Online Supplement for "Appointment Scheduling and the Effects of Customer Congestion on Service"

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Proof of Lemma 1

Proof: Here and throughout the appendix, we let $C_i(\omega)$ denote the completion time for customer i in scenario ω and $\triangle$ is the difference operator. We have $|\triangle C_1(\omega)| = 0$ as $a_1 = 0$, $W_1(\omega) = 0$ and $\delta_1(W_1(\omega)) = 0$ by assumption. Assuming $||\triangle a|| \leq h$ where h is a constant number, we have the following inequalities:

$$|\triangle W_2(\omega)| \leq |\triangle C_1(\omega) - \triangle a_2| \leq |\triangle C_1(\omega)| + |\triangle a_2| \leq h, \tag{EC-1a}$$

$$|\triangle \delta_2(W_2(\omega))| \leq L(\omega)|\triangle W_2(\omega)| \leq L(\omega)h, \tag{EC-1b}$$

$$\begin{aligned}|\triangle C_2(\omega)| &= |\triangle a_2 + \triangle W_2(\omega) - \triangle \delta_2(W_2(\omega))| \\ &\leq |\triangle a_2| + |\triangle W_2(\omega)| + |\triangle \delta_2(W_2(\omega))| \leq (L(\omega)+2)h,\end{aligned} \tag{EC-1c}$$

where inequality (EC-1a) is from the definition of waiting time, inequality (EC-1b) is from Assumption 1 where $L(\omega)$ is a constant defined in Assumption 1, and inequality (EC-1c) is from the definition of completion time. By induction, we have

$$|\triangle W_i(\omega)| \leq |\triangle C_{i-1}(\omega)| + |\triangle a_i| \leq (L(\omega)+3)^{i-2}\, h, \tag{EC-2a}$$

$$|\triangle \delta_i(W_i(\omega))| \leq L(\omega)|\triangle W_i(\omega)| \leq L(\omega)\,(L(\omega)+3)^{i-2}\, h, \tag{EC-2b}$$

$$|\triangle C_i(\omega)| \leq |\triangle a_2| + |\triangle W_2(\omega)| + |\triangle \delta_2(W_2(\omega))| \leq (L(\omega)+1)\,(L(\omega)+3)^{i-2}\, h. \tag{EC-2c}$$

Thus, $W_i(\omega)$ and $\delta_i(W_i(\omega))$ are Lipschitz-continuous in $\mathbf{a}$. Letting $K=(L(\omega)+3)^{n-1}$, we have $\triangle W_i(\omega)\le Kh$, $\triangle\delta_i(W_i(\omega))\le Kh$, and $\triangle C_i(\omega)\le Kh$ for all i and ω and thus $O(\omega)\le Kh$. As a result, we have the following inequality:

$$|\triangle f(\mathbf{a},\omega)|\le\sum_{i=2}^{n}\alpha_i|\triangle W_i(\omega)|+\sum_{i=2}^{n}\beta_i|\triangle\delta_i(W_i(\omega))|+|\triangle O(\omega)|, \quad \text{(EC-3a)}$$

$$\le\left(\sum_{i=2}^{n}\alpha_i+\sum_{i=2}^{n}\beta_i+1\right)Kh. \quad \text{(EC-3b)}$$

Therefore, $f(\mathbf{a},\omega)$ defined as (2a) is Lipschitz-continuous in $\mathbf{a}$. □

Proof of Lemma 2

Proof: The sample path cost function, $f(\mathbf{a},\omega)$, is differentiable everywhere except at points with one of the following conditions:

- *(i)* customer i arrives at exactly when $i-1$ completes, i.e., $a_{i-1}+W_{i-1}(\omega)+\xi_{i-1}(\omega)-\delta_{i-1}(W_{i-1}(\omega))=a_i$,
- *(ii)* the last customer, n, completes at exactly when the session ends, i.e., $a_n+W_n(\omega)+\xi_n(\omega)-\delta_n(W_n(\omega))=d$,
- *(iii)* $\delta_i(W_i(\omega))$ is nondifferentiable at $W_i(\omega)$.

Conditions *(i)* and *(ii)* occur with probability zero because $\xi_i(\omega)-\delta_i(W_i(\omega))$ is a continuous random variable with finite density according to Assumption 3. $W_i(\omega)$ is also a random variable that is independent of a_i and d. Condition *(iii)* also occurs with probability zero because the set of nondifferential points of $\delta_i(W_i(\omega))$ is finite according to Assumption 2. As a result, $f(\mathbf{a},\omega)$ is differentiable everywhere except at finite saddle points at measure 0. □

Proofs of Theorem 1 and Lemma 3

We omit to prove Theorem 1 and Lemma 3 as similar proofs can be found in the literature, e.g., proofs of Theorem 1 and Lemma 4 in Zhang and Xie (2015).

Proof of Theorem 2

Proof: We first prove equation (3): we let ϵ denote a sufficiently small positive constant. When $W_i(\omega)=0$ and a_i is increased by ϵ, $W_i(\omega)$ and $\delta_i(W_i(\omega))$ are unchanged, but $C_i(\omega)$ is increased by ϵ, and thus $f(\mathbf{a},\omega)$ is increased by $\lambda_{i+1}\epsilon$. When $W_i(\omega)>0$ and a_i is increased by ϵ, $W_i(\omega)$ and $\delta_i(W_i(\omega))$ are decreased by ϵ and $\delta_i'(\omega)\epsilon$, respectively, the start time of customer i is unchanged, and thus $C_i(\omega)$ is increased by $\delta_i'(\omega)\epsilon$. As a result, $f(\mathbf{a},\omega)$ is decreased by $[\alpha_i+\delta_i'(\omega)(\beta_i-\lambda_{i+1})]\,\epsilon$. We next prove equation (4): when $W_i(\omega)=0$ and $C_{i-1}(\omega)$ is increased by ϵ, $W_i(\omega)$ is unchanged and thus $f(\mathbf{a},\omega)$ is unchanged. When $W_i(\omega) > 0$ and $C_{i-1}(\omega)$ is increased, $W_i(\omega)$ and $\delta_i(W_i(\omega))$ are increased by ϵ and $\delta_i'(\omega)\epsilon$, respectively, the start time of customer i is increased by ϵ, and thus $C_i(\omega)$ is increased by $(1-\delta_i'(\omega))\,\epsilon$. As a result, $f(\mathbf{a},\omega)$ is increased by $[\alpha_i + (1-\delta_i'(\omega))\lambda_{i+1} + \delta_i'(\omega)\beta_i]\,\epsilon$. When $i = n+1$ and $C_{i-1}(\omega)$ is increased by ϵ, the overtime, $O(\omega)$, is increased by ϵ if $O(\omega) > 0$, and unchanged if $O(\omega) = 0$. □

Proof of Theorem 3

Proof: When a_i is increased, $\triangle C_i(\omega) \geq 0$ and thus $\triangle W_{i+1}(\omega) \geq 0$. According to Assumption 4, $\triangle\delta_{i+1}(\omega) \leq \triangle W_{i+1}(\omega)$, and thus $\triangle C_{i+1}(\omega) = \triangle W_{i+1}(\omega) - \triangle\delta_{i+1}(\omega) \geq 0$. By induction, we have $\triangle C_j(\omega) \geq 0$, $\triangle W_j(\omega) \geq 0$, $\forall j > i$, and $\triangle O(\omega) > 0$. □

Proof of Lemma 4

Proof: When both $W_i(\omega)$ and $I_i(\omega)$ are positive, the decrease of both $W_i(\omega)$ and $I_i(\omega)$ results in less waiting time and nonincreasing completion time for customer i according to Assumption 4, and thus it does not increase waiting time for the remaining customers. As a result, without increasing the total cost, $W_i(\omega)$ and $I_i(\omega)$ can be decreased until either of them reaches zero, suggesting the complementary slackness constraints (8j)-(8k) automatically hold when the optimal solution is achieved. □

Proof of Theorem 4

Proof: Under Assumptions 4-5, increasing the service time for customer i does not reduce waiting time for the remaining customers and thus there is no reduction in the total cost. Therefore, $\hat{\delta}_i(\omega) \geq \delta_i(W_i(\omega))$ when the optimal solution is achieved. On the other hand, constraints (8d)-(8e) jointly enforce $\hat{\delta}_i(\omega) \leq \delta_i(W_i(\omega))$; as a result, $\hat{\delta}_i(\omega) = \delta_i(W_i(\omega))$ automatically holds for each i and ω. Under Assumptions 4, constraints (8j)-(8k) can be removed according to Lemma 4. Therefore, model (8) reduces to the convex program (9). □

References

Zhang, Zheng, Xiaolan Xie. 2015. Simulation-based optimization for surgery appointment scheduling of multiple operating rooms. *IIE Transactions* **47**(9) 998–1012.