A Complete Classification of Topological Relations Using 9-Intersection Method *

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Abstract

Formalization of topological relations between spatial objects is an important aspect of spatial representation and reasoning. The well-known 9-Intersection Method (9IM) was previously used to characterize topological relations between *simple regions*, i.e. regions with connected boundary and exterior. This simplified abstraction of spatial objects as simple regions cannot model the variety and complexity of spatial objects. For example, countries like Italy may contain islands and holes. It is necessary that existing formalisms, 9IM in particular, cover this variety and complexity.

This paper generalizes the 9IM to cope with general regions, where a (general) region is a nonempty proper regular closed subset of the Euclidean plane. We give a complete classification of topological relations between plane regions. For each possible relation we either show that it violates some topological constraints and hence is non-realizable or find two plane regions it relates. Altogether 43 (out of 512) relations are identified as realizable. Among these, five can be realized only between *exotic* (plane) regions, where a region is exotic if there is another region which has the same boundary but is not its complement. For all the remaining 38 relations, we construct configurations by using sums, differences, and complements of disks.

Keywords: Qualitative Spatial Reasoning; Geographic Information Science; Topological relation; 9-Intersection Method

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1 Introduction

Representation of and reasoning about spatial knowledge are important in many application areas such as geographic information systems, robotics, and computer vision. While quantitative, numeric-based approach to spatial reasoning has been popular in computer graphics and computer vision, in many cases a purely qualitative approach can be beneficial. (Kuipers 1978) has pointed out that quantitative approach is an inappropriate representation of human cognition and spatial reasoning. This is partially because humans typically use qualitative knowledge such as “San Marino is *surrounded by* Italy,” “Germany is *north of* Italy” to represent and communicate spatial knowledge. Qualitative representation also makes reasoning easy: from above facts, for example, we know “San Marino is *south of* Germany.”

Spatial relations can be roughly classified into three categories: topology, direction, and distance. Topological relations are the most important spatial relations. Examples are terms like “inside,” “equal,” and “disjoint.” Topological relations are invariant under topological transformations, such as translation, scaling, and rotation.

Several topological formalisms have been proposed in the literature (Egenhofer & Franzosa 1991, Egenhofer & Herring 1990, Randell, Cui & Cohn 1992, Clementini, Di Felice & van Oosterom 1993). The point-based 9-Intersection Method (9IM) is perhaps the most well-known topological formalisms in geographic information science. The 9IM, developed in (Egenhofer & Herring 1990), is based on point-set topology, where the topological relation between two regions is characterized by the 9 intersections of interiors, boundaries, and exteriors of the two regions. A region is said to be *simple* if it has a connected boundary and a connected exterior. Then, a set of 8 *jointly exhaustive and pairwise disjoint* (JEPD) relations can be characterized between simple regions by using the 9IM. We call these *Egenhofer relations*, and call the collection of simple regions, together with Egenhofer relations, *Egenhofer model* (Li & Ying 2003*a*).

Simplified abstraction of spatial objects as simple regions cannot model the variety and complexity of spatial objects. For example, countries like Italy may contain islands and holes. It is necessary that the 9IM and other topological formalisms cover this variety and complexity.

Several works have been carried out towards this direction. (Egenhofer, Clementini & Di Felice 1994) generalize the 4-Intersection Method (4IM) (Egenhofer & Franzosa 1991), a restricted form of 9IM, to cover topological relations between regions with holes. (Clementini, Di Felice & Califano 1995) extend the Calculus-Based Method (CBM) (Clementini et al. 1993) for simple regions to cover topological relations between composite regions, i.e. regions that are composed of more than one disjoint simple regions. These works, however, only consider quite special complex regions. We need a formalism that can cover all possible configurations of spatial entities, including objects with holes which have islands to any finite level (Worboys & Bofakos 1993).

The Region Connection Calculus (RCC) developed by Cohn and colleagues (Randell & Cohn 1989, Randell et al. 1992) is such an example. RCC is a first-order theory based on one primitive contact relation. Using this primitive relation, we can define a collection of topological relations. Among all relations defined in RCC, RCC8, which contains 8 JEPD relations, is of particular importance and has been investigated extensively in the field of qualitative spatial reasoning (QSR). We invite the readers to consult (Renz 2002) for more information of RCC8.

RCC8 relations looks very similar to Egenhofer relations. Indeed, they are identical as far as simple regions are concerned. Moreover, the same composition table has been constructed independently in (Cui, Cohn & Randell 1993) and (Egenhofer 1991). There are nevertheless some significant dissimilarities. First, their domains of discourse are different: Egenhofer relations are between simple regions, while RCC takes the most general definition of a region as a nonempty regular closed set. Second, the semantics of the two composition tables are different. All compositions of Egenhofer relations are *extensional* (Li & Ying 2003*a*), but not all RCC8 compositions are extensional (Li & Ying 2003*b*), where by "extensional" we mean the composition is in the sense of set theory.

This paper generalizes the 9IM to cope with general regions, where a (general) region is a nonempty proper regular closed set. We first show that the RCC8 relations can be characterized by 4 out of the 9 intersections. This can be compared to the 4IM for determining topological relations between simple regions. Based on this observation, we give a complete classification of topological relations between two complex regions.

For each possible relation we either show that it violates some topological constraints and hence is non-realizable or find two plane regions it relates. Altogether 43 (out of 512) relations are identified as realizable. Among these, five can be realized only between *exotic* (plane) regions, where a region is exotic if there is another region which has the same boundary but is not its complement. For all the remaining 38 relations, we construct configurations by using sums, differences, and complements of disks.

The rest of this paper is structured as follows. In Section 2 we recall some basic topological notions. Section 3 then introduces the RCC8 relations. In Section 4, after a simple description of the 9IM, we discuss symmetries among 3×3 Boolean matrices. We then give 9IM-characterizations of RCC8 relations. The 9IM matrices between regions with the same boundary is also discussed here. The remaining three sections examine all possible topological relations. Further discussions and related works are given in Section 8, and Section 9 concludes the paper.

2 Preliminaries

In this section we recall some basic topological notions. More information can be find in (Alexandroff 1961, Kelley 1975).

2.1 Basic topological notions

The usual definitions of open and closed sets in a topological space X are assumed. For a subset $A \subseteq X$, we write A° for the *interior* of A in X, i.e.

$$A^\circ = \bigcup\{U \subseteq A : U \text{ is open}\} \tag{1}$$

which is the largest open set contained in A. Similarly, we write $\overline{A}$ for the *closure* of A in X, i.e.

$$\overline{A} = \bigcap\{A \subseteq F : F \text{ is closed}\} \tag{2}$$

which is the smallest closed set containing A. The *boundary* of A, written ∂A, is defined to be the set difference of $\overline{A}$ and A°, i.e.

$$\partial A = \overline{A} \setminus A^\circ \tag{3}$$

The *exterior* of A, written A^e, is defined as

$$A^e = X \setminus \overline{A} = (X \setminus A)^\circ \tag{4}$$

Then we have the following results:

Lemma 2.1. *Let X be a topological space. For two open sets U, V in X, if $U \cap V = \varnothing$, then $U \cap \overline{V} = \overline{U} \cap V = \varnothing$.*

Lemma 2.2. *Let X be a topological space, and let A be a subset in X. Then A° and A^e are open sets, $\partial A = \overline{A} \setminus A^\circ$ is a closed set. Moreover,*

$$A^\circ \cap A^e = A^\circ \cap \partial A = A^e \cap \partial A = \varnothing, \quad A^\circ \cup \partial A \cup A^e = X$$

i.e. $\{A^\circ, \partial A, A^e\}$ forms a partition of X.

A closed set $A \subseteq X$ is called *regular* if $A = \overline{A^\circ}$. For each set A, $\overline{A^\circ}$ is the smallest regular closed set containing A°. We call $\overline{A^\circ}$ the *regularization* of A.

The collection of regular closed sets of X, written $\mathsf{RC}(X)$, is a complete Boolean algebra. Given two regular closed sets A, B in $\mathsf{RC}(X)$, $A + B$, the sum of A, B, is the union of A and B; $A \cdot B$, the product of A, B, is the regularization of $A \cap B$; $-A$, the complement of A, is the regularization of $X \setminus A$.

Lemma 2.3. *Let X be a topological space, and let A be a regular closed set in X. Then $-A = \overline{(X \setminus A)^\circ} = X \setminus A^\circ = \partial A \cup A^e$, $(-A)^\circ = A^e$, $\partial(-A) = \partial A$, $(-A)^e = A^\circ$, and $A \cap -A = \partial A$.*

2.2 Plane regions

The Euclidean plane $\mathbb{R}^2$ is the most important spatial model. In this paper, we are mainly concerned with regions in $\mathbb{R}^2$.

A *plane region* (or *region*) is a nonempty proper regular subset of the real plane. As any other topological spaces, the collection of all regions in the real plane, together with $\varnothing$ and $\mathbb{R}^2$, forms a complete Boolean algebra.

We call a plane region *simple* if it is homeomorphic to the unit closed disk $\mathbb{D}$. Clearly, a simple region has connected boundary and connected exterior. Furthermore, it is bounded, connected, and has no holes.

Not all plane regions are simple. In the following we call a region *complex* if it is not simple. A complex region may be unbounded, may contain several (possibly infinite) connected components, and may contain holes which have islands to any finite level. This is in accord with the variety and complexity of spatial entities. For example, Italy is a country that has a hole (San Marino) and two main islands.

For a simple region A, we know no region other than $-A$, which is the complement of A, has the same boundary as A. This property seems to be true for all regions.

Definition 2.1. A region A is *exotic* if there is a region B such that $\partial A = \partial B$ and $B \neq A$, $B \neq -A$.

The following theorem, which shows that there are k plane regions having the same boundary for any $k \geq 3$, guarantees the existence of exotic regions. This theorem is due to the famous Netherland topologist Brouwer. [1]

Lemma 2.4. *Let $\mathbb{D}$ be the unit closed disk. For any $k \geq 3$, there are k regions $U_1, \cdots, U_k$ such that*

- $U_1 \cup \cdots \cup U_k = \mathbb{D}$*;*
- $(U_i \cap U_j)^\circ = \varnothing$ *for any* $1 \leq i \neq j \leq k$*;*

[1] An informal description can be found in `http://www.cut-the-knot.org/do_you_know/brouwer.shtml`. Interested readers can get a formal proof from the author.

- $\partial U_1 = \cdots = \partial U_k \neq \varnothing$

3 RCC8 topological relations

The RCC theory is a first-order theory based on one primitive contact relation $\mathbf{C}$. Using the contact relation $\mathbf{C}$, we can define a collection of other relations. In particular, the part-of relation $\mathbf{P}$ can be defined as follows:

$$x\mathbf{P}y \Leftrightarrow (\forall z)(z\mathbf{C}x \rightarrow z\mathbf{C}y) \tag{5}$$

Write $\leq$ for $\mathbf{P}$. Then the following relations can be defined in RCC by $\mathbf{C}$ and $\mathbf{P}$. Note that relations $\mathbf{EQ}$,$\mathbf{PO}$,$\mathbf{O}$,$\mathbf{DR}$,$\mathbf{DC}$,$\mathbf{EC}$ are symmetrical, and relations $\mathbf{P}$,$\mathbf{PP}$,$\mathbf{TPP}$,$\mathbf{NTPP}$ are asymmetrical. For an asymmetrical relation $\mathbf{R}$, we write $\mathbf{R}^\sim$ for its converse. Relations

$$\mathbf{EQ}, \mathbf{DR}, \mathbf{PO}, \mathbf{PP}, \mathbf{PP}^\sim \tag{6}$$

form a JEPD set of relations, which is known as RCC5. Note that $\mathbf{DR}$ can be divided into $\mathbf{EC}$ and $\mathbf{DC}$, $\mathbf{PP}$ ($\mathbf{PP}^\sim$, resp.) can be divided into $\mathbf{TPP}$ and $\mathbf{NTPP}$ ($\mathbf{TPP}^\sim$ and $\mathbf{NTPP}^\sim$, resp.). RCC5 can be refined to the following JEPD set of relations, known as RCC8:

$$\mathbf{EQ}, \mathbf{DC}, \mathbf{EC}, \mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^\sim, \mathbf{NTPP}, \mathbf{NTPP}^\sim \tag{7}$$

RCC8 is of significant importance in spatial reasoning (see (Renz 2002)).

RCC8 relations can also be interpreted over either the collection of closed disks or the collection of simple regions. Under each interpretation, RCC8 forms a relation algebra (Düntsch 2005, Li & Ying 2003*a*). Moreover, when interpreted over simple regions, RCC8 can also be determined by the 9-Intersection Method (9IM) of (Egenhofer & Herring 1990).

4 The 9-Intersection Method

In this section, after a simple description of the principle of 9IM, we discuss various symmetries among topological 9IM relations, and then show how to

Table 1: Relations defined in RCC

Relation	Interpretation	Definition
EQ	a is *identical with* b	$a = b$
DR	a is *discrete from* b	$a \cdot b = 0$
PP	a is a *proper part of* b	$a < b$
O	a *overlaps* b	$a \cdot b > 0$
PO	a *partially overlaps* b	$(a\mathbf{O}b) \wedge (a \nleq b) \wedge (a \ngeq b)$
DC	a is *disconnected with* b	$\neg(a\mathbf{C}b)$
EC	a is *externally connected with* b	$(a\mathbf{C}b) \wedge (a\mathbf{DR}b)$
TPP	a is a *tangential proper part of* b	$(a\mathbf{PP}b) \wedge (a\mathbf{EC} - b)$
NTPP	a is a *non-tangential proper part of* b	$a\mathbf{DC} - b$

determine the RCC5 or RCC8 relations (over general regions) by the nine intersections.

4.1 The principle of 9IM

The topological relation between two regions can be characterized by considering intersections of interiors, boundaries, and exteriors of the two regions. The results can be concisely summarized in a 3×3 matrix:

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^e \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^e \\ A^e \cap B^\circ & A^e \cap \partial B & A^e \cap B^e \end{pmatrix}.$$

In this paper we only consider the *content* of these 9 intersections. In other words, we decide for each intersection whether it is empty or not. If an intersection is empty, we write 0 for the entry in the corresponding matrix, and write 1 otherwise. In this way, the topological relation between any two regions, A, B, can be represented as a 3×3 Boolean matrix $M(A, B)$. Note that there are altogether $2^9 = 512$ such matrices. A question that arises naturally is which matrix represents a *genuine* topological relation?

Given a 3×3 Boolean matrix M, we say M is *realizable* if there are two (*possibly exotic*) regions A, B such that $M = M(A, B)$. Not all 3×3 Boolean matrices are realizable. For example, if all entries are 0, then the matrix cannot be realizable. More constraints are summarized in the following lemma:

Lemma 4.1. *For any two regions A, B, let M be the 9IM matrix of (A, B). Suppose α_i is the i-th row, and β_j is the j-th column of M. Then*

- $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ *cannot be* $(0, 0, 0)$*;*

- $\alpha_1, \alpha_3, \beta_1, \beta_3$ *cannot be* $(0, 1, 0)$.

Proof. The first result follows from the fact that $A^\circ, \partial A, A^e$ and $B^\circ, \partial B, B^e$ are partitions of the real plane. The second result follows from the fact that the interior of a boundary is empty, hence no open set can be contained in a boundary. □

If we restrict the domain of discourse to the collection of simple regions, then there will only be 8 realizable matrices. In other words, there are only 8 topological relations between simple regions that can be characterized by using the 9IM. Interestingly, these 8 topological relations, as far as simple regions are concerned, are precisely the RCC8 relations (see Table 2 for illustrations).

In the next subsection we consider symmetries among these 9IM matrices.

4.2 Symmetries among 3×3 Boolean matrices

Given a 3×3 Boolean matrix M, the *transpose* of M is a matrix, written M^t, formed from M by interchanging the rows and columns so that row i of M becomes column i of the transpose matrix. Matrix M is called *symmetric* if $M = M^t$.

The *row* (*column*) *transpose* of M is a matrix, written M^r (M^c), formed from M by interchanging the first and third rows (columns) of M. We call M *row* (*column*) *symmetric* if its first and third rows (columns) are identical.

We have the following lemma.

Table 2: Illustrations and matrix representations of Egenhofer relations

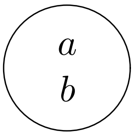
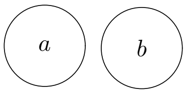
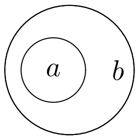
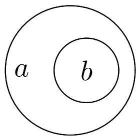
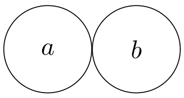
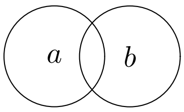
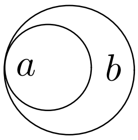
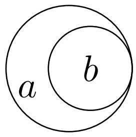
EQ	**DC**	**NTPP**	**NTPP$^{\sim}$**
			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
equal	disjoint	inside	contains
EC	**PO**	**TPP**	**TPP$^{\sim}$**
			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
meet	overlap	coveredBy	covers

Table 3: Variant forms of a 9-Intersection Matrix

$$\underset{M}{\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}} \quad \underset{M^r}{\begin{pmatrix} t_{31} & t_{32} & t_{33} \\ t_{21} & t_{22} & t_{23} \\ t_{11} & t_{12} & t_{13} \end{pmatrix}} \quad \underset{M^c}{\begin{pmatrix} t_{13} & t_{12} & t_{11} \\ t_{23} & t_{22} & t_{21} \\ t_{33} & t_{32} & t_{31} \end{pmatrix}} \quad \underset{M^{rc} = M^{cr}}{\begin{pmatrix} t_{33} & t_{32} & t_{31} \\ t_{23} & t_{22} & t_{21} \\ t_{13} & t_{12} & t_{11} \end{pmatrix}}$$

$$\underset{M^t}{\begin{pmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}} \quad \underset{M^{tr} = M^{ct}}{\begin{pmatrix} t_{13} & t_{23} & t_{33} \\ t_{12} & t_{22} & t_{32} \\ t_{11} & t_{21} & t_{31} \end{pmatrix}} \quad \underset{M^{tc} = M^{rt}}{\begin{pmatrix} t_{31} & t_{21} & t_{11} \\ t_{32} & t_{22} & t_{12} \\ t_{33} & t_{23} & t_{13} \end{pmatrix}} \quad \underset{M^{trc} = M^{tcr}}{\begin{pmatrix} t_{33} & t_{23} & t_{13} \\ t_{32} & t_{22} & t_{12} \\ t_{31} & t_{21} & t_{11} \end{pmatrix}}$$

Lemma 4.2. *Given two regions A, B and a 3×3 Boolean matrix M, if $M(A,B) = M$, then $M(B,A) = M^t$, $M(-A,B) = M^r$, $M(A,-B) = M^c$.*

These basic operations can be combined. Given a 3×3 Boolean matrix M, a matrix M' is called a *variant form* of M if M' can be obtained from M by a finite sequence of the three basic operations.

For each Boolean matrix of order 3, M has, including itself, at most 8 different variant forms (see Table 3).

Now by Lemma 4.2, we know

Proposition 4.1. *Given two regions A, B and a 3×3 Boolean matrix M, if $M = M(A,B)$, then $M^{rc} = M(-A,-B)$, $M^{tr} = M(-B,A)$, $M^{tc} = M(B,-A)$, $M^{trc} = M(-B,-A)$.*

4.3 Determining RCC5 and RCC8 relations by the 9IM

RCC5 and RCC8 (see (6) and (7)) can be determined by the 9IM. Suppose A, B are two regions in $\mathsf{RC}(\mathbb{R}^2)$. Then the RCC5 relation between A and B is determined by the 3 intersections

$$(A^\circ \cap B^\circ, A^\circ \cap B^e, A^e \cap B^\circ) \tag{8}$$

In fact, we have

- $A\mathbf{EQ}B$ iff $A^\circ \cap B^\circ \neq \varnothing$, $A^\circ \cap B^e = \varnothing$, and $A^e \cap B^\circ = \varnothing$;

- $A\mathbf{PP}B$ iff $A^\circ \cap B^\circ \neq \varnothing$, $A^\circ \cap B^e = \varnothing$, and $A^e \cap B^\circ \neq \varnothing$;
- $A\mathbf{PO}B$ iff $A^\circ \cap B^\circ \neq \varnothing$, $A^\circ \cap B^e \neq \varnothing$, and $A^e \cap B^\circ \neq \varnothing$;
- $A\mathbf{DR}B$ iff $A^\circ \cap B^\circ = \varnothing$, $A^\circ \cap B^e \neq \varnothing$, and $A^e \cap B^\circ \neq \varnothing$.

To determine the topological RCC8 relation, we need to consider the intersection $\partial A \cap \partial B$.

- $A\mathbf{TPP}B$ iff $A\mathbf{PP}B$ and $\partial A \cap \partial B \neq \varnothing$;
- $A\mathbf{NTPP}B$ iff $A\mathbf{PP}B$ and $\partial A \cap \partial B = \varnothing$;
- $A\mathbf{EC}B$ iff $A\mathbf{DR}B$ and $\partial A \cap \partial B \neq \varnothing$;
- $A\mathbf{DC}B$ iff $A\mathbf{DR}B$ and $\partial A \cap \partial B = \varnothing$.

Therefore, the RCC8 relation between two regions A, B can be uniquely determined by the four intersections

$$(A^\circ \cap B^\circ, A^\circ \cap B^e, A^e \cap B^\circ, \partial A \cap \partial B) \tag{9}$$

This means, given the content of the four intersections, we can tell in which RCC8 relation A, B are related.

On the other hand, suppose we know A, B are related by a particular RCC8 relation $\mathbf{R}$. From the above characterization of RCC8 by using the 4 intersections, the content of some other intersections may also be determined.

Take $\mathbf{EC}$ for example. If $A\mathbf{EC}B$, then $A^\circ \cap B^\circ = \varnothing$, $A^\circ \cap B^e \neq \varnothing$, $A^e \cap B^\circ \neq \varnothing$, and $\partial A \cap \partial B \neq \varnothing$. By $A^\circ \cap B^\circ = \varnothing$ we know $A^\circ \cap \partial B = \varnothing$ and $\partial A \cap B^\circ = \varnothing$. The intersections $\partial A \cap B^e$, $A^e \cap \partial B$, and $A^e \cap B^e$ are undetermined.

Table 4 summarizes the results. A question mark (?) appears whenever the content of the corresponding intersection is undetermined. We regard each matrix in Table 4 as a constraint on 9IM matrices. For example, $C_{\mathbf{EC}}$ is the constraint $(I_{\circ\circ} = 0) \wedge (I_{\circ\partial} = 0) \wedge (I_{\circ e} = 1) \wedge (I_{\partial\circ} = 0) \wedge (I_{\partial\partial} = 1) \wedge (I_{e\circ} = 1)$, where $I_{\partial\circ}$, say, denotes the intersection of the boundary of the first region and the interior of the second region. Similarly, we can define $C_{\mathbf{R}}$ for any other RCC8 relation $\mathbf{R}$.

Table 4: 9-Intersection Matrices of the RCC8 relations

$$\begin{array}{cccc}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
\mathbf{EQ} & \mathbf{DC} & \mathbf{NTPP} & \mathbf{NTPP}^{\sim} \\
\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & ? \\ 1 & ? & ? \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ 1 & ? & 1 \end{pmatrix} & \begin{pmatrix} 1 & ? & 1 \\ 0 & 1 & ? \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & ? & 1 \\ ? & ? & ? \\ 1 & ? & ? \end{pmatrix} \\
\mathbf{EC} & \mathbf{TPP} & \mathbf{TPP}^{\sim} & \mathbf{PO}
\end{array}$$

Proposition 4.2. *For any two regions A, B, the 9IM matrix $M(A, B)$ satisfies one and only one of the constraints in*

$$\{C_{\mathbf{EQ}}, C_{\mathbf{DC}}, C_{\mathbf{NTPP}}, C_{\mathbf{NTPP}^{\sim}}, C_{\mathbf{EC}}, C_{\mathbf{TPP}}, C_{\mathbf{TPP}^{\sim}}, C_{\mathbf{PO}}\}. \tag{10}$$

Proof. Since RCC8 is a JEPD set of relations, any two regions A, B are related by one and only one RCC8 relation. Therefore, $M(A, B)$ satisfies one and only one constraint in (10). □

Corollary 4.1. *A 3×3 matrix M is realizable only if M satisfies one constraint in (10).*

The "if" part of Corollary 4.1 is, however, not right. Take $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ for example. Clearly M satisfies the constraint $C_{\mathbf{PO}}$. But by $A^e \cap B^e = \varnothing$ we know $A^e \subset B$. Since A^e is open, we have $A^e \subset B^\circ$, hence $A^e \cap \partial B = \varnothing$. This is a contradiction. Therefore, M is non-realizable.

In the following we say a matrix M *represent*s an RCC8 relation $\mathbf{R}$, or M is a *representation* of $\mathbf{R}$, if M is realizable and satisfies the constraint $C_{\mathbf{R}}$. Note that a matrix is realizable if and only if it is a representation of some RCC8 relation. In order to give a complete classification of topological relations, we need to find all matrix representations of the RCC8 relations.

Table 5: 9IM matrices between regions with same boundary

$$\underset{M_1}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{M_2}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{M_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{M_4}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

$$\underset{M_5}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{M_6}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{M_7}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

By Table 4, each of $\mathbf{EQ}, \mathbf{DC}, \mathbf{NTPP}, \mathbf{NTPP}^\sim$ has a unique 9IM matrix representation. In the following sections we will find all matrix representations of $\mathbf{EC}, \mathbf{TPP}, \mathbf{TPP}^\sim$ and $\mathbf{PO}$.

Before carrying out this work, we first consider the realizable 9IM matrices between regions with the same boundary.

4.4 Matrices between regions with the same boundary

This subsection concerns 9IM matrices between regions that share the same boundary. We first note that if A and B have the same boundary, then the second row and the second column of $M(A, B)$ are both $(0, 1, 0)$. Therefore, the matrix $M(A, B)$ has the form

$$\begin{pmatrix} ? & 0 & ? \\ 0 & 1 & 0 \\ ? & 0 & ? \end{pmatrix}.$$

Since no row and no column can be $(0, 0, 0)$, only 7 cases (given in Table 5) are possible.

Clearly, M_3 holds if and only if $A = B$, i.e. $M(A, A) = M_3$. This is precisely the $\mathbf{EQ}$ relation. Similarly, M_1 holds if and only if $B = -A$, the complement of A. In this case, we have $M(A, -A) = M_1$.[2]

By Lemma 2.4 (taking $k = 3$) we have three exotic regions U_1, U_2, U_3 that satisfy the conditions given there. Then since $U_1^\circ \cap U_2^\circ = \varnothing$ and $U_1^e \cap U_2^e \neq \varnothing$, we have $M(U_1, U_2) = M_2$. Similarly, we have $M(U_1, U_1 + U_2) = M_4$, $M(U_1 + U_2, U_1) = M_5$. Since $M_4^r = M_6$, we have, by Lemma 4.2, $M(-U_1, U_1 + U_2) = M_6$. Finally, it is also straightforward to show $M(U_1 + U_2, U_1 + U_3) = M_7$.

In summary, there are 7 realizable matrices between regions with the same boundary. Given two regions A, B, suppose $\partial A = \partial B$. If $A \neq B$ and $A \neq -B$, then, by Definition 2.1, we know A, B are exotic regions. Therefore, the five realizable matrices M_2, M_4, M_5, M_6, M_7 can be realized only between exotic regions. In the following we also call these *exotic* relations.

5 Matrix representations of EC

If $A\mathbf{EC}B$, i.e. $A \cap B \neq \varnothing$, $A^\circ \cap B^\circ = \varnothing$, then we have $A^\circ \cap B^\circ$, $A^\circ \cap \partial B$, and $\partial A \cap B^\circ$ are empty, while $A^\circ \cap B^e$, $\partial A \cap \partial B$, and $A^e \cap B^\circ$ are nonempty. The remaining 3 intersections are undetermined. This means that the 9IM matrix representations of **EC** have the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & ? \\ 1 & ? & ? \end{pmatrix}.$$

We divide the examination into two cases according to whether the intersection of the two exteriors is empty or not.

Suppose $A^e \cap B^e = \varnothing$. By Lemma 2.1 we have $\overline{A^e} \cap B^e = A^e \cap \overline{B^e} = \varnothing$. Recall $\overline{A^e} = A^e \cup \partial A$, $\overline{B^e} = B^e \cup \partial B$. We have $\partial A \cap B^e = \varnothing$ and $A^e \cap \partial B = \varnothing$. Therefore, there is only one possible matrix in this case. We write this relation $\mathbf{EC}_1$ (see Table 6). Two regions A, B are related by $\mathbf{EC}_1$ if and only if B is the complement of A. This is precisely the matrix M_3 given in Table 5.

Next, suppose $A^e \cap B^e \neq \varnothing$. There are two places that are undetermined

[2] This relation can also be defined in the RCC theory (Düntsch, Schmidt & Winter 2001).

Table 6: 9IM matrix representation of **EC**

$$\underset{\mathbf{EC}_1}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{\boxed{\mathbf{EC}_2}}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{EC}_3}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{EC}_4}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{EC}_5}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}$$

and hence at most 4 matrices (see Table 6). We next show that these 4 matrices are all realizable.

First, note that $\mathbf{EC}_2$ is precisely the relation M_2 given in Table 5, hence it is realizable. Second, the following example gives configurations of the other **EC**-relations.

Example 5.1 ($\mathbf{EC}_1$,$\mathbf{EC}_3$,$\mathbf{EC}_4$,$\mathbf{EC}_5$). Take disks D_i $(0 \leq i \leq 2)$ such that $D_0\mathbf{EC}D_2$, and $D_1\mathbf{NTPP}D_0$ (see Figure 1).

Take $A_1 = D_0$, and $B_1 = -D_0$. Then $M(A_1, B_1) = \mathbf{EC}_1$.

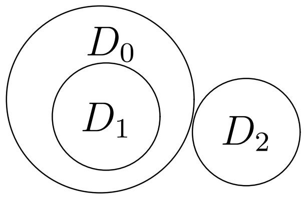
$\mathbf{EC}_3$ corresponds to the case where the boundary of A is a proper part of that of B. Take $A_2 = D_1$, $B_2 = D_0 - D_1$. Then $M(A_2, B_2) = \mathbf{EC}_3$. Since $\mathbf{EC}_4$ is the converse of $\mathbf{EC}_3$, we have $M(B_2, A_2) = \mathbf{EC}_4$.

$\mathbf{EC}_5$ corresponds to the case where ∂A and ∂B are incomparable. In this case both the second row and the second column are $(0, 1, 1)$. Clearly, $\mathbf{EC}_5$ is the Egenhofer relation "*meet*" when interpreted over simple regions. Set $A_3 = D_0$, $B_3 = D_2$. Then $M(A_3, B_3) = \mathbf{EC}_5$.

In summary, there are altogether five 9IM matrices that represent **EC**. Recall that $\mathbf{EC}_2$ can be realized only between exotic regions. In the following we will draw a frame box around the name of the matrix if it can be realized only between exotic regions.

6 Matrix representations of TPP

For two regions A, B, if A is a tangential proper part of B, then $A \subset B$ and $\partial A \cap \partial B \neq \varnothing$. By $A \subset B$, we know $A^\circ \subset B^\circ$. Therefore, the first row of the 9IM



$$\begin{aligned} M(D_0, -D_0) &= \mathbf{EC}_1 \\ M(D_1, D_0 - D_1) &= \mathbf{EC}_3 \\ M(D_0 - D_1, D_1) &= \mathbf{EC}_4 \\ M(D_0, D_2) &= \mathbf{EC}_5 \end{aligned}$$

Figure 1: Illustrations of **EC**-relations

Table 7: 9IM matrix representations of **TPP**

$$\underset{\boxed{\mathbf{TPP}_1}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{TPP}_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{TPP}_3}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{TPP}_4}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

is $(1,0,0)$. By $\partial A \subset B$, we know $\partial A \cap B^e = \varnothing$. Also note that $A^e \cap B^e \neq \varnothing$, since $B \neq \mathbb{R}^2$. Moreover, we claim $A^e \cap B^\circ \neq \varnothing$. Suppose that this does not happen. Then by $A^e \cap B^\circ = \varnothing$, we have $B^\circ \subseteq \mathbb{R}^2 \setminus A^e = A$, which is a contradiction. Consequently, each 9IM matrix of **TPP** has the form

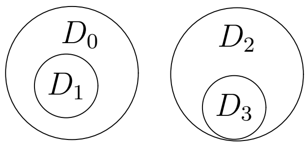
$$\begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ 1 & ? & 1 \end{pmatrix}.$$

Since only two places are undetermined, **TPP** has at most 4 9IM matrix representations. We list the 4 matrices in Table 7. All these matrices are realizable.

First, note that $\mathbf{TPP}_1$ is precisely the relation M_4 in Table 5. Second, the next example gives configurations of the other **TPP**-relations.

Example 6.1 ($\mathbf{TPP}_2$-$\mathbf{TPP}_4$). Take disks D_i $(0 \leq i \leq 3)$ such that $D_0 \cap D_2 = \varnothing$, $D_1 \mathbf{NTPP} D_0$, and $D_3 \mathbf{TPP} D_2$ (see Figure 2).

$\mathbf{TPP}_2$ corresponds to the case where the boundary of A is a proper part of that of B. Take $A_1 = D_0$, $B_1 = D_0 + D_2$. Then $M(A_1, B_1) = \mathbf{TPP}_2$.



$$
\begin{aligned}
M(D_0, D_0 + D_2) &= \mathbf{TPP}_2 \\
M(D_0 - D_1, D_0) &= \mathbf{TPP}_3 \\
M(D_3, D_2) &= \mathbf{TPP}_4
\end{aligned}
$$

Figure 2: Illustrations of **TPP**-relations

$\mathbf{TPP}_3$ corresponds to the case where the boundary of A contains that of B as a proper part. Take $A_2 = D_0 - D_1$, $B_2 = D_0$. Then $M(A_2, B_2) = \mathbf{TPP}_3$.

$\mathbf{TPP}_4$ corresponds to the case where the boundaries of A and B are incomparable. Take $A_3 = D_3$, $B_3 = D_2$. Then $M(A_3, B_3) = \mathbf{TPP}_4$. In fact, $\mathbf{TPP}_4$ is the Egenhofer relation "*cover*" when interpreted over simple regions.

7 Matrix representations of PO

Given two regions A, B, A partially overlaps B if and only if $A \not\subseteq B$, $B \not\subseteq A$, and $A^\circ \cap B^\circ \neq \varnothing$. By $A \not\subseteq B$, we have $A^\circ \not\subseteq B$, or equivalently $A^\circ \cap B^e \neq \varnothing$. Symmetrically, we have $A^e \cap B^\circ \neq \varnothing$. Consequently, each 9IM matrix representation of **PO** has the form

$$
\begin{pmatrix}
1 & ? & 1 \\
? & ? & ? \\
1 & ? & ?
\end{pmatrix}.
$$

There are 6 undetermined places in the matrix, and hence $2^6 = 64$ possible matrices should be examined. Given two regions A, B so that A partially overlaps B, we divide the examination into 4 cases according to the mereological (part-whole) relation between the boundaries of A and B.

7.1 A and B have the same boundary

Suppose A and B have the same boundary, i.e. $\partial A = \partial B$. In this case, the second row and the second column are both $(0, 1, 0)$. So only $A^e \cap B^e$ is not

Table 8: **PO**-matrices with $\partial A = \partial B$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boxed{\mathbf{PO}_1} \qquad \boxed{\mathbf{PO}_2}$$

Table 9: **PO**-matrices with $\partial A \subset \partial B$

$$\underset{\mathbf{PO}_3}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_4}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_5}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_6}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

determined, and there are two possible matrices as given in Table 8. Clearly, $\mathbf{PO}_1$ and $\mathbf{PO}_2$ are, respectively, M_7 and M_6 given in Table 5.

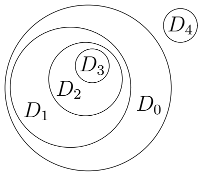
7.2 The boundary of A is a proper part of that of B

Suppose the boundary of A is strictly contained in that of B, i.e. $\partial A \subsetneqq \partial B$. In this case, the second row, denoted by α_2, is $(0, 1, 0)$, and the second column, denoted by β_2, is a 3-tuple $(u, 1, v)$ where $u = 1$ or $v = 1$. We divide this situation into two cases according to the content of $A^e \cap B^e$.

If $A^e \cap B^e \neq \varnothing$, then only the two places in the second column β_2 are undetermined. But since $\partial B \not\subseteq \partial A$, β_2 cannot be $(0, 1, 0)$. We have three matrices to check, viz. $\mathbf{PO}_3$, $\mathbf{PO}_4$ and $\mathbf{PO}_5$ in Table 9.

If $A^e \cap B^e = \varnothing$, then we have $A^e \subseteq B$ and $B^e \subseteq A$, hence $A^e \subseteq B^\circ$ and $B^e \subseteq A^\circ$. This shows that the third row and the third column are both $(1, 0, 0)$. Moreover, the second column, β_2, must be $(1, 1, 0)$ since $\beta_2 \neq (0, 1, 0)$. This 9IM matrix is $\mathbf{PO}_6$ in Table 9.

The 4 possible matrices in Table 9 are all realizable. We give configurations in the following example.



$$
\begin{aligned}
M(D_1 + D_4, D_4 + (D_0 - D_1)) &= \mathbf{PO}_3 \\
M((D_0 - D_2) + D_3, (D_0 - D_1) + (D_2 - D_3)) &= \mathbf{PO}_4 \\
M(D_1, (D_0 - D_1) + D_2) &= \mathbf{PO}_5 \\
M(D_0, -D_0 + D_1) &= \mathbf{PO}_6
\end{aligned}
$$

Figure 3: Illustrations of $\mathbf{PO}_3$ to $\mathbf{PO}_6$

Example 7.1 ($\mathbf{PO}_3$-$\mathbf{PO}_6$). Take disks D_i $(0 \le i \le 4)$ such that $D_0 \cap D_4 = \varnothing$, and $D_3\mathbf{NTPP}D_2\mathbf{NTPP}D_1\mathbf{NTPP}D_0$ (see Figure 3).

$\mathbf{PO}_3$ corresponds to the case where $\beta_2 = (0,1,1)$. Let $A_1 = D_1 + D_4$, and let $B_1 = D_4 + (D_0 - D_1)$. Then $M(A_1, B_1) = \mathbf{PO}_3$.

$\mathbf{PO}_4$ corresponds to the case where $\beta_2 = (1,1,0)$. Set $A_2 = (D_0 - D_2) + D_3$ and set $B_2 = (D_0 - D_1) + (D_2 - D_3)$. Then $M(A_2, B_2) = \mathbf{PO}_4$.

$\mathbf{PO}_5$ corresponds to the case where $\beta_2 = (1,1,1)$. Set $A_3 = D_1$, $B_3 = (D_0 - D_1) + D_2$. Then $M(A_3, B_3) = \mathbf{PO}_5$.

$\mathbf{PO}_6$ corresponds to the case where $\alpha_3 = \beta_3 = (1,0,0)$ and $\beta_2 = (1,1,0)$. Set $A_4 = D_0$, $B_4 = -D_0 + D_1$. Then $M(A_4, B_4) = \mathbf{PO}_6$.

7.3 The boundary of B is a proper part of that of A

Suppose the boundary of B is strictly contained in that of A, i.e. $\partial B \subsetneqq \partial A$. Then the second column is $(0,1,0)$ and the second row is a 3-tuple $(u,1,v)$ where $u = 1$ or $v = 1$. This case is the converse of the case where $\partial A \subsetneqq \partial B$. There are four realizable 9IM matrices (see Table 10), which are, respectively, the transposes of the matrices given in Table 9.

7.4 The boundary of A is incomparable to that of B

Suppose the boundary of A is incomparable to that of B. This means that neither the second row α_2 nor the second column β_2 is $(0,1,0)$. We divide the discussion into several subcases.

Table 10: **PO**-matrices with $\partial A \supset \partial B$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{PO}_7 \qquad \mathbf{PO}_8 \qquad \mathbf{PO}_9 \qquad \mathbf{PO}_{10}$$

Table 11: **PO**-matrices with $A^e \cap B^e \neq \varnothing$, $\partial A \cap \partial B \neq \varnothing$, $\partial A \not\subseteq \partial B$ and $\partial A \not\supseteq \partial B$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{PO}_{11} \qquad \mathbf{PO}_{12} \qquad \mathbf{PO}_{13} \qquad \mathbf{PO}_{14} \qquad \mathbf{PO}_{15}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{PO}_{16} \qquad \mathbf{PO}_{17} \qquad \mathbf{PO}_{18} \qquad \mathbf{PO}_{19}$$

7.4.1 The case where $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B \neq \varnothing$

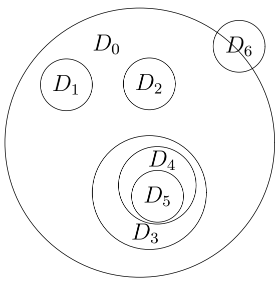
If $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B \neq \varnothing$, then the 9IM matrix has the form

$$\begin{pmatrix} 1 & ? & 1 \\ ? & 1 & ? \\ 1 & ? & 1 \end{pmatrix}.$$

Since neither α_2 nor β_2 can be $(0,1,0)$, there are 9 possible matrices (see Table 11). We next show these 9 matrices are all realizable.

Example 7.2 ($\mathbf{PO}_{11}$-$\mathbf{PO}_{19}$). Take closed disks D_i ($0 \leq i \leq 7$) such that (i) D_1, D_2, D_3, D_6 are pairwise disjoint; (ii) $D_5\mathbf{TPP}D_4\mathbf{NTPP}D_3$; (iii) $D_i\mathbf{NTPP}D_0$ for $1 \leq i \leq 5$; and (iv) $D_6\mathbf{PO}D_0$ (see Figure 4).

$\mathbf{PO}_{11}$ corresponds to the case where $\alpha_2 = \beta_2 = (0,1,1)$. Set $A_1 = D_1 + D_2$, $B_1 = D_1 + D_3$. Then $M(A_1, B_1) = \mathbf{PO}_{11}$.



$$
\begin{aligned}
M(D_1 + D_2, D_1 + D_3) &= \mathbf{PO}_{11} \\
M(D_1 + D_2, D_0 - (D_1 + D_3)) &= \mathbf{PO}_{12} \\
M(D_1 + D_5, D_2 + D_4) &= \mathbf{PO}_{13} \\
M(D_0 - (D_1 + D_3), D_1 + D_2) &= \mathbf{PO}_{14} \\
M(D_2 + D_4, D_1 + D_5) &= \mathbf{PO}_{15} \\
M(D_0 - (D_1 + D_2), D_0 - (D_1 + D_3)) &= \mathbf{PO}_{16} \\
M(D_3 - D_5, (D_0 - D_3) + D_4) &= \mathbf{PO}_{17} \\
M((D_0 - D_3) + D_4, D_3 - D_5) &= \mathbf{PO}_{18} \\
M(D_0, D_6) &= \mathbf{PO}_{19}
\end{aligned}
$$

Figure 4: Illustrations of $\mathbf{PO}_{11}$ to $\mathbf{PO}_{19}$

$\mathbf{PO}_{12}$ corresponds to the case where $\alpha_2 = (1,1,0)$ and $\beta_2 = (0,1,1)$. Set $A_2 = D_1 + D_2$, $B_2 = D_0 - (D_1 + D_3)$. Then $M(A_2, B_2) = \mathbf{PO}_{12}$.

$\mathbf{PO}_{14}$ corresponds to the case where $\alpha_2 = (0,1,1)$ and $\beta_2 = (1,1,0)$. Note that $\mathbf{PO}_{14}$ is the transpose of $\mathbf{PO}_{12}$. We have $M(B_2, A_2) = \mathbf{PO}_{14}$.

$\mathbf{PO}_{16}$ corresponds to the case where $\alpha_2 = \beta_2 = (1,1,0)$. Set $A_3 = D_0 - (D_1 + D_2)$, $B_3 = D_0 - (D_1 + D_3)$. Then $M(A_3, B_3) = \mathbf{PO}_{16}$.

$\mathbf{PO}_{13}$ is the case where $\alpha_2 = (1,1,1)$ and $\beta_2 = (0,1,1)$, $\mathbf{PO}_{15}$ is the case where $\alpha_2 = (0,1,1)$ and $\beta_2 = (1,1,1)$. Clearly, $\mathbf{PO}_{15}$ is the transpose of $\mathbf{PO}_{13}$. Set $A_4 = D_1 + D_5$, $B_4 = D_2 + D_4$. Then $M(A_4, B_4) = \mathbf{PO}_{13}$ and $M(B_4, A_4) = \mathbf{PO}_{15}$.

$\mathbf{PO}_{17}$ is the case where $\alpha_2 = (1,1,0)$ and $\beta_2 = (1,1,1)$, $\mathbf{PO}_{18}$ is the case where $\alpha_2 = (1,1,1)$ and $\beta_2 = (1,1,0)$. Hence, $\mathbf{PO}_{18}$ is the transpose of $\mathbf{PO}_{17}$. Set $A_5 = D_3 - D_5$, $B_5 = (D_0 - D_3) + D_4$. Then $M(A_5, B_5) = \mathbf{PO}_{17}$ and $M(B, A) = \mathbf{PO}_{18}$.

Note that $\mathbf{PO}_{19}$ is the case where $\alpha_2 = \beta_2 = (1,1,1)$. Then $M(D_0, D_6) = \mathbf{PO}_{19}$. In other words, $\mathbf{PO}_{19}$ is precisely the Egenhofer relation "*overlap*" when interpreted over simple regions.

7.4.2 The case where $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B = \varnothing$

If $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B = \varnothing$, then the 9IM matrix has the form

$$\begin{pmatrix} 1 & ? & 1 \\ ? & 0 & ? \\ 1 & ? & 1 \end{pmatrix}.$$

Note that neither α_2 nor β_2 can be $(0,0,0)$. There are 9 matrices left. The following lemma shows that the 4 matrices with $\alpha_2, \beta_2 \in \{(1,0,0),(0,0,1)\}$ are impossible.

Lemma 7.1. *If $\partial A \subset B^\circ$ and $\partial B \subset A^\circ$, then $A^e \cap B^e = \varnothing$.*

Proof. Suppose $\partial A \subset B^\circ$ and $\partial B \subset A^\circ$. If $A^e \cap B^e \neq \varnothing$, then there is a point $p \notin A \cup B$. Set $r = d(p, A)$, $s = d(p, B)$, where $d(p, X) = \inf\{d(p, x) : x \in X\}$ for $X \subseteq \mathbb{R}^2$. Since A and B are two closed sets, we have $a \in \partial A$ and $b \in \partial B$ such that $r = d(p, a)$ and $s = d(p, b)$.

Since $\partial A \subset B^\circ$, we know $a \in B^\circ$. So there is $\epsilon > 0$ such that $B(a, \epsilon) \subseteq B^\circ$, where $B(a, \epsilon)$ is the closed disk centered at a with radius ϵ. Clearly, there exists a point a' in $B(a, \epsilon)$ such that $d(p, a') < d(p, a)$. This shows $d(p, b) = d(p, B) \leq d(p, a') < d(p, a)$. A similar argument shows $d(p, a) < d(p, b)$, which is a contradiction. Therefore, $A^e \cap B^e$ is empty. $\square$

The following lemmas can be obtained from Lemma 7.1 by replacing A and/or B with their complements $-A$ and/or $-B$.

Lemma 7.2. *If $\partial A \subset B^\circ$ and $\partial B \subset A^e$, then $A^\circ \cap B^e = \varnothing$.*

Lemma 7.3. *If $\partial A \subset B^e$ and $\partial B \subset A^\circ$, then $A^e \cap B^\circ = \varnothing$.*

Lemma 7.4. *If $\partial A \subset B^e$ and $\partial B \subset A^e$, then $A^\circ \cap B^\circ = \varnothing$.*

By these lemmas, we have the following corollary.

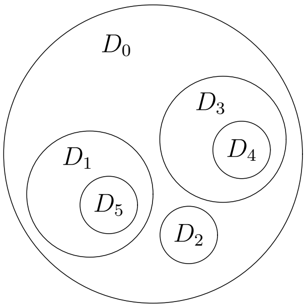
Corollary 7.1. *For two regions $A\mathbf{PO}B$, suppose $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B = \varnothing$. Denote α_2 the second row, and β_2 the second column of $M(A, B)$. Then*

$$\alpha_2 = (1,0,0) \quad \Rightarrow \quad \beta_2 = (1,0,1) \tag{11}$$

$$\alpha_2 = (0,0,1) \quad \Rightarrow \quad \beta_2 = (1,0,1) \tag{12}$$

Table 12: **PO**-matrices with $A^e \cap B^e \neq \varnothing$ and $\partial A \cap \partial B = \varnothing$

$$\underset{\mathbf{PO}_{20}}{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_{21}}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_{22}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_{23}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_{24}}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}$$



$$\begin{aligned} M(D_1 + D_4, D_2 + D_3) &= \mathbf{PO}_{20} \\ M(D_2 + D_3, D_1 + D_4) &= \mathbf{PO}_{21} \\ M(D_2 + D_3, D_0 - (D_1 + D_4)) &= \mathbf{PO}_{22} \\ M(D_0 - (D_1 + D_4), D_2 + D_3) &= \mathbf{PO}_{23} \\ M(D_1 + D_4, D_3 + D_5) &= \mathbf{PO}_{24} \end{aligned}$$

Figure 5: Illustrations of $\mathbf{PO}_{20}$ to $\mathbf{PO}_{24}$

Only 5 matrices (given in Table 12) satisfy the constraints given in Corollary 7.1. We next show these matrices are all realizable.

Example 7.3 ($\mathbf{PO}_{20}$-$\mathbf{PO}_{24}$). Take disks D_i $(0 \leq i \leq 5)$ such that D_1, D_2, D_3 are pairwise disjoint, and $D_4\mathbf{NTPP}D_3$, $D_5\mathbf{NTPP}D_1$, $D_i\mathbf{NTPP}D_0$ $(1 \leq i \leq 5)$ (see Figure 5).

Note that $\mathbf{PO}_{20}$ is the matrix where $\alpha_2 = (1, 0, 1)$ and $\beta_2 = (0, 0, 1)$, and $\mathbf{PO}_{21}$ is its converse. Set $A_1 = D_1 + D_4$, $B_1 = D_2 + D_3$. Then A_1 and B_1 has a common part D_4. Moreover, $\partial B_1 \cap A_1$ is empty. This shows $M(A_1, B_1) = \mathbf{PO}_{20}$ and $M(B_1, A_1) = \mathbf{PO}_{21}$.

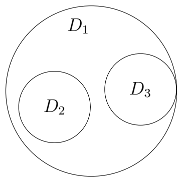
As for $\mathbf{PO}_{23}$, note that it is the matrix where $\alpha_2 = (1, 0, 1)$ and $\beta_2 = (1, 0, 0)$, and its converse is $\mathbf{PO}_{22}$. Set $A_2 = D_0 - (D_1 + D_4)$, $B_2 = D_2 + D_3$. Then A_2 and B_2 overlap, and $\partial B_1 \subset A_2^\circ$. Therefore, $M(A_2, B_2) = \mathbf{PO}_{23}$ and $M(B_2, A_2) = \mathbf{PO}_{22}$.

The matrix $\mathbf{PO}_{24}$ is a symmetric one where $\alpha_2 = \beta_2 = (1, 0, 1)$. Set $A_3 = D_1 + D_4$, $B_3 = D_3 + D_5$. Then $M(A_3, B_3) = \mathbf{PO}_{24}$.

Table 13: **PO**-matrices with $A^e \cap B^e = \varnothing$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\mathbf{PO}_{25}$ $\qquad$ $\mathbf{PO}_{26}$



$$\begin{aligned} M(-D_2, D_1) &= \mathbf{PO}_{25} \\ M(-D_3, D_1) &= \mathbf{PO}_{26} \end{aligned}$$

Figure 6: Illustrations of $\mathbf{PO}_{25}$ and $\mathbf{PO}_{26}$

7.4.3 The case where $A^e \cap B^e = \varnothing$

Last, we consider the situation where $A^e \cap B^e = \varnothing$, i.e. $A \cup B = \mathbb{R}^2$. By $A^e \subseteq B$ and $B^e \subseteq A$, we have $A^e \subseteq B^\circ$ and $B^e \subseteq A^\circ$. Hence $\alpha_3 = (1,0,0)$, $\beta_3 = (1,0,0)$.

Moreover, since we assume that ∂A and ∂B are incomparable, $\alpha_2, \beta_2 \notin \{(0,1,0),(0,0,0)\}$. This suggests both $\partial A \cap B^\circ$ and $A^\circ \cap \partial B$ are nonempty. Therefore, the matrix has the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & ? & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are only two possible matrices in this situation (see Table 13). The following example shows that both are realizable.

Example 7.4 ($\mathbf{PO}_{25}, \mathbf{PO}_{26}$). Take three disks D_1, D_2, D_3 such that $D_2\mathbf{NTPP}D_1$, $D_3\mathbf{TPP}D_1$ (see Figure 6).

If $\partial A \cap \partial B = \varnothing$, then we have $\partial A \subset B^0$ and $\partial B \subset A^0$. The unique 9IM matrix is $\mathbf{PO}_{25}$. Then $M(-D_2, D_1) = \mathbf{PO}_{25}$.

If $\partial A \cap \partial B \neq \varnothing$, then by $\partial A \not\subseteq \partial B$ and $\partial B \not\subseteq \partial A$, we have $\alpha_2 = \beta_2 = (1,1,0)$.

Then $M(-D_3, D_1) = \mathbf{PO}_{26}$.

8 Further discussions and related works

We have shown above that altogether 43 9IM matrices (see Table 14) are realizable in the real plane. While the 5 matrices given in Table 5 are only realizable between exotic regions, for all the remaining 38 relations, we construct configurations by using sums, differences, and complements of disks.

8.1 Raster regions in digital plane

In practice the most used discrete space is the *digital plane* (or *raster space*) $\mathbb{Z}^2$, which is defined as a rectangular array of points or pixels. Each point p is addressed by a pair of integers (p_1, p_2). For each point p, let S_p be the square in the real plane centered at p with length 1 (Li & Ying 2004, pages 18-19). We call each S_p a *pixel*. In this way we associate to each point in $\mathbb{Z}^2$ a region in $\mathbb{R}^2$.

In general, for a nonempty proper subset $X \subset \mathbb{Z}^2$, we define $\widehat{X} = \bigcup\{S_p : p \in X\}$, and call it a *raster region*. Note that $\widehat{X}$ is also a region in $\mathbb{R}^2$.

We now consider the topological relations between raster regions.

First, since raster regions are also plane regions, there are at most 43 9IM relations between raster regions. Second, we note that for two raster regions A, B, $\partial A = \partial B$ if and only if $A = B$ or $A = -B$. In other words, no raster region can be exotic. Therefore, the 5 exotic 9IM relations are non-realizable in the digital plane. Finally, it is straightforward to adapt the illustrations given in Figures 1-6 to construct configurations of the 38 non-exotic relations by using raster regions.

Therefore, altogether 38 9IM relations (those in Table 14 without frame box) can be realized in the digital plane.

Table 14: All possible 9IM relations

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
$\mathbf{EQ}$	$\mathbf{DC}$	$\mathbf{NTPP}$	$\mathbf{NTPP}^{\sim}$	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
$\mathbf{EC}_1$	$\boxed{\mathbf{EC}_2}$	$\mathbf{EC}_3$	$\mathbf{EC}_4$	$\mathbf{EC}_5$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	
$\boxed{\mathbf{TPP}_1}$	$\mathbf{TPP}_2$	$\mathbf{TPP}_3$	$\mathbf{TPP}_4$	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
$\boxed{\mathbf{TPP}_1^{\sim}}$	$\mathbf{TPP}_2^{\sim}$	$\mathbf{TPP}_3^{\sim}$	$\mathbf{TPP}_4^{\sim}$	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$
$\boxed{\mathbf{PO}_1}$	$\boxed{\mathbf{PO}_2}$	$\mathbf{PO}_3$	$\mathbf{PO}_4$	$\mathbf{PO}_5$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\mathbf{PO}_6$	$\mathbf{PO}_7$	$\mathbf{PO}_8$	$\mathbf{PO}_9$	$\mathbf{PO}_{10}$
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
$\mathbf{PO}_{11}$	$\mathbf{PO}_{12}$	$\mathbf{PO}_{13}$	$\mathbf{PO}_{14}$	$\mathbf{PO}_{15}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
$\mathbf{PO}_{16}$	$\mathbf{PO}_{17}$	$\mathbf{PO}_{18}$	$\mathbf{PO}_{19}$	$\mathbf{PO}_{20}$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\mathbf{PO}_{21}$	$\mathbf{PO}_{22}$	$\mathbf{PO}_{23}$	$\mathbf{PO}_{24}$	$\mathbf{PO}_{25}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$				
$\mathbf{PO}_{26}$				

8.2 Bounded regions

Now we restrict our discussion on bounded regions. For two bounded regions A, B, by $A \cup B \neq \mathbb{R}^2$ we know $A^e \cap B^e \neq \varnothing$, i.e. $I_{ee} = 1$. Matrices that do not satisfy this constraint cannot be realized between bounded regions. There are 6 such matrices in Table 14, viz. $\mathbf{EC}_1$, $\boxed{\mathbf{PO}_2}$, $\mathbf{PO}_6, \mathbf{PO}_{10}, \mathbf{PO}_{25}, \mathbf{PO}_{26}$.

How about other matrices with $I_{ee} = 1$? Can we find bounded configurations for these relations? Checking over the illustrations given in Figures 1-6, we find that all our configurations for these relations are bounded. Therefore, there are altogether 37 (4 exotic and 33 non-exotic) 9IM relations that can be realized between bounded regions.

8.3 Worboys-Bofakos model

Using simple regions as atomic regions, one can construct some (not all) complex regions. (Worboys & Bofakos 1993) propose a model for a large class of complex regions, where each complex region can be uniquely expressed as finite combinations of simple regions. This model is constructed in three stages: firstly, *atoms* are simple regions, then *base areas*, which contain atoms as components, and *generic areas*, which are allowed to have holes and can be represented as trees. We call this model the *Worboys-Bofakos model*. Note that each region in this model is bounded and non-exotic. If we restrict our discussion on regions in this model, there are at most 33 realizable 9IM matrices. Checking over Figures 1-6, we find that all these relations can be realized in this model.

8.4 Related works

Two particular kinds of complex regions are *regions with holes* (Egenhofer et al. 1994) and *composite regions* (Clementini et al. 1995). In (Egenhofer et al. 1994), each region with holes is represented by its generalized region – the union of the object and its holes – and each hole, where the generalized region and each hole are simple regions. The topological relation between two regions with holes is described by the Egenhofer relations between the generalized regions and holes of the two regions. Composite regions are those made up of several components, where each component is a simple region. Topological relations between composite regions are represented in (Clementini et al. 1995), which is based on the CBM (Clementini et al. 1993) for simple regions.

Our approach is very general and applicable for arbitrary plane regions. The approaches by (Egenhofer et al. 1994) and (Clementini et al. 1995), however, are limited to quite special complex regions, where the topological relation between two complex regions is characterized by the relations between involved simple regions. This means that the topological relation is indeed described by a binary constraints network. As the number of holes/components increases, the number of relations increases quickly. It is no surprise that some topologically distinct relations between, say regions with holes (e.g. the three configurations given in (Egenhofer et al. 1994, Fig. 2)), cannot be distinguished by the 9IM.

(Schneider & Behr 2005) provide a complete classification of topological relations between complex regions and complex lines. It is also based on the 9IM, where a complex region is defined as a bounded regular closed set with finite connected components.

As we have shown in Section 8.2, there are 37 (4 exotic and 33 non-exotic) realizable 9IM relations between bounded regions. The same result is applicable to complex regions of (Schneider & Behr 2005) since each complex region in their sense is bounded.

9 Conclusions and further work

A complete classification of topological relations using the 9-Intersection Method has been carried out. Unlike (Egenhofer & Herring 1990), which is restricted to simple regions, we apply the 9IM to cope with general plane regions. We have shown that there are all together 43 topological relations that can be realized in the real plane. This set of relations refines the well-known RCC8 topological relations of (Randell et al. 1992).

In Section 4.3 we showed that 4 intersections are enough to determine the

Table 15: Four pairs of realizable 9IM matrices

$$\underset{\mathbf{EC}_3}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_3}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \Bigg\| \underset{\mathbf{TPP}_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_3}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \Bigg\| \underset{\mathbf{TPP}_2^{\sim}}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{PO}_7}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \Bigg\| \underset{\mathbf{NTPP}}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \quad \underset{\mathbf{TPP}_4}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}$$

RCC8 relations. A question that arises naturally is how many intersections are needed to determine the RCC8 relations. As a byproduct of our complete classification, we claim that the 4 intersections given in (9) is the smallest set of intersections needed to determine the RCC8 relations. This is because, for each of the four intersections in (9), we have two realizable matrices so that they differ only at the value of this intersection (see Table 15).

(Düntsch et al. 2001) have investigated relations that can be defined by the connectedness relation in an RCC model. It would be interesting to compare these RCC relations with the 38 topological relations classified by the 9IM. For example, the RCC11 relations defined in (Düntsch 2005) can be completely characterized by the 9IM in the *complemented* Egenhofer model (Li & Li 2006), which contains all simple regions and their complements. As a matter of fact, RCC11 contains RCC8 relations **EQ**,**DC**,**TPP**,**NTPP**,$\mathbf{TPP}^{\sim}$, $\mathbf{NTPP}^{\sim}$, it splits **EC** into **ECN** and **ECD**, and splits **PO** into **PON**, **PODY**, **PODZ**. Applying the 9IM on the complemented Egenhofer model, we can see that the matrix representations of **ECN**, **ECD**, **PON**, **PODY**, **PODZ** are respectively $\mathbf{EC}_5, \mathbf{EC}_1$, $\mathbf{PO}_{19}$, $\mathbf{PO}_{26}, \mathbf{PO}_{25}$. RCC11, when interpreted over the complemented Egenhofer model, is highly related to the 11 spherical relations defined by Egenhofer (Egenhofer 2005): these two systems of relations have identical 9IM matrix representations.

This paper considers each object as a whole, and overlooks the internal relations between holes and components. It is reasonable to extend the work reported here to a more detailed formalization, where internal as well external relations are expressed.

Another question concerns the compositions of these 9IM relations. Unlike the 8 Egenhofer relations between simple regions, the complete set of relations defined in this paper does not form a relation algebra. This is because some compositions are *weak*, i.e. non-extensional.

Take the composition $\mathbf{TPP}_2 \circ \mathbf{TPP}_2$ for example. Recall $A\mathbf{TPP}_2 B$ iff $A \subset B$ and $\partial A \subset \partial B$. For three regions A, B, C such that $A\mathbf{TPP}_2 B$ and $B\mathbf{TPP}_2 C$, by $A \subset B \subset C$ and $\partial A \subset \partial B \subset \partial C$, we know $A\mathbf{TPP}_2 C$. This means $\mathbf{TPP}_2 \circ \mathbf{TPP}_2 \subseteq \mathbf{TPP}_2$. This composition, however, is not extensional, i.e. $\mathbf{TPP}_2 \circ \mathbf{TPP}_2 \neq \mathbf{TPP}_2$. For example, take two disjoint disks D_1, D_2, set $A = D_1$ and $C = D_2 + D_2$. Then $A\mathbf{TPP}_2 C$. But there exists no region B such that $A\mathbf{TPP}_2 B$ and $B\mathbf{TPP}_2 C$ hold.

Compared with RCC8, the 38 (non-exotic) 9IM relations are more elaborate in spatial representation. This could be helpful also in reasoning. For example, from $A\mathbf{EC}_4 B$ and $B\mathbf{EC}C$ we know that A cannot be discrete from C, i.e. $A \cap C \neq \varnothing$. This is because, by $A\mathbf{EC}_4 B$, the boundary of A contains that of B, and by $B\mathbf{EC}C$, we know B and C (hence A and C) meet at the boundary. This information, however, cannot be deduced by the RCC8 compositions.

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