Supplement

This supplement describes the deduction of the formula for approximating the theoretical BAEE model.

Supplement A. Derivation process of the approximated ellipse for BAEE model

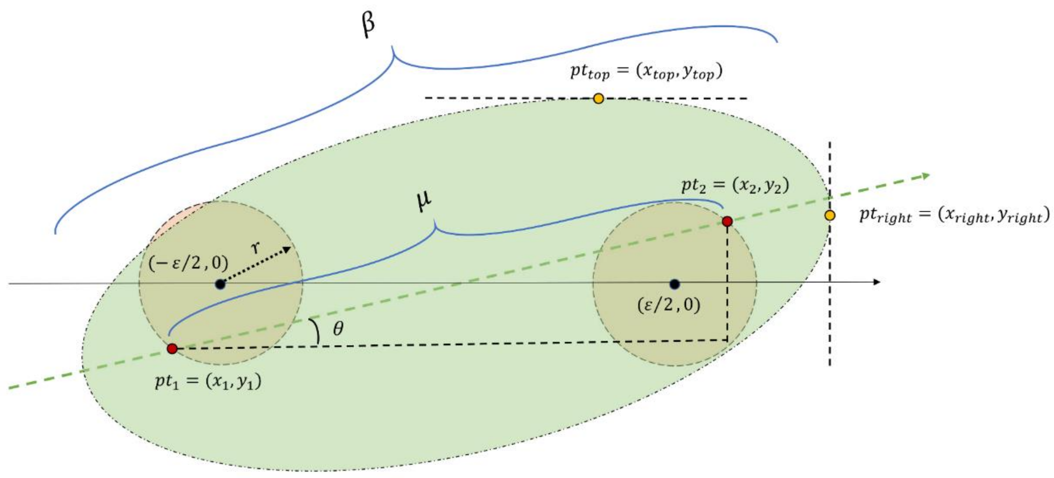

Figure A.1. Diagram for finding the maximum y or x of BAEE model.

Figure A.1 shows a general example that illustrates the calculation of extreme values of the BAEE model. Given two sampled points $(-\varepsilon/2, 0)$ and $(\varepsilon/2, 0)$, inferring that the theoretical BAEE model is an axisymmetric shape is easy. Therefore, determining the range for the arbitrary point (x, y) on BAEE boundary only requires the calculation of $\max(x)$ and $\max(y)$, whereas $\min(x) = -\max(x)$ and $\min(y) = -\max(y)$.

1) Limits of a rotated ellipse:

An ellipse centring at the origin with a rotation of θ can be expressed as:

$$\frac{(x\cos\theta - y\sin\theta)^2}{a^2} + \frac{(x\sin\theta + y\cos\theta)^2}{b^2} = 1, \tag{A.1}$$

where a and b denote the length of its semi-major and semi-minor axes, respectively. Through considerable algebra and tears, the limits of this ellipse can be easily obtained as follows:

$$\max(x) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}, \max(y) = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}. \tag{A.2}$$

Therefore, the limits of an arbitrary ellipse with a rotation of θ and $pt_1 = (x_1, y_1)$ and $pt_2 = (x_2, y_2)$ as foci, such as the ellipse in Figure A.1, can be expressed as:

$$\begin{aligned} \max(x) = x_{right} &= \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \frac{x_1 + x_2}{2}, \\ \max(y) = y_{top} &= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} + \frac{y_1 + y_2}{2} \end{aligned} . \tag{A.3}$$

2) Possible locations of pt_1 and pt_2:

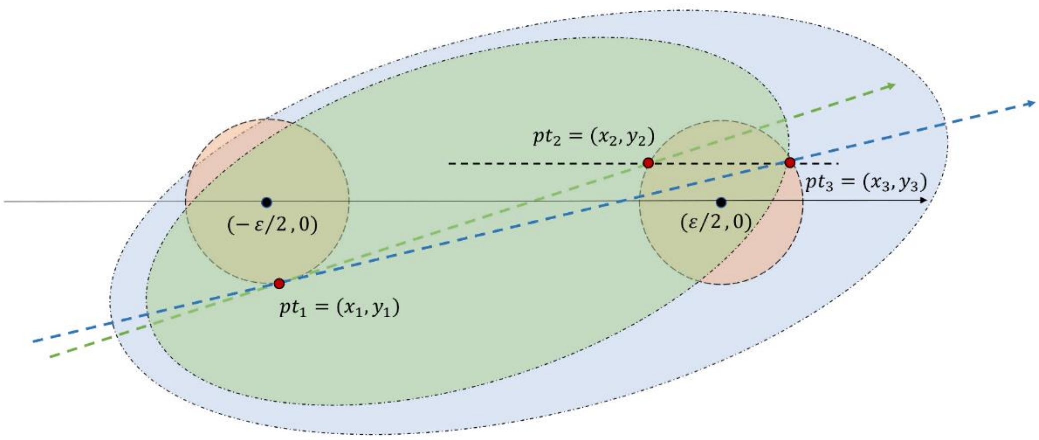

Figure A.2. Diagram for exploring the possible locations of point pair.

Given a certain $p_1 = (x_1, y_1)$, for any $p_2 = (x_2, y_2)$ on the left semi-circle as shown in Figure A.2, we can find a point $p_3 = (x_3, y_3)$ on the right semi-circle where $y_3 = y_2$.

Note that the rotation θ can be represented by pt_1, pt_2, that is, $\cos\theta = \frac{x_2-x_1}{u}$, $\sin\theta = \frac{y_2-y_1}{u}$, and $a = \beta/2 = c\mu/2$, $b = \sqrt{c^2-1} * u/2$, Therefore, the Equation (A.2) can be simplified as:

$$\begin{aligned} \max(x) = x_{right} = \frac{1}{2}\sqrt{c^2(x_2-x_1)^2+(c^2-1)*(y_2-y_1)^2}+\frac{x_1+x_2}{2}, \\ \max(y) = y_{top} = \frac{1}{2}\sqrt{c^2(y_2-y_1)^2+(c^2-1)*(x_2-x_1)^2}+\frac{y_1+y_2}{2} \end{aligned}. \tag{A.4}$$

Given $x_3 \geq x_2$, on the basis of Equation (A.4), we have $x_{right}(pt_1, pt_2) \leq x_{right}(pt_1, pt_3)$. For this reason, extrapolating the possible domain for pt_1, pt_2 corresponding to the maximum x value of BAEE model is easy. The domain is shown in Figure A.3 (a). Similarly, the domain for pt_1, pt_2 corresponding to the maximum y value of the BAEE model is shown in Figure A.3 (b).

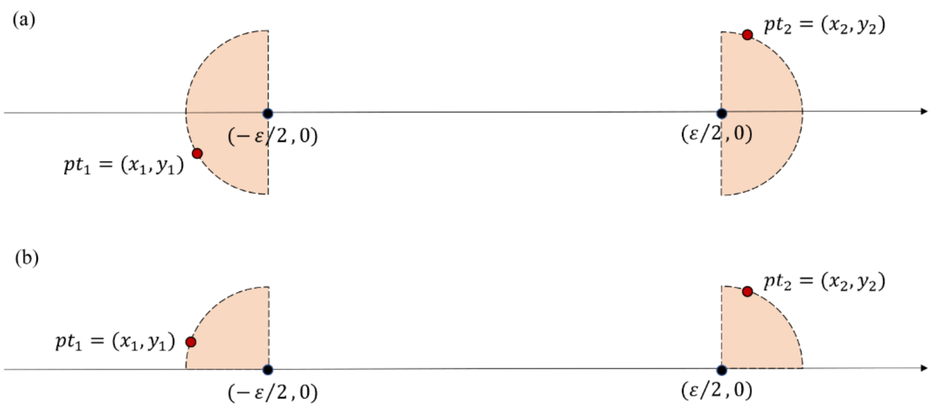

Figure A.3. Possible domain for pt_1, pt_2 corresponding to the limits of BAEE model.

3) Limits of BAEE (y-axis):

Let $E = \{e_0, e_1, e_2, \dots\}$ denote an infinite set composed of the ellipses between all possible point pairs, where the maximum coordinate for $e \in E$ is expressed as x_e and y_e.

$$\forall k \in N, \exists e^y \in E \quad s.t. \quad y_{e^y} \geq y_{e_k} \,. \tag{A.5}$$

Therefore, the computation of maximum y for theoretical BAEE model can be transformed into an objective function, which can be expressed as follows:

$$\max Z = \frac{1}{2}\sqrt{c^2(y_2-y_1)^2+(c^2-1)*(x_2-x_1)^2}+\frac{y_1+y_2}{2}$$
$$s.t. \quad \begin{cases} (x_1+\frac{\varepsilon}{2})^2+y_1{}^2=r^2 \\ (x_2-\frac{\varepsilon}{2})^2+y_2{}^2=r^2 \end{cases} \,. \tag{A.6}$$

For the example in Figure A.1, the objective function Z can be rewritten as:

$$Z=\frac{y_1+y_2}{2}+\frac{1}{2}\sqrt{c^2(y_1-y_2)^2+(c^2-1)(\sqrt{r^2-y_1^2}+\varepsilon+\sqrt{r^2-y_2^2})^2} \,. \tag{A.7}$$

The limits of Z appear at the stationary point (y_1^0, y_2^0), which should satisfy:

$$Z_{y_1}(y_1^0,y_2^0)=\left.\frac{\partial Z}{\partial y_1}\right|_{\substack{y_1=y_1^0\\ y_2=y_2^0}}=0 \quad \text{and} \quad Z_{y_2}(y_1^0,y_2^0)=\left.\frac{\partial Z}{\partial y_2}\right|_{\substack{y_1=y_1^0\\ y_2=y_2^0}}=0 \,. \tag{A.8}$$

After the differential, the partial derivative, $Z_{y_1}(y_1^0, y_2^0)$, for instance, can be expressed as

$$Z_{y_1}(y_1^0,y_2^0)=\frac{c^2(y_1^0\text{-}y_2^0)-\dfrac{(c^2-1)*y_1^0*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})}{\sqrt{r^2-(y_1^0)^2}}}{2\sqrt{(c^2-1)*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})^2+c^2(y_1^0-y_2^0)^2}}+1, \tag{A.9}$$

Similarly, we can compute $Z_{y_2}(y_1^0, y_2^0)$ as follows:

$$Z_{y_2}(y_1^0,y_2^0)=\frac{c^2(y_2^0\text{-}y_1^0)-\dfrac{(c^2-1)*y_2^0*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})}{\sqrt{r^2-(y_2^0)^2}}}{2\sqrt{(c^2-1)*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})^2+c^2(y_1^0-y_2^0)^2}}+1 \cdot \tag{A.10}$$

Combining Equation (A-8)–(A-10) and after considerable algebra, we can obtain:

$$y_1^0=y_2^0=\frac{r}{c} \quad \text{and} \quad Z_{\max}=r*c+\frac{\varepsilon}{2}\sqrt{c^2-1} \,. \tag{A.11}$$

4) Limits of BAEE (x-axis):

Similar to the derivation process in calculating the maximum y, the objective function Z can be written as:

$$Z=\frac{\sqrt{r^2-y_2^2}-\sqrt{r^2-y_1^2}}{2}+\frac{1}{2}\sqrt{(c^2\text{-}1)*(y_1-y_2)^2+c^2(\sqrt{r^2-y_1^2}+\varepsilon+\sqrt{r^2-y_2^2})^2}\ . \quad \text{(A.12)}$$

The partial derivative, $Z_{y_1}(y_1^0, y_2^0)$ can be expressed as:

$$Z_{y_1}(y_1^0,y_2^0)=\frac{(c^2\text{-}1)*(y_1^0\text{-}y_2^0)-\dfrac{c^2y_1^0*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})}{\sqrt{r^2-(y_1^0)^2}}}{2\sqrt{c^2*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})^2+(c^2-1)*(y_1^0-y_2^0)^2}}+\frac{y_1^0}{2\sqrt{r^2-(y_1^0)^2}}, \quad \text{(A.13)}$$

The partial derivative, $Z_{y_2}(y_1^0, y_2^0)$ can be expressed as:

$$Z_{y_2}(y_1^0,y_2^0)=\frac{(c^2\text{-}1)*(y_2^0\text{-}y_1^0)-\dfrac{c^2y_2^0*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})}{\sqrt{r^2-(y_2^0)^2}}}{2\sqrt{c^2*(\sqrt{r^2-(y_1^0)^2}+\varepsilon+\sqrt{r^2-(y_2^0)^2})^2+(c^2-1)*(y_1^0-y_2^0)^2}}-\frac{y_2^0}{2\sqrt{r^2-(y_2^0)^2}}, \quad \text{(A.14)}$$

By equating Equation (A-13) and (A-14) to zero, we have $y_1^0 = y_2^0 = 0$. Therefore, the maximum x can be calculated as $\mathrm{Z}_{max} = c * (\frac{\varepsilon}{2} + r)$.