Online Companion: Driving Inventory System Simulations with Limited Demand Data: Insights from the Newsvendor Problem

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This is a supporting document that contains the presentation of a proof for each approximation presented in the paper (Appendix A).

Appendix A: Proofs of the Approximations

Proof of Approximation 1. We first obtain the maximum a posteriori (MAP) estimates of the unknown demand parameters μ and σ^2. Specifically, the MAP estimate for μ, which we denote by $\tilde{\mu}$, maximizes the posterior density function $p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)$ with respect to the parameter μ; i.e., $\tilde{\mu} = \text{argmax}_\mu \log p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)$. Since the logarithm of the posterior density function, $\log p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)$ is proportional to

$$-\frac{(\nu_n + 3)}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \left(\zeta_n^2 + \kappa_n (\mu - \mu_n)^2 \right),$$

we identify $\tilde{\mu}$ as a result of solving the following equation for μ:

$$\frac{\partial \log p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)}{\partial \mu} = -\frac{1}{\sigma^2} \kappa_n (\mu - \mu_n) = 0.$$

Because this equation is satisfied for $\mu - \mu_n = 0$, we obtain $\tilde{\mu}$ as equivalent to μ_n.

Similarly, the MAP estimate for the parameter σ^2, which we denote by $\tilde{\sigma}^2$, is given by argmax_{σ^2} $\log p(\mu,\sigma^2 | x_1, x_2, \ldots, x_n)$. Therefore, $\tilde{\sigma}^2$ solves the following equation for σ^2:

$$\frac{\partial \log p(\mu, \sigma^2 | x_1, x_2, \ldots, x_n)}{\partial \sigma^2} = \frac{1}{2\sigma^2} \left(-(\nu_n + 3) + \frac{\zeta_n^2}{\sigma^2} \right) = 0.$$

Since this equation is equal to zero for $\sigma^2 = \zeta_n^2 / (\nu_n + 3)$, we obtain $\tilde{\sigma}^2$ as equivalent to $\zeta_n^2 / (\nu_n + 3)$.

We are now ready to prove Approximation 1. We know that the mean simulation output response $g(\mu, \sigma^2)$ is asymptotically normal with mean $g(\mu_n, \zeta_n^2 / (\nu_n + 3))$ and variance $\boldsymbol{\beta} \boldsymbol{\Sigma} \boldsymbol{\beta}'$, where $\boldsymbol{\beta}$ contains

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the first-order derivatives of $g(\mu, \sigma^2)$ with respect to μ and σ^2, which are evaluated at their MAP estimates. Therefore, it will be sufficient to obtain $\boldsymbol{\beta\Sigma\beta}'$ for the proof of the approximation.

We first start with obtaining $\boldsymbol{\beta}$. Taking the derivative of $g(\mu, \sigma^2) := \Phi[(I-\mu)/\sigma]$ with respect to the demand parameters μ and σ^2 provides

$$\frac{\partial g(\mu,\sigma)}{\partial \mu} = -\frac{1}{\sigma}\phi\left(\frac{I-\mu}{\sigma}\right) \text{ and } \frac{\partial g(\mu,\sigma)}{\partial \sigma^2} = \frac{\mu - I}{2\sigma^3}\phi\left(\frac{I-\mu}{\sigma}\right).$$

As a result of evaluating these expressions at the MAP estimates of μ and σ^2, we obtain

$$\begin{aligned}
\frac{\partial g(\mu,\sigma)}{\partial \mu}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} &= -\frac{\sqrt{\nu_n+3}}{\zeta_n}\phi\left(\frac{(I-\mu_n)\sqrt{\nu_n+3}}{\zeta_n}\right), \\
\frac{\partial g(\mu,\sigma)}{\partial \sigma^2}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} &= \frac{(\mu_n - I)(\nu_n+3)^{3/2}}{2\zeta_n^3}\phi\left(\frac{(I-\mu_n)\sqrt{\nu_n+3}}{\zeta_n}\right).
\end{aligned}$$

We continue with the derivation of $\boldsymbol{\Sigma}$ as follows:

$$\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} -\frac{\partial^2 \log\left(p(\mu,\sigma^2|\mathbf{x})\right)}{\partial^2\mu}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} & -\frac{\partial^2 \log\left(p(\mu,\sigma^2|\mathbf{x})\right)}{\partial\mu\partial\sigma^2}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} \\ -\frac{\partial^2 \log\left(p(\mu,\sigma^2|\mathbf{x})\right)}{\partial\mu\partial\sigma^2}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} & -\frac{\partial^2 \log\left(p(\mu,\sigma^2|\mathbf{x})\right)}{\partial^2\sigma^2}\Big|_{\mu=\tilde{\mu},\sigma^2=\tilde{\sigma}^2} \end{pmatrix}.$$

Because

$$\begin{aligned}
\frac{\partial \log p(\mu,\sigma^2|x_1,x_2,\ldots,x_n)}{\partial\mu} &= -\frac{1}{\sigma^2}\kappa_n(\mu-\mu_n), \\
\frac{\partial \log p(\mu,\sigma^2|x_1,x_2,\ldots,x_n)}{\partial^2\mu} &= -\frac{1}{\sigma^2}\kappa_n, \\
\frac{\partial \log p(\mu,\sigma^2|x_1,x_2,\ldots,x_n)}{\partial\mu\partial\sigma^2} &= \frac{1}{(\sigma^2)^2}\kappa_n(\mu-\mu_n), \\
\frac{\partial \log p(\mu,\sigma^2|x_1,x_2,\ldots,x_n)}{\partial\sigma^2} &= -\frac{(\nu_n+3)}{2\sigma^2} + \frac{(\zeta_n^2+\kappa_n(\mu-\mu_n)^2)}{2(\sigma^2)^2}, \\
\frac{\partial \log p(\mu,\sigma^2|x_1,x_2,\ldots,x_n)}{\partial^2\sigma^2} &= -\frac{\nu_n+3}{2(\sigma^2)^2} - \frac{(\zeta_n^2+\kappa_n(\mu-\mu_n)^2)}{(\sigma^2)^3},
\end{aligned}$$

we identify $\boldsymbol{\Sigma}$ as equivalent to $[\zeta_n^2/[\kappa_n(\nu_n+3)]\ 0;\ 0\ 2\zeta_n^4/(\nu_n+3)^3]$. Thus, we obtain

$$\begin{pmatrix} -\frac{\sqrt{\nu_n+3}}{\zeta_n}\phi\left(\frac{\sqrt{\nu_n+3}(I-\mu_n)}{\zeta_n}\right) \\ \frac{(\nu_n+3)^{3/2}(\mu_n-I)}{2\zeta_n^3}\phi\left(\frac{\sqrt{\nu_n+3}(I-\mu_n)}{\zeta_n}\right) \end{pmatrix}' \begin{pmatrix} \frac{\zeta_n^2}{\kappa_n(\nu_n+3)} & 0 \\ 0 & \frac{2\zeta_n^4}{(\nu_n+3)^3} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{\nu_n+3}}{\zeta_n}\phi\left(\frac{\sqrt{\nu_n+3}(I-\mu_n)}{\zeta_n}\right) \\ \frac{(\nu_n+3)^{3/2}(\mu_n-I)}{2\zeta_n^3}\phi\left(\frac{\sqrt{\nu_n+3}(I-\mu_n)}{\zeta_n}\right) \end{pmatrix},$$

which further simplifies to

$$\phi^2\left(\frac{(I-\mu_n)(\nu_n+3)^{1/2}}{\zeta_n}\right)\left(\frac{1}{\kappa_n} + \frac{(I-\mu_n)^2}{2\zeta_n^2}\right),$$

completing the proof of the approximation. $\square$

Proof of Approximation 2. Since the posterior density function of the lognormal demand random variable is identical to the posterior density function of the normal demand random variable, the derivation of Approximation 1 follows except with a mean simulation response function of $g(\mu, \sigma^2) := \Phi[(\log(I) - \mu)/\sigma]$. $\square$

Proof of Approximation 3. In a similar manner, the asymptotic approximation to the amount of demand parameter uncertainty in the Type-2 service-level variance is given by $\boldsymbol{\beta}\boldsymbol{\Sigma}\boldsymbol{\beta}'$ with $\boldsymbol{\beta} := [\beta_1\ \beta_2]$. Since the posterior density function of the unknown demand parameters μ and σ^2 is independent of the type of the service level chosen, we again identify $\boldsymbol{\Sigma}$ as $[\zeta_n^2/[\kappa_n(\nu_n+3)]\ 0;\ 0\ 2\zeta_n^4/(\nu_n+3)^3]$. Thus, the amount of demand parameter uncertainty in the Type-2 service-level variance is given by $\beta_1^2\zeta_n^2/[\kappa_n(\nu_n+3)] + 2\beta_2^2\zeta_n^4/(\nu_n+3)^3$ with β_1 and β_2 derived as follows:

First, we characterize the mean simulation response function of the Type-2 service level by

$$\begin{aligned} g(\mu, \sigma^2) &= \mathbb{E}\left\{\frac{\min\{X, I\}}{X}\right\} \\ &= \int_{-\infty}^{I} f(x; \mu, \sigma^2) dx + \int_{I}^{\infty} f(x; \mu, \sigma^2) dx, \end{aligned}$$

where $f(x; \mu, \sigma^2)$ is defined by $\sigma^{-1}\phi((x-\mu)/\sigma)$.

The first-order derivative of $g(\mu, \sigma^2)$ with respect to μ is given by

$$\frac{\partial g(\mu, \sigma^2)}{\partial \mu} = -\frac{1}{\sigma}\phi\left(\frac{I-\mu}{\sigma}\right) + \frac{I}{\sigma^3}\int_I^\infty \left(\frac{x-\mu}{x}\right)\phi\left(\frac{x-\mu}{\sigma}\right) dx.$$

The evaluation at the MAP estimates of μ and σ^2 using the definition of $\alpha_1(I; \mu, \sigma)$ as $\int_I^\infty (x-\mu)/x\ \phi((x-\mu)/\sigma)\ dx$ provides

$$\beta_1 = -\frac{(\nu_n+3)^{1/2}}{\zeta_n}\ \phi\left[\frac{(I-\mu_n)\ (\nu_n+3)^{1/2}}{\zeta_n}\right] + \frac{I\ (\nu_n+3)^{3/2}}{\zeta_n^3}\ \alpha_1(I; \mu_n, \zeta_n/\sqrt{\nu_n+3}).$$

The first-order derivative of $g(\mu, \sigma^2)$ with respect to σ^2 is given by

$$\begin{aligned} \frac{\partial g(\mu, \sigma^2)}{\partial \sigma^2} = &-\frac{1}{(\sqrt{2\pi\sigma^2})^3}\int_{-\infty}^{I} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx + \frac{1}{2\sqrt{2\pi}\sigma^5}\int_{-\infty}^{I}(x-\mu)^2 \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\ &- \frac{I}{(\sqrt{2\pi\sigma^2})^3}\int_I^\infty \frac{1}{x}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} dx + \frac{I}{2\sqrt{2\pi}\sigma^5}\int_I^\infty \frac{(x-\mu)^2}{x}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx. \end{aligned}$$

Using the definition of $\alpha_2(I; \mu, \sigma) := 2\pi^{-1/2}\int_0^{\sqrt{2}(I-\mu)/(2\sigma)} e^{-t^2} dt$, $\alpha_3(I; \mu, \sigma) := \int_I^\infty x^{-1}\exp\{-(x-\mu)^2/(2\sigma^2)\}dx$, and $\alpha_4(I; \mu, \sigma) := \int_I^\infty (x-\mu)^2 x^{-1}\exp\{-(x-\mu)^2/(2\sigma^2)\}dx$, this expression can be further simplified as

$$-\frac{1}{4\pi\sigma^2}\left(1-\alpha_2(I; \mu, \sigma)\right) + \frac{1}{4\sigma^2}\left(1+\alpha_2(I; \mu, \sigma)\right) + \frac{(\mu-1)}{2\sqrt{2\pi}\sigma^3}\exp\left\{-\frac{(I-\mu)^2}{2\sigma^2}\right\}$$

$$-\frac{I}{(\sqrt{2\pi\sigma^2})^3}\alpha_3(I;\mu,\sigma)+\frac{I}{2\sqrt{2\pi}\sigma^5}\alpha_4(I;\mu,\sigma).$$

By evaluating this expression at the MAP estimates of μ and σ^2, we obtain

$$\begin{aligned}\beta_2 = &-\frac{(\nu_n+3)}{4\pi\zeta_n^2}\left(1-\alpha_2(I;\mu_n,\zeta_n/\sqrt{\nu_n+3})\right)+\frac{(\nu_n+3)}{4\zeta_n^2}\left(1+\alpha_2(I;\mu_n,\zeta_n/\sqrt{\nu_n+3})\right)\\ &+\frac{(\mu_n-1)(\nu_n+3)^{3/2}}{2\sqrt{2\pi}\zeta_n^3}\exp\left\{\frac{-(I-\mu_n)^2(\nu_n+3)}{2\zeta_n^2}\right\}-\frac{I(\nu_n+3)^{3/2}}{(\sqrt{2\pi})^3\zeta_n^3}\alpha_3(I;\mu_n,\zeta_n/\sqrt{\nu_n+3})\\ &+\frac{I(\nu_n+3)^{5/2}}{2\sqrt{2\pi}\zeta_n^5}\alpha_4(I;\mu_n,\zeta_n/\sqrt{\nu_n+3}).\end{aligned}$$

which is the expression given in Approximation 3. $\square$