The knowledge of knots: an interdisciplinary literature review

Paulo Santos, Pedro Cabalar, Roberto Casati

The Knowledge of Knots: an interdisciplinary literature review

Paulo E. Santos* Pedro Cabalar† Roberto Casati‡

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Abstract

Knots can be found and used in a variety of situations in the 3D world, such as in vines, in the DNA, polymer chains, electrical wires, in mountaneering, seamanship and when ropes or other flexible objects are involved for exerting forces and holding objects in place. Research on knots as topological entities has contributed with a number of findings, not only of interest to pure mathematics, but also to statistical mechanics, quantum physics, genetics and chemistry. Yet, the cognitive (or algorithmic) aspects involved in the act of tying a knot are a largely uncharted territory. This paper presents a review of the literature related to the investigation of knots from the topological, physical, cognitive and computational (including robotics) standpoints, aiming at bridging the gap between pure mathematical work on knot theory and macroscopic physical knots, with an eye to applications and modelling.

1 Introduction

Indirect evidence in the form of perforated beads, fishing net weights and water marks on artifacts, suggests that the first use of ropes and knots happened some time between 250,000 to 2.5 million years ago in the Pleistocene period. This development has probably caused a positive impact (if not a revolution) in the major activities of the prehistoric hunter-gatherer societies, facilitating the assembly of complex weapons, stable huts, fishing nets or carrier bags (Warner and Bednarik, 1998; Hardy, 2008). Ropes and knots were also part of some of the major technological advancements pre-dating the 20th century (Decker, 2010), in particular in seamanship, and today knots are so integrated in our everyday actions (such as tying shoe laces, sewing, cabling, packaging, holding objects in places) that they are taken for granted even outside the sailing or climbing circles. Knots have attracted an enormous amount of theoretical attention for

*Electrical Eng. Dep., Centro Universitário da FEI, SP, Brazil
†Computing Dep., Corunna University, Spain
‡Institut Jean Nicod, Ecole Normale Supérieure, France

their abstract complexity. Modern mathematical knot theory is a thriving field involving topology, algebra and combinatorics, finding important applications in theoretical physics (Kauffman, 2005), chemistry (Horner et al., 2016), biology (especially genetics) (Price, 2016; Summers, 2011) among other fields (Adams, 1994; Kauffman, 2006). Besides this, knots as objects of scientific inquiry have also been considered in physics, philosophy, cognitive science, and computer science. The present paper brings together these strands of research as an interdisciplinary literature review, clarifying the similarities and major distinctions between the concept of knot as assumed in these various disciplines. To the best of our knowledge this is the first work that tackles this issue in a single article. A similar venture was undertaken by Turner and Van de Griend (1996), where the various disciplines related to knots are described in 18 chapters written by distinct authors. Although the present paper does not claim to be as extensive as an entire book on the subject, we believe it is comprehensive enough to serve as an up-to-date critical review. A complete description of knots as a scientific subject would be a task for a full encyclopedia, as claimed in the preface of (Turner and Van de Griend, 1996).

The scientific literature on knots usually distinguishes three basic types of string entanglements: *hitches*, *braids* and *knots*. Usually they are (informally) defined as follows: *hitches* are a special kind of knots used to fasten a rope around another object (usually a post or another rope); *braids* are entanglements of a number of strings generated by twisting motions, so that the direction of each string remains the same. The general term for *knot* is used to represent entanglements of strings capable of holding their own shape, regardless of their relation with external objects. We also find the term *link* to represent an intertwined, but non-intersecting, collection of knots. Distinct basic definitions of these concepts are assumed by the distinct disciplines that investigate knots.

Topological knot theory, or simply *knot theory*, studies mathematical knots defined as instances of a circle in the 3D Euclidean space, up to continuous deformations (isotopies) (Kauffman, 2005). A fundamental problem in knot theory is to determine when distinct mathematical descriptions or diagrams of such embeddings of a 3D circle represent the same knot. An *unknot* is any such instance that is ambient-isotoped to a round circle. There is currently a fair amount of issues discussed in elementary knot theory (Lackenby, 2016) and in the theory of braids in topology (Birman and Brendle, 2005)), also with respect to its relations with theoretical physics (Kauffman, 2005). Section 2 describes the main concepts of topological knot theory, without dwelling into its various application domains.

The concept of knot in topology, has an insufficiently studied connection with the everyday use of knots as stable configurations of a rope. In this work, we call *physical knots* the filamentary structures whose shapes are governed by an interplay of friction and bending, being of special interest to climbing, sailing, and surgery (Jawed et al., 2015; Audoly et al., 2007). Much investigation on physical knots rely on the derivation of predictive models for the knot's mechanical equilibrium (Bayman, 1977; Maddocks and Keller, 1987), as described in Section 3.

A different aspect about knots, not covered by the topological or physical approaches, is their cognitive dimension. Knots are the result of carefully planned human actions. When studied under this perspective, we talk instead about *ecological knots* (Casati, 2013) and consider different issues such as: the modelling of the actions involved in knot tying; the way our body interacts with a string in order to produce a knot; whether there is (and what would be) a cognitive representation of knot-like structures; how this representation is updated according to the various ways a knot may be constructed; how we understand pictures of knots, and, finally, how these representations could be used by an artificial autonomous agent to solve tasks that involve reasoning (and manipulating) the states of a flexible rope. The literature on ecological knots is reviewed in Section 4, whereas Section 5 considers computational aspects related to reasoning about knots, including autonomous knot making in robotics research.

2 Knot Theory

Topological Knot Theory (or simply *Knot Theory*) is a well-established discipline in mathematics (Menasco, 2005) that has a long track of theoretical results derived from the fields of combinatorics, topology and group theory, and has provided important contributions to theoretical physics, biology and chemistry (Murasugi, 1996; Kauffman, 2005). Topological problems considering strings and knots have been studied since Gauss in the early 19th century (van de Griend, 1998). Nowadays, Knot Theory constitutes a very prolific area in topology, currently represented by one specialised journal (the *Journal of Knot Theory and its Ramifications*, since in 1992) and several textbooks (Adams, 1994; Kauffman, 2006; Lackenby, 2016; Crowell and Fox, 2008)[1] making a complete up-to-date review of this area virtually impossible. Still, in the interest of contextualising this research, the present section relies on existing reviews (Lackenby, 2016; van de Griend, 1998). Further literature surveys are given in (Menasco, 2005; Kawauchi, 1996) and the excellent historical overview in (van de Griend, 1998).

The fundamental issue in knot theory is deciding whether two knots or links are equivalent: this is known as the *equivalence problem*. Most work on this topic is related to the search of knot invariants, which are formal expressions that uniquely represent a knot, independently of any particular depiction of it.

Within the myriad of possible invariants proposed in the past century, two are worth mentioning in any review of knot theory due to their influence in other fields: the knot group and the knot polynomials. The former is defined as the fundamental group of the knot complement in R^3. Thus, the concept of knot group makes possible to relate algebraically two knots by the mathematical descriptions of their complement. The definition of the knot group was a fundamental breakthrough in knot theory; however, it is an incomplete knot invariant, as the group of a knot determines its complement up to the homotopy

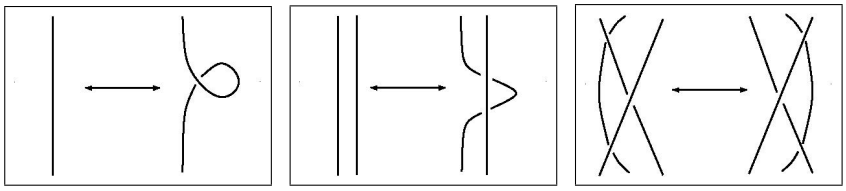
[1] A quick search on MIT library brought over one million references of books and half-a-million of articles related to Knot Theory.

type. In other words, distinct diagrammatic representations of equivalent knots may result in distinct representations of their knot groups. In principle, the distinct group representations of equivalent knots should be isomorphic groups. However, it has been proven that the problem of assessing if two finite groups are isomorphic is undecidable (Johnson, 1997; van de Griend, 1998).

The search for stronger invariants led to development of a number of methods for describing a knot in terms of polynomials (such as the Alexander, Conway, HOMFLY or Jones polynomials, among others (Adams, 1994)) whose coefficients encode properties of the knot. These polynomials proved to be good invariants for certain kinds of knots. For instance, the Alexander polynomials are able to distinguish certain classes of knots, defined up to 9 crossings (van de Griend, 1998).

Another algebraic way of investigating knots was derived from assuming smaller pieces of knots that respect a set of conditions. This led to the definition of *braids* or, more formally, of the braid group (i.e. the set of all braids on n strings, whose operation is the composition of braids). The connection between the braid group and knot theory is given by the Alexander's theorem that states that every knot can be represented as a closed braid (Alexander, 1923). Many current applications of knot theory in theoretical physics seem to gravitate around knot (including braid) groups or knot polynomials (Kauffman, 2005).

Perhaps the approach to the equivalence problem that permeates the other fields of interest in the present paper (as we shall see below) is the method for deciding the equivalence by means of reducing one knot representation to another using Reidemeister Moves (Reidemeister, 1983; Coward and Lackenby, 2011). The equivalence problem has a diagrammatic representation, whereby a "knot" is actually assumed to be an equivalence class of knot diagrams. Some operations on the syntactic elements of diagrams allow establishing equivalences in an intuitive and straightforward way. Reidemeister Moves (Reidemeister, 1983) are local modifications in the link diagrams representing knots that preserve the topological properties of the knot. Figure 1 shows the three basic Reidemeister moves, that can be described as follows: Reidemeister move I (Figure 1(a)) adds or deletes a simple twist in the string; Reidemeister move II (Figure 1(b)) allows the inclusion (or exclusion) of two crossings in the string; Reidemeister move III (Figure 1(c)) slides a strand of the string from one side of a crossing to the other.



(a) (b) (c)

Figure 1: The Reidemeister moves

Solving the equivalence problem by finding a sequence of Reidemeister moves from one diagram to another is clearly a semi-decidable process, since if two diagrams represent distinct knots, the process does not terminate. This fact has been observed by Turing (1955) with respect to entanglement puzzles (as presented in Section 5 below). However, if the maximum number of steps for finding the equivalence is known, this information could be used as a stopping criterion in the decision of whether two link diagrams represent distinct knots. This leads to another fundamental problem, i.e. finding good upper and lower bounds on the number of moves when deciding whether two diagrams are equivalent of not (Lackenby, 2016).

Recent solutions to the equivalence problem include, for instance, the definition of hierarchies of incompressible surfaces associated to the knot's boundaries upon which (triangular) cell structures are defined and are used to distinguish knots (Matveev, 2007). There are also algorithmic solutions based on the decompositions of knots into geometrical pieces by means of normal surface theory (Matveev, 2007; Lackenby, 2016), or solutions related to proving the isomorphism between two manifolds (representing the knots) using their related fundamental groups. The complexity class of the equivalence problem, however, is still an open issue in the field (Lackenby, 2016); some results for particular cases are proposed in (Hass et al., 1999; Hass and Lagarias, 2001).

Related to the search for an efficient algorithmic solution to the equivalence problem is the *recognition problem*, whose goal is to find a method for deciding whether a link type is represented by a given diagram. A somewhat related open problem is the computation of a minimum number of crossing changes (the *unknotting number*) needed to transform a set of links on a string into an unknot, whenever it is possible. The question of whether the unknotting number is algorithmically computable (or whether there is an algorithm to decide if a given knot has an unknotting number equal to one) is still unresolved.

Given any natural number c, it is possible to compute the number of possible knot types with c crossings, $N(c)$. However, the asymptotic behaviour of $N(c)$ as c grows to infinity is another issue of interest still under investigation in the field.

The problems briefly described above are only a small portion of the research questions tackled in knot theory alone. They do not try to cover all the fundamental issues in the discipline, let alone the possible applications of this theory in various other fields (Kauffman, 2005, 2006; Horner et al., 2016; Summers, 2011)). However, this discussion provides some elements of interest with respect to other disciplines investigating knots, as presented in the remainder of this paper.

3 The Physics of Knots

Analytical theories of physical (or mechanical) knots aim at defining a model that allows the prediction of the mechanical equilibrium of a knot. The first such theories were mostly based on the mathematical modelling of *hitches* (Figure

2), which are knots in which a string is tied to a pole or other cylindrical object (Bayman, 1977). In this case, the task is to predict whether a hitch is going to hold or to slip from the object it is tied on, given the hitches' topology and the frictional forces involved. Analytical results for hitches are simpler to obtain than those for knots, since the interaction of the string with a rigid (cylindrical) object imposes a circular shape to the string, whereas in analysing a knot one has to take into account the complicated 3D interactions of the string with itself.

Bayman (1977) presents a seminal theory of hitches where only the frictional forces between the string and the rigid object are modelled, the friction between the string on itself is considered as negligible. The model developed in (Bayman, 1977) defines a (string) segment of a hitch as the piece of the string separated by two over-crossings, or between the free ends of the string and the nearest over-crossing (following along the string). The theory then introduces conditions for maintaining a hitch of any given configuration as inequations between the tensions in the various string segments. This is accomplished by taking into account the coefficient of friction between the string and the pole and also the number of times the string crosses over itself. Each such crossover introduces an increase in the string tension in the lower segments, proportional to the tension on the upper segments (Maddocks and Keller, 1987). This model is initially developed for the clove hitch (shown in Figure 2), where the condition for the knot not to slip (i.e. the condition for the mechanical equilibrium of the hitch) is given by Equation 1:

$$t_4 \leq k \times t_0 \tag{1}$$

where t_0 and t_4 are the tensions at the end segments of the knot (segments 0 and 4 in Figure 2) and k is a constant based on two attributes: (1) the coefficient of friction between the pole and the string; (2) the ratio of the string and pole diameters.

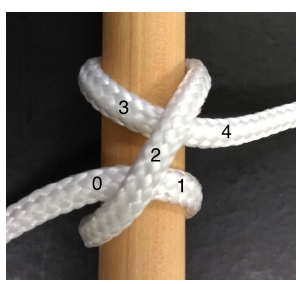


Figure 2: Clove Hitch.

This model is further defined in (Bayman, 1977) for a general hitch composed of an arbitrary number of crossings. In the general case, mechanical equilibrium equations, analogous to (1), are drawn to every segment of the hitch, involving the tensions at the beginning and at the end of the segment. The equilibrium condition is then based on the solution of a system of inequalities representing

the equilibrium at each segment. The generalised equilibrium condition can, thus, be reduced to a condition on the determinant of a matrix of the system's coefficients. This general method is illustrated with the modelling of the clove hitch in (Bayman, 1977).

Maddocks and Keller (1987) extend the work of Bayman (1977) investigating the mechanical equilibrium of hitches as well as knots considering also the friction between parts of the string on itself, the diameter of the string and the shape of the surface around which the hitch is tied. With this extended theory, they derive equilibrium conditions for various knots (such as the square and the sheepshank knots), as well as for a rope lying on an arbitrary curved surface.

Neither Maddocks and Keller (1987) nor Bayman (1977), however, present any experimental verification measuring the extent of which their analytical model predictions are in agreement with real hitches and knots.

Although experimental work on the mechanical equilibrium of cables with loops (or hockles) has been of interest for some time (Lu and Perkins, 1995; Thompson et al., 2003; Warner, 1998), only recently the analytical models relating the configuration of the string and the braid geometry, where shown to predict with great accuracy the experimental mechanical response of a knot (Audoly et al., 2007; Pieranski et al., 2001; Jawed et al., 2015).

Pieranski et al. (2001) describes several finite-element numerical simulations of knots with the particular interest of localising the breakage points in knotted strings, since knots introduce weak points in the medium in which they are tied. The predictions provided by the simulations were compatible with the breakage of distinct macroscopic knots as observed by a high-speed camera. The results presented in (Pieranski et al., 2001) show some similarities with the molecular dynamics simulations of knotted polymer chains or of DNA filaments.

Audoly et al. (2007) describe an analytical and experimental investigation of the limits of an applied tension on open trefoil and cinquefoil knots bent with elastic rods. The analytical solution is based on a model with two straight, half-infinite tails connected by one circular loop with a fixed radius. The equilibrium is achieved by minimising the solution of an energy function. This function is determined on the variational structure of a set of Kirchhoff equations defined on the knotted configurations under consideration. The predictions obtained with this theoretical model had a good agreement with experiments conducted with real knots. An analogous solution to tight (instead of open) knots is suggested as future work in that paper.

More recently, Jawed et al. (2015) investigated how the mechanic response of elastic knots under tension is influenced by their geometrical configuration. Analogous to the work described in (Audoly et al., 2007), the analytical model proposed in (Jawed et al., 2015) is based on a non-linear Kirchhoff model for elastic rods. Jawed et al. (2015) consider the geometry of the braids defining the knots and incorporate parameters for the number of crossings, the bending rigidity and the effect of friction. This theory is used to show how adding or subtracting a crossing in the knot affects the knot's mechanical equilibrium. The experiments conducted with real knots, presented in that paper, show a good agreement with the theoretical predictions for overhand knots defined over a

range of crossings.

Both (Audoly et al., 2007) and Jawed et al. (2015) present the result of experiments and of analytical models for simple knots defined by multiple twists (like braids) of the open ends of a single string. The application of these ideas to more complex knots is still work in progress.

Research studies on topological knots and physical knots are developed upon distinct abstractions. Knot theory analyses the shapes and representations of mathematical entities that resemble real knots, whereas physical knots are represented as equations relating the dynamic properties of an elastic rod. None of them, however, allows the representation of the actions (and their effects) involved in the act of tying knots, let alone the mental processes involved in this process. The next sections are dedicated to the discussion of these issues.

4 Ecological Knots

Casati (2013); Casati and Santos (2018) explore the relationship between compositionaly, lexical and normative elements in natural knots aiming at the investigation of the interface between human reasoning, perception and action. In order to achieve this goal, Casati (2013) suggests a research agenda to tackle the structure of underlying competence for performing a knotting task by, first, looking into cases of performance (i.e., understanding equivalences between distinct knots and taking into consideration the verbal descriptions of the same knot from various practitioners) and, second, the explanation of the performances by means of the theory of Graphic Schemes (Pignocchi, 2010). This explanation should take into account the definition of the mental lexicon for the basic operations of tying a knot (the knowledge involved in knotting tasks) in sensory-motor terms, that would allow a process-sensitive mental representation. This is motivated by the idea that the basic elements of a knot, loops in a string segment, store the energy that stabilises the knot's shape. Thus, knots could be seen as shapes on a string that store the steps (or the actions) that were taken in their construction. These *shapes in actions* should be the constituent part of the representation of knots, therefore integrating a dynamic representation of the way our body interacts with the string. Since tying a knot depends on a continuous sensory-motor feedback, the study of ecological knots is a province of embodied and object-dependent cognition.

Knots have different pragmatics, i.e. uses in different conditions. Some knots are suited to situations in which it is important that they last indefinitely; others to situations in which it is important that they be subsequently untied, possibly quickly. Specialised handbooks (e.g. mountaineering, sailing) describe in detail the different situations and the appropriate knots for each of them; the distinction is part of the specialised lore of knotters. It should be observed that the problem of tying a knot has its dual in the problem of untying a knot (see below the Fisherman's Folly problem as an instance). This is different from reducing a knot representation through the Reidemeister moves, and it present both a physical and a cognitive side. A further problem is that of assessing knot

equivalences cognitively; as it turns out, knotters give different names, and have different practices, for what turn out to be the same knots.

Another issue to consider in the study of ecological knots is whether the object used for stabilising the knot (if distinct from the string itself) should be considered as part of the knot or not. Current formalisations (such as those introduced in (Cabalar and Santos, 2016)) assume that the stabilising object (usually a post) is considered as part of the knot, otherwise the knot wouldn't hold. In fact, Cabalar and Santos (2016) develop a unified representation of actions involving strings and rigid objects, so that every element of the domain is modeled as a string.

Besides stabilisation, an important characteristic of a knot is that it allows itself to be untied through the execution of specific actions. If only stabilisation were of interest, any disorganised stable entanglement of a rope wound be used to bind things together. This is related to qualities in the loops that constitute a knot, which are usually neglected in the study of abstract (topological) or physical knots.

When considering the verbal descriptions of knotting tasks, one essential point to notice is that loops on the string are used (and referred to) as "holes". Motivated by this, (Cabalar and Santos, 2016) makes no distinction between actions applied on a holed (rigid) object and on string loops: they are equally used to pass objects through. However, it is worth pointing out that loops do not satisfy any of the physical or geometrical properties of a hole, although they do satisfy some of its functional properties (Casati and Santos, 2018).

Sloman (2014) describes a set of examples, related to the detection and reasoning about knots, where humans seem to reason intuitively in a mathematical way. In (Sloman, 2014) we are confronted with a number of pictures of an open-ended string under various configurations of (self) crossings, where the task is to decide using only visual inspection: (i) which of the string configurations leads to a knot if the ends of the string are pulled apart; (ii) which pictures represent the same configurations, but depicted from distinct viewpoints; and, (iii) which configurations are simple modifications of others. Regarding these scenarios, Sloman considers two fundamental issues: the first is whether there is (and what would be) the fundamental knowledge about space and object properties necessary to decide on the formation of knots (or not) from reasoning about the effects of pulling the two ends of a string; the second is whether it is possible to develop a formal system (logical/algebraic/computational) capable of solving this task. The first inquiry is related to Casati's search for a structure of underlying competence whereas the second is closely related to the automated reasoning about knots (described in the next section) and to the knot equivalence problem in knot theory (discussed in Section 2).

Toffoli and Giardino (2014) studied the importance of the use of knot diagrams in mathematical practice. Although algebraic solutions to equivalence problems are arguably the final word, the heuristics mathematicians use strongly rely on diagrammatic knot depictions. The cognitive mechanism underlying the interpretation of knot diagrams are argued to mobilise motor competence.

Interestingly, the computational complexity of determining whether a rope is

knotted or not has been related to the human ability of processing linguistic expressions (Camps and Uriagereka, 2006; Balari et al., 2011), which suggests the existence of a common computational structure underlying both knot-making abilities and linguistic capabilities. The consequences of this hypothesis are intriguing since early human knotting evidences could be linked to early human linguistic competence (Casati, 2013). This idea met serious objections: the main criticism against it is that, although the complexity of knotting may be related to the complexity of processing a string of symbols, there is no evidence supporting that the same result could be applied to the human capacity of language processing (Lobina, 2012). This debate goes on further (Balari et al., 2012; Lobina and Brenchley, 2012), and hard evidence to support either side of the argument is still to be provided.

There is strong archeological evidence that the ancient Inkas recorded information about censuses and tributes (including arithmetical paradigms and algebraic formulae) in a complex system of distinct knots on coloured strings called Khipus (or *talking knots*) (Ascher and Ascher, 2013; Urton, 2017). Urton (2017) argues that Khipus store information in a semasiographic form, where concepts are stored as symbols without a direct relation with words or with a phonetic alphabet. However, recent findings suggest that (at least) some of these khipus record information in terms of a combination of phonetic and ideographic symbols and could have been used to record narratives of historical events (Hyland, 2017). Whether or not the information in khipus is stored in a semasiographic or a phonographic form seems to be a secondary issue in cognitive science when compared to the fact that the information is stored in a three-dimensional form, accessed by touch as well as sight. Therefore, the recognition of structural stability and structural distinctiness in the knot's properties, which are key aspects of the investigation of ecological knots, are at the very heart of decoding khipus (van de Griend, 1998).

The recognition and representation of the multiple possible states of a string, involved or not with other rigid objects, are the building blocks of automated systems capable of reasoning about flexible objects, as discussed in the next section.

5 Knots in Automated Reasoning and Robotics

Bridging the gap between ecological knots and automated reasoning about knot-like structures, Freksa et al. (2018) propose the use of strings and pins, instead of compass and straightedge, for problem solving in Euclidean geometry. The motivating argument for this investigation is the idea that the spatial properties, that are essential parts of mathematical proofs, are implicit in diagrammatic representations, whereas they must be spelled out in the analytical (non-visual) derivations. In this sense, reasoning about geometrical/spatial domains is facilitated by the medium where the inferences are conducted. It is argued that this metaphor extends the range of constructive solutions, as exemplified with three strings-and-pins constructions that cannot be represented with compass

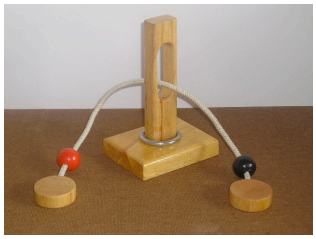
and straightedge: the construction of an ellipse, the solution for the shortest path problem and the angle trisection problem.

The automated solution of domains involving strings and rigid objects (entanglement puzzles), and the problem of finding a finite number of transformations for solving the knot equivalence problem, were cited in (Turing, 1955) as examples of mathematical problems that cannot be handled by any systematic method (i.e., cannot be solved by a universal Turing machine). The argument is based on translating these problems as equivalent *substitution puzzles*, i.e. puzzles where the states are represented as strings of symbols, and actions are substitutions of subsets of each of these symbols with other strings. Turing then argues that any systematic method for solving these domains can also be replaced by an equivalent substitution puzzle and, thus, this process does not terminate if the original problem has no solution. This argument parallels the Church-Turing thesis (Copeland, 2004).

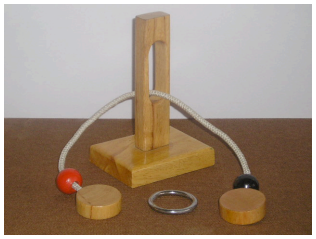
Nevertheless, if the problem has a solution, there should be an algorithm capable of finding it, regardless of the number of states forced by the possibility of winding (or knotting) the string. To the best of our knowledge, within the current automated reasoning literature the only investigation on the representation and reasoning about domains containing strings were those reported in a series of papers by the same authors (Cabalar and Santos, 2006; Santos and Cabalar, 2008; Cabalar and Santos, 2011, 2016; Santos and Cabalar, 2016). In the research reported in these papers, formal representations and automated solutions are proposed for domains (puzzles) defined by an entanglement of strings, holes and rigid objects, whose goal is to release a ring from the system (which is akin to the untying problem in knot theory Santos and Cabalar (2016)). This work has started on the formalisation of the Fisherman's Folly puzzle (shown in Figure 3), but has evolved with the consideration of more complex puzzles, where reasoning about string loops and knots are relevant to the solution (Cabalar and Santos, 2016; Santos and Cabalar, 2016). A discussion of the technical evolution of this line of research is outside the scope of this paper, however, a brief summary of the solution to the Fisherman's Folly puzzle is in order to illustrate the main elements of this work.

The Fisherman's Folly puzzle is composed of a holed post ($Post$) fixed to a wooden base ($Base$), a string (Str), a ring ($Ring$), a pair of spheres ($Sphere1, Sphere2$) and a pair of disks ($Disk1, Disk2$). The spheres can be moved along the string, whereas the disks are fixed at each string endpoint. The string passes through the post's hole in a way that one sphere and one disk remain on each side of the post. The spheres are larger than the post's hole, thus the string cannot be separated from the post without cutting either the post, or the string, or destroying one of the spheres. The disks and the ring can pass through the post's hole.

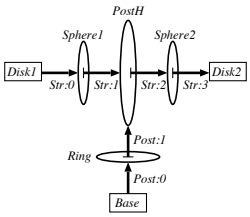
In the initial state (Figure 3(a)) the post is in the middle of the ring, which in its turn is supported on the post's base. The goal of this puzzle is to find a sequence of (non-destructive) transformations that, when applied on the domain objects, frees the ring from the other objects, regardless of their final configuration. Figure 3(b) shows one possible goal state.



(a) S_0



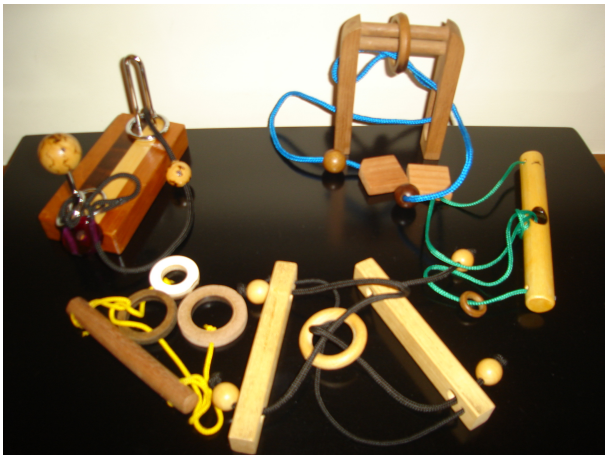
(b) Goal



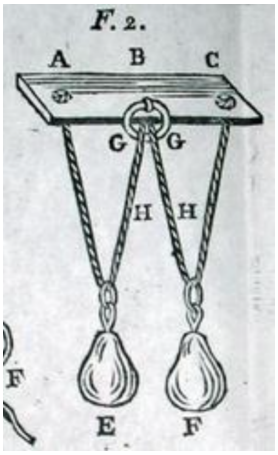
(c) Diagram of S_0

Figure 3: A spatial puzzle: the Fisherman's Folly.

The Fisherman's Folly puzzle is just one example of a set of (so called) entanglement puzzles that can be formalised and solved in a similar way, a few other examples are shown in Figure 4(a). Some of these puzzles have been proposed as challenges for human problem solving since the 16th century, e.g. Figure 4(b)(Pacioli, 2009; Rusca, 1743; dos Santos Hirth, 2015).



(a) Various string puzzles



(b) Solomon's Seal (Rusca, 1743), as cited in (dos Santos Hirth, 2015).

Figure 4: Puzzles with strings, holes and rigid objects.

The formalisation of the Fisherman's Folly puzzle (as well as of other puzzles) presented in Cabalar and Santos (2011, 2016) is based on a list data structure named $chain(X)$, and resembles the substitution puzzle suggested in (Turing, 1955), mentioned above. This data structure represents the sequence of all hole crossings on a long object X, when traversing X from its negative tip to its positive one. For instance, the state shown in Figure 3(c) is represented by the following two chains: $chain(P) = [Ring^+]$ for the post object P; and $chain(Str) = [Sphere1^+, PostH^+, Sphere2^+]$ for the string object Str. The

state	$chain(P)$	$chain(Str)$
S_0	$[Ring^+]$	$[Sphere1^+, PostH^+, Sphere2^+]$
s_1	$[Ring^+]$	$[Sphere1^+, PostH^+, Sphere2^+, PostH^-]$
s_2	[ ]	$[Sphere1^+, Ring^-, PostH^+, Ring^+, Ring^-, Sphere2^+, Ring^+, PostH^-]$
s_3	[ ]	$[Sphere1^+, Ring^-, PostH^+, Sphere2^+, PostH^-, Ring^+]$
s_4	[ ]	$[Sphere1^+, PostH^+, Ring^-, Sphere2^+, Ring^+, PostH^-]$
s_5	[ ]	$[Sphere1^+, PostH^+, Sphere2^+, PostH^-]$

Figure 5: A formal solution for the Fisherman's puzzle.

former represents that the post P crosses the ring hole whereas the latter states that the string Str crosses the hole on sphere 1, the post hole and the hole on sphere 2, respectively. Note that, for brevity, only the outgoing hole faces are shown, following the direction from the negative to the positive tip.

An action *pass* was defined to represent the movements of puzzle objects: $pass(Obj, Hole^i)$ represents the action of passing an object Obj towards the i face of a hole $Hole$, where $i \in \{+, -\}$. The effects of *pass* either add or delete hole crossings from the *chain* on which it is applied. Using these definitions, a solution to the Fisherman's Folly puzzle can be represented by the sequence of chains shown on Figure 5.

A simple planning system capable of finding a solution to a number of such puzzles was described in (Cabalar and Santos, 2011).

Robotics research on the hand-eye coordination capabilities for handling flexible objects (aiming at knot-tying tasks) has started in the 1980s with the work of Inaba and Inoue (1985) and has a long list of contributions (as reviewed in (Bell, 2010)). The problem of incorporating explicit knowledge about strings and string manipulation has been tackled in (Morita et al., 2003; Takamatsu et al., 2006) where a robotic system capable of learning to tie a knot from visual observation is proposed. In this system, each state of a string is represented by a matrix encoding the string segments, that are defined as the portion of the string separated by two over-crossings, or between the free ends of the string and the nearest over-crossing, following along the string (analogously to the definition used in (Bayman, 1977)). Actions on flexible objects in this context were defined as an extension of the Reidemeister moves in knot theory (Reidemeister, 1983). A similar representation was used in (Wakamatsu et al., 2005) for the task of manipulation planning for knotting and unknotting deformable linerar objects. This representation is suitable for the identification of string states from a computer vision system. Analogously, Yamakawa et al. (2017) defines a set of manipulation skills (resembling Reidemeister moves) on top of a list representation of string crossings that is used by a robot manipulator to execute a number of distinct knots (including a clove hitch). Although the knot representation is given as input to the robot, it is worth noting that this representation includes rope crossings that must be fixed by the robot hand in order

to successfully tie the knot. Wang and Balkcom (2016) presents an approach of knot-tying tasks for robot manipulation in which the knots are tightened on an (automatically generated) fixture board, which is a set of rods whose locations are obtained algorithmically in terms of the knot shape. This approach takes into account distance constraints in the string segments in order to enforce *friction locks* stabilising the knot parts.

Much work on the robotic knot-tying tasks are strongly influenced by the topological description of knots (as developed in knot theory, cf. Section 2 above). In contrast, the work reported by Onda et al. (2016) considers the autonomous robotic construction of knots, with one or more strings, in terms of metric information (alongside topology) in order to facilitate the adjustments of string's lengths and shapes within a knot. This is accomplished by updating two interrelated lists, one representing string crossings (points where a string crosses itself) and the other representing the length of the segment between adjacent crossings. This work, however, does not consider explicitly the relations between the robot hand and the rope, whereas the research reported in (Vinh et al., 2017) uses the positions of the rope relative to the robot fingers at key points in the knot construction as conditions for performing a knotting task. These relations are conjunctions of the intersections between parts of the fingers and the rope and are manually constructed from a sequence of snapshots of a single (human) hand tying an overhand knot.

Lee et al. (2015) introduces a method for learning force-based manipulation of deformable objects assuming multiple demonstrations, where non-rigid registration is used to compute warping functions that map poses and forces in the demonstrations with the current state of the knot. From these warping functions, a trajectory consistent with the demonstrations is obtained to perform the knotting task.

Inspired by traditional work on using magnetic field representation for robot navigation (Koren and Borenstein, 1991), Marzinotto and Stork (2016) solves the manipulation of deformable objects for knotting tasks by means of parameterised magnetic fields. In this solution a virtual magnetic field is defined through the interior of loops defined on strings and the Biot-Savart law[2] is used to guide the robot arm in the task of passing a tip of the string through the loop.

Research on the robot manipulation of flexible objects has been receiving increasing attention lately due to some highly relevant application problems. One prominent example is autonomous robotic surgery where analogous problems have to be solved, or in the field of autonomous manufacture, where robots are currently unable to manipulate, and compute, actions related to flexible objects. The current literature in the field (reported above) seems to be tackling the problem from a fragmented perspective, each contribution concentrated on a single modality of the issue. It seems, however, that a more robust solution would be obtained by combining some of the elements described above, in order

[2]The Biot-Savart law expresses the relation between the magnetic field generated by a constant electric current.

to account simultaneously for the topological descriptions of knots (Takamatsu et al., 2006), the friction locks and stabilising points in knotting (Wang and Balkcom, 2016), the relations between hand and rope (Vinh et al., 2017), the tensions and forces involved in making a knot (Lee et al., 2015). Besides, the development of a reasoning/problem-solving algorithm for knotting tasks (Cabalar and Santos, 2016; Santos and Cabalar, 2016) is an issue yet to be tackled in robotics research.

The experiments with real robots reported in the literature start the knotting tasks from a straight rope. It is unclear, however, how the vision systems used in these experiments would perform, or how they could be adapted to perform, with entanglements of strings, where self-occlusion is a rule, not an exception. A challenge to the field that intersects robotics, computer vision and automated reasoning (and that also takes into account commonsense reasoning) is the task of untying a knot given to the robot at first hand, i.e. without assuming previous knowledge of its construction.

6 Discussion and Concluding Remarks

Gauss, Turing and Sloman, among the other authors cited in this literature review, share an interest on finding methods (computational or not) capable of solving problems involving strings and knots that have been solved naturally by humans since the Pleistocene, and that probably gave the homo sapiens an advantage over other species (Warner and Bednarik, 1998; Hardy, 2008). Defining formally (or computationally) such problem-solving methods, however, has been an elusive task mainly due to the great number of states implied by the string's flexibility, and its various potential functions: e.g., as binding/locking structures (knots), as holes (Cabalar and Santos, 2016; Santos and Cabalar, 2016), or as measuring devices (Freksa et al., 2018).

Knot theory conceptualises knots as embeddings of a circle and aims at investigating the topological structure and the belated representations of these as mathematical entities. It is not surprising that similar knot-like structures have been found in various other fields such as statistical physics, genetics and chemistry; since, it is conceivable that any linear (flexible) structure would exist in a topological space that is isotopic to a knot.

In contrast, the physics of real knots is mainly concerned with the derivation of analytical models capable of defining conditions for the mechanical equilibrium of knot structures and of predicting potential breakage points. Therefore, basic physics models knots as entities represented by equations relating the dynamic properties of elastic rods with their shapes. The forces involved in a knot structure (the tensions and frictions that hold its shape) are made explicit in these mechanical models. However, these models do not explain how the knot is constructed, what actions should be taken in their constructions and why a particular knot would be useful for a given task, and not another. On the other hand, automated reasoning and robotics in this context have considered the formal definitions of actions on a rope as their major subject of investigation,

largely neglecting the forces involved in a knotting task (with some exceptions (Lee et al., 2015)).

An extension of the research agenda proposed in (Casati, 2013) would include the study of cognitive precursors of knotting abilities, relying on animal, developmental, intercultural and pathological studies. Its results may provide guidelines to the development of automated reasoning and robotics systems capable of dealing with knot-like structures in a more integral way. The description of a *mental lexicon in sensory-motor terms*, a dynamic representation of how the body interacts with the object, could provide a common ground for the interplay between string manipulation actions, as investigated in robotics research, and the automated reasoning about strings states. This could also show an avenue for including the physical models for expressing the stabilisation of a knot structure, as it makes the energy stored in the knot's sections (that is used to stabilise the knot (Casati and Santos, 2018)) as part of the knot-tying process. Research on the robotic manipulation of ropes may in turn provide the appropriate tools for the empirical study of embodied cognition, facilitating the development and empirical evaluation of models that not only solve problems in space, but that also take into account the body actions and perceptual feedback when doing so.

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