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Secrecy Coverage

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Abstract. Motivated by information-theoretic secrecy, geometric models for secrecy in wireless networks have begun to receive increased attention. The general question is how the presence of eavesdroppers affects the properties and performance of the network. Previously, the focus has been mostly on connectivity. Here we study the impact of eavesdroppers on the coverage of a network of base stations. The problem we address is the following. Let base stations and eavesdroppers be distributed as stationary Poisson point processes in a disk of area n. If the coverage of each base station is limited by the distance to the nearest eavesdropper, what is the maximum density of eavesdroppers that can be accommodated while still achieving full coverage, asymptotically as $n \to \infty$?

1. Introduction

1.1. Motivation and Related Work

While coverage problems have been studied for several decades from a purely mathematical perspective, they have recently begun to attract significant attention by the engineering and computer science communities due to the advent of wireless networks, in particular sensor networks. The standard problem formulation is the following. Place a number n of nodes randomly in a certain set $B \subset \mathbb{R}^d$ and equip each node with the capability of covering (sensing) a disk or sphere of radius r around itself. How large should n be to guarantee coverage of

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B with probability $1-\epsilon$? Or if the area or volume of B is scaled in proportion to n, what $r(n)$ guarantees coverage with high probability as $n \to \infty$?

In the mathematical literature, one of the early and now classical coverage problems is the coverage of a sphere with circular caps introduced in [Moran and de St. Groth 62] and solved in [Gilbert 65]. Extensions to k dimensions were considered in [Hall 85] and [Janson 86]. Hall later provided a detailed account of coverage processes in his book [Hall 88]. Many generalizations have been considered since; see, e.g., [Athreya et al. 04, Roy 06] for coverage problems in $\mathbb{R}^d$.

In the sensor networking literature, the case of interest is mostly coverage of (part of) the plane with disks of fixed or random radii or more random shapes. To provide redundancy for robustness and also to let some sensor nodes be asleep to save energy, single coverage is sometimes not sufficient, but k-coverage, $k > 1$, is required. Consequently, [Xing et al. 05] studies the relationship between k-connectivity and k-coverage and suggests a sleep-scheduling algorithm to maintain coverage, while [Miorandi 08] considers coverage and connectivity in the presence of channel randomness. A detailed analysis of conditions for k-coverage (of a unit square) of a sensor network in which most nodes are asleep is provided in [Kumar et al. 08], and [Lazos and Poovendran 06] studies coverage problems in which the sensing areas are not restricted to disks, using results from integral geometry.

Here we focus on a coverage problem that is inspired by secrecy constraints. We assume an information-theoretic model for secrecy in which a communication is secure from eavesdroppers if the intended receiver is closer to the transmitter than all eavesdroppers. The information-theoretic foundation for this model is the so-called *wiretap channel*, whereby a source sends information to a legitimate receiver in the presence of an eavesdropper. It was shown in [Leung-Yan-Cheong and Hellman 78] that the *secrecy capacity*, the maximum achievable rate of the transmission to the legitimate receiver while guaranteeing that the eavesdropper cannot decode the message, is the difference of the two channel capacities. Hence if the transmitter is located at x, the legitimate receiver at y, and the eavesdropper at z, the secrecy capacity is $C_{\mathrm{s}} = \max\{0, R_{\mathrm{s}}\}$ for

$$R_{\mathrm{s}} = C - C_{\mathrm{e}} = \log_2\left(1 + \frac{P\ell(\|x-y\|)}{W}\right) - \log_2\left(1 + \frac{P\ell(\|x-z\|)}{W}\right), \qquad (1.1)$$

where P is the transmit power, ℓ is a (strictly) monotonically decreasing path loss function, and W is the power of the Gaussian noise at the receivers. So whenever $\|x-y\| < \|x-z\|$, the secrecy capacity is positive, and secure communication is possible, albeit at a possibly small rate.

Based on this model, the *secrecy graph*, a random geometric graph that includes only edges along which secure communication is possible, was introduced and studied in [Haenggi 08]. More elaborate physical layer models are considered in [Pinto et al. 10], while [Goel et al. 11] studies the effect of eavesdroppers on network connectivity if the locations of the eavesdroppers are not known precisely. In [Koyluoglu et al. 12], it is shown that as long as the density of eavesdroppers goes to zero faster than $(\log n)^{-2}$ for a network with n legitimate nodes, secrecy does not affect the $1/\sqrt{n}$ capacity scaling law. The first rigorous bounds on the percolation threshold in the secrecy graph were established in [Sarkar and Haenggi 11]. This line of work is based on graph models and focuses on connectivity. In contrast, there is no prior work on the related coverage problem, which is the theme of the present paper.

Base stations and eavesdroppers are distributed randomly in the plane, and the base stations can cover circular areas with radii determined by the distance to the nearest eavesdroppers. While these assumptions result in an analytically tractable model, they are quite realistic. In fact, cellular networks are now undergoing a major transition from carefully planned base station deployments to an irregular heterogeneous infrastructure that includes micro-base stations and femtocells [Dhillon et al. 12]. In [Andrews et al. 11] it is shown that even without such small base stations, the results obtained from a random model are as accurate or better than those obtained from a lattice-based model for the base stations. The reason that a base station can cover the disk-shaped area up to the nearest eavesdropper is that given the distance to the farthest legitimate node within the disk, it can always choose a positive rate of transmission such that the secrecy capacity (1.1) is positive. The locations of the eavesdroppers remain uncovered, i.e., the disks are open disks.

The main question is this: what density of eavesdroppers can be accommodated while still guaranteeing that the entire area or volume of interest is covered securely? Such a density would ensure that mobile stations could roam around everywhere and be reached securely by a base station. Hence the downlink is intrinsically secure, while the uplink (from the mobile to the base station) has to be secured by transmission of a one-time pad via the downlink.

1.2. Problem Formulation

To make the problem concrete, we assume that the base stations and eavesdroppers form independent Poisson point processes of intensities 1 and λ, respectively, in $\mathbb{R}^d$. We will denote the process of base stations by $\mathcal{P}$ and call its points *black points*, and the process of eavesdroppers by $\mathcal{P}'$ and call its points *red points*. Now place an open ball $D(p, r_p)$ of radius r_p around each black point $p \in \mathcal{P}$, where r_p is maximal so that $D(p, r_p) \cap \mathcal{P}' = \varnothing$. In other words, r_p is the distance from the black point p to the nearest red point $p' \in \mathcal{P}'$. We thus obtain a random set

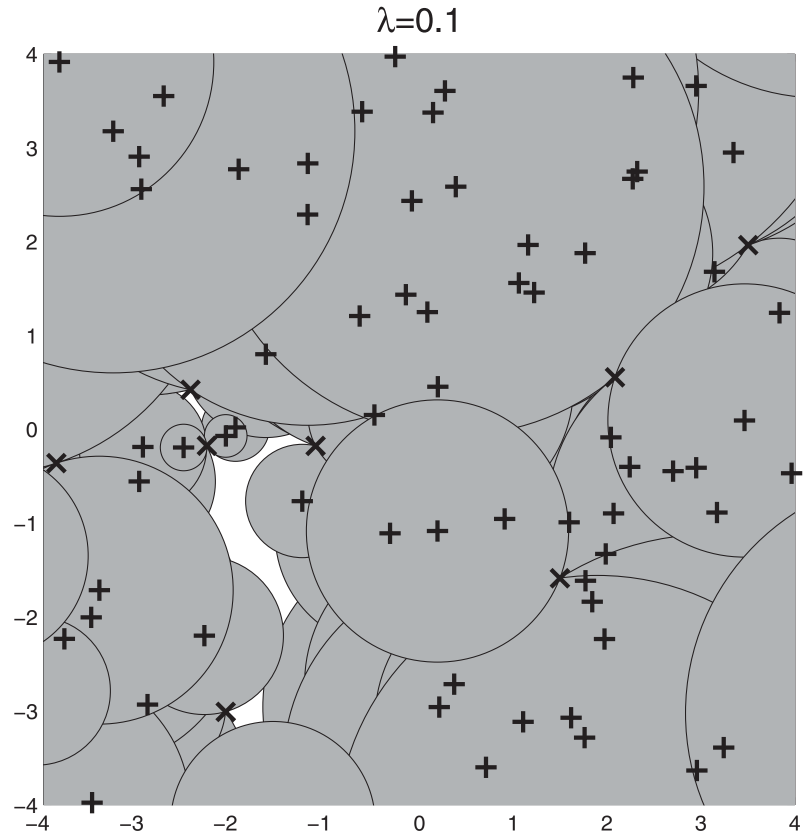

Figure 1. Example for coverage of an 8×8 square for $\lambda = 0.1$. The base stations are marked by $+$, the eavesdroppers by $\times$, and the covered area is shaded gray. The locations of the eavesdroppers themselves cannot be covered.

$\mathcal{A}_\lambda^d \subset \mathbb{R}^d$, which is the union of balls centered at the points of $\mathcal{P}$. Figure 1 shows a 2-dimensional example for $\lambda = 0.1$. Our aim is to study properties of $\mathcal{A}_\lambda^d$, in particular the covered volume fraction (Section 2) and the asymptotic conditions for complete coverage in one (Section 3) and two (Section 4) dimensions.

In two dimensions, the radius R of each disk is given by the nearest-neighbor distance, distributed as

$$f_R(x) = 2\pi x \lambda \exp(-\lambda \pi x^2)\,.$$

Note that this coverage problem is rather different from that in which the covering disks have *independent* radii drawn from f_R. The difference is that in our case, the disk radii of nearby nodes are strongly correlated, which leads to drastically different conditions for coverage compared with the independent case. Indeed, for the standard model with random independent disk radii and intensity 1, a disk of area n is covered almost surely (a.s.) if

$$\mathbb{E}\left(\pi R^2\right) = (1+\varepsilon)\log n$$

for every $\varepsilon > 0$.[1] Since $\mathbb{E}\left(\pi R^2\right) = \lambda^{-1}$, this translates to

$$\lambda = \left[(1+\varepsilon)\log n\right]^{-1},$$

which indicates that λ may decrease rather slowly with n while still achieving full coverage. In the secrecy case, however, λ has to decrease much faster, at a rate of about $n^{-1/3}$, as we will show.

1.3. Summary of Main Results

Our main results, in Sections 3 and 4, concern the probability $p_\lambda^d(n)$ that a fixed ball $B_n^d \subset \mathbb{R}^d$ of volume n is covered by $\mathcal{A}_\lambda^d$ (except for the points of $\mathcal{P}'$). In Section 3, we take $d = 1$, and in Section 4, we take $d = 2$. The results are as follows:

Theorem 1.1. *If $n \to \infty$ with $\lambda n \to \infty$, then $p_\lambda^1(n) \sim e^{-4n\lambda^2}$. Consequently, if $\lambda^2 n \to \infty$, then $p_\lambda^1(n) \to 0$, and if $\lambda^2 n \to 0$, then $p_\lambda^1(n) \to 1$.*

Theorem 1.2. *If $f(n) = \lambda^3 n \to \infty$, then $p_\lambda^2(n) \to 0$.*

Theorem 1.3. *If $g(n) = \lambda^3 n (\log n)^3 \to 0$, then $p_\lambda^2(n) \to 1$.*

Note that in contrast to the situation for percolation [Sarkar and Haenggi 11], we need λ to tend to 0 at a certain rate (as $n \to \infty$) in order to achieve coverage with high probability. Furthermore, it is entirely possible that Theorem 1.3 gives a better indication of the threshold required for coverage than Theorem 1.2. If so, this would run counter to our initial intuition regarding the obstructions to coverage in two dimensions.

2. Covered Volume Fraction

For $\lambda > 0$, write

$$C^d(\lambda) = \mathbb{P}\left(O \in \mathcal{A}_\lambda^d\right).$$

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[1] This condition is not the sharpest possible.

By stationarity of the model, $C^d(\lambda)$ can also be interpreted as the fraction of $\mathbb{R}^d$ that is covered by $\mathcal{A}_\lambda^d$, known as the *covered volume fraction.*

Theorem 2.1.

$$C^d(\lambda) = 1 - \mathbb{E}\left(e^{-V_d/\lambda}\right) = 1 - \int_0^\infty f_d(t) e^{-t/\lambda}\, dt,$$

where $f_d(t)$ is the probability density function for the volume V_d of a randomly chosen cell in a Voronoi tessellation associated with a unit-intensity Poisson process in $\mathbb{R}^d$.

Proof. We rescale the model so that $\mathcal{P}$ and $\mathcal{P}'$ have intensities $1/\lambda$ and 1 respectively. This does not affect $C^d(\lambda)$. Now, $O \notin \mathcal{A}_\lambda^d$ if and only if there are no points of $\mathcal{P}$ in the Voronoi cell C defined by $\mathcal{P}' \cup \{O\}$ containing O. If C has volume V, then $\mathbb{P}(C \cap \mathcal{P} = \varnothing) = e^{-V/\lambda}$. □

Corollary 2.2. *In one dimension, we have*

$$C^1(\lambda) = \frac{1 + 4\lambda}{(1 + 2\lambda)^2}\,.$$

Proof. Let $\mathcal{P}$ be a unit-intensity Poisson process in $\mathbb{R}$. The distribution of the gap lengths between points of $\mathcal{P}$ has density e^{-t}, but the distribution of the length of the gap containing a fixed point, such as the origin O, has density te^{-t}. (This is known as the *waiting time paradox.*) Consequently, the density function for the length of the Voronoi cell defined by $\mathcal{P} \cup \{O\}$ containing the origin is $4te^{-2t}$, so that by Theorem 2.1,

$$C^1(\lambda) = 1 - \int_0^\infty 4te^{-2t-t/\lambda}\, dt = \frac{1 + 4\lambda}{(1 + 2\lambda)^2}.$$

□

Remark 2.3. This corollary can also be proved as follows. Let L be the event that the origin O is covered by points of $\mathcal{P}$ lying only to the left of O, and let R be the event that O is covered by points of $\mathcal{P}$ lying only to the right of O. Then L and R are independent, and

$$C^1(\lambda) = 1 - (1 - \mathbb{P}(L))(1 - \mathbb{P}(R)) = 1 - (1 - \mathbb{P}(R))^2 = 2\mathbb{P}(R) - \mathbb{P}(R)^2,$$

by symmetry. Now, R occurs if and only if the closest black point to the right of O is at distance t and there are no red points in the interval $[0, 2t]$ for some

$t > 0$. Thus

$$\mathbb{P}(R) = \int_0^\infty e^{-t} e^{-2\lambda t}\, dt = \frac{1}{2\lambda + 1},$$

which gives the desired result.

3. Probability of Total Coverage in One Dimension

In this and the next section, we study the following problem. With $\mathcal{P}$, $\mathcal{P}'$, and $\mathcal{A}_\lambda^d$ as before, let $B_n^d \subset \mathbb{R}^d$ be a fixed ball of volume n, and set $\mathcal{A}_\lambda^d(B_n^d) = \mathcal{A}_\lambda^d \cap B_n^d$. Write $B_\lambda^d(n)$ for the event that $\mathcal{A}_\lambda^d(B_n^d)$ covers B_n^d (except for the points of $\mathcal{P}'$), and set $p_\lambda^d(n) = \mathbb{P}(B_\lambda^d(n))$. Our principal goal is to estimate $p_\lambda^d(n)$ for arbitrary d.

Let us first consider the case $d = 1$. In this case, $I = B_n^1$ is simply an interval of length n containing some black and red points. We place an interval centered at each black point of maximal length subject to containing no red points, and ask for the probability that I is covered by such intervals. Clearly, a necessary condition for coverage is that I contain no two consecutive red points, for then the entire interval between these points will not be covered. (This condition is not, however, sufficient.) Writing X for the number of pairs of consecutive red points (where, for instance, three consecutive points count as two pairs), it is easy to see that if $\lambda = o(1)$, then

$$\mathbb{E}(X) \sim \lambda n \cdot \frac{\lambda}{1+\lambda} \sim \lambda^2 n.$$

Now, $p_\lambda^1(n) = \mathbb{P}(B_\lambda^1(n)) \le \mathbb{P}(X = 0)$, and a standard coupling argument shows that $\mathbb{P}(X = 0)$ is decreasing in λ. Suppose that $\lambda^2 n \to c$. For fixed instances of $\mathcal{P}$ and $\mathcal{P}'$, number the points of $\mathcal{P} \cup \mathcal{P}'$ inside B_n^1 as $p_1, p_2, \ldots, p_M$ from left to right. Write X_O for the number of pairs of red points of the form (p_{2i-1}, p_{2i}), and X_E for the number of pairs of red points of the form (p_{2i}, p_{2i+1}). With high probability, M satisfies $|M - n| \le n^{2/3}$. Using the coloring theorem [Kingman 93], we may obtain $\mathcal{P}$ and $\mathcal{P}'$ by first placing the points p_i according to a Poisson process of intensity $1 + \lambda$, and then coloring the points red or black independently, with respective probabilities

$$p = \frac{\lambda}{1+\lambda} \quad \text{and} \quad 1 - p = \frac{1}{1+\lambda}.$$

Since $\lambda^3 n \to 0$, it should follow that

$$p_\lambda^1(n) \le \mathbb{P}(X_O = 0)\mathbb{P}(X_E = 0)(1 + o(1)) = \left(\left(1 - p^2\right)^{n/2}\right)^2 (1 + o(1)) \to e^{-c},$$

and indeed the conclusion that $p^1_\lambda(n)$ is asymptotically at most e^{-c} follows from the Janson inequality (see, for instance, [Alon and Spencer 08, Chapter 8]). (The above inequality is hard to establish, since the random variables X_O and X_E are not quite independent.) In fact, one can use the Stein–Chen method [Barbour and Chen 05] to show that $X \to \mathrm{Po}(c)$ in distribution. Consequently, if now $\lambda^2 n \to \infty$, then $p^1_\lambda(n) \to 0$.

Next suppose that $\lambda^2 n \to 0$. Our aim is to show that in this case, $p^1_\lambda(n) \to 1$. To simplify our analysis, let us suppose that $\mathcal{P}$ and $\mathcal{P}'$ are placed on a circle T of circumference n rather than an interval of length n; there is asymptotically no difference. Our strategy is to place the red points $\mathcal{P}'$ first, partitioning the circle T into $M \sim \mathrm{Po}(n\lambda)$ arcs A_i. Now place the black points $\mathcal{P}$. For each arc A_i, let C_i be the event that A_i is covered by the smaller arcs associated with the black points in A_i. The events C_i are independent, and this will enable us to estimate $p^1_\lambda(n)$.

Suppose A_i has length ℓ, and let m_i be the midpoint of A_i. Let x be the distance of the closest black point to m_i lying to the left of m_i, and let y be the distance of the closest black point to m_i lying to the right of m_i. Whether C_i occurs, i.e., whether A_i is covered by small arcs, is determined solely by x and y. In fact, it is easy to see that

$$C_i \text{ occurs if and only if } x + y \le \ell/2.$$

Now, $x + y$ has the gamma distribution with density function te^{-t}, and consequently,

$$\mathbb{P}\left(C_i \mid A_i \text{ has length } \ell\right) = \int_0^{\ell/2} te^{-t}\,dt = 1 - e^{-\ell/2}(1 + \ell/2). \tag{3.1}$$

Further, since the black and red points are independent, and the length ℓ has an exponential distribution with density function $\lambda e^{-\lambda t}$, we have

$$\mathbb{P}\left(C_i\right) = \int_0^\infty \lambda e^{-\lambda\ell}\left(1 - e^{-\ell/2}\left(1 + \frac{\ell}{2}\right)\right)d\ell = \frac{1}{(1+2\lambda)^2}.$$

Conditioning on the number of arcs M changes the distribution of the lengths of the A_i, but the difference is asymptotically negligible. Consequently, as $n \to \infty$ with $\lambda^2 n \to 0$ but $\lambda n \to \infty$, we have

$$p^1_\lambda(n) \sim \sum_{m=0}^{\infty} \mathbb{P}\left(M = m\right)(2\lambda+1)^{-2m} \sim e^{-4n\lambda^2(\lambda+1)/(2\lambda+1)^2} \sim e^{-4n\lambda^2} \to 1,$$

as required.

Remark 3.1. In the light of the second half of the proof ($\lambda^2 n \to 0$), the first half is superfluous, but it illustrates an alternative method.

We summarize these results as a theorem.

Theorem 1.1. *If $n \to \infty$ with $\lambda n \to \infty$, then $p_\lambda^1(n) \sim e^{-4n\lambda^2}$. Consequently, if $\lambda^2 n \to \infty$, then $p_\lambda^1(n) \to 0$, and if $\lambda^2 n \to 0$, then $p_\lambda^1(n) \to 1$.*

4. Probability of Coverage in Two Dimensions

4.1. Analysis

Our next aim is to generalize these results to arbitrary d. Simple heuristics (see below) suggest that if $\lambda^{d+1}n \to 0$, then $p_\lambda^d(n) \to 1$, and if $\lambda^{d+1}n \to \infty$, then $p_\lambda^d(n) \to 0$. Unfortunately, attempts to generalize the above arguments run into difficulties, mainly due to the lack of an order structure in $\mathbb{R}^d$ for $d \geq 2$.

The most natural analogue of the above argument (for $\lambda^2 n \to 0$) involves the Delaunay simplices corresponding to the process $\mathcal{P}'$. Points $p_1, \ldots, p_{d+1} \in \mathcal{P}'$ form a simplex in the Delaunay tessellation for $\mathcal{P}'$ if their closed circumhypersphere (circumcircle for $d = 2$) contains no other points of $\mathcal{P}'$. The useful feature when $d = 1$, where the Delaunay simplices are just intervals, is that events corresponding to the coverage of distinct simplices are independent. When $d \geq 2$, it is entirely possible for a Delaunay simplex S to remain uncovered by balls originating from black points inside S but to be covered by $\mathcal{A}_\lambda^d$ nonetheless. In other words, balls originating from black points *outside* S might be needed to cover S. This destroys the independence. One might hope to prove a bound by restoring independence and considering the event that S is covered by balls originating inside S. However, it is the balls originating near the circumcenter of S that are likely to be most useful in covering S, and the circumcenter of S may well lie outside S.

Consequently, we turn instead to the Voronoi tessellation corresponding to $\mathcal{P}'$.

Lemma 4.1. *With $f_d(t)$ as in Theorem 2.1, if*

$$n\lambda \int_0^\infty f_d(t) e^{-t/\lambda}\, dt \to \infty,$$

then $p_\lambda^d(n) \to 0$.

Proof. As in Theorem 2.1, we rescale the model so that $\mathcal{P}$ and $\mathcal{P}'$ have intensities $1/\lambda$ and 1 respectively. Consider the Voronoi tessellation $\mathcal{V}$ formed from the red points $\mathcal{P}'$. If there is a single Voronoi cell $C_p \in \mathcal{V}$ containing no black points, i.e., if $C_p \cap \mathcal{P} = \varnothing$ for some $C_p \in \mathcal{V}$, then the red point p corresponding to C_p, and a small open neighborhood of p, will not be covered by $\mathcal{A}_\lambda^d$. Consequently, the probability that an arbitrary Voronoi cell is not completely covered is at least the probability that its red point is not covered, which is just $1 - C^d(\lambda) = \int_0^\infty f_d(t)e^{-t/\lambda}\,dt$. Writing X for the number of Voronoi cells that are not covered by $\mathcal{A}_\lambda^d$, and Y for the number of Voronoi cells whose red points are not covered by $\mathcal{A}_\lambda^d$, we have

$$\mathbb{E}(X) \geq \mathbb{E}(Y) \sim n\lambda \int_0^\infty f_d(t)e^{-t/\lambda}\,dt \to \infty,$$

by hypothesis. (Note that due to the rescaling, B_n^d now has volume $n\lambda$.) A simple application of the second moment method now shows that $p_\lambda^d(n) = \mathbb{P}(X = 0) \to 0$. □

There are two problems with this lemma. First, $f_d(t)$ is known only when $d = 1$ (although see [Tanemura 03] for simulation results, and [Zuyev 92] for rigorous partial results). Second, when $d = 1$, the bound misses the truth by an order of magnitude. Indeed, when $d = 1$, we have

$$n\lambda \int_0^\infty f_d(t)e^{-t\lambda}\,dt = n\lambda \int_0^\infty 4te^{-2t-t/\lambda}\,dt = \frac{4n\lambda^3}{(1+2\lambda)^2},$$

so that we would need $\lambda^3 n \to \infty$, rather than the weaker $\lambda^2 n \to \infty$, to deduce $p_\lambda^1(n) \to 0$. The reason for this is that the obstructions to coverage are not empty Voronoi cells, even when $d = 1$. For instance, a more likely obstruction when $d = 1$ is formed by two consecutive red points.

We turn to the issue of obtaining a useful sufficient condition for coverage. A first step in this direction is given by the following lemma, which for simplicity's sake we state and prove for the case $d = 2$.

Lemma 4.2. *Let $d = 2$. Let $\mathcal{V}$ be the Voronoi tessellation formed from the red points $\mathcal{P}'$. Let $\mathcal{S}$ be the set of vertices of $\mathcal{V}$ (these are the corners of the Voronoi cells). If $\mathcal{S} \subset \mathcal{P}$, i.e., if there is a black point at each vertex of $\mathcal{V}$, then full coverage occurs.*

Proof. Suppose the hypothesis holds, and let $C_p \in \mathcal{V}$ be a fixed Voronoi cell. Let $v_1, \ldots, v_m$ be the vertices of C_p, and divide C_p into triangles

$$v_1pv_2,\ \ v_2pv_3,\ \ \ldots,\ \ v_mpv_1.$$

Since there are black points at each of the v_i, it follows that each triangle v_ipv_{i+1} is covered by the disks $D(v_i, \|v_i - p\|)$ and $D(v_{i+1}, \|v_{i+1} - p\|)$. Consequently, $C_p \subset \bigcup_i D(v_i, \|v_i - p\|)$, and since C_p was arbitrary, full coverage occurs. □

Of course, the event that $\mathcal{S} \subset \mathcal{P}$ has probability zero, but one might hope that the existence within each cell of a black point sufficiently close to each vertex of that cell would suffice for coverage. Indeed, only the "nearest" few points to each vertex matter, as in the case $d = 1$. However, how close "sufficiently close" is depends on random variables connected with both the size and shape of a typical cell, and little is known about these.

For the rest of the paper, we restrict attention to the case $d = 2$. It turns out to be useful to consider the *Gilbert disk model* on the red points $\mathcal{P}'$, with an appropriately chosen radius. This model is constructed by simply joining two points of $\mathcal{P}'$ if the distance between them is less than some specified threshold.

Theorem 1.2. *If $f(n) = \lambda^3 n \to \infty$, then $p_\lambda^2(n) \to 0$.*

Proof. Suppose that $n \to \infty$ and also that $\lambda^3 n \to \infty$. Let $R > 0$ be a large constant. Construct the Gilbert disk model $G = G_R(\mathcal{P}')$ on the red points $\mathcal{P}'$, with radius R (i.e., we join two points of $\mathcal{P}'$ if they are within distance R). Let T be the (random) number of triangles in G that lie entirely within B_n^2. Then for some absolute constant C_1,

$$\mathbb{E}\,(T) \sim C_1(\lambda \pi R^2)^2 \lambda n = \left(C_1 \pi^2 R^4\right) \lambda^3 n \to \infty.$$

Similarly, if T_1 denotes the (random) number of triangles in G that lie entirely inside B_n^2 and have all angles between $\pi/6$ and $\pi/2$, then also $\mathbb{E}\,(T_1) \to \infty$. Finally, putting in the black points $\mathcal{P}$ and writing T_2 for the number of triangles counted in T_1 that are not within distance $1000R$ of a black point, we have that $\mathbb{E}\,(T_2) \to \infty$. A simple application of the second moment method shows that $\mathbb{P}\,(T_2 \geq 1) \to 1$ (this is intuitively obvious from $\mathbb{E}\,(T_2) \to \infty$ due to the long-range independence of the model). However, every red triangle counted in T_2 will have points in its interior that are not covered by black disks. Since with high probability, $T_2 \geq 1$, we have that $p_\lambda^2(n) \to 0$. $\square$

The other direction seems to require a more elaborate argument (and a stronger hypothesis).

Theorem 1.3. *If $g(n) = \lambda^3 n (\log n)^3 \to 0$, then $p_\lambda^2(n) \to 1$.*

Proof. Suppose that $n \to \infty$ and also that $g(n) = \lambda^3 n (\log n)^3 \to 0$. Once again, we construct the Gilbert disk model $G = G_R(\mathcal{P}')$ on the red points $\mathcal{P}'$, but this time $R = R(n)$ will be a function of n. As long as $R(n)$ is not too large, there will be (up to a constant) λn vertices, $R^2 \lambda^2 n$ edges, and $R^4 \lambda^3 n$ triangles in G inside B_n^2. We will show that $R(n)$ can be chosen such that:

(I) The maximum degree of G is one.

(II) The region of B_n^2 close to points of $\mathcal{P}'$ is covered (by $\mathcal{A}_\lambda^2$).

(III) The rest of B_n^2, far from points of $\mathcal{P}'$, is covered (by $\mathcal{A}_\lambda^2$).

Condition (I) simply states that G consists of isolated vertices and isolated edges. It is (II) and (III) that necessitate the stronger hypothesis. We will define a set of *bad events*, which will depend on $\mathcal{P}$ and $\mathcal{P}'$, and show that coverage (by $\mathcal{A}_\lambda^2$) occurs in the absence of bad events. We will then show that the probability of at least one bad event occurring tends to zero.

First, we overlay a grid of squares of side length $r = \sqrt{\log n}$ onto B_n^2. The probability that a given small square of the grid contains no point of $\mathcal{P}$ is $e^{-\log n} = n^{-1}$. Since there are $\sim n/\log n$ such squares, the expected number of them containing no black points is asymptotically $1/\log n \to 0$. Consequently, with high probability, every small square contains a black point. Now fix a small square S. If no point of S is within distance $\sqrt{2\log n}$ of a red point, and if S contains a black point, then all of S will be covered by $\mathcal{A}_\lambda^2$. Consequently, with high probability, every point of B_n^2 at distance more than $\sqrt{8\log n}$ from all red points will be covered by $\mathcal{A}_\lambda^2$. Our first type of bad event will be that some square S of the grid contains no black points: if no such event occurs, then we may assume that points at distance $\sqrt{8\log n}$ from $\mathcal{P}'$ are covered. This deals with (III).

Take $R(n) = 1000\sqrt{\log n}$. Our second bad event will be that G contains vertices of degree at least 2. Write W for the number of such vertices in G (within B_n^2). Then

$$\mathbb{E}(W) \sim \frac{1}{2} e^{-\lambda\pi R^2} (\lambda\pi R^2)^2 \lambda n = \left(5 \cdot 10^{11} \pi^2 (\log n)^2 + o(1)\right) \lambda^3 n \to 0,$$

so that $\mathbb{P}(W = 0) \to 1$. This deals with (I).

It remains to deal with (II). From now on, we may assume that G has only isolated vertices and isolated edges (inside B_n^2). Recall that we need only worry about the coverage of points within distance $\sqrt{8\log n}$ from a red point: we will color all such points yellow.

First we deal with the isolated vertices. Consider the circles of radii $10\sqrt{\log n}$ and $11\sqrt{\log n}$ around each isolated red point, and divide the annulus between these circles into ten equal "sectors." With high probability, there is a black point inside each sector. But then the yellow region surrounding the red point is covered.

Next we turn to the edges. Consider a fixed red edge p_1p_2, where we may assume $p_1 = (0, 0)$ and $p_2 = (0, t)$. A similar argument to that in the above paragraph takes care of the yellow points whose x-coordinates lie outside the interval $[0, t]$. However, coverage of the yellow points in the rectangle

$[0,t] \times [-\sqrt{8\log n}, \sqrt{8\log n}]$ is not guaranteed. In particular, we need to show that the yellow points lying on the edge p_1p_2 itself are covered. Write

$$M = \max\left\{\frac{50\log n}{t}, 10\sqrt{\log n}\right\},$$

and consider the region

$$R^+ = [0,t] \times [M, M + 10t^{-1}\log n]$$

and its reflection R^- in the x-axis. Then R^+ and R^- each have area $10\log n$. For the purposes of covering the yellow points in the strip $[0,t] \times [-\sqrt{8\log n}, \sqrt{8\log n}]$, only the x-coordinates of black points in R^+ and R^- matter. (The reason behind the choice of M is that we want to ensure that disks originating from black points inside R^+ and R^- have almost equal radii, and these radii should also be at least $10\sqrt{\log n}$. This second requirement is so that coverage of the yellow regions follows from coverage of $p_1p_2 = [0,t]$ from both sides; see Figure 2.) Assuming for the moment that all such black points give rise to disks that are stopped by either p_1 or p_2, we can proceed as follows. We project the points of R^+ and of R^- to the edge $p_1p_2 = [0,t]$, where they form two separate Poisson processes, each of intensity $10t^{-1}\log n$, on an interval of length t. For coverage purposes, these are equivalent to two processes of intensity 1 on an interval of length $10\log n$, and so from (3.1), we see that coverage of the relevant yellow points occurs with probability

$$\begin{aligned}\left(1 - e^{-5\log n}(1 + 5\log n)\right)^2 = \left(1 - n^{-5}(1 + 5\log n)\right)^2 > 1 - 12n^{-5}\log n \\ > 1 - n^{-4},\end{aligned}$$

for sufficiently large n. In other words, coverage fails with probability at most n^{-4}, and since with high probability there are fewer than $\sqrt{n}$ edges in G, the expected number of edges with uncovered yellow points tends to zero. Consequently, with high probability, all yellow points near all edges of G inside B_n^2 are covered.

However, we still have to justify the assumption that disks from black points in R^+ and R^- are stopped by either p_1 or p_2. Write $t_0 = 1$ and $t_{i+1} = 2t_i$ for all $i \in \mathbb{Z}$. If there were an edge p_1p_2 inside B_n^2 and a third red point stopping a disk originating in either R^+ or R^- (each corresponding to p_1p_2), then for some $i \in \mathbb{Z}$, there would be a red point (p_1), another red point within distance t_i of the first (p_2), and a further red point, the "stopping point," within distance $200t_i^{-1}\log n$ of the first point, all inside B_n^2. But the probability of this occurring for some i is at most

$$\sum_{i=-C\log n}^{C'\log\log n} \lambda n \cdot \lambda\pi t_i^2 \cdot 40000\lambda\pi t_i^{-2}(\log n)^2 \le C''\lambda^3 n(\log n)^3 \to 0,$$

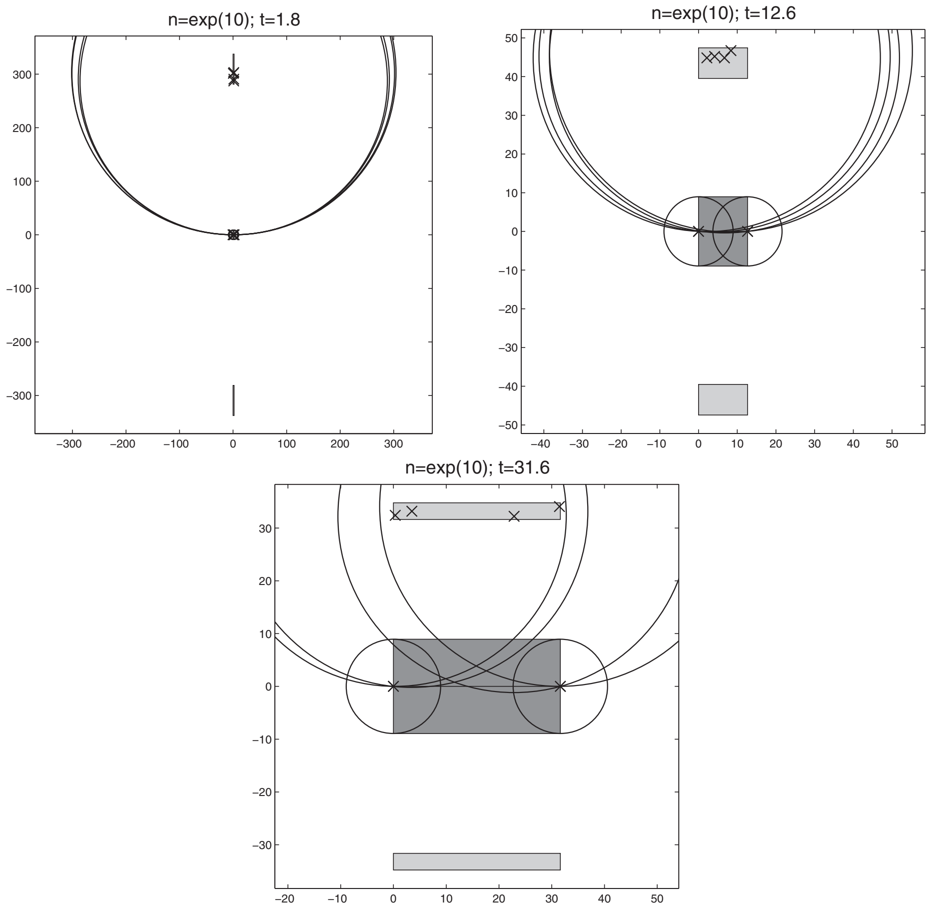

Figure 2. Illustration for Theorem 1.3. Shown are the rectangle $[0, t] \times [-\sqrt{8 \log n}, \sqrt{8 \log n}]$ (at the center of each subfigure) and R^+ and R^- for $n = e^{10}$ and $t = 1.8, 12.6, 31.6$. For small t (top left subfigure), the regions R^+ and R^- are essentially line segments, perpendicular to $p_1 p_2$. Also included are four randomly placed black points in R^+ with their coverage disks.

by hypothesis. (In bounding the number of terms in the above sum, we used the fact that with high probability, all edges in G inside B_n^2 have length at least $n^{-1/5}$.) $\square$

4.2. Simulation Results

Here we provide two simulation results, see Figure 3, that give an indication of the fraction of the area that remains uncovered if the condition in Theorem 1.2

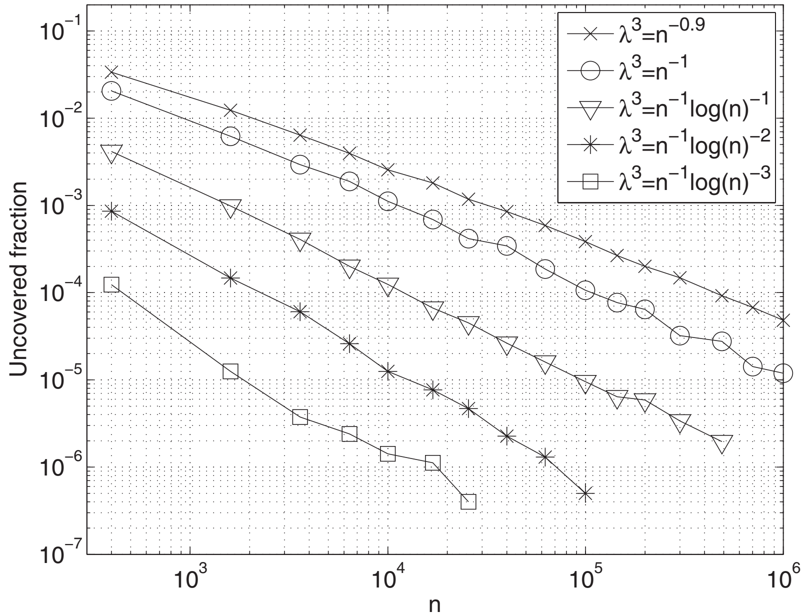

Figure 3. Simulation results for Theorems 1.2 and 1.3. The top curve, marked with $\times$, is for $\lambda^3 = n^{-0.9}$. The bottom curve, marked with $\square$, is for $\lambda^3 = n^{-1}(\log n)^{-3}$. In this case, the function $g(n)$ in Theorem 1.3 is a constant. The curves in between are $\lambda^3 = n^{-1}$, $\lambda^3 = n^{-1}(\log n)^{-1}$, and $\lambda^3 = n^{-1}(\log n)^{-2}$, marked with $\circ$, ∇, and $*$, respectively.

holds, and how quickly this fraction goes to zero if the condition in Theorem 1.3 holds. Although these are estimates of the covered volume fraction, rather than the probability of coverage, there does appear to be a noticeable difference in behavior between the cases $\lambda^3 = n^{-0.9}$ and $\lambda^3 = n^{-1}(\log n)^{-3}$. We hope to investigate this further in future work.

5. Conclusions

We have introduced a novel class of coverage problems whereby the size of the covering disks is determined by the distance to the nearest point in a second point process. In the Poisson–Poisson case, in which black and red points are independent Poisson point processes, we have provided expressions for the covered volume fraction and the probability of complete coverage in the one- and two-dimensional cases.

The main result is the asymptotic threshold for coverage in two dimensions. For

$$\lambda^3 n(\log n)^3 \to 0\,, \quad n \to \infty\,,$$

full coverage is achieved with probability tending to 1. On the other hand, if

$$\lambda^3 n \to \infty\,, \quad n \to \infty\,,$$

then the probability of full coverage tends to 0.[1]

The model can be viewed as a germ–grain model with germs of random and *correlated* size.

The results have applications in secure wireless networking. If the red points are eavesdroppers and the black points base stations, then full coverage in our model implies that from all points of the plane, messages can be received from at least one base station securely, without any eavesdropping.

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