# Online Supplement for ”Approximate key performance indicator joint distribution through ordered block model and pair copula construction”

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### A. The bi-variate Gaussian copula

The density of the bi-variate Gaussian copula is given by

$$c(u,v;\rho)=\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{\rho^2(x_1^2+x_2^2)-2\rho x_1x_2}{2(1-\rho^2)}\right\}, \tag{1}$$

where $\rho\in(-1,1)$ is the parameter of the copula, $x_1=\Phi^{-1}(u)$, $x_2=\Phi^{-1}(v)$ and $\Phi^{-1}(\cdot)$ is the inverse of the standard univariate Gaussian distribution function. For this copula, the $h$-function is given by

$$h_{uv}(u,v;\rho)=\Phi\left(\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(v)}{\sqrt{1-\rho^2}}\right), \tag{2}$$

and the inverse of the $h$-function is :

$$h_{uv}^{-1}(u,v;\rho)=\Phi\big(\Phi^{-1}(u)\sqrt{1-\rho^2}+\rho\Phi^{-1}(v)\big), \tag{3}$$

The dependence represented by a bi-variate Gaussian copula is $\frac{2}{\pi}\arcsin(\rho)$.

### B. The bi-variate Student copula

The density of the bi-variate Student copula is given by

$$c(u,v;\rho,\tau)=\frac{\Gamma(\frac{\tau+2}{2})/\Gamma(\frac{\tau}{2})}{\tau\pi\mathrm{d}t(x_1,\tau)\mathrm{d}t(x_2,\tau)\sqrt{1-\rho^2}}\left\{1+\frac{x_1^2+x_2^2-2\rho x_1x_2}{\tau(1-\rho^2)}\right\}^{-\frac{\tau+1}{2}}, \tag{4}$$

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where $\rho \in (-1, 1)$ and $\tau \in \mathbb{N}$ are the parameters of the copula, $x_1 = t_\tau^{-1}(u)$, $x_2 = t_\tau^{-1}(v)$, and $\mathrm{d}t(\cdot, \tau)$ and $t_\tau^{-1}(\cdot)$ are the probability density and the quantile function, respectively, for the univariate standard Student-distribution with $\tau$ degrees of freedom. For this copula, the $h$-function is given by

$$h_{uv}(u, v; \rho, \tau) = t_{\tau+1}\left\{ \frac{t_\tau^{-1}(u) - \rho t_\tau^{-1}(v)}{\sqrt{\frac{\tau + (t_\tau^{-1}(v))^2 (1-\rho^2)}{\tau+1}}} \right\}, \tag{5}$$

and the inverse of the $h$-function is given by

$$h_{uv}^{-1}(u, v; \rho, \tau) = t_\tau\left\{ t_{\tau+1}^{-1}(u) \sqrt{\frac{\tau + (t_\tau^{-1}(v))^2 (1-\rho^2)}{\tau+1}} + \rho t_\tau^{-1}(v) \right\}, \tag{6}$$

The dependence represented by a bi-variate student copula is $\frac{2}{\pi} \arcsin(\rho)$.

### *C. The bi-variate Clayton copula*

The density of the bi-variate Clayton copula is given by

$$c(u, v; \delta) = (1 + \delta)(u \cdot v)^{-1-\delta}(u^{-\delta} + v^{-\delta} - 1)^{-1/\delta - 2}, \tag{7}$$

where $\delta \in (0, \infty)$ is the parameter of the copula. For this copula, the $h$-function is given by

$$h_{uv}(u, v; \delta) = v^{-\delta-1}(u^{-\delta} + v^{-\delta} - 1)^{-1-1/\delta}, \tag{8}$$

and the inverse of the $h$-function is given by

$$h_{uv}^{-1}(u, v; \delta) = \left( (u \cdot v^{\delta+1})^{-\frac{\delta}{\delta+1}} + 1 - v^{-\delta} \right)^{-1/\delta}, \tag{9}$$

The dependence represented by a bi-variate Clayton copula is $\frac{\delta}{\delta+2}$.

### *D. The bi-variate Gumbel copula*

Let $\tilde{u} = -\log u$ and $\tilde{v} = -\log v$. The density of the bi-variate Gumbel copula is given by

$$\begin{aligned} c(u, v; \delta) &= C(u, v; \delta)(u \cdot v)^{-1} \frac{(\tilde{u}\tilde{v})^{\delta-1}}{(\tilde{u}^\delta + \tilde{v}^\delta)^{2-1/\delta}} \\ C(u, v; \delta) &= \exp\{-(\tilde{u}^\delta + \tilde{v}^\delta)^{1/\delta}\}, \end{aligned} \tag{10}$$

where $\delta \in [1, \infty)$ is the parameter of the copula. For this copula, the $h$-function is given by

$$h_{uv}(u, v; \delta) = v^{-1} \exp\big(-(\tilde{u}^{\delta} + \tilde{v}^{\delta})^{1/\delta}\big) \cdot \left(1 + \left(\frac{\tilde{u}}{\tilde{v}}\right)^{\delta}\right)^{-1+1/\delta}, \tag{11}$$

and the inverse of the $h$ function must be obtained numerically using, for example, Newton-Raphson method (Lee, 1993).

The dependence represented by a bi-variate Gumbel copula is $1 - \frac{1}{\delta}$.

### *E. The bi-variate Frank copula*

The density of the bi-variate Frank copula is given by

$$\begin{aligned} c(u, v; \delta) = & \frac{\delta\eta \exp\big(-\delta(u+v)\big)}{\Big(\eta - \big(1 - \exp(\delta u)\big)\big(1 - \exp(\delta v)\big)\Big)^2} \\ & \eta = 1 - \exp(-\delta), \end{aligned} \tag{12}$$

where $\delta \in [0, \infty)$ is the parameter of the copula. For this copula, the $h$-function is given by

$$h_{uv}(u, v; \delta) = \frac{e^{-\delta v}}{\frac{1-e^{-\delta}}{1-e^{-\delta u}} + e^{-\delta v} - 1}, \tag{13}$$

and the inverse of the $h$-function is given by

$$h_{uv}^{-1}(u, v; \delta) = -\log\left\{1 - \frac{1 - e^{-\delta}}{(u^{-1} - 1)e^{\delta v} + 1}\right\} \Big/ \delta, \tag{14}$$

the According to (Bauer et al., 2012), the joint pdf can be decomposed based on the OBM-PCC model as follows:

$$f(\mathbf{x}) = \prod_{i \in V} \prod_{w \in pa(i)} c_{iw|pa(i;w)}\big(F_{i|pa(i;w)}(x_i|\mathbf{x}_{pa(i;w)}), F_{w|pa(i;w)}(x_w|\mathbf{x}_{pa(i;w)})\big) \prod_{k \in V} f_k(x_k), \tag{15}$$

The dependence represented by a bi-variate Frank copula is $1+\frac{4\left(D(\delta)-1\right)}{\delta}$, where $D(\delta) = \frac{1}{\delta}\int_0^{\delta} \frac{t}{\exp(t)-1}dt$.

## References


Bauer, A., Czado, C. and Klein, T. (2012), ‘Pair-copula constructions for non-gaussian dag models’, *Canadian Journal of Statistics* **40**(1), 86–109.

Lee, A. (1993), ‘Generating random binary deviates having fixed marginal distributions and specified degrees of association’, *The American Statistician* **47**(3), 209–215.