{C\}\}}}{[[\perp]]}}{\neg(B \wedge C)}\ 1$$

For the triple $(\{B^*, C^*\}, (B \vee C)^*)$, the grounding derivation is the following:

$$\dfrac{B^* \quad C^* \quad \dfrac{\{B \vee C\}^1 \quad \dfrac{\dfrac{[B^*]^1}{[B^\perp]} \quad \{\{B\}\}^2}{[[\perp]]} \quad \dfrac{\dfrac{[C^*]^1}{[C^\perp]} \quad \{\{C\}\}^2}{[[\perp]]}}{[[\perp]]}\,{}_2}{\neg(A \vee B)}\,{}_1$$

Finally, for the triple $(\{B, C^*\}, (B \to C)^*)$, the grounding derivation is the following:

$$\dfrac{B \quad C^* \quad \dfrac{\dfrac{[C^*]^1}{[C^\perp]} \quad \dfrac{\{B \to C\}^1 \quad [B]^1}{\{\{C\}\}}}{[[\perp]]}}{\neg(B \to C)}\,{}_1$$

$\square$

Corollary 3.10. *The calculus* $\mathrm{Gr}^{\triangleright}$ *is sound and complete with respect to the notion of complete and immediate grounding of Definition 3.5 over the language* $\mathcal{L}_1$.

Proof. By Theorems 3.8 and 3.9. $\square$

Corollary 3.11. *The calculus* Gr, *over the language* $\mathcal{L}_1$, *is sound and complete with respect to the calculus* PGr

Proof. Theorem 5.4 in Poggiolesi (2018) establishes that PGr is sound and complete with respect to the notion of complete and immediate grounding of Definition 3.7. Corollary 3.10 establishes that Gr is sound and complete with respect to the notion of complete and immediate grounding of Definition 3.5. Since Definition 3.7 in Poggiolesi (2018) and Definition 3.5 are equivalent, we have that the claim holds. $\square$

Even though the calculus Gr is equivalent to the calculus PGr, in Gr we avoid side conditions by explicitly introducing the $(\ )^\perp$ operator for converse formulae in the language of the calculus and $\alpha\kappa$ rules for the a-c equivalence. This does not only provide an explicit procedural interpretation of a-c equivalence but also constitutes a central simplification that enables us to prove a normalisation result for the calculus. Indeed, the immediate subformulae of the conclusion of the grounding rules in Poggiolesi (2018) are not identical to the corresponding premisses but only a-c equivalent to them, and this would hinder the definition of reductions corresponding to those that we present in Tables 6 and 7.

3.2. Classical soundness and completeness

We show now that, if we employ the grounding rules as introduction rules for the connectives of classical logic, then the calculus Gr is also a suitable calculus for classical logic. To do so, we prove that it is sound and complete with respect to the traditional natural deduction calculus NC for this logic, see, for instance, Prawitz (1971). In order to do so, we restrict ourselves to the language $\mathcal{L}_{\mathrm{CL}}$ of classical logic.

Definition 3.12 (Language $\mathcal{L}_{\mathrm{CL}}$)**.** The language $\mathcal{L}_{\mathrm{CL}}$ is defined by the following grammar:

$$\begin{aligned}\varphi &::= \xi \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \\ \xi &::= p \mid q \mid r \mid \ldots\end{aligned}$$

where $p, q, r, \ldots$ are all propositional variables of the language.

We adopt the usual notation for derivability.

Definition 3.13 (Derivability)**.** For any calculus χ, set of formulae Γ, and expression E, the relation $\Gamma \vdash_\chi E$ holds if there is a derivation of E from hypotheses Γ constructed using exclusively rules of the calculus χ.

Before proving soundness and completeness, we state some simple lemmata which will enable us to work with the formula interpretation $f(A^\perp)$ of expressions of the form $A^\perp$ without bothering about the internal structure of A.

Lemma 3.14. *For any formula A, $A \vee f(A^\perp)$ is derivable in* Gr *and in* NC*.*

Proof. See the appendix. ☐

In the next lemma we establish the fact that a formula A and its converse formula $f(A^\perp)$ are always contradictory.

Lemma 3.15. *For any A, one of the following is both a* Gr *and an* NC *derivation:*

$$\dfrac{f(A^\perp) \quad A}{\bot} \qquad \dfrac{A \quad f(A^\perp)}{\bot}$$

Proof. See the appendix. ☐

$$\dfrac{A \quad B}{A \wedge B} \qquad \dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B} \qquad \dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B} \qquad \dfrac{A \vee B \quad \begin{array}{c} A^n \\ \vdots \\ C \end{array} \quad \begin{array}{c} B^n \\ \vdots \\ C \end{array}}{C}\, n$$

$$\dfrac{\begin{array}{c} A^n \\ \vdots \\ B \end{array}}{A \to B}\, n \qquad \dfrac{A \to B \quad A}{B} \qquad \dfrac{\begin{array}{c} A^n \\ \vdots \\ \bot \end{array}}{\neg A}\, n \qquad \dfrac{\neg A \quad A}{\bot} \qquad \dfrac{\bot}{A} \qquad \dfrac{\begin{array}{c} \neg A^n \\ \vdots \\ \bot \end{array}}{A}\, n$$

where $n \in \mathbb{N}$

Table 5. The Calculus NC

We also establish that $\alpha\kappa$ rules are classically sound. In order to do so, we show that for each instance of an $\alpha\kappa$ rule with premiss $\mathcal{C}[H]$ and conclusion $\mathcal{C}[K]$, we can derive $\mathcal{C}[K]$ from $\mathcal{C}[H]$ by only using NC rules. Since deriving K from H in NC always only requires a few rule applications, the proof boils down to a simple induction on the number of symbols of the context $\mathcal{C}[x]$.

Lemma 3.16. *The $\alpha\kappa$ rules in Table 1 are derivable in* NC.

Proof. See the appendix. $\square$

We can finally prove that Gr is sound and complete with respect to classical logic.

Theorem 3.17 (Classical Soundness)**.** *The calculus* Gr*, over the language $\mathcal{L}_{CL}$, is sound with respect to the calculus* NC*.*

Proof. We show, in particular, that, for any set of hypotheses Γ and expression E, if $\Gamma \vdash_{\mathrm{Gr}} E$ then $f(\Gamma) \vdash_{\mathrm{NC}} f(E)$. The proof is by induction on the number of rule applications in the Gr derivation of E.

If no rule is applied in the NC derivation of E, $f(\Gamma) = \{f(E)\}$ and the statement trivially holds.

Assume then that the Gr derivation of F contains $n > 0$ rule applications and that if E has a Gr derivation containing m rule applications, for $m < n$, then $f(E)$ has an NC derivation. We consider the last rule applied in the Gr derivation of E.

We only present some exemplar cases, see the appendix for an unabridged version of the proof.

- $\dfrac{A^{\perp} \quad | \; B^{\perp}}{A \to B}$ By induction hypothesis, $f(A^{\perp})$ is derivable in NC. By Lemma 3.15, one of the following is an NC derivation

$$\dfrac{\dfrac{\dfrac{A^1 \quad f(A^{\perp})}{\perp}}{B}}{A \to B}\,1 \qquad\qquad \dfrac{\dfrac{\dfrac{f(A^{\perp}) \quad A^1}{\perp}}{B}}{A \to B}\,1$$

  Therefore, the conclusion of the rule is derivable in NC as well.

- $\dfrac{A \quad B \qquad \begin{array}{c}[A]^n \; [B]^n \; \{C\}^n \\ \vdots \\ [[\perp]]\end{array}}{\neg C}\,n$ By induction hypothesis, $f(A)$ and $f(B)$ are derivable in NC. Moreover, again by induction hypothesis, there is a derivation of $\perp$ from the hypotheses $f(A), f(B)$ and $f(\{C\}) = C$. Hence, we can derive the conclusion $\neg C$ in NC by

$$\dfrac{\begin{array}{c} f(A) \; f(B) \; C^n \\ \vdots \\ \perp \end{array}}{\neg C}\,n$$

  where $f(A)$ and $f(B)$ are not cancelled by the negation introduction but derived by an NC derivation.

- $\dfrac{\neg\neg A}{A}$ Since $\neg\neg A$ is a formula, also A is one. Moreover, by induction hypothesis,

$\neg\neg A$ is derivable in NC . By

$$\dfrac{\dfrac{\dfrac{\neg(A\vee\neg A)^2 \qquad \dfrac{\dfrac{\dfrac{\neg(A\vee\neg A)^2 \quad \dfrac{A^3}{A\vee\neg A}}{\bot}}{\neg A}\,{\scriptstyle 3}}{A\vee\neg A}}{\bot}}{A\vee\neg A}\,{\scriptstyle 2} \qquad \dfrac{A^1}{A} \quad \dfrac{\neg\neg A \quad \neg A^1}{A}}{A}\,{\scriptstyle 1}$$

we can show that also the conclusion is derivable in NC.

□

Theorem 3.18 (Classical Completeness)**.** *The calculus* Gr *is complete with respect to the calculus* NC*.*

Proof. We show, in particular, that, for any set of hypotheses Γ and formula F, $\Gamma \vdash_{\mathrm{NC}} F$ then $\Gamma \vdash_{\mathrm{Gr}} F$. The proof is by induction on the number of rule applications in the NC derivation of F.

If no rule is applied in the NC derivation of F, the statement trivially holds.

Assume then that the NC derivation of F contains $n > 0$ rule applications and that if a formula has an NC derivation containing m rule applications, for $m < n$, then it has also an Gr derivation. We consider the last rule applied in the NC derivation of F.

We only present some exemplar cases, see the appendix for an unabridged version of the proof.

- $\dfrac{A}{A\vee B}$ By induction hypothesis, the premiss is derivable in Gr. By Lemma 3.14 and Proposition 2.8 we have that the following is a $\mathrm{Gr}^{\triangleright}$ derivation:

$$\dfrac{\overset{\cdots}{B\vee f(B^\perp)} \qquad \dfrac{A \quad B^1}{A\vee B} \qquad \dfrac{A \quad \mid \quad \dfrac{f(B^\perp)^1}{B^\perp}}{A\vee B}}{A\vee B}\,{\scriptstyle 1}$$

- $\dfrac{\begin{matrix}A^n\\ \vdots\\ B\end{matrix}}{A\to B}\,{\scriptstyle n}$ By induction hypothesis, the premiss B is derivable in Gr fom the hypothesis A. By Lemma 3.14 and Proposition 2.8 the following is a $\mathrm{Gr}^{\triangleright}$ derivation:

$$\dfrac{\overset{\cdots}{A\vee f(A^\perp)} \qquad \dfrac{A^1 \quad \begin{matrix}A^1\\ \vdots\\ B\end{matrix}}{A\to B} \qquad \dfrac{B\vee f(B^\perp) \qquad \dfrac{\dfrac{f(A^\perp)^1}{A^\perp} \quad B^2}{A\to B} \qquad \dfrac{\dfrac{f(A^\perp)^1}{A^\perp} \quad \mid \quad \dfrac{f(B^\perp)^2}{B^\perp}}{A\to B}}{A\to B}\,{\scriptstyle 2}}{A\to B}\,{\scriptstyle 1}$$

□

Corollary 3.19. *The calculus* Gr*, over the language* $\mathcal{L}_{CL}$*, is sound and complete with respect to the calculus* NC *for classical logic.*

Proof. By Theorems 3.17 and 3.18. $\square$

4. Normalisation of the calculus Gr

In the previous sections we showed that the calculus Gr captures the notion of grounding defined by Poggiolesi (2016, 2018) and is sound and complete with respect to classical logic. This means that grounding rules can be used instead of introduction rules to fully characterise classical logic. What we still have to show is that grounding rules are as balanced as logical introduction rules with respect to elimination rules. In order to show this, we define a normalisation procedure for Gr derivations. Normalising a derivation means making it more direct by removing the redundant steps occurring inside the derivation. These steps, if we consider Gr, are those in which we introduce a connective by a grounding rule just to immediately eliminate it by a logical elimination rule. If we are able to remove all these redundant steps from all Gr derivations—as shown by the reductions in Tables 6 and 7—then we know that grounding rules are balanced with respect to elimination rules in the sense that by eliminating a connective we do not obtain more than what we had before introducing it. In other terms, we know that the premisses of a grounding rule contain all the information that we can extract, by elimination rules, from its conclusion. After the normalisation proof we will show that any Gr derivation in normal form enjoys the subformula property. Formally, this means that such a derivation only contains subformulae of its conclusion or of its hypotheses; intuitively, this indicates that the derivation proceeds from the hypotheses directly to the conclusion without going through formulae of unnecessarily high complexity. This result is interesting for us here because it guarantees that our reductions are strong and thorough enough: they really eliminate all redundant steps in the derivation.

We leave for the moment the rules for the grounding operator $\triangleright$ aside, since we focus here on the properties of grounding rules for logical connectives. The tasks of defining a normalisation procedure also involving the rules for $\triangleright$ and of studying the proof-theoretical properties of these rules is certainly interesting, but for the sake of clarity and not to burden the preset work too much, we leave them for future work.

Before introducing the specific machinery required for the normalisation procedure, we divide grounding and logical rules into introduction rules and elimination rules.

Definition 4.1 (Introduction and Elimination Rules)**.** The introduction rules are: the grounding rules for $\wedge$, $\vee$, $\rightarrow$ and $\neg$; the logical introduction rule for $\neg$; both rules for introducing $(\ )^{\perp}$ and the rule for introducing $\neg\neg$.

The elimination rules are: the elimination rules for $\wedge$, $\vee$, $\rightarrow$ and $\neg$, the rule for eliminating $(\ )^{\perp}$ and that for eliminating $\neg\neg$.

The reduction rules for Gr derivations are presented in Tables 6, 7, 8 and 9.

We precisely define what a reduction of a Gr derivation is and some related terminology.

Definition 4.2 (Reductions Redexes and Critical Rules)**.** For any four derivations s, s', d and d', if $s \mapsto s'$ according to the reduction rules shown in Tables 6, 7, 8 and 9, d contains s as a subderivation, and d' can be obtained by replacing s with s' in d, then the relation $d \mapsto d'$ holds and we say that d reduces to d'.

$$\dfrac{\dfrac{A \quad B}{A \wedge B}}{A} \mapsto A \qquad\qquad \dfrac{\dfrac{A \quad B}{A \wedge B}}{B} \mapsto B$$

$$\dfrac{\dfrac{B \quad \mid A}{A \to B} \quad A}{B} \mapsto B \qquad \dfrac{\dfrac{A^{\perp} \quad B}{A \to B} \quad A}{B} \mapsto B \qquad \dfrac{\dfrac{A^{\perp} \quad \mid B^{\perp}}{A \to B} \quad A}{B} \mapsto \dfrac{\dfrac{A^{\perp} \quad A}{\bot}}{B}$$

$$\dfrac{\dfrac{A \quad B}{A \vee B} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n \mapsto \begin{matrix} A \\ \vdots \\ C \end{matrix}$$

$$\dfrac{\dfrac{A \quad \mid B^{\perp}}{A \vee B} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n \mapsto \begin{matrix} A \\ \vdots \\ C \end{matrix} \qquad \dfrac{\dfrac{B \quad \mid A^{\perp}}{A \vee B} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n \mapsto \begin{matrix} B \\ \vdots \\ C \end{matrix}$$

Table 6. Reductions, Part 1

We denote by $\mapsto^*$ the reflexive and transitive closure of $\mapsto$.

As usual, if the bottom-most rule of a derivation d and one of the rules applied immediately above it form one of the configurations shown in Tables 6, 7, 8 and 9 to the left of $\mapsto$, then we say that d is *a redex*. We call the *critical rules* of the redex the two rule applications that form one of the configurations shown in Tables 6, 7, 8 and 9.

We provide some simple and rather usual definitions that will be used for the normalisation of the calculus.

Definition 4.3 (Logical Complexity)**.** The logical complexity of formulae is defined as usual. The logical complexity of an expression $A^{\perp}$ is the logical complexity of A plus 1.

Definition 4.4 (Redex Complexity)**.** The complexity of a redex r is defined as the logical complexity of the formula, or expression, introduced by the uppermost critical rule of r.

Definition 4.5 (Normal Form)**.** We say that a Gr derivation d is normal, or in normal form, if there is no derivation d' such that $d \mapsto d'$ holds.

Obviously, being normal and not containing redexes are equivalent conditions.

The normalisation proof for Gr will follow the method employed by Troelstra and Schwichtenberg (1996). The basic idea behind this proof is that, generally, by applying a reduction rule, we eliminate a redex of a certain complexity and, possibly, generate new redexes of smaller complexity. For most reduction rules, there is nothing more to say. If we apply them to a suitably selected redex in our derivation, either the maximal redex complexity decreases, or the number of redexes with maximal complexity decreases. If all our reduction rules were of this kind, we could just prove the normalisation by induction on a pair of values representing the maximal complexity of the redexes in the derivation and the number of redexes with maximal complexity occurring in the derivation. Nevertheless, not all reduction rules have such a smooth behaviour because some of them implement permutations between rules, and a permutation does not have any effect on the complexity of redexes. We need therefore a method to keep track of permutations and to account for them in the complexity measure that we adopt. In order to do so, we borrow the notion of *segment* from (Troelstra & Schwichtenberg, 1996, Def. 6.1.1.). A segment is a path inside the derivation tree which connects two rule applications. But not any path is a segment. Intuitively, a path is a segment only if it meets two conditions: first, the path must connect two rule applications that would form a redex if they occurred one immediately after the other—where this redex must be different from a permutation redex—and, second, it must be possible to shorten the path by using permutations and eventually obtain the redex formed by the two rule applications.

Definition 4.6 (Segment and Segment Complexity)**.** For any Gr derivation d, a segment of length n in d is a sequence $A_1, \ldots, A_n$ of formula occurrences in d such that the following holds.

(1) For $1 < i < n$, one of the following holds:
  - A_i is a minor premiss of an application of $\vee$ elimination in d with conclusion $A_{i+1} = A_i$,
  - A_i is the premiss of a converse rule with conclusion A_{i+1} and the logical complexity of A_i is the same as that of A_{i+1},
  - A_i is the premiss of an $\alpha\kappa$ rule.

(2) A_n is not the minor premiss of a $\vee$ elimination, not the premiss of a converse rule the conclusion of which has the same logical complexity as A_n, and not the premiss of an $\alpha\kappa$ rule.

(3) A_1 is not the conclusion of a $\vee$ elimination, not the conclusion of a converse rule the premiss of which has the same logical complexity as A_1, and not the conclusion of an $\alpha\kappa$ rule.

For any segment, if A_n is the major premiss of an elimination rule and

- $n > 1$ or
- $n = 1$ and A_1 is the conclusion of an introduction rule

then the complexity of the segment is the logical complexity of A. Otherwise, the complexity of the segment is 0.

Notice that all formulae in a segment have the same logical complexity. This is obvious for the case of $\vee$ eliminations, assumed for the case of converse rules, and easy to see for the case of $\alpha\kappa$ rules.

We introduce some terminology to describe the relative position of two segments in a derivation and prove a simple fact about the arrangement of segments in a derivation which will be used in the normalisation proof.

Definition 4.7 (Terminology for Segments)**.** If a segment contains only one formula occurrence, by *reducing the segment* we mean reducing—if possible—the non-permutation redex the critical rules of which are applied immediately above and immediately below the formula; if, otherwise, the segment contains more than one formula occurrence, by reducing the segment we mean reducing the permutation redex which has as bottommost formula the bottommost formula of the segment.

A segment r *occurs above* a segment s if the bottommost formula of r occurs above the bottommost formula of s.

A segment r *occurs to the right* of a segment s if there are derivations ρ and σ such that some formula of r occurs in ρ, some formula of s occurs in σ, the root of ρ and the root of σ are premisses of the same rule application, and the root of ρ occurs to the right of the root of σ with respect to such rule application.

Lemma 4.8. *For any two distinct segments in a derivation d, if neither is to the right of the other, then one is above the other.*

Proof. See the appendix. □

We now prove that all Gr derivations normalize.

Theorem 4.9 (Normalisation)**.** *For any* Gr *derivation d, there is a derivation d' such that d can be reduced to d' in a finite number of reductions and d' is normal.*

Proof. We employ the following reduction strategy. We reduce a rightmost segment of maximal complexity that does not occur below any other maximal segment. By Lemma 4.8, we can always find such a segment.

We prove that this reduction strategy always produces a series of reductions which is of finite length and which results in a normal form.

We define the complexity of a derivation d to be the triple of natural numbers (m, n, u), where m is the maximal complexity of the segments in d, n is the sum of the lengths of the segments in d with segment complexity m, and u is the number of rule applications in d. We then fix a generic derivation d and reason by induction on the lexicographic order on triples of natural numbers.

We moreover suppose that the outermost negation of all hypotheses of the form $\neg A$ discharged by negation introductions is immediately eliminated. If this is not the case, we transform all relevant subderivations as follows:

$$\begin{array}{c} \neg A^n \\ \vdots \\ \dfrac{\bot}{\neg\neg A}\, n \end{array} \quad \mapsto \quad \begin{array}{c} \dfrac{\dfrac{\neg A^n \quad A^m}{\bot}}{\neg A}\, m \\ \vdots \\ \dfrac{\bot}{\neg\neg A}\, n \end{array}$$

Notice moreover that if a derivation complies with this assumption, no reduction can produce a derivation that does not comply with it.

If the complexity of d is $(0, 0, u)$ then d is normal and the claim holds.

Suppose now that the complexity of d is (m, n, u), that $m + n > 0$, and that for each derivation simpler than d the claim holds. Since $m + n > 0$, there must be at least one maximal segment in d. By Lemma 4.8, we can always find one which does not occur below any other maximal segment. We reduce a maximal segment which does not occur below any other segment of this kind, and we reason on the obtained derivation d' by cases on the shape of the reduction—we use lowercase Greek letters to denote subderivations of d.

We only present some exemplar cases, see the appendix for an unabridged version of the proof.

- $$\dfrac{\dfrac{\overset{\alpha}{A \vee B} \quad \overset{\beta}{C} \quad \overset{\gamma}{C}}{C}}{D}\,\text{elim.} \quad \mapsto \quad \dfrac{\overset{\alpha}{A \vee B} \quad \dfrac{\overset{\beta}{C}}{D} \quad \dfrac{\overset{\gamma}{C}}{D}}{D}$$

  We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of C and D, (ii) the segment contains the displayed D, (iii) the segment contains the displayed C. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one since D is obtained by eliminating some connectives of C. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \to B}}{A \to B'}\,\alpha\kappa \quad \overset{\beta}{A}}{B'} \quad \mapsto \quad \dfrac{\dfrac{A \to B \quad \overset{\beta}{A}}{B}}{B'}\,\alpha\kappa$$

  We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain neither the displayed occurrence of B' nor the displayed occurrence of $A \to B'$; (ii) the segment contains the displayed occurrence of B'; (iii) the segment contains the displayed occurrence of $A \to B'$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since B' is less complex than $A \to B$ and $A \to B'$. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A^{\perp}} \quad | \, \overset{\beta}{B^{\perp}}}{A \to B} \quad \overset{\gamma}{A}}{B} \quad \mapsto \quad \dfrac{\dfrac{\overset{\alpha}{A^{\perp}} \quad \overset{\gamma}{A}}{\perp}}{B}$$

  By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$

since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain neither the displayed occurrence of $A^\bot$ nor the displayed occurrence of $A \to B$, (ii) the segment contains the displayed occurrence of $A^\bot$, (iii) the segment contains the displayed occurrence of $A \to B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might have increased the complexity of the segment since $A^\bot$ was the premiss of an introduction rule and now is the premiss of an elimination rule, but the resulting complexity is still less than the complexity of the reduced segment since $A^\bot$ is less complex than $A \to B$. We just eliminated the only segment for which (iii) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A}\quad \overset{\beta}{B}\quad \begin{matrix}[A]^n\ [B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix}}{\neg C}\,n \qquad \overset{\gamma}{C}}{\bot} \mapsto \begin{matrix}\alpha\ \beta\ \gamma\\ A\ B\ C\\ \vdots\\ \bot\end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed expression occurrences, (ii) the segment contains some of the displayed occurrences of A, B or C (iii) the segment contains some of the displayed occurrences of $\bot$, (iv) the segment contains the displayed occurrence of $\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since C, A and B are less complex than $\neg C$; C for obvious reasons, and A and B because of the restrictions of the $\neg$ introduction rule. If (iii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since $\bot$ is less complex than $\neg C$. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{\neg A}}{A^\bot}\quad \overset{\beta}{A}}{\bot} \mapsto \dfrac{\overset{\alpha}{\neg A}\quad \overset{\beta}{A}}{\bot}$$

By the reduction we decrease the length of one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m, n', u') < (m, n, u)$ since we decreased the length of a maximal segment. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed $A^\bot$ and $\bot$, (ii) the segment contains the displayed $A^\bot$. If

(i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction decreased the length of the segment—notice indeed that the complexity of $\neg A$ and $A^{\perp}$ is the same.

□

4.1. The subformula property for Gr

We prove now that the normalisation procedure for Gr yields derivations that enjoy the subformula property: each formula occurring in a normal derivation either already appears in some hypothesis of the derivation or appears inside its conclusion. Since in Gr we allow rules for commutativity and associativity to be applied inside formulae because we do not want to distinguish between a-c equivalent formulae, we need to accordingly adapt our formulation of the subformula property. Therefore, in stating it, we will consider different a-c equivalent formulae as the same formula, or to be more precise as undistinguishable formulae. Thus, for instance, if the formula $A \wedge B$ is a subformula of F, then we also consider $B \wedge A$ a subformula of F.

Since, moreover, expressions which are not formulae might occur in a derivation, we strengthen the statement of the subformula property in order to include these expressions as well. If an expression is not a formula, though, we certainly cannot expect it to be a subformula of some hypothesis or of the conclusion of the derivation. Therefore, instead of requiring this of the expressions themselves, we require it of their formula interpretation. As a consequence, also negation plays a particular role. Indeed, the formula interpretation of $A^{\perp}$ might be $\neg A$.

Theorem 4.10 (Subformula Property)**.** *For any normal* Gr *derivation d of an expression G from hypotheses Γ, for any expression E occurring in d, one of the following holds:*

(1) $f(E) = \perp$,
(2) $f(E)$ *is a-c equivalent to a subformula of the formula interpretation of an element of* $\Gamma \cup \{G\}$.
(3) $f(E)$ *is a-c equivalent to the negation of a subformula of the formula interpretation I of an element of* $\Gamma \cup \{G\}$ *which does not have two negations as outermost connectives.*
(4) $f(E)$ *is a-c equivalent to the double negation of a subformula of the formula interpretation I of an element of* $\Gamma \cup \{G\}$ *which does not have a negation as outermost connective.*[8]

Proof. We prove a stronger statement:

For any normal Gr derivation d of an expression G from hypotheses Γ, for any expression E occurring in d, one of the following holds:

(1) $f(E) = \perp$,
(2) $f(E)$ is a-c equivalent to a subformula of the formula interpretation of an element of $\Gamma \cup \{G\}$.
(3) $f(E)$ is a-c equivalent to the negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have two negations as outermost connectives.

[8] The last two conditions are required because of the double negation elimination rule and since, according to Definition 3.4, the negations $\neg A$ and $\neg B$ are contained in some of the multisets which are completely and immediately less g-complex than $A \wedge B$, $A \vee B$ and $A \rightarrow B$. Which means that, with respect to our formalisation of grounding, we must regard also $\neg A$ and $\neg B$ as immediate subformulae of $A \wedge B$, $A \vee B$ and $A \rightarrow B$.

(4) $f(E)$ is a-c equivalent to the double negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have a negation as outermost connective.

Moreover, if G is obtained by an elimination rule which is not a $\vee$ or $\neg\neg$ elimination, G is a subformula of an element of Γ.

Consider any normal Gr derivation d of G from hypotheses Γ. We show by induction on the number of rules applied in d that the statement holds for d. If d contains no rule application, then d consists only in the hypothesis G and the statement trivially holds. Suppose now that d contains $n+1$ rule applications and that the statement holds for all derivations containing n or less rule applications. We show that the statement holds for d as well. We reason on the last rule applied in d.

We only present a few exemplar cases, see the appendix for an unabridged version of the proof.

- $\dfrac{A \quad \mid B^{\perp}}{A \vee B}$ By inductive hypothesis, the statement holds for the derivations of A and $B^{\perp}$. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well. The formula $f(B^{\perp})$ is indeed either a subformula of B, and 2 holds, or the negation of B, and thus the negation of a proper subformula of $A \vee B$, and 3 holds.

- $\dfrac{A \quad B \qquad \begin{matrix}[A]^n \ [B]^n \ \{C\}^n \\ \vdots \\ [[\perp]]\end{matrix}}{\neg C}\, n$ By inductive hypothesis, the statement holds for the derivations of A and B and for the derivation of $[[\perp]]$ from $[A]$, $[B]$ and $\{C\}$. By the restrictions on the form of the derivation of $[[\perp]]$, we know that A and B are proper subformulae of C. Therefore, the statement holds also for d.

- $\dfrac{A \vee B \quad \begin{matrix}A^n \\ \vdots \\ C\end{matrix} \quad \begin{matrix}B^n \\ \vdots \\ C\end{matrix}}{C}\, n$ Since d is normal, $A \vee B$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $A \vee B$ is a subformula of a hypothesis H. Since all subformulae of A and B are also proper subformulae of H, we have, by induction hypothesis on the derivation of $A \vee B$ and one the two derivations of C, that the statement holds for d as well.

□

4.2. Formal explanations and proofs

We have thus proven that Gr is a suitable grounding calculus and a complete one for classical logic too. We can consider now the kind of analysis that Gr enables us to conduct on specific derivations.

First of all, in Gr we can both construct grounding derivations and classical derivations. Since, moreover, Gr is a fully modular calculus, we can also interleave these two kinds of derivations. Interleaving them, though, does not imply a loss of information: we can still immediately determine what parts of a derivation are explanatory and what parts are purely logical—and this can be simply done by noticing that formal explanations only contain rule applications denoted by a double inference line. The fact that formal explanations in Gr are not obtained by applying additional redundant rules expressly introduced to construct formal explanations, but are obtained by applying the same rules that are required to formalise regular logical reasoning, shows that Gr implements a perspective on logical reasoning according to which formal explanations are a non-redundant subset of logical derivations, and thus can be really considered as logical derivations of a particular kind. The normalisation and analyticity results for Gr, moreover, guarantee that Gr is a reasonable and natural way to formalise logical reasoning. Hence, Gr constitutes a rather minimal framework formally embodying the idea that we can see formal explanations as logical derivations of a particular kind. Hence, if we restrict ourselves to the context of pure logic, the calculus Gr strongly corroborates the idea, shared for instance by Aristotele (Barnes, 1984, Post. An. I, 2–8) and Bolzano (2014), that the general notion of proof covers both the notion of proof-that—a proof that just guarantees that a statement is true—and the notion of proof-why—a proof that also provides the reason why the statement is true.

Let us exemplify how Gr displays this relationship between formal explanations and proofs by considering a simple example. Consider, for instance, the classical tautology $P \to P \vee Q$ where P and Q are atomic formulae. We can prove it in Gr as follows:

$$\dfrac{\overset{\alpha}{P\vee\neg P} \qquad \dfrac{\overset{\beta}{Q\vee\neg Q}\quad \dfrac{\dfrac{P^1\quad Q^2}{P\vee Q}\quad \mid P^1}{P\to P\vee Q} \quad \dfrac{\dfrac{P^1 \ \mid \dfrac{\neg Q^2}{Q^\perp}}{P\vee Q}\quad \mid P^1}{P\to P\vee Q}}{P\to P\vee Q}\,{\scriptstyle 2} \quad \dfrac{\overset{\beta}{Q\vee\neg Q}\quad \dfrac{\dfrac{\neg P^1}{P^\perp}\quad \dfrac{Q^3 \mid \dfrac{\neg P^1}{P^\perp}}{P\vee Q}}{P\to P\vee Q}\quad \overset{\gamma}{P\to P\vee Q}}{P\to P\vee Q}\,{\scriptstyle 3}}{P\to P\vee Q}\,{\scriptstyle 1}$$

where α and β are proofs of the relevant instances of the excluded middle for atoms, and γ is the derivation

$$\dfrac{\dfrac{\neg P^1}{P^\perp}\ \mid \qquad \dfrac{\dfrac{\neg P^1\quad \neg Q^3 \qquad \dfrac{\{P\vee Q\}\quad \dfrac{\dfrac{[\neg P]^4}{[P^\perp]}\quad \{\{P\}\}^5}{[[\perp]]}\quad \dfrac{\dfrac{[\neg Q]^4}{[Q^\perp]}\quad \{\{Q\}\}^5}{[[\perp]]}}{[[\perp]]}\,{\scriptstyle 5}}{\neg(P\vee Q)}\,{\scriptstyle 4}}{(P\vee Q)^\perp}}{P\to P\vee Q}$$

Now, the whole derivation is clearly not an explanation since several logical rules are used in it. At the same time, it contains several applications of grounding rules. Hence we can say that it is a logical proof of $P \to P \vee Q$—a proof *that* $P \to P \vee Q$ holds—which also contains formal explanations of certain formulae. One of these is the following:

$$(1) \qquad \dfrac{\dfrac{P^1\quad Q^2}{P\vee Q}\quad \mid P^1}{P\to P\vee Q}$$

Derivation (1) is a formal explanation of the truth of $P \to P \vee Q$ employing the truth of P and Q. We can immediately see that (1) is a formal explanation because it only contains grounding rule applications, which are recognisable by the double inference line.

This simple example is already enough to prove several interesting points concerning

Gr. It shows us that (*i*) a formal explanation can just be seen as a derivation of a particular kind, (*ii*) we can combine logical derivations and formal explanations, and (*iii*) we can immediately distinguish the explanatory parts of a logical derivation from its non-explanatory parts. Explanation (1) shows moreover that (*iv*) we can compose formal explanations in order to construct chains of grounding steps mounting up from consequences to simpler and simpler grounds. Indeed in (1) we have that $P \to P \vee Q$ is explained by $P \vee Q$ under the condition that P is true, and, in turn, $P \vee Q$ is explained by P and Q. And this chain of explanations is entirely contained in a single Gr derivation, making the calculus also a suitable means for formalising the notion of mediate grounding.

More general conclusions on logical grounding can also be drawn from the analysis of Gr. The calculus, indeed, enables us to see that for any formula F containing at least one conjunction, disjunction or implication we can find a grounding derivation with conclusion F—it is enough to apply suitable grounding rules of Gr backwards starting from F. By such a derivation, we can formally explain the truth of F under the hypothesis that its relevant subformulae are true. Nevertheless, we can also see that no formula can have a proof entirely composed of grounding steps, that is, a formal explanation with no undischarged hypotheses. Indeed, grounding rule applications in Gr always depend on undischarged hypotheses, which reflects the fact that all elements of the ground of a formula are supposed to be true—and hence, proof-theoretically speaking, either derived or assumed as hypotheses.

Finally, as far as the relation between formal explanations and normal proofs is concerned, Gr provides us with strong evidence in favour of the existence of an essential connection between explanatoriness and normality as far as logical proofs are concerned. Indeed, formal explanations in Gr are not modified by the normalisation procedure, and we can hence consider them as derivations which are normal by nature. Technically, individual grounding rule applications might be eliminated during the normalisation, if they constitute redundant steps in the derivation, but a series of grounding rule applications is never restructured by normalisation reductions. This is due to the fact that all grounding rules are introduction rules and perfectly matches the idea that a formal explanation is already supposed to be a logical derivation in which all relevant information about the conclusion of each rule application occurring in it is orderly displayed in the clearest and most direct way. It is moreover easy to see that if a derivation only contains grounding rule applications, it already enjoys the subformula property—modulo proof-theoretic equivalences and converses—and is thus analytic. This relation between formal explanations and normal proofs seems also to be in line with the comparison proposed by Rumberg (2013) between *grounding trees* for conceptual truths—that is, a tree representing the ascension from a conceptual truth to simpler and simpler grounds for it—and *canonical normal proofs* as defined in Dummett (1991).

5. Conclusions

In the present work we investigated grounding rules, and formal explanations constructed by using them, from a proof-theoretical perspective. First of all, we showed that grounding rules can be employed as logical introduction rules, and thus we corroborated the view that a formal explanation by grounding rules can be seen as a logical derivation of a particular kind. We proved, indeed, that grounding rules behave rather satisfactorily as introduction rules and are balanced with respect to elimination rules from a proof-theoretical perspective since they admit a normalisation procedure that yields analytic derivations. Moreover, we showed that the introduced calculus constitutes a significant improvement with respect to the calculus presented by Poggiolesi (2018), since it enables us to combine logical derivations and explanations, to distinguish the explanatory parts of derivations from their non-explanatory parts, and to compose explanations in order to construct chains of consecutive grounding steps and thus formalise the notion of mediate grounding.

Many questions concerning the notion of logical grounding have not yet been answered. Among the several obscure areas that still surround the notion of logical grounding, we point at those that constitute natural extensions of the present work.

First of all, the calculus Gr does not only contain grounding rules for introducing connectives, but also contains one logical introduction rule for negation. While this enables us to prove a relatively simple normalisation procedure yielding proofs that enjoy the subformula property, one might wonder whether it is possible to define a calculus that exclusively uses grounding rules as introduction rules. Such a formal system could be key for understanding in more depth the relationship between classical logic and grounding rules. On a similar line, since grounding rules are sound with respect to intuitionistic logic and since formal explanation and constructive reasoning seem to have strong connections, it would be extremely interesting to endeavour also in an investigation of the relation between grounding and intuitionistic logic, which could be developed on the basis of the connection between classical logic and grounding displayed in the present work.

A study of the proof-theoretical properties of the rules governing the grounding operator is still missing as well. It would be interesting, in particular, to investigate whether introduction and elimination rules for this operator display proof-theoretical balance to some degree and, if not, what features of this relation interfere with the proof-theoretical properties of its logical rules. There are, moreover, two main ways to generalise the grounding relation in order to capture different notions of explanation: on the one hand, we can consider transitive grounding, which enables us to directly relate a truth to any of the simpler—and not necessarily immediately simpler—truths on which it depends; on the other hand, we can consider explanation trees, which are complex objects constructed by chaining individual grounding steps one after the other, without loosing any information on each individual grounding step. It would be of great technical and philosophical interest to develop proof-theoretical frameworks in which suitable grounding operators can be used to represent these objects in the logical language. These frameworks would enable an extensive proof-theoretical analysis of a logic of formal explanations and might provide useful philosophical insights on grounding itself. Finally, since grounding and explanation are typical examples of hyper-intensional notions, such an endeavour could constitute a considerable step forward for the proof-theory of hyper-intensional operators, a widely unexplored field of great interest for both logics and philosophy.

Acknowledgements

We thank the anonymous reviewers for their extensive and detailed comments, which greatly helped to improve this article. Moreover, we thank Francesca Poggiolesi very much for a great number of long and fruitful discussions and for very useful comments on the first draft of this article.

Declaration of interest

The author declares that there are no known conflicts of interest associated with this work and that there has been no significant financial support that could have influenced its outcome.

References

Barnes, J. (Ed.). (1984). *The complete works of aristotle.* Princeton University Press.

Betti, A. (2010). Explanation in metaphysics and bolzano's theory of ground and consequence. *Logique & Analyse*, *211*, 281–316.

Bolzano, B. (2014). *Theory of science.* Oxford: Oxford University Press. (Translated by Rolf George and Paul Rusnok)

Correia, F. (2010). Grounding and truth-functions. *Logique et Analyse*, *53*(211), 251–279.

Correia, F. (2014). Logical grounds. *The review of symbolic logic*, *7*(1), 31–59.

Correia, F., & Schnieder, B. (2012). *Methaphysical grounding. understanding the structure of reality.* Cambridge University Press.

Dummett, M. (1991). *The logical basis of metaphysics.* Harvard university press.

Fine, K. (2012a). Guide to ground. In *Methaphysical grounding. understanding the structure of reality* (pp. 37–80). Cambridge University Press.

Fine, K. (2012b). The pure logic of ground. *The Review of Symbolic Logic*, *5*(1), 1–25.

Korbmacher, J. (2018a). Axiomatic theories of partial ground i. *Journal of Philosophical Logic*, *47*(2), 161–191.

Korbmacher, J. (2018b). Axiomatic theories of partial ground ii. *Journal of Philosophical Logic*, *47*(2), 193–226.

McSweeney, M. M. (2020). Debunking logical ground: Distinguishing metaphysics from semantics. *Journal of the American Philosophical Association*, *6*(2), 156–170.

Poggiolesi, F. (2016). On defining the notion of complete and immediate formal grounding. *Synthese*, *193*, 3147–3167.

Poggiolesi, F. (2018). On constructing a logic for the notion of complete and immediate formal grounding. *Synthese*, *195*, 1231–1254.

Prawitz, D. (1971). Ideas and results in proof theory. In *Proceedings of the second scandinavian logic symposium.*

Prawitz, D. (2015). Explaining deductive inference. In H. Wansing (Ed.), *Dag prawitz on proofs and meaning* (pp. 65–100). Springer International Publishing.

Prawitz, D. (2019). The concepts of proof and ground. In *Mathesis universalis, computability and proof* (pp. 291–309). Springer.

Rosen, G. (2010). Metaphysical dependence: Grounding and reduction explanation in mathematics. In B. Hale & A. Hoffmann (Eds.), *Modality: Metaphysics, logic, and epistemology* (pp. 109–136). Oxford University Press.

Rumberg, A. (2013). Bolzano's concept of grounding (abfolge) against the background of normal proofs. *Review of Symbolic Logic*, *6*(3), 424–459.

Schnieder, B. (2011). A logic for because. *Review of Symbolic Logic*, *4*(3), 445–465.

Sider, T. (2018). Ground grounded. *Philosophical Studies*, 1–21.

Troelstra, A. S., & Schwichtenberg, H. (1996). *Basic proof theory.* Cambridge University Press.

Appendix. Unabridged Proofs

Proposition 2.5 The rules. $\dfrac{\mathcal{C}[(A \star B) \star C]}{\mathcal{C}[A \star (B \star C)]}\ \alpha\kappa$ for $\star \in \{\wedge, \vee\}$ are derivable by using applications of rules in Table 1 for the same context $\mathcal{C}[x]$.

Proof. The rule

$$\frac{\mathcal{C}[(A \star B) \star C]}{\mathcal{C}[A \star (B \star C)]}\ \alpha\kappa$$

for $\star \in \{\wedge, \vee\}$ can be derived by the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{C}[(A \star B) \star C]}{\mathcal{C}[C \star (A \star B)]}}{\mathcal{C}[(C \star A) \star B]}}{\mathcal{C}[B \star (C \star A)]}}{\mathcal{C}[(B \star C) \star A]}}{\mathcal{C}[A \star (B \star C)]}$$

$\square$

Proposition 2.8. For any A, the following is a $\mathrm{Gr}^{\triangleright}$ derivation:

$$\frac{f(A^{\perp})}{A^{\perp}}$$

Proof. If $A = \neg^{2n}B$, then $f(A^{\perp}) = \neg A$ and

$$\frac{\neg\neg^{2n}B}{(\neg^{2n}B)^{\perp}}$$

If $A = \neg\neg^{2n}B$, then $f(A^{\perp}) = \neg^{2n}B$ and

$$\frac{\neg^{2n}B}{(\neg\neg^{2n}B)^{\perp}}$$

$\square$

Lemma 3.7. For any formula A, there is a derivation of A in NC from the hypothesis $\neg f(A^{\perp})$.

Proof. If $A = \neg^{2n}B$, then $f(A^{\perp}) = \neg A$ and $\neg f(A^{\perp}) = \neg\neg A$. Hence, we can derive A in NC as follows:

$$\cfrac{\cfrac{\neg\neg A \quad \neg A^{1}}{\bot}}{A}\ 1$$

If $A = \neg\neg^{2n}B$, then $f(A^{\perp}) = \neg^{2n}B$ and $\neg f(A^{\perp}) = \neg\neg^{2n}B$. Thus, $\neg f(A^{\perp}) = A$. $\square$

Theorem 3.8 (Ground Soundness)**.** For any consistent set of formulae $\{G_1, \ldots, G_n, D, C\} \subseteq \mathcal{L}_{\mathrm{CL}}$, if we can derive

$$G_1, \ldots, G_n \mid C \triangleright D$$

in $\mathrm{Gr}^{\triangleright}$ from the hypotheses $G_1, \ldots, G_n, D$, then $\{G_1, \ldots, G_n\}$ completely and immediately grounds D under the, possibly empty, robust condition C according to Definition 3.5.

Proof. Since $G_1, \ldots, G_n, D, C \in \mathcal{L}_{\mathrm{CL}}$, the connective $\triangleright$ is introduced immediately below a grounding rule application—see Table 4. We reason on the grounding rule which is applied immediately above the introduction of $\triangleright$ and on the derivations $\alpha\kappa_1 \ldots, \alpha\kappa_n$ of its premises: $\dfrac{\genfrac{}{}{0pt}{}{G_1}{\genfrac{}{}{0pt}{}{\alpha\kappa_1}{G'_1}} \quad \ldots \quad \genfrac{}{}{0pt}{}{G_n}{\genfrac{}{}{0pt}{}{\alpha\kappa_n}{G'_n}} \quad \mid \quad \genfrac{}{}{0pt}{}{C}{\genfrac{}{}{0pt}{}{\alpha\kappa_{n+1}}{C'}}}{D}$. In the following we denote by A' the premiss of the grounding rule which is derived by $\alpha\kappa$ rules from the ground, or robust condition, A.

- $\dfrac{A' \quad B'}{A' \wedge B'}$ First, we can derive the conclusion of this inference from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and conjunction introduction.

  Second, we can derive $\neg(A' \wedge B')$ from $\neg A$ and $\neg B$ since we can derive $\neg A'$ and $\neg B'$ from these two formulae by $\alpha\kappa$ rules and then construct the following derivations:

$$\dfrac{\dfrac{\neg A' \qquad \dfrac{A' \wedge B'^{1}}{A'}}{\bot}}{\neg(A' \wedge B')}\,1 \qquad \text{and} \qquad \dfrac{\dfrac{\neg B' \qquad \dfrac{A' \wedge B'^{1}}{B'}}{\bot}}{\neg(A' \wedge B')}\,1$$

  Finally, $\{A, B\}$ is less g-complex than $A' \wedge B'$ because $A \wedge B$ and $A' \wedge B'$ are a-c equivalent.

- $\dfrac{A' \quad B'}{A' \vee B'}$ First, we can derive the conclusion of this inference from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and disjunction introduction.

  Second, we can derive $\neg(A' \vee B')$ from $\neg A$ and $\neg B$ since we can derive $\neg A'$ and $\neg B'$ from these two formulae by $\alpha\kappa$ rules and then construct the following derivation:

$$\dfrac{\dfrac{A' \vee B'^{1} \qquad \dfrac{\neg A' \quad A'^{2}}{\bot} \qquad \dfrac{\neg B' \quad B'^{2}}{\bot}}{\bot}\,2}{\neg(A' \vee B')}\,1$$

  Finally, $\{A, B\}$ is less g-complex than $A' \vee B'$ because $A \vee B$ and $A' \vee B'$ are clearly a-c equivalent.

- $\dfrac{A' \quad \mid (B')^{\perp}}{A' \vee B'}$ First, we can derive the conclusion of this inference from the hypothesis A by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and disjunction introduction.

Second, we can derive $\neg(A' \vee B')$ from $\neg A$ and the robust condition $f(B^{\perp})$ since we can derive $\neg A'$ and $f(B'^{\perp})$ from these two formulae by $\alpha\kappa$ rules and then, by Lemma 3.15, construct the following derivation:

$$\dfrac{\dfrac{A' \vee B'^{1} \quad \dfrac{\neg A' \quad A'^{2}}{\perp} \quad \dfrac{f(B'^{\perp}) \quad B'^{2}}{\perp}}{\perp}\,2}{\neg(A' \vee B')}\,1$$

Finally, $\{A, f(B^{\perp})\}$ is less g-complex than $A' \vee B'$ because $A' \vee B' \cong A \vee B$, since they are a-c equivalent, and $\{A, f(B^{\perp})\} = \{A, B^{*}\}$.

- $\dfrac{B' \quad | \, (A')^{\perp}}{A' \vee B'}$ This case is symmetric to the previous one.
- $\dfrac{B' \quad | \, A'}{A' \to B'}$ First, we can derive the conclusion of this inference from the hypothesis B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and implication introduction.

Second, we can derive $\neg(A' \to B')$ from $\neg B$ and the robust condition A since we can derive $\neg B'$ and A' from these two formulae by $\alpha\kappa$ rules and then construct the following derivation:

$$\dfrac{\dfrac{\neg B' \quad \dfrac{A' \quad A' \to B'^{1}}{B'}}{\perp}}{\neg(A' \to B')}\,1$$

Finally, $\{B, A\}$ is clearly less g-complex than $A' \to B'$ since $A' \wedge B'$ is a-c equivalent to $A \vee B$.

- $\dfrac{(A')^{\perp} \quad B'}{A' \to B'}$ First, we can derive the conclusion of this inference from the hypothesis B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17—and implication introduction.

Second, we can derive $\neg(A' \to B')$ from $\neg f(A^{\perp})$ and $\neg B$ since we can derive $\neg f(A'^{\perp})$ and $\neg B'$ from these two formulae by $\alpha\kappa$ rules and then, by Lemma 3.7, construct the following derivation:

$$\dfrac{\dfrac{\neg B' \quad \dfrac{\begin{matrix}\neg f(A'^{\perp}) \\ \vdots \\ A'\end{matrix} \quad A' \to B'^{1}}{B'}}{\perp}}{\neg(A' \to B')}\,1$$

Finally, $\{f(A^{\perp}), B\}$ is less g-complex than $A' \to B'$ because $A' \to B' \cong A \to B$, since they are a-c equivalent, and $\{f(A^{\perp}), B\} = \{A^{*}, B\}$.

- $\dfrac{(A')^{\perp} \quad | \, (B')^{\perp}}{A' \to B'}$ First, we can derive $f((A')^{\perp})$ from the hypothesis $f(A^{\perp})$ by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. Then, by Lemma 3.15, we can derive the conclusion of this rule from $f((A')^{\perp})$as follows:

$$\dfrac{\dfrac{\dfrac{f((A')^{\perp}) \quad (A')^1}{\perp}}{B'}}{A' \to B'}\,1$$

Second, we can derive $\neg(A' \to B')$ from $\neg f(A^{\perp})$ and the robust condition $f(B^{\perp})$ since we can derive $\neg f(A'^{\perp})$ and $f(B'^{\perp})$ from these two formulae by $\alpha\kappa$ rules and then construct the following derivation, by also using Lemmata 3.7 and 3.15:

$$\dfrac{\dfrac{f(B'^{\perp}) \qquad \dfrac{\begin{array}{c}\neg f(A'^{\perp})\\ \vdots \\ A'\end{array} \quad A' \to B'^1}{B'}}{\perp}}{\neg(A' \to B')}\,1$$

Finally, $\{f(A^{\perp}), f(B^{\perp})\}$ is less g-complex than $A' \to B'$ because $A' \to B' \cong A \to B$, since they are a-c equivalent, and $\{f(A^{\perp}), f(B^{\perp})\} = \{A^*, B^*\}$.

- $$\dfrac{A' \quad B' \qquad \begin{array}{c}[A']^n\ [B']^n\ \{C'\}^n\\ \vdots \\ [[\perp]]\end{array}}{\neg C'}\,n$$ First of all, we notice that the hypotheses $[A']$, $[B']$ and $\{C'\}$ are between parentheses. Now, the only rules that can be applied to formulae between parentheses are $\alpha\kappa$ rules, converse rules and the following five rules:

$$\dfrac{\{A \wedge B\}}{\{\{A\}\}} \qquad \dfrac{\{A \wedge B\}}{\{\{B\}\}} \qquad \dfrac{\{A \vee B\} \quad \begin{array}{c}\{\{A\}\}^n\\ \vdots \\ [[\perp]]\end{array} \quad \begin{array}{c}\{\{B\}\}^n\\ \vdots \\ [[\perp]]\end{array}}{[[\perp]]}\,n \qquad \dfrac{\{A \to B\} \quad [A]}{\{\{B\}\}} \qquad \dfrac{[A^{\perp}] \quad \{\{A\}\}}{[[\perp]]}$$

Let us call the first two rules displayed here *bracketed conjunction eliminations*, the third *bracketed disjunction elimination*, the fourth *bracketed implication elimination*, and the fifth *bracketed converse elimination*. We argue, considering the restrictions on the applicability of these five rules and considering that the hypotheses $[A']$, $[B']$ and $\{C'\}$ must appear exactly once in the derivation of $[[\perp]]$, that C' must be either of the form $C_1 \vee C_2$ or of the form $C_1 \to C_2$. Indeed, $[[\perp]]$ can only be obtained by bracketed eliminations applied to $\{C'\}$. Moreover, only bracketed implication and bracketed disjunction elimination enable us to use both hypotheses $[A']$ and $[B']$ exactly once, as required by the grounding rule for negation. Indeed, bracketed disjunction elimination enables us to obtain $[[\perp]]$ twice—once for each disjunct, once in combination with $[A']$ and once in combination with $[B']$; and bracketed implication elimination enables us to use one among $[A']$ and $[B']$ to eliminate the implication and the other one to obtain $[[\perp]]$ in combination with the consequent of the implication. If we used bracketed conjunction elimination, on the other hand, we would obtain only one formula from $\{C'\}$ and we would not be able to use it in combination with both $[A']$ and

$[B']$ to obtain $[[\perp]]$. If we consider moreover that the $\alpha\kappa$ rules do not change the main connective of their premiss and that, if we applied a converse introduction rule to $\{C'\}$, we would obtain an expression that cannot be used as premiss of any rule—since there are no rules that act on expression of the form $\{F^\perp\}$—we can conclude that C' must be either of the form $C_1 \lor C_2$ or of the form $C_1 \to C_2$. We reason then by cases on the form of C'.

If $C' = C_1 \lor C_2$, without loss of generality, we have that A' and $C_1^\perp$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules; and B' and $C_2^\perp$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules. Therefore, we need to show that we can derive $\neg(C_1 \lor C_2)$ from the hypotheses A and B. We know that we can derive $f(A')$ and $f(C_1^\perp)$, and $f(B')$ and $f(C_2^\perp)$ from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following NC derivation:

$$\dfrac{\dfrac{C_1 \lor C_2{}^{1} \quad \dfrac{f(C_1^\perp) \quad C_1{}^{2}}{\perp} \quad \dfrac{f(C_2^\perp) \quad C_2{}^{2}}{\perp}}{\perp}\,{\scriptstyle 2}}{\neg(C_1 \lor C_2)}\,{\scriptstyle 1}$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

If $C' = C_1 \to C_2$, without loss of generality, A' and C_1 can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules; and B' and $C_2^\perp$ can be obtained from each other by a—possibly empty—series of applications of $\alpha\kappa$ rules. Therefore, we need to show that we can derive $\neg(C_1 \to C_2)$ from the hypotheses A and B. We know that we can derive A' and C_1, and $f(B')$ and $f(C_2^\perp)$ from the hypotheses A and B by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following NC derivation:

$$\dfrac{\dfrac{f(C_2^\perp) \quad \dfrac{C_1 \to C_2{}^{1} \quad C_1}{C_2}}{\perp}}{\neg(C_1 \lor C_2)}\,{\scriptstyle 1}$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Second, we need to show that if $C' = C_1 \lor C_2$ we can derive $\neg\neg(C_1 \lor C_2)$ from the hypotheses $\neg A$ and $\neg B$, and if $C' = C_1 \to C_2$ we can derive $\neg\neg(C_1 \to C_2)$ from the hypotheses $\neg A$ and $\neg B$. In the first case, we can derive $\neg f(A') = \neg f(C_1^\perp)$ and $\neg f(B') = \neg f(C_2^\perp)$ from $\neg A$ and $\neg B$ by $\alpha\kappa$ rules. Hence, by Lemma 3.7, we can construct the following derivations:

$$\dfrac{\dfrac{\neg(C_1 \lor C_2)^1 \quad \dfrac{\begin{matrix}\neg f(C_1^\perp)\\ \vdots \\ C_1\end{matrix}}{(C_1 \lor C_2)}}{\perp}\,{\scriptstyle 2}}{\neg\neg(C_1 \lor C_2)}\,{\scriptstyle 1} \qquad \text{and} \qquad \dfrac{\dfrac{\neg(C_1 \lor C_2)^1 \quad \dfrac{\begin{matrix}\neg f(C_2^\perp)\\ \vdots \\ C_2\end{matrix}}{(C_1 \lor C_2)}}{\perp}\,{\scriptstyle 2}}{\neg\neg(C_1 \lor C_2)}\,{\scriptstyle 1}$$

or similar ones also including some derivation steps translating the $\alpha\kappa$ rule applications. In the second case, we can derive $\neg A' = \neg C_1$ and $\neg f(B') = \neg f(C_2^{\perp})$ from $\neg A$ and $\neg B$ by $\alpha\kappa$ rules. Hence, by Lemma 3.7, we can construct the following derivations:

$$\cfrac{\cfrac{\neg(C_1 \to C_2)^1 \qquad \cfrac{\cfrac{\cfrac{\neg C_1 \quad C_1{}^{2}}{\perp}}{C_2}}{C_1 \to C_2}\,{\scriptstyle 2}}{\perp}}{\neg\neg(C_1 \to C_2)}\,{\scriptstyle 1} \qquad \text{and} \qquad \cfrac{\cfrac{\neg(C_1 \to C_2)^1 \qquad \cfrac{\begin{matrix}\neg f(C_2^{\perp}) \\ \vdots \\ C_2\end{matrix}}{(C_1 \to C_2)}}{\perp}\,{\scriptstyle 2}}{\neg\neg(C_1 \to C_2)}\,{\scriptstyle 1}$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Finally, we show that $\{f(A), f(B)\}$ is less g-complex than $\neg C'$. Now, if $C' = C_1 \vee C_2$. we have that $A' = C_1^{\perp}$ and $B' = C_2^{\perp}$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there are formulae D and E such that

- ∘ $A = D^{\perp}$ and $B = E^{\perp}$, and thus $D^* = f(A)$ and $E^* = f(B)$,
- ∘ D is a-c equivalent to C_1 and E is a-c equivalent to C_2.

Therefore, $\neg C' = \neg(C_1 \vee C_2) \cong (D \vee E)$ and $\{f(A), f(B)\} = \{D^*, E^*\}$. In conclusion, $\{f(A), f(B)\}$ is less g-complex than $\neg C'$. If, on the other hand, $C' = C_1 \to C_2$. we have that $A' = C_1$ and $B' = C_2^{\perp}$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there is a formula E such that

- ∘ $B = E^{\perp}$, and thus $E^* = f(B)$,
- ∘ E is a-c equivalent to C_2.

Therefore, $\neg C' = \neg(C_1 \to C_2) \cong (A \to E)$ and $\{A, f(B)\} = \{A, E^*\}$. In conclusion, also in this case, $\{A, f(B)\}$ is less g-complex than $\neg C'$.

- $$\dfrac{A' \quad B' \qquad \begin{matrix}[A']^n\ \{C'\}_1^n \\ \vdots \\ [[\perp]]\end{matrix} \qquad \begin{matrix}[B']^n\ \{C'\}_2^n \\ \vdots \\ [[\perp]]\end{matrix}}{\neg C'}\,{\scriptstyle n}$$ By using an argument similar to the one used for the previous case, since the hypotheses $[A']$ and $[B']$ must appear exactly once in the derivations of $[[\perp]]$, we know that the only rules that can be used on the hypotheses $\{C'\}$ are

$$\frac{\{A \wedge B\}_1}{\{\{A\}\}} \qquad \frac{\{A \wedge B\}_2}{\{\{B\}\}}$$

and that C' must be of the form $C_1 \wedge C_2$. Without loss of generality, A' and $C_1^{\perp}$ can be obtained from each other by a—possibly empty—series of $\alpha\kappa$ rule applications, and B' and $C_2^{\perp}$ can be obtained from each other by a—possibly empty—series of $\alpha\kappa$ rule applications. Therefore, we need to show that we can derive $\neg(C_1 \wedge C_2)$ from the hypotheses A and B. We know that we can derive $f(A') = f(C_1^{\perp})$ and $f(B') = f(C_2^{\perp})$ from the hypotheses A and B by $\alpha\kappa$ rules – which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following derivations:

$$\cfrac{\cfrac{f(C_1^{\perp}) \qquad \cfrac{C_1 \wedge C_2{}^{1}}{C_1}}{\perp}}{\neg(C_1 \wedge C_2)}\,{\scriptstyle 1} \qquad \text{and} \qquad \cfrac{\cfrac{f(C_2^{\perp}) \qquad \cfrac{C_1 \wedge C_2{}^{1}}{C_2}}{\perp}}{\neg(C_1 \wedge C_2)}\,{\scriptstyle 1}$$

or similar ones also including some derivation steps translating the $\alpha\kappa$ rule applications.

Second, we need to show that we can derive $\neg\neg(C_1 \wedge C_2)$ from the hypotheses $\neg A$ and $\neg B$. Since we can derive $\neg f(A')$ and $\neg f(C_1^{\perp})$, and $\neg f(B')$ and $\neg f(C_2^{\perp})$ from $\neg A$ and $\neg B$ by $\alpha\kappa$ rules, we can construct the following derivation, by also using Lemma 3.7:

$$\dfrac{\dfrac{\neg(C_1 \wedge C_2)^1 \qquad \dfrac{\dfrac{\neg f(C_1^{\perp})}{\vdots \atop C_1} \qquad \dfrac{\neg f(C_2^{\perp})}{\vdots \atop C_2}}{(C_1 \wedge C_2)}\, 2}{\perp}}{\neg\neg(C_1 \wedge C_2)}\, 1$$

or similar ones also including some derivation steps translating the $\alpha\kappa$ rule applications.

Finally, we show that $\{f(A), f(B)\}$ is less g-complex than $\neg C'$. Now, $A' = C_1^{\perp}$ and $B' = C_2^{\perp}$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there are formulae D and E such that

- $A = D^{\perp}$ and $B = E^{\perp}$, and thus $D^* = f(A)$ and $E^* = f(B)$,
- D is a-c equivalent to C_1 and E is a-c equivalent to C_2.

Therefore, $\neg C' = \neg(C_1 \wedge C_2) \cong (D \wedge E)$ and $\{f(A), f(B)\} = \{D^*, E^*\}$. In conclusion, $\{f(A), f(B)\}$ is less g-complex than $\neg C'$.

- $$\dfrac{A' \qquad \begin{matrix}[A']^n\ \{C'\}_1^n \\ \vdots \\ [[\perp]]\end{matrix} \qquad \begin{matrix}[B'^{\perp}]^n\ \{C'\}_2^n \\ \vdots \\ [[\perp]]\end{matrix} \qquad \mid B'}{\neg C'}\, n$$ By using an argument similar to the one used in the two cases above, since the hypotheses $[A']$ and $[B'^{\perp}]$ must appear exactly once in the derivations of $[[\perp]]$, we know that the only rules that can be used on the hypotheses $\{C'\}$ are

$$\frac{\{A \wedge B\}_1}{\{\{A\}\}} \qquad \frac{\{A \wedge B\}_2}{\{\{B\}\}}$$

and that C' must be of the form $C_1 \wedge C_2$. Without loss of generality, A' and $C_1^{\perp}$ can be obtained from each other by a—possibly empty—series of $\alpha\kappa$ rules applications, and that B' and C_2 can be obtained from each other by a—possibly empty—series of $\alpha\kappa$ rules applications. Therefore, we need to show that we can derive $\neg(C_1 \wedge C_2)$ from the hypotheses A and B. We know that we can derive $f(A')$ and $f(C_1^{\perp})$ from the hypothesis A by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. But then, by Lemma 3.15, we can construct the following derivation:

$$\dfrac{\dfrac{f(C_1^{\perp}) \qquad \dfrac{C_1 \wedge C_2{}^1}{C_1}}{\perp}}{\neg(C_1 \wedge C_2)}\, 1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Second, we need to show that we can derive $\neg\neg(C_1 \wedge C_2)$ from the hypothesis

$\neg A$ and the robust condition B. Since we can derive $\neg f(A')$ and $\neg f(C_1^\perp)$, and B' and C_2 from $\neg A$ and B by $\alpha\kappa$ rules, we can construct the following derivation, by also using Lemma 3.7:

$$\dfrac{\dfrac{\neg(C_1 \wedge C_2)^1 \qquad \dfrac{\dfrac{\neg f(C_1^\perp)}{\vdots \atop C_1} \quad C_2}{(C_1 \wedge C_2)}\,2}{\bot}}{\neg\neg(C_1 \wedge C_2)}\,1$$

or a similar one also including some derivation steps translating the $\alpha\kappa$ rule applications.

Finally, we show that $\{f(A), B\}$ is less g-complex than $\neg C'$. Now, $A' = C_1^\perp$ and $B' = C_2$ can be obtained from $\alpha\kappa$ rules from A and B. Hence, there is a formula D such that

- ◦ $A = D^\perp$, and thus $D^* = f(A)$,
- ◦ D is a-c equivalent to C_1.

Therefore, $\neg C' = \neg(C_1 \wedge C_2) \cong (D \wedge B)$ and $\{f(A), B\} = \{D^*, B\}$. In conclusion, $\{f(A), B\}$ is less g-complex than $\neg C'$.

- $\dfrac{A'}{\neg\neg A'}$ First, we can derive A from A' by $\alpha\kappa$ rules—which are sound with respect to classical logic, see Theorem 3.17. Then, from A' we can derive the conclusion of the rule as follows:

$$\dfrac{\dfrac{A' \quad (\neg A')^1}{\bot}}{\neg\neg A'}\,1$$

Second, we can derive $\neg A'$ from $\neg A$ by $\alpha\kappa$ rules. Then, we can derive $\neg\neg\neg A'$ as follows:

$$\dfrac{\dfrac{\neg A' \quad (\neg\neg A')^1}{\bot}}{\neg\neg\neg A'}\,1$$

Finally, $\{A\}$ is less g-complex than $\neg\neg A'$ because $\neg\neg A \cong \neg\neg A'$. Which in turn is true since A and A' are a-c equivalent.

□

Lemma 3.14. For any formula A, $A \vee f(A^\perp)$ is derivable in Gr and in NC.

Proof. If $A = \neg^{2n} B$, then $f(A^\perp) = \neg A$ and $f((\neg A)^\perp) = A$. Thus, we can derive

$A \vee f(A^{\perp}) = A \vee \neg A$ in Gr as follows:

$$
\begin{prooftree}
\AxiomC{$\neg(A \vee \neg A)^1$}
\AxiomC{$\neg(A \vee \neg A)^1$}
\AxiomC{A^2}
\AxiomC{A^2}
\UnaryInfC{$(\neg A)^{\perp}$}
\doubleLine
\BinaryInfC{$A \vee \neg A$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\AxiomC{$\neg(A \vee \neg A)^1$}
\AxiomC{A^2}
\AxiomC{A^2}
\UnaryInfC{$(\neg A)^{\perp}$}
\doubleLine
\BinaryInfC{$A \vee \neg A$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\UnaryInfC{$A^{\perp}$}
\doubleLine
\BinaryInfC{$A \vee \neg A$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$\neg\neg(A \vee \neg A)$}
\UnaryInfC{$A \vee \neg A$}
\end{prooftree}
$$

If, on the other hand, $A = \neg\neg^{2n}B$, then $f(A^{\perp}) = \neg^{2n}B$ and $A \vee f(A^{\perp}) = \neg\neg^{2n}B \vee \neg^{2n}B$ is of the form $\neg C \vee C$ where $f(C^{\perp}) = \neg C$ and $f((\neg C)^{\perp}) = C$. Thus, we can derive $A \vee f(A^{\perp}) = \neg C \vee C$ in Gr as follows:

$$
\begin{prooftree}
\AxiomC{$\neg(\neg C \vee C)^1$}
\AxiomC{$\neg(\neg C \vee C)^1$}
\AxiomC{C^2}
\AxiomC{C^2}
\UnaryInfC{$(\neg C)^{\perp}$}
\doubleLine
\BinaryInfC{$\neg C \vee C$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg C$}
\AxiomC{$\neg(\neg C \vee C)^1$}
\AxiomC{C^2}
\AxiomC{C^2}
\UnaryInfC{$(\neg C)^{\perp}$}
\doubleLine
\BinaryInfC{$\neg C \vee C$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg C$}
\UnaryInfC{$C^{\perp}$}
\doubleLine
\BinaryInfC{$\neg C \vee C$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$\neg\neg(\neg C \vee C)$}
\UnaryInfC{$\neg C \vee C$}
\end{prooftree}
$$

As for NC, we just have to notice—as we have argued above—that $A \vee f(A^{\perp})$ has always either the form $C \vee \neg C$ or the form $\neg C \vee C$. Hence, we can always derive $A \vee f(A^{\perp})$ in NC by one of the following derivations:

$$
\begin{prooftree}
\AxiomC{$\neg(C \vee \neg C)^1$}
\AxiomC{$\neg(C \vee \neg C)^1$}
\AxiomC{C^2}
\UnaryInfC{$C \vee \neg C$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg C$}
\UnaryInfC{$C \vee \neg C$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$C \vee \neg C$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$\neg(\neg C \vee C)^1$}
\AxiomC{$\neg(\neg C \vee C)^1$}
\AxiomC{C^2}
\UnaryInfC{$\neg C \vee C$}
\BinaryInfC{$\perp$}
\RightLabel{2}
\UnaryInfC{$\neg C$}
\UnaryInfC{$\neg C \vee C$}
\BinaryInfC{$\perp$}
\RightLabel{1}
\UnaryInfC{$\neg C \vee C$}
\end{prooftree}
$$

$\square$

Lemma 3.15. For any A, one of the following is both a Gr and an NC derivation:

$$
\dfrac{f(A^{\perp}) \quad A}{\perp} \qquad \dfrac{A \quad f(A^{\perp})}{\perp}
$$

Proof. If $A = \neg^{2n}B$, then $f(A^{\perp}) = \neg A$ and

$$\dfrac{\neg A \quad A}{\bot}$$

If $A = \neg\neg^{2n}B$, then $f(A^{\perp}) = \neg^{2n}B$ and

$$\dfrac{\neg\neg^{2n}B \quad \neg^{2n}B}{\bot}$$

$\square$

Lemma 3.16. The $\alpha\kappa$ rules in Table 1 are derivable in NC.

Proof. We show in particular that each instance $\dfrac{\mathcal{C}[H]}{\mathcal{C}[K]}\,\alpha\kappa$ of some $\alpha\kappa$ rule can be simulated by an NC derivation of $\mathcal{C}[K]$ from the hypothesis $\mathcal{C}[H]$. The proof is by induction on the number of symbols in the context $\mathcal{C}[x]$.

If $\mathcal{C}[x] = x$, we have the following NC derivations:

$$\dfrac{\dfrac{A\wedge B}{B} \quad \dfrac{A\wedge B}{A}}{B\wedge A} \qquad \dfrac{\dfrac{\dfrac{A\wedge(B\wedge C)}{A} \quad \dfrac{\dfrac{A\wedge(B\wedge C)}{B\wedge C}}{B}}{A\wedge B} \quad \dfrac{\dfrac{A\wedge(B\wedge C)}{B\wedge C}}{C}}{(A\wedge B)\wedge C} \qquad \dfrac{A\vee B \quad \dfrac{A^1}{B\vee A} \quad \dfrac{B^1}{B\vee A}}{B\vee A}\,{\scriptstyle 1}$$

$$\dfrac{A\vee(B\vee C) \quad \dfrac{\dfrac{A^1}{A\vee B}}{(A\vee B)\vee C} \quad \dfrac{B\vee C^1 \quad \dfrac{\dfrac{B^2}{A\vee B}}{(A\vee B)\vee C} \quad \dfrac{C^2}{(A\vee B)\vee C}}{(A\vee B)\vee C}\,{\scriptstyle 2}}{(A\vee B)\vee C}\,{\scriptstyle 1}$$

Suppose now that $\mathcal{C}[x]$ contains at least one symbol. Suppose moreover, that for each context $\mathcal{D}[x]$ containing less symbols than $\mathcal{C}[x]$ and for any formulae D, E, F, it holds that the inferences

$$\dfrac{\mathcal{D}[D\wedge E]}{\mathcal{D}[E\wedge D]}\,\alpha\kappa \qquad \dfrac{\mathcal{D}[D\wedge(E\wedge F)]}{\mathcal{D}[(D\wedge E)\wedge F]}\,\alpha\kappa \qquad \dfrac{\mathcal{D}[D\vee E]}{\mathcal{D}[E\vee D]}\,\alpha\kappa \qquad \dfrac{\mathcal{D}[D\vee(E\vee F)]}{\mathcal{D}[(D\vee E)\vee F]}\,\alpha\kappa$$

are derivable in NC. We show that, for any formulae A, B, C, the inferences

$$\dfrac{\mathcal{C}[A\wedge B]}{\mathcal{C}[B\wedge A]}\,\alpha\kappa \qquad \dfrac{\mathcal{C}[A\wedge(B\wedge C)]}{\mathcal{C}[(A\wedge B)\wedge C]}\,\alpha\kappa \qquad \dfrac{\mathcal{C}[A\vee B]}{\mathcal{C}[B\vee A]}\,\alpha\kappa \qquad \dfrac{\mathcal{C}[A\vee(B\vee C)]}{\mathcal{C}[(A\vee B)\vee C]}\,\alpha\kappa$$

are derivable in NC as well.

We consider a specific application of $\alpha\kappa$ rule of the form $\dfrac{\mathcal{C}[H]}{\mathcal{C}[K]}\,\alpha\kappa$, and show that we can derive $\mathcal{C}[K]$ in NC from $\mathcal{C}[H]$. We distinguish several cases with respect to the shape of $\mathcal{C}[H]$.

- $\mathcal{C}[H] = \neg\mathcal{E}[H]$. By induction hypothesis, we have that $\dfrac{\mathcal{E}[H]}{\mathcal{E}[K]}\ \alpha\kappa$ is derivable in NC. By Proposition 2.5, $\dfrac{\mathcal{E}[K]}{\mathcal{E}[H]}\ \alpha\kappa$ is also derivable in NC. Therefore, we can derive $\mathcal{C}[K] = \neg\mathcal{E}[K]$ from the hypothesis $\neg\mathcal{E}[H]$ in NC as follows:

$$\dfrac{\dfrac{\neg\mathcal{E}[H] \quad \begin{array}{c}\mathcal{E}[K]^1\\ \vdots\\ \mathcal{E}[H]\end{array}}{\bot}}{\neg\mathcal{E}[K]}\ 1$$

- $\mathcal{C}[H] = \mathcal{E}[H] \wedge J$ for some formula J. By induction hypothesis, we have that $\dfrac{\mathcal{E}[H]}{\mathcal{E}[K]}\ \alpha\kappa$ is derivable in NC. Therefore, we can derive $\mathcal{C}[K] = \mathcal{E}[K] \wedge J$ from the hypothesis $\mathcal{E}[H] \wedge J$ in NC as follows:

$$\dfrac{\begin{array}{c}\dfrac{\mathcal{E}[H] \wedge J}{\mathcal{E}[H]}\\ \vdots\\ \mathcal{E}[K]\end{array} \quad \dfrac{\mathcal{E}[H] \wedge J}{J}}{\mathcal{E}[K] \wedge J}$$

- $\mathcal{C}[H] = J \wedge \mathcal{E}[H]$ for some formula J. This case is symmetric to the previous case.
- $\mathcal{C}[H] = \mathcal{E}[H] \vee J$ for some formula J. By induction hypothesis, we have that $\dfrac{\mathcal{E}[H]}{\mathcal{E}[K]}\ \alpha\kappa$ is derivable in NC. Therefore, we can derive $\mathcal{C}[K] = \mathcal{E}[K] \vee J$ from the hypothesis $\mathcal{E}[H] \vee J$ in NC as follows:

$$\dfrac{\mathcal{E}[H] \vee J \quad \begin{array}{c}\mathcal{E}[H]\\ \vdots\\ \dfrac{\mathcal{E}[K]^1}{\mathcal{E}[K] \vee J}\end{array} \quad \dfrac{J^1}{\mathcal{E}[K] \vee J}}{\mathcal{E}[K] \vee J}\ 1$$

- $\mathcal{C}[H] = J \vee \mathcal{E}[H]$ for some formula J. This case is symmetric to the previous case.
- $\mathcal{C}[H] = \mathcal{E}[H] \to J$ for some formula J. By induction hypothesis, we have that $\dfrac{\mathcal{E}[H]}{\mathcal{E}[K]}\ \alpha\kappa$ is derivable in NC. By Proposition 2.5, $\dfrac{\mathcal{E}[K]}{\mathcal{E}[H]}\ \alpha\kappa$ is also derivable in NC. Therefore, we can derive $\mathcal{C}[K] = \mathcal{E}[K] \to J$ from the hypothesis $\mathcal{E}[H] \to J$ in NC as follows:

$$\dfrac{\dfrac{\mathcal{E}[H] \to J \quad \begin{array}{c}\mathcal{E}[K]^1\\ \vdots\\ \mathcal{E}[H]\end{array}}{J}}{\mathcal{E}[K] \to J}\ 1$$

- $\mathcal{C}[H] = J \to \mathcal{E}[H]$ for some formula J. By induction hypothesis, we have that

$\dfrac{\mathcal{E}[H]}{\mathcal{E}[K]}\ \alpha\kappa$ is derivable in NC. Therefore, we can derive $\mathcal{C}[K] = J \to \mathcal{E}[K]$ from the hypothesis $J \to \mathcal{E}[H]$ in NC as follows:

$$\dfrac{\dfrac{J \to \mathcal{E}[H] \quad J^1}{\mathcal{E}[H]}}{\dfrac{\vdots}{\dfrac{\mathcal{E}[K]}{J \to \mathcal{E}[K]}\ 1}}$$

□

Theorem 3.17 (Classical Soundness)**.** The calculus Gr is sound with respect to the calculus NC.

Proof. We show, in particular, that, for any set of hypotheses Γ and expression E, if $\Gamma \vdash_{\mathrm{Gr}} E$ then $f(\Gamma) \vdash_{\mathrm{NC}} f(E)$. The proof is by induction on the number of rule applications in the Gr derivation of E.

If no rule is applied in the NC derivation of E, $f(\Gamma) = \{f(E)\}$ and the statement trivially holds.

Assume then that the Gr derivation of F contains $n > 0$ rule applications and that if E has a Gr derivation containing m rule applications, for $m < n$, then $f(E)$ has an NC derivation. We consider the last rule applied in the Gr derivation of E.

- $\dfrac{\mathcal{C}[A \wedge B]}{\mathcal{C}[B \wedge A]}\ \alpha\kappa$ By Lemma 3.16, we know that the rule application is derivable in NC. Hence, by induction hypothesis, we have that the conclusion is derivable in NC as well.
- $\dfrac{\mathcal{C}[A \wedge (B \wedge C)]}{\mathcal{C}[(A \wedge B) \wedge C]}\ \alpha\kappa$ By Lemma 3.16, we know that the considered rule application is derivable in NC. Hence, by induction hypothesis, we have that the conclusion is derivable in NC as well.
- $\dfrac{\mathcal{C}[A \vee B]}{\mathcal{C}[B \vee A]}\ \alpha\kappa$ By Lemma 3.16, we know that the considered rule application is derivable in NC. Hence, by induction hypothesis, we have that the conclusion is derivable in NC as well.
- $\dfrac{\mathcal{C}[A \vee (B \vee C)]}{\mathcal{C}[(A \vee B) \vee C]}\ \alpha\kappa$ By Lemma 3.16, we know that the considered rule application is derivable in NC. Hence, by induction hypothesis, we have that the conclusion is derivable in NC as well.
- $\dfrac{\neg\neg^{2n} A}{(\neg^{2n} A)^{\perp}}$ By Definition 2.7, $\neg\neg^{2n} A$ and $f((\neg^{2n} A)^{\perp})$ are the same formula. Therefore, we can clearly derive in NC the conclusion if the premiss is derivable in NC. But the premiss is derivable in NC.
- $\dfrac{\neg^{2n} A}{(\neg\neg^{2n} A)^{\perp}}$ By Definition 2.7, $\neg^{2n} A$ and $f((\neg\neg^{2n} A)^{\perp})$ are the same formula. Therefore, we can clearly derive in NC the conclusion if the premiss is derivable in NC. But the premiss is derivable in NC.
- $\dfrac{A \quad B}{\overline{A \wedge B}}$ Since the conclusion of this rule application is a formula, also the premises must be formulae. By induction hypothesis, moreover, the premisses

are derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.

- $\dfrac{A \quad B}{A \vee B}$ Since the conclusion of this rule application is a formula, also the premises must be formulae. By induction hypothesis, moreover, the premisses are derivable in NC. By one of the following inferences

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

we can construct an NC derivation of the conclusion.

- $\dfrac{A \quad \mid B^{\perp}}{A \vee B}$ Since the conclusion of this rule application is a formula, A must be a formula. By induction hypothesis, moreover, A is derivable in NC. By

$$\frac{A}{A \vee B}$$

we can construct an NC derivation of the conclusion.

- $\dfrac{B \quad \mid A^{\perp}}{A \vee B}$ Since the conclusion of this rule application is a formula, B must be a formula. By induction hypothesis, moreover, B is derivable in NC. By

$$\frac{B}{A \vee B}$$

we can construct an NC derivation of the conclusion.

- $\dfrac{B \quad \mid A}{A \to B}$ Since the conclusion of this rule application is a formula, also the premises must be formulae. By induction hypothesis, moreover, the premisses are derivable in NC. By

$$\frac{B}{A \to B}$$

we can construct an NC derivation of the conclusion.

- $\dfrac{A^{\perp} \quad B}{A \to B}$ Since the conclusion of this rule application is a formula, B must be a formula. By induction hypothesis, moreover, B is derivable in NC. By

$$\frac{B}{A \to B}$$

we can construct an NC derivation of the conclusion.

- $\dfrac{A^{\perp} \quad \mid B^{\perp}}{A \to B}$ By induction hypothesis, $f(A^{\perp})$ is derivable in NC. By Lemma 3.15, one of the following is an NC derivation

$$\dfrac{\dfrac{\dfrac{A^1 \quad f(A^{\perp})}{\perp}}{B}}{A \to B}\,1 \qquad\qquad \dfrac{\dfrac{\dfrac{f(A^{\perp}) \quad A^1}{\perp}}{B}}{A \to B}\,1$$

Therefore, the conclusion of the rule is derivable in NC as well.

- $$\dfrac{A \quad B \quad \begin{array}{c}[A]^n\ [B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{array}}{\neg C}\,n$$ By induction hypothesis, $f(A)$ and $f(B)$ are derivable in NC. Moreover, again by induction hypothesis, there is a derivation of $\bot$ from the hypotheses $f(A), f(B)$ and $f(\{C\}) = C$. Hence, we can derive the conclusion $\neg C$ in NC by

$$\dfrac{\begin{array}{c}f(A)\ f(B)\ C^n\\ \vdots\\ \bot\end{array}}{\neg C}\,n$$

where $f(A)$ and $f(B)$ are not cancelled by the negation introduction but derived by an NC derivation.

- $$\dfrac{A \quad \begin{array}{c}[A]^n\ \{C\}_1^n\\ \vdots\\ [[\bot]]\end{array} \quad \begin{array}{c}[B^\bot]^n\ \{C\}_2^n\\ \vdots\\ [[\bot]]\end{array} \quad \mid B}{\neg C}\,n$$ By induction hypothesis, $f(A)$ and $f(B)$ are derivable in NC. Moreover, again by induction hypothesis, there is a derivation of $\bot$ from the hypotheses $f(A)$ and $f(\{C\}) = C$. Hence, we can derive the conclusion $\neg C$ in NC by

$$\dfrac{\begin{array}{c}f(A)\ C^n\\ \vdots\\ \bot\end{array}}{\neg C}\,n$$

where $f(A)$ is not cancelled by the negation introduction but derived by an NC derivation.

- $\dfrac{A}{\neg\neg A}$ Since the conclusion of the rule is a formula, also A must be a formula. By induction hypothesis, A ia derivable in NC. Therefore, by

$$\dfrac{\dfrac{\neg A^1 \quad A}{\bot}}{\neg\neg A}\,1$$

we can derive the conclusion of the rule in NC.

- $\dfrac{\{A \wedge B\}_1}{\{\{A\}\}}$ By induction hypothesis, $f(\{A \wedge B\}) = A \wedge B$ is derivable in NC. By

$$\dfrac{A \wedge B}{A}$$

we can derive $A = f(\{\{A\}\})$ in NC.

- $\dfrac{\{A \wedge B\}_2}{\{\{B\}\}}$ By induction hypothesis, $f(\{A \wedge B\}) = A \wedge B$ is derivable in NC. By

$$\dfrac{A \wedge B}{B}$$

we can derive $B = f(\{\{B\}\})$ in NC.

- $\dfrac{\{A \vee B\} \quad \begin{matrix}\{\{A\}\}^n \\ \vdots \\ [[\perp]]\end{matrix} \quad \begin{matrix}\{\{B\}\}^n \\ \vdots \\ [[\perp]]\end{matrix}}{[[\perp]]}\, n$ By induction hypothesis, in NC , there is a derivation of $\perp$ both from the hypothesis $f(\{\{A\}\}) = A$ and from the hypothesis $f(\{\{B\}\}) = B$. Moreover, there is an NC derivation of $f(\{A \vee B\}) = A \vee B$. Hence, by

$$\dfrac{A \vee B \quad \begin{matrix}A^n \\ \vdots \\ \perp\end{matrix} \quad \begin{matrix}B^n \\ \vdots \\ \perp\end{matrix}}{\perp}\, n$$

we can construct an NC derivation of the conclusion as well.
- $\dfrac{\{A \to B\} \quad [A]}{\{\{B\}\}}$ By induction hypothesis, $f([A]) = A$ and $f(\{A \to B\}) = A \to B$ are derivable in NC. By

$$\dfrac{A \to B \quad A}{B}$$

we can derive $B = f(\{\{B\}\})$ in NC.
- $\dfrac{[A^\perp] \quad \{\{A\}\}}{[[\perp]]}$ By induction hypothesis, $f(\{\{A\}\}) = A$ and $f([A^\perp])$ are derivable in NC as well. By Lemma 3.15, also the conclusion can be derived in NC.
- $\dfrac{A \wedge B}{A}$ The premiss of this rule is necessarily a formula and, by induction hypothesis, it is derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.
- $\dfrac{A \wedge B}{B}$ The premiss of this rule is necessarily a formula and, by induction hypothesis, it is derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.
- $\dfrac{A \vee B \quad \begin{matrix}A^n \\ \vdots \\ C\end{matrix} \quad \begin{matrix}B^n \\ \vdots \\ C\end{matrix}}{C}\, n$ Since the major premiss $A \vee B$ of this rule is a formula, also A and B must be formulae. By induction hypothesis, $A \vee B$ is derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.
- $\dfrac{A \to B \quad A}{B}$ The premisses of this rule and its conclusion are necessarily formulae. By induction hypothesis, moreover, the premisses are derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.
- $\dfrac{\begin{matrix}A^n \\ \vdots \\ \perp\end{matrix}}{\neg A}\, n$ The conclusion of this rule is necessarily a formula. By induction hypothesis, there is an NC derivation of $\perp$ from the hypothesis A. Since this rule belongs to NC as well, also the conclusion can be derived in NC.
- $\dfrac{\neg A \quad A}{\perp}$ Since $\neg A$ is a formula, also A must be one. By induction hypothesis, moreover, both $\neg A$ and A are derivable in NC. Since this rule belongs to NC as well, also the conclusion can be derived in NC.

- $\dfrac{A^{\perp} \quad A}{\perp}$ By induction hypothesis, $f(A^{\perp})$ and A are derivable in NC as well. By Lemma 3.15, also the conclusion can be derived in NC.
- $\dfrac{\perp}{A}$ By induction hypothesis there is a derivation of $\perp$ in NC. By

$$\frac{\perp}{f(A)}$$

  we can derive $f(A)$ in NC.
- $\dfrac{\neg\neg A}{A}$ Since $\neg\neg A$ is a formula, also A is one. Moreover, by induction hypothesis, $\neg\neg A$ is derivable in NC . By

$$\dfrac{\dfrac{\dfrac{\neg(A \vee \neg A)^2 \qquad \dfrac{\dfrac{\dfrac{\neg(A \vee \neg A)^2 \qquad \dfrac{A^3}{A \vee \neg A}}{\perp}\,3}{\neg A}}{A \vee \neg A}}{\perp}}{A \vee \neg A}\,2 \qquad \dfrac{A^1}{A} \qquad \dfrac{\neg\neg A \quad \neg A^1}{A}}{A}\,1$$

  we can show that also the conclusion is derivable in NC.

$\square$

Theorem 3.18 (Classical Completeness)**.** The calculus Gr is complete with respect to the calculus NC.

Proof. We show, in particular, that, for any set of hypotheses Γ and formula F, $\Gamma \vdash_{\mathrm{NC}} F$ then $\Gamma \vdash_{\mathrm{Gr}} F$. The proof is by induction on the number of rule applications in the NC derivation of F.

If no rule is applied in the NC derivation of F, the statement trivially holds.

Assume then that the NC derivation of F contains $n > 0$ rule applications and that if a formula has an NC derivation containing m rule applications, for $m < n$, then it has also an Gr derivation. We consider the last rule applied in the NC derivation of F.

- $\dfrac{A \quad B}{A \wedge B}$ By induction hypothesis, the premisses are derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.
- $\dfrac{A \wedge B}{A}$ By induction hypothesis, the premiss is derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.
- $\dfrac{A \wedge B}{B}$ This case is symmetric to the previous one.
- $\dfrac{A}{A \vee B}$ By induction hypothesis, the premiss is derivable in Gr. By Lemma 3.14 and Proposition 2.8 we have that the following is a $\mathrm{Gr}^{\triangleright}$ derivation:

$$\dfrac{\overset{\dots}{B \vee f(B^{\perp})} \qquad \dfrac{A \quad B^1}{A \vee B} \qquad \dfrac{A \;\mid\; \dfrac{f(B^{\perp})^1}{B^{\perp}}}{A \vee B}}{A \vee B}\,1$$

- $\dfrac{B}{A \vee B}$ This case is symmetric to the previous one.

- $\dfrac{A \vee B \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n$ By induction hypothesis, the premisses are derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.

- $\dfrac{\begin{matrix} A^n \\ \vdots \\ B \end{matrix}}{A \to B}\, n$ By induction hypothesis, the premiss B is derivable in Gr fom the hypothesis A. By Lemma 3.14 and Proposition 2.8 the following is a $\mathrm{Gr}^{\triangleright}$ derivation:

$$\dfrac{\begin{matrix} \dots \\ A \vee f(A^{\perp}) \end{matrix} \quad \dfrac{A^1 \quad \begin{matrix} A^1 \\ \vdots \\ B \end{matrix}}{A \to B} \quad \dfrac{B \vee f(B^{\perp}) \quad \dfrac{\dfrac{f(A^{\perp})^1}{A^{\perp}} \quad B^2}{A \to B} \quad \dfrac{\dfrac{f(A^{\perp})^1}{A^{\perp}} \quad | \quad \dfrac{f(B^{\perp})^2}{B^{\perp}}}{A \to B}}{A \to B}\, 2}{A \to B}\, 1$$

- $\dfrac{A \to B \quad A}{B}$ By induction hypothesis, the premisses are derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.

- $\dfrac{\begin{matrix} A^n \\ \vdots \\ \perp \end{matrix}}{\neg A}\, n$ By induction hypothesis, the premiss $\perp$ is derivable in Gr from the hypothesis A. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.

- $\dfrac{\neg A \quad A}{\perp}$ By induction hypothesis, the premisses are derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.

- $\dfrac{\perp}{A}$ By induction hypothesis, the premiss is derivable in Gr. Since this rule belongs to Gr as well, also the conclusion can be derived in Gr.

- $\dfrac{\begin{matrix} \neg A^n \\ \vdots \\ \perp \end{matrix}}{A}\, n$ By induction hypothesis, the premiss $\perp$ is derivable in Gr from the hypothesis $\neg A$. By the rules of negation introduction and double negation elimination, we can derive A in $\mathrm{Gr}^{\triangleright}$ as follows:

$$\dfrac{\dfrac{\begin{matrix} \neg A^n \\ \vdots \\ \perp \end{matrix}}{\neg\neg A}\, n}{A}$$

□

Lemma 4.8. For any two distinct segments in a derivation d, if neither is to the right of the other, then one is above the other.

Proof. Suppose that neither r is to the right of s nor s is to the right of r. This means that all subderivations of d that contain some element of one of the two segments, say r, also contain all elements of s. Because if this were not the case we could find a subderivation α of d that contains some element of r but does not contain some element

S of s. Since α does not contain S, there must be some rule application in d such that above one of its premisses occurs α and above another premise occurs S. But this would give us that either r is to the right of s or s is to the right of r, which contradicts the assumption. Therefore, all subderivations of d that contain some element of r also contain all elements of s. But this means that the bottommost element of s is above the bottommost element of r. $\square$

Theorem 4.9 (Normalisation)**.** For any Gr derivation d, there is a derivation d' such that d can be reduced to d' in a finite number of reductions and d' is normal.

Proof. We employ the following reduction strategy. We reduce any rightmost segment of maximal complexity that does not occur below any other segment of maximal complexity. By Lemma 4.8, we can always find such a segment.

We prove that this reduction strategy always produces a series of reductions which is of finite length and which results in a normal form.

We define the complexity of a derivation d to be the triple of natural numbers (m, n, u), where m is the complexity of the segments in d with maximal segment complexity, n is the sum of the lengths of the segments in d with segment complexity m, and u is the number of rule applications in d. We then fix a generic derivation d and reason by induction on the lexicographic order on triples of natural numbers.

We moreover suppose that the outermost negation of all hypotheses of the form $\neg A$ discharged by negation introductions is immediately eliminated. If this is not the case, we transform all relevant subderivations as follows:

$$\dfrac{\begin{array}{c}\neg A^n \\ \vdots \\ \bot\end{array}}{\neg\neg A}\,n \quad \mapsto \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\neg A^n \quad A^m}{\bot}}{\neg A}\,m \\ \vdots \\ \bot\end{array}}{\neg\neg A}\,n$$

Notice moreover that if a derivation complies with this assumption, no reduction can produce a derivation that does not comply with it.

If the complexity of d is $(0, 0, u)$ then d is normal and the claim holds.

Suppose now that the complexity of d is (m, n, u), that $m + n > 0$, and that for each derivation simpler than d the claim holds. Since $m + n > 0$, there must be at least one maximal segment in d that does not occur below any other maximal segment. We reduce one of such segments and reason on the obtained derivation d' by cases on the shape of the reduction—we use lowercase Greek letters to denote subderivations of d.

- $$\dfrac{\dfrac{\overset{\alpha}{A \vee B} \quad \overset{\beta}{C} \quad \overset{\gamma}{C}}{C}}{D}\,\text{elim.} \quad \mapsto \quad \dfrac{\overset{\alpha}{A \vee B} \quad \dfrac{\overset{\beta}{C}}{D} \quad \dfrac{\overset{\gamma}{C}}{D}}{D}$$

  We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of C and D, (ii) the segment contains the displayed D, (iii) the segment contains the displayed C. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one since D is obtained by eliminating some connectives of C. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \vee B} \quad \overset{\beta}{C} \quad \overset{\gamma}{C}}{C} \qquad \overset{\delta}{D}}{E}\,\text{elim.} \quad \mapsto \quad \dfrac{\overset{\alpha}{A \vee B} \qquad \dfrac{\overset{\beta}{C} \quad \overset{\delta}{D}}{E} \qquad \dfrac{\overset{\gamma}{C} \quad \overset{\delta}{D}}{E}}{E}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of C, D and E and does not occur in δ, (ii) the segment contains the displayed E, (iii) the segment contains the displayed D or occurs in δ, (iv) the segment contains the displayed C. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one since E is either obtained by eliminating some connective of C or by $(\ )^{\perp}$ elimination. All segments for which (iii) holds have been duplicated by the reduction, but their segment complexity is not maximal in d since the segment that we reduced is rightmost among the segments of maximal complexity. The lengths of all segments for which (iv) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \vee B} \quad \overset{\gamma_1}{C} \quad \overset{\gamma_2}{C}}{C} \qquad \overset{\delta_1}{D} \quad \overset{\delta_2}{D}}{D}\,\text{elim.} \quad \mapsto \quad \dfrac{\overset{\alpha}{A \vee B} \qquad \dfrac{\overset{\gamma_1}{C} \quad \overset{\delta_1}{D} \quad \overset{\delta_2}{D}}{D} \qquad \dfrac{\overset{\gamma_2}{C} \quad \overset{\delta_1}{D} \quad \overset{\delta_2}{D}}{D}}{D}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of C and D, and does not occur in δ_1 or δ_2; (ii) the segment contains some of the displayed occurrences of D, or occurs in δ_1 or δ_2; (iii) the segment contains some of the displayed occurrences of C. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since otherwise there would be in d some segments of maximal complexity to the right of the reduced one, which contradicts the assumptions. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \to B}}{A \to B'}\,\alpha\kappa \qquad \overset{\beta}{A}}{B'} \quad\mapsto\quad \dfrac{\dfrac{A \to B \qquad \overset{\beta}{A}}{B}}{B'}\,\alpha\kappa$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain neither the displayed occurrence of B' nor the displayed occurrence of $A \to B'$; (ii) the segment contains the displayed occurrence of B'; (iii) the segment contains the displayed occurrence of $A \to B'$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since B' is less complex than $A \to B$ and $A \to B'$. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \to B}}{A' \to B}\,\alpha\kappa \qquad \overset{\beta}{A'}}{B} \quad\mapsto\quad \dfrac{\overset{\alpha}{A \to B} \qquad \dfrac{\overset{\beta}{A'}}{A}\,\alpha\kappa}{B}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain neither the displayed occurrence of A' nor the displayed occurrence of $A' \to B$; (ii) the segment contains the displayed occurrence of A'; (iii) the segment contains the displayed occurrence of $A' \to B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since B' is less complex than $A \to B$ and $A' \to B$. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A \vee B}}{A' \vee B'}\,\alpha\kappa \qquad \begin{matrix} A'^n \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} B'^n \\ \vdots \\ C \end{matrix}}{C}\,n \quad\mapsto\quad \dfrac{\overset{\alpha}{A \vee B} \qquad \begin{matrix} \dfrac{A^n}{A'}\,\alpha\kappa \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \dfrac{B^n}{B'}\,\alpha\kappa \\ \vdots \\ C \end{matrix}}{C}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed occurrences of A', B' and $A' \vee B'$; (ii) the segment contains the displayed occurrence of A' or B'; (iii) the segment contains the displayed occurrence of $A' \vee B'$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since A' and B' are less complex than $A \vee B$ and $A' \vee B'$. The lengths of all segments for which (iii) holds have been reduced.

•

$$\dfrac{\dfrac{\overset{\alpha}{A \vee B}}{B \vee A}\,\alpha\kappa \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix}}{C} \quad \mapsto \quad \dfrac{\overset{\alpha}{A \vee B} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed occurrence of $B \vee A$; (ii) otherwise. If (i) the segment has neither been modified nor been duplicated by the reduction. The lengths of all segments for which (ii) holds have been reduced.

•

$$\dfrac{\dfrac{\overset{\alpha}{A \wedge B}}{A' \wedge B'}\,\alpha\kappa}{A'} \quad \mapsto \quad \dfrac{\dfrac{\overset{\alpha}{A \wedge B}}{A}}{A'}\,\alpha\kappa \qquad\qquad \dfrac{\dfrac{\overset{\alpha}{A \wedge B}}{A' \wedge B'}\,\alpha\kappa}{B'} \quad \mapsto \quad \dfrac{\dfrac{\overset{\alpha}{A \wedge B}}{B}}{B'}\,\alpha\kappa$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed occurrences of A', B' and $A' \wedge B'$; (ii) the segment contains the displayed occurrences of A' or B'; (iii) the segment contains the displayed occurrence of $A' \wedge B'$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since A' and B' are less complex than $A \wedge B$ and $A' \wedge B'$. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{\neg A}}{\neg A'}\ \alpha\kappa \qquad \overset{\beta}{A'}}{\bot} \quad \mapsto \quad \dfrac{\overset{\alpha}{\neg A} \qquad \dfrac{\overset{\beta}{A'}}{A}\ \alpha\kappa}{\bot}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed occurrences of A' and $\neg A'$; (ii) the segment contains the displayed occurrence of A'; (iii) the segment contains the displayed occurrence of $\neg A'$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, since A' is less complex than $\neg A'$. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A^{\bot}}}{A'^{\bot}}\ \alpha\kappa \qquad \overset{\beta}{A'}}{\bot} \quad \mapsto \quad \dfrac{\overset{\alpha}{A^{\bot}} \qquad \dfrac{\overset{\beta}{A'}}{A}\ \alpha\kappa}{\bot}$$

We reduced the complexity of the considered maximal segment because its length has been reduced. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we reduced the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed occurrences of A' and $A'^{\bot}$; (ii) the segment contains the displayed occurrence of A'; (iii) the segment contains the displayed occurrence of $A'^{\bot}$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction increases the length of the segment, but the resulting segment is still less complex than the reduced one, because if A' were more complex or as complex than $A'^{\bot}$, then the reduced segment would not be rightmost among the maximal ones, and this is against the assumptions. The lengths of all segments for which (iii) holds have been reduced.

- $$\dfrac{\dfrac{\overset{\alpha}{A} \quad \overset{\beta}{B}}{\dfrac{}{A \wedge B}}}{A} \quad \mapsto \quad \overset{\alpha}{A}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of A and $A \wedge B$, (ii) the segment contains some of the displayed occurrences of A. (iii) the segment contains the displayed occurrence of $A \wedge B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since A is less complex than $A \wedge B$. We just eliminated the only segment for which (iii) holds.

-
$$\dfrac{\dfrac{\overset{\alpha}{A} \quad \overset{\beta}{B}}{A \wedge B}}{B} \mapsto \overset{\beta}{B}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of B and $A \wedge B$, (ii) the segment contains some of the displayed occurrences of B. (iii) the segment contains the displayed occurrence of $A \wedge B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since B is less complex than $A \wedge B$. We just eliminated the only segment for which (iii) holds.

-
$$\dfrac{\dfrac{\overset{\beta}{B} \quad \overset{\alpha}{\mid A}}{A \to B} \quad \overset{\gamma}{A}}{B} \mapsto \overset{\beta}{B}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of B and $A \to B$, (ii) the segment contains some of the displayed occurrences of B. (iii) the segment contains the displayed occurrence of $A \to B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since B is less complex than $A \to B$. We just eliminated the only segment for which (*iii*) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A^{\perp}} \quad \overset{\beta}{B}}{A \to B} \quad \overset{\gamma}{A}}{B} \mapsto \overset{\beta}{B}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (*i*) the segment does not contain any of the displayed occurrences of B and $A \to B$, (*ii*) the segment contains some of the displayed occurrences of B. (*iii*) the segment contains the displayed occurrence of $A \to B$. If (*i*) the segment has neither been modified nor been duplicated by the reduction. If (*ii*) the reduction might join the segment with another one for which (*ii*) holds, but the resulting segment is still less complex than the reduced one since B is less complex than $A \to B$. We just eliminated the only segment for which (*iii*) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A^{\perp}} \quad \mid \overset{\beta}{B^{\perp}}}{A \to B} \quad \overset{\gamma}{A}}{B} \mapsto \dfrac{\dfrac{\overset{\alpha}{A^{\perp}} \quad \overset{\gamma}{A}}{\perp}}{B}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (*i*) the segment does not contain neither the displayed occurrence of $A^{\perp}$ nor the displayed occurrence of $A \to B$, (*ii*) the segment contains the displayed occurrence of $A^{\perp}$, (*iii*) the segment contains the displayed occurrence of $A \to B$. If (*i*) the segment has neither been modified nor been duplicated by the reduction. If (*ii*) the reduction might have increased the complexity of the segment since $A^{\perp}$ was the premiss of an introduction rule and now is the premiss of an elimination rule, but the resulting complexity is still less than the complexity of the reduced segment since $A^{\perp}$ is less complex than $A \to B$. We just eliminated the only segment for which (*iii*) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A} \quad \overset{\beta}{B}}{A \vee B} \quad \begin{matrix} A^n \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B^n \\ \vdots \\ C \end{matrix}}{C}\, n \mapsto \begin{matrix} \alpha \\ A \\ \vdots \\ C \end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of A, C and $A \vee B$, (ii) the segment contains some of the displayed occurrences of A, (iii) the segment contains some of the displayed occurrences of C, (iv) the segment contains the displayed occurrence of $A \vee B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since A is less complex than $A \vee B$. If (iii) the length of the segment has been reduced by the reduction. We just eliminated the only segment for which (iv) holds.

•

$$\dfrac{\dfrac{\overset{\alpha}{A} \;\mid\; \overset{\beta}{B^{\perp}}}{A \vee B} \qquad \begin{array}{c} A^n \\ \vdots \\ C \end{array} \qquad \begin{array}{c} B^n \\ \vdots \\ C \end{array}}{C}\, n \quad \mapsto \quad \begin{array}{c} \alpha \\ A \\ \vdots \\ C \end{array}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of A, C and $A \vee B$, (ii) the segment contains some of the displayed occurrences of A, (iii) the segment contains some of the displayed occurrences of C, (iv) the segment contains the displayed occurrence of $A \vee B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since A is less complex than $A \vee B$. If (iii) the length of the segment has been reduced by the reduction. We just eliminated the only segment for which (iv) holds.

•

$$\dfrac{\dfrac{\overset{\beta}{B} \;\mid\; \overset{\alpha}{A^{\perp}}}{A \vee B} \qquad \begin{array}{c} A^n \\ \vdots \\ C \end{array} \qquad \begin{array}{c} B^n \\ \vdots \\ C \end{array}}{C}\, n \quad \mapsto \quad \begin{array}{c} \beta \\ B \\ \vdots \\ C \end{array}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of B, C and $A \vee B$, (ii) the segment contains some of the displayed occurrences of B, (iii) the segment contains some of the displayed occurrences of C, (iv) the segment contains the displayed occurrence of $A \vee B$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since B is less complex than $A \vee B$. If (iii) the length of the segment has been reduced by the reduction. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\begin{matrix}A^n\\ \vdots\\ \bot\end{matrix}}{\neg A}\,n \qquad \begin{matrix}\alpha\\ A\end{matrix}}{\bot} \quad\mapsto\quad \begin{matrix}\alpha\\ A\\ \vdots\\ \bot\end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of A, $\bot$ and $\neg A$, (ii) the segment contains some of the displayed occurrences of A, (iii) the segment contains some of the displayed occurrences of $\bot$, (iv) the segment contains the displayed occurrence of $\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since A is less complex than $\neg A$. If (iii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since $\bot$ is less complex than $\neg A$. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\begin{matrix}\alpha\\ A\end{matrix}\quad\begin{matrix}\beta\\ B\end{matrix}\quad\begin{matrix}[A]^n\ [B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix}}{\neg C}\,n \qquad \begin{matrix}\gamma\\ C\end{matrix}}{\bot} \quad\mapsto\quad \begin{matrix}\alpha\ \beta\ \gamma\\ A\ B\ C\\ \vdots\\ \bot\end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed expression occurrences, (ii) the segment contains some of the displayed occurrences of A, B or C (iii) the segment contains some of the displayed occurrences of $\bot$, (iv) the segment contains the displayed occurrence of $\neg A$. If

(i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since C, A and B are less complex than $\neg C$; C for obvious reasons, and A and B because of the restrictions of the $\neg$ introduction rule. If (iii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since $\perp$ is less complex than $\neg C$. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A} \quad \overset{\beta}{B} \quad \begin{matrix}[A]^n\ \{C\}_1^n\\ \vdots\\ [[\perp]]\end{matrix} \quad \begin{matrix}[B]^n\ \{C\}_2^n\\ \vdots\\ [[\perp]]\end{matrix}}{\neg C}\,n \qquad \overset{\gamma}{C}}{\perp} \quad \mapsto \quad \begin{matrix}\overset{\alpha}{A}\ \overset{\gamma}{C}\\ \vdots\\ \perp\end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed expression occurrences except for B, (ii) the segment contains some of the displayed occurrences of A or C (iii) the segment contains some of the displayed occurrences of $\perp$, (iv) the segment contains the displayed occurrence of $\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since C and A are less complex than $\neg C$; C for obvious reasons and A because of the restriction of the $\neg$ introduction rules. If (iii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since $\perp$ is less complex than $\neg C$. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A} \quad \begin{matrix}[A]^n\ \{C\}_1^n\\ \vdots\\ [[\perp]]\end{matrix} \quad \begin{matrix}[B^{\perp}]^n\ \{C\}_2^n\\ \vdots\\ [[\perp]]\end{matrix} \quad \mid \overset{\beta}{B}}{\neg C}\,n \qquad \overset{\gamma}{C}}{\perp} \quad \mapsto \quad \begin{matrix}\overset{\alpha}{A}\ \overset{\gamma}{C}\\ \vdots\\ \perp\end{matrix}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed expression occurrences except for B, (ii) the segment contains some of the displayed occurrences of A or C (iii) the segment contains some of the displayed occurrences of $\perp$, (iv) the segment contains the displayed occurrence of $\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might join the segment with another one for which (ii) holds, but the resulting segment is still less complex than the reduced one since C and A are less complex than $\neg C$; C for obvious reasons and A because of the restriction of the $\neg$ introduction rules. If (iii) the reduction might join the segment with another one for which (iii) holds, but the resulting segment is still less complex than the reduced one since $\perp$ is less complex than $\neg C$. We just eliminated the only segment for which (iv) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{\neg A}}{A^{\perp}} \quad \overset{\beta}{A}}{\perp} \mapsto \dfrac{\overset{\alpha}{\neg A} \quad \overset{\beta}{A}}{\perp}$$

  By the reduction we decrease the length of one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m, n', u') < (m, n, u)$ since we decreased the length of a maximal segment. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed $A^{\perp}$ and $\perp$, (ii) the segment contains the displayed $A^{\perp}$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction decreased the length of the segment—notice indeed that the complexity of $\neg A$ and $A^{\perp}$ is the same.

- $$\dfrac{\dfrac{\overset{\alpha}{A}}{(\neg A)^{\perp}} \quad \overset{\beta}{\neg A}}{\perp} \mapsto \dfrac{\overset{\beta}{\neg A} \quad \overset{\alpha}{A}}{\perp}$$

  By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain the displayed $\neg A$ and $(\neg A)^{\perp}$, (ii) the segment contains the displayed $\neg A$, (iii) the segment contains the displayed $(\neg A)^{\perp}$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might have increased the complexity of the segment—because now $\neg A$ is the major premiss of an elimination rule—but this is still simpler than the eliminated segment since $\neg A$ is simpler than $(\neg A)^{\perp}$. We just eliminated the only segment for which (iii) holds.

- $$\dfrac{\dfrac{\overset{\alpha}{A}}{\overline{\neg\neg A}}}{A} \mapsto \overset{\alpha}{A}$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of $\neg\neg A$ and A, (ii) the segment contains some of the displayed occurrences of A, (iii) the segment contains the displayed occurrence of $\neg\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the reduction might have joined the segment with another one for which (ii) holds, but the resulting segment is still simpler than the eliminated one since A is simpler than $\neg\neg A$.

•

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\neg A^n \quad \begin{array}{c}\vdots\\ A\end{array}}{\bot}\\ \vdots\\ \bot\end{array}}{\neg\neg A}\,n}{A}\quad \cdots}{A'}\;elim.
\qquad\mapsto\qquad
\dfrac{\dfrac{\begin{array}{c}\dfrac{\neg A'^n \quad \dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad \cdots}{A'}\;elim.}{\bot}\\ \vdots\\ \bot\end{array}}{\neg\neg A'}\,n}{A'}
$$

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\neg A^n \quad \begin{array}{c}\vdots\\ A\end{array}}{\bot}\\ \vdots\\ \bot\end{array}}{\neg\neg A}\,n}{A}\quad \cdots}{\neg B}\;elim.
\qquad\mapsto\qquad
\dfrac{\begin{array}{c}\dfrac{\dfrac{B^n}{\neg\neg B} \quad \dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad \cdots}{\neg B}\;elim.}{\bot}\\ \vdots\\ \bot\end{array}}{\neg B}\,n
$$

By the reduction we eliminate one maximal segment. We show now that no segment of maximal complexity has been duplicated, the length of no segment of maximal complexity has been increased, and no segment has become as complex as the reduced one; and hence that the complexity of d' is $(m', n', u') < (m, n, u)$ since we either reduced the maximal complexity of the segments or the sum of the lengths of the segments with maximal complexity. For each segment in d exactly one of the following holds: (i) the segment does not contain any of the displayed occurrences of A and A' (or $\neg B$), (ii) the segment contains the displayed occurrence of A, (iii) the segment contains the displayed occurrence of A' (or $\neg B$), (iv) the segment contains the occurrence of $\neg\neg A$. If (i) the segment has neither been modified nor been duplicated by the reduction. If (ii) the segment has neither been modified nor been duplicated by the reduction. If (iii) the length of the segment might have been modified by the reduction, but the segment was not maximal because on the assumption on the choice of the segment which we have just reduced. We have just reduced the complexity of the only segment for which (iv) holds.

□

Theorem 4.10 (Subformula Property)**.** For any normal Gr derivation d of an expression G from hypotheses Γ, for any expression E occurring in d, one of the following holds:

(1) $f(E) = \bot$,
(2) $f(E)$ is a-c equivalent to a subformula of the formula interpretation of an element of $\Gamma \cup \{G\}$.
(3) $f(E)$ is a-c equivalent to the negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have two negations as outermost connectives.
(4) $f(E)$ is a-c equivalent to the double negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have a negation as outermost connective.[9]

Proof. We prove a stronger statement:

> For any normal Gr derivation d of an expression G from hypotheses Γ, for any expression E occurring in d, one of the following holds:
>
> (1) $f(E) = \bot$,
> (2) $f(E)$ is a-c equivalent to a subformula of the formula interpretation of an element of $\Gamma \cup \{G\}$.
> (3) $f(E)$ is a-c equivalent to the negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have a two negations as outermost connectives.
> (4) $f(E)$ is a-c equivalent to the double negation of a subformula of the formula interpretation I of an element of $\Gamma \cup \{G\}$ which does not have a negation as outermost connective.
>
> Moreover, if G is obtained by an elimination rule which is not a $\vee$ or $\neg\neg$ elimination, either G is a subformula of an element of Γ.

Consider any normal Gr derivation d of G from hypotheses Γ. We show by induction on the number of rules applied in d that the statement holds for d. If d contains no rule application, then d consists only in the hypothesis G and the statement trivially holds. Suppose now that d contains $n + 1$ rule applications and that the statement holds for all derivations containing n or less rule applications. We show that the statement holds for d as well. We reason on the last rule applied in d.

- $\dfrac{\mathcal{C}[A \wedge B]}{\mathcal{C}[B \wedge A]}\,\alpha\kappa$ By inductive hypothesis, the statement holds for the derivation of $\mathcal{C}[A \wedge B]$. Since no hypothesis is discharged by the last rule applied in d and since G is a-c equivalent to $\mathcal{C}[A \wedge B]$, the statement holds for d as well.
- $\dfrac{\mathcal{C}[A \wedge (B \wedge C)]}{\mathcal{C}[(A \wedge B) \wedge C]}\,\alpha\kappa$ By inductive hypothesis, the statement holds for the derivation of $\mathcal{C}[A \wedge (B \wedge C)]$. Since no hypothesis is discharged by the last rule applied in d and since G is a-c equivalent to $\mathcal{C}[A \wedge (B \wedge C)]$, the statement holds for d as well.
- $\dfrac{\mathcal{C}[A \vee B]}{\mathcal{C}[B \vee A]}\,\alpha\kappa$ By inductive hypothesis, the statement holds for the derivation of $\mathcal{C}[A \vee B]$. Since no hypothesis is discharged by the last rule applied in d and since G is a-c equivalent to $\mathcal{C}[A \vee B]$, the statement holds for d as well.

[9]The last two conditions are required because of the double negation elimination rule and since, according to Definition 3.4, the negations $\neg A$ and $\neg B$ are contained in some of the multisets which are completely and immediately less g-complex than $A \wedge B$, $A \vee B$ and $A \to B$. Which means that, with respect to our formalisation of grounding, we must regard also $\neg A$ and $\neg B$ as immediate subformulae of $A \wedge B$, $A \vee B$ and $A \to B$.

- $\dfrac{\mathcal{C}[A \vee (B \vee C)]}{\mathcal{C}[(A \vee B) \vee C]}\ \alpha\kappa$ By inductive hypothesis, the statement holds for the derivation of $\mathcal{C}[A \vee (B \vee C)]$. Since no hypothesis is discharged by the last rule applied in d and since G is a-c equivalent to $\mathcal{C}[A \vee (B \vee C)]$, the statement holds for d as well.
- $\dfrac{\neg\neg^{2n} A}{(\neg^{2n} A)^{\perp}}$ By inductive hypothesis, the statement holds for the derivation of $\neg\neg^{2n} A$. Since no hypothesis is discharged by the last rule applied in d and since by Definition 2.7, $\neg\neg^{2n} A$ and $f((\neg^{2n} A)^{\perp})$ are the same formula, the statement holds for d as well.
- $\dfrac{\neg^{2n} A}{(\neg\neg^{2n} A)^{\perp}}$ By inductive hypothesis, the statement holds for the derivation of $\neg^{2n} A$. Since no hypothesis is discharged by the last rule applied in d and since by Definition 2.7, $\neg^{2n} A$ and $f((\neg\neg^{2n} A)^{\perp})$ are the same formula, the statement holds for d as well.
- $\dfrac{A \quad B}{A \wedge B}$ By inductive hypothesis, the statement holds for the derivations of A and B. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well.
- $\dfrac{A \quad B}{A \vee B}$ By inductive hypothesis, the statement holds for the derivations of A and B. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well.
- $\dfrac{A \quad \mid B^{\perp}}{A \vee B}$ By inductive hypothesis, the statement holds for the derivations of A and $B^{\perp}$. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well. The formula $f(B^{\perp})$ is indeed either a subformula of B, and 2 holds, or the negation of B, and thus the negation of a proper subformula of $A \vee B$, and 3 holds.
- $\dfrac{B \quad \mid A^{\perp}}{A \vee B}$ By inductive hypothesis, the statement holds for the derivations of $A^{\perp}$ and B. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well. The formula $f(A^{\perp})$ is indeed either a subformula of A, and 2 holds, or the negation of A, and thus the negation of a proper subformula of $A \vee B$, and 3 holds.
- $\dfrac{B \quad \mid A}{A \to B}$ By inductive hypothesis, the statement holds for the derivations of A and B. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well.
- $\dfrac{A^{\perp} \quad B}{A \to B}$ By inductive hypothesis, the statement holds for the derivations of $A^{\perp}$ and B. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well. The formula $f(A^{\perp})$ is indeed either a subformula of A, and 2 holds, or the negation of A, and thus the negation of a proper subformula of $A \to B$, and 3 holds.
- $\dfrac{A^{\perp} \quad \mid B^{\perp}}{A \to B}$ By inductive hypothesis, the statement holds for the derivations of $A^{\perp}$ and $B^{\perp}$. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well. The formulae $f(A^{\perp})$ and $f(B^{\perp})$ are either subformulae of A respectively B, and 2 holds, or the negation of A respectively B, and thus the negation of a proper subformula of $A \to B$, and 3 holds.

- $$\dfrac{A \quad B \qquad \begin{array}{c}[A]^n\ [B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{array}}{\neg C}\,n$$ By inductive hypothesis, the statement holds for the derivations of A and B and for the derivation of $[[\bot]]$ from $[A]$, $[B]$ and $\{C\}$. By the restrictions on the form of the derivation of $[[\bot]]$, we know that A and B are proper subformulae of C. Therefore, the statement holds also for d.
- $$\dfrac{A \qquad \begin{array}{c}[A]^n\ \{C\}^n_1\\ \vdots\\ [[\bot]]\end{array} \qquad \begin{array}{c}[B^\bot]^n\ \{C\}^n_2\\ \vdots\\ [[\bot]]\end{array} \qquad \mid B}{\neg C}\,n$$ By inductive hypothesis, the statement holds for the derivations of A and B and for the derivations of $[[\bot]]$ from $[A]$ and $\{C\}$, and $[B^\bot]$ and $\{C\}$. By the restrictions on the form of the derivations of $[[\bot]]$, we know that A and $f(B^\bot)$ are proper subformulae of C. Therefore, the statement holds also for d.
- $\dfrac{A}{\neg\neg A}$ By inductive hypothesis, the statement holds for the derivation of A. Since no hypothesis is discharged by the last rule applied in d, the statement holds for d as well.
- $\dfrac{\{A \wedge B\}_1}{\{\{A\}\}}$ The restrictions on this rule application imply that $A \wedge B$ is a hypothesis. Hence, the statement holds.
- $\dfrac{\{A \wedge B\}_2}{\{\{B\}\}}$ The restrictions on this rule application imply that $A \wedge B$ is a hypothesis. Hence, the statement holds.
- $$\dfrac{\{A \vee B\} \qquad \begin{array}{c}\{\{A\}\}^n\\ \vdots\\ [[\bot]]\end{array} \qquad \begin{array}{c}\{\{B\}\}^n\\ \vdots\\ [[\bot]]\end{array}}{[[\bot]]}\,n$$ The restrictions on this rule application imply that $A \vee B$ is a hypothesis. Hence, the statement holds.
- $\dfrac{\{A \to B\} \quad [A]}{\{\{B\}\}}$ The restrictions on this rule application imply that $A \to B$ is a hypothesis. Hence, the statement holds.
- $\dfrac{[A^\bot] \quad \{\{A\}\}}{[[\bot]]}$ The restrictions on this rule application imply that $A^\bot$ is a hypothesis and that A is a proper subformula of a hypothesis. Hence, the statement holds.
- $\dfrac{A \wedge B}{A}$ Since d is normal, $A \wedge B$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $A \wedge B$ is a subformula of a hypothesis. Therefore, and by inductive hypothesis on the derivation of $A \wedge B$, the statement holds for d as well.
- $\dfrac{A \wedge B}{B}$ Since d is normal, $A \wedge B$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $A \wedge B$ is a subformula of a hypothesis. Therefore, and by inductive hypothesis on the derivation of $A \wedge B$, the statement holds for d as well.
- $$\dfrac{A \vee B \quad \begin{array}{c}A^n\\ \vdots\\ C\end{array} \quad \begin{array}{c}B^n\\ \vdots\\ C\end{array}}{C}\,n$$ Since d is normal, $A \vee B$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $A \vee B$ is a subformula of a hypothesis H. Since all subformulae of A and B are

also proper subformulae of H, we have, by induction hypothesis on the derivation of $A \vee B$ and one the two derivations of C, that the statement holds for d as well.

- $\dfrac{A \to B \quad A}{B}$ Since d is normal, $A \to B$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $A \to B$ is a subformula of a hypothesis. Therefore, and by inductive hypothesis on the derivations of $A \to B$ and A, the statement holds for d as well.
- $\dfrac{\begin{matrix} A^n \\ \vdots \\ \perp \end{matrix}}{\neg A}\,n$ By induction hypothesis on the derivation of $\perp$ and since all subformulae of A are proper subformulae of $\neg A$, the statement holds for d as well.
- $\dfrac{\neg A \quad A}{\perp}$ Since d is normal, $\neg A$ can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $\neg A$ is a subformula of a hypothesis. Therefore, and by inductive hypothesis on the derivations of $\neg A$ and A, the statement holds for d as well.
- $\dfrac{A^{\perp} \quad A}{\perp}$ Since d is normal, the most complex among $A^{\perp}$ and A can only be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination. Thus, by inductive hypothesis, $f(A^{\perp})$ or A is a subformula of a hypothesis. Therefore, and by inductive hypothesis on the derivations of $A^{\perp}$ and A, the statement holds for d as well.
- $\dfrac{\perp}{A}$ Since the last rule applied in d does not discharge any hypothesis, by inductive hypothesis, the statement holds for d.
- $\dfrac{\neg\neg A}{A}$ Since d is normal, $\neg\neg A$ can either be the conclusion of an elimination rule which is not a $\vee$ or $\neg\neg$ elimination or the conclusion of a negation introduction rule. In the first case, by inductive hypothesis, $\neg\neg A$ is a subformula of a hypothesis. Therefore, the statement holds for d as well. In the second case, we have a derivation d_0 of $\perp$, with $\neg A$ among its hypotheses—otherwise, by inductive hypothesis, $\neg\neg A$ complies with the statement with respect to the hypotheses and A too since no other hypothesis has been discharged. By inductive hypothesis, the statement holds for d_0. Now, according to the statement, the formula interpretation of any expression occurring in d_0 either complies with the statement with respect to A and the hypotheses of d_0, or is of the form $\neg\neg A$ or $\neg A$. Indeed, $\neg\neg$ elimination cannot be applied to negated formulae. Since also the cases in which the formula interpretation of an expression occurring in d_0 is of the form $\neg\neg A$ or $\neg A$ comply with the statement with respect to A and any set of hypotheses, we have that the statement holds for d as well.

$\square$

$$\dfrac{\dfrac{\begin{matrix}A^n\\ \vdots\\ \bot\end{matrix}}{\neg A}\,n \qquad A}{\bot} \quad\mapsto\quad \begin{matrix}A\\ \vdots\\ \bot\end{matrix} \qquad\qquad \dfrac{\dfrac{A \quad B \qquad \begin{matrix}[A]^n\ [B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix}}{\neg C}\,n \qquad C}{\bot} \quad\mapsto\quad \begin{matrix}A\ B\ C\\ \vdots\\ \bot\end{matrix}$$

$$\dfrac{\dfrac{A \quad B \qquad \begin{matrix}[A]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix} \qquad \begin{matrix}[B]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix}}{\neg C}\,n \qquad C}{\bot} \quad\mapsto\quad \begin{matrix}A\ C\\ \vdots\\ \bot\end{matrix}$$

$$\dfrac{\dfrac{A \qquad \begin{matrix}[A]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix} \qquad \begin{matrix}[B^{\bot}]^n\ \{C\}^n\\ \vdots\\ [[\bot]]\end{matrix} \qquad \mid B}{\neg C}\,n \qquad C}{\bot} \quad\mapsto\quad \begin{matrix}A\ C\\ \vdots\\ \bot\end{matrix}$$

$$\dfrac{\dfrac{\neg A}{A^{\bot}} \quad A}{\bot} \mapsto \dfrac{\neg A \quad A}{\bot} \qquad\qquad \dfrac{\dfrac{A}{(\neg A)^{\bot}} \quad \neg A}{\bot} \mapsto \dfrac{\neg A \quad A}{\bot} \qquad\qquad \dfrac{\dfrac{A}{\neg\neg A}}{A} \quad\mapsto\quad A$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{\neg A^n \quad \begin{matrix}\vdots\\ A\end{matrix}}{\bot}\\ \vdots\\ \bot\end{matrix}}{\neg\neg A}\,n}{A} \qquad \dots}{A'}\ elim. \quad\mapsto\quad \dfrac{\dfrac{\begin{matrix}\dfrac{\neg A'^{\,n} \quad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \dots}{A'}\ elim.}{\bot}\\ \vdots\\ \bot\end{matrix}}{\neg\neg A'}\,n}{A'}$$

where A' is not negated

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{\neg A^n \quad \begin{matrix}\vdots\\ A\end{matrix}}{\bot}\\ \vdots\\ \bot\end{matrix}}{\neg\neg A}\,n}{A} \qquad \dots}{\neg B}\ elim. \quad\mapsto\quad \dfrac{\begin{matrix}\dfrac{\dfrac{B^n}{\neg\neg B} \quad \dfrac{\begin{matrix}\vdots\\ A\end{matrix} \quad \dots}{\neg B}\ elim.}{\bot}\\ \vdots\\ \bot\end{matrix}}{\neg B}\,n$$

Table 7. Reductions, Part 2: Negation and Double Negation

$$\dfrac{\dfrac{A \lor B \quad C \quad C}{C}}{D}\,\text{elim.} \quad\mapsto\quad \dfrac{A \lor B \quad \dfrac{C}{D} \quad \dfrac{C}{D}}{D}$$

$$\dfrac{\dfrac{A \lor B \quad C \quad C}{C} \qquad D}{E}\,\text{elim.} \quad\mapsto\quad \dfrac{A \lor B \quad \dfrac{C \quad D}{E} \quad \dfrac{C \quad D}{E}}{E}$$

$$\dfrac{\dfrac{A \lor B \quad C \quad C}{C} \qquad D \quad D}{D}\,\text{elim.} \quad\mapsto\quad \dfrac{A \lor B \quad \dfrac{C \quad D \quad D}{D} \quad \dfrac{C \quad D \quad D}{D}}{D}$$

Table 8. Reductions, Part 3: Disjunction Elimination Permutations

$$\dfrac{\dfrac{A \to B}{A \to B'}\,\alpha\kappa \qquad A}{B'} \quad\mapsto\quad \dfrac{\dfrac{A \to B \quad A}{B}}{B'}\,\alpha\kappa \qquad\qquad \dfrac{\dfrac{A \to B}{A' \to B}\,\alpha\kappa \qquad A'}{B} \quad\mapsto\quad \dfrac{A \to B \quad \dfrac{A'}{A}\,\alpha\kappa}{B}$$

$$\dfrac{\dfrac{A \lor B}{A' \lor B'}\,\alpha\kappa \quad \begin{matrix} A' \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B' \\ \vdots \\ C \end{matrix}}{C} \quad\mapsto\quad \dfrac{A \lor B \quad \begin{matrix} \dfrac{A}{A'}\,\alpha\kappa \\ \vdots \\ C \end{matrix} \quad \begin{matrix} \dfrac{B}{B'}\,\alpha\kappa \\ \vdots \\ C \end{matrix}}{C}$$

$$\dfrac{\dfrac{A \lor B}{B \lor A}\,\alpha\kappa \quad \begin{matrix} B \\ \vdots \\ C \end{matrix} \quad \begin{matrix} A \\ \vdots \\ C \end{matrix}}{C} \quad\mapsto\quad \dfrac{A \lor B \quad \begin{matrix} A \\ \vdots \\ C \end{matrix} \quad \begin{matrix} B \\ \vdots \\ C \end{matrix}}{C}$$

$$\dfrac{\dfrac{A \land B}{A' \land B'}\,\alpha\kappa}{A'} \quad\mapsto\quad \dfrac{\dfrac{A \land B}{A}}{A'}\,\alpha\kappa \qquad\qquad \dfrac{\dfrac{A \land B}{A' \land B'}\,\alpha\kappa}{B'} \quad\mapsto\quad \dfrac{\dfrac{A \land B}{B}}{B'}\,\alpha\kappa$$

$$\dfrac{\dfrac{\neg A}{\neg A'}\,\alpha\kappa \qquad A'}{\bot} \quad\mapsto\quad \dfrac{\neg A \quad \dfrac{A'}{A}\,\alpha\kappa}{\bot} \qquad\qquad \dfrac{\dfrac{A^{\perp}}{A'^{\perp}}\,\alpha\kappa \qquad A'}{\bot} \quad\mapsto\quad \dfrac{A^{\perp} \quad \dfrac{A'}{A}\,\alpha\kappa}{\bot}$$

Table 9. Reductions, Part 4: $\alpha\kappa$ Rules Permutations