Technical Appendix
Global Sourcing Decisions for a Multinational Firm With Foreign Tax Credit Planning

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1. Definitions of Δs in Figures 1 and 2 and Derivation of Figure 2

1.1. Definitions of Δs in Figure 1

$$\Delta(O) = Pr\{D_1 < \theta_1 Q_1, D_2 < \theta_2 Q_2\}$$

$$\Delta(C_1) = Pr\{\theta_1 Q_1 < D_1 < Q_1, D_2 < \theta_2 Q_2\}$$

$$\Delta(C_2) = Pr\{\theta_1 Q_1 < D_1 < Q_1, D_2 < \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2} \frac{1}{r_2 - s_2}[(r_1 - s_1)D_1 - (c_1 - s_1)Q_1]$$

$$\Delta(C_3) = Pr\{D_1 > Q_1, \theta_2 Q_2 < D_2 < \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2} \gamma_1 Q_1\}$$

$$\Delta(C_4) = Pr\{D_1 > Q_1, D_2 < \theta_2 Q_2\}$$

$$\Delta(L_1) = Pr\{D_1 < \theta_1 Q_1, \theta_2 Q_2 < D_2 < Q_2\}$$

$$\Delta(L_2) = Pr\{D_1 < \theta_1 Q_1, D_2 > Q_2\}$$

$$\Delta(L_3) = Pr\{\theta_1 Q_1 < D_1 < Q_1, D_2 > Q_2\}$$

$$\Delta(L_4) = Pr\{D_1 > Q_1, D_2 > Q_2\}$$

$$\Delta(L_5) = Pr\{D_1 > Q_1, \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2} \gamma_1 Q_1 < D_2 < Q_2\}$$

$$\Delta(L_6) = Pr\{\theta_1 Q_1 < D_1 < Q_1, \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2} \gamma_1 Q_1 < D_2 < Q_2\}$$

$$\Delta(L_7) = Pr\{\theta_1 Q_1 < D_1 < Q_1, D_2 > \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2} \frac{1}{r_2 - s_2}[(r_1 - s_1)D_1 - (c_1 - s_1)Q_1]\}$$

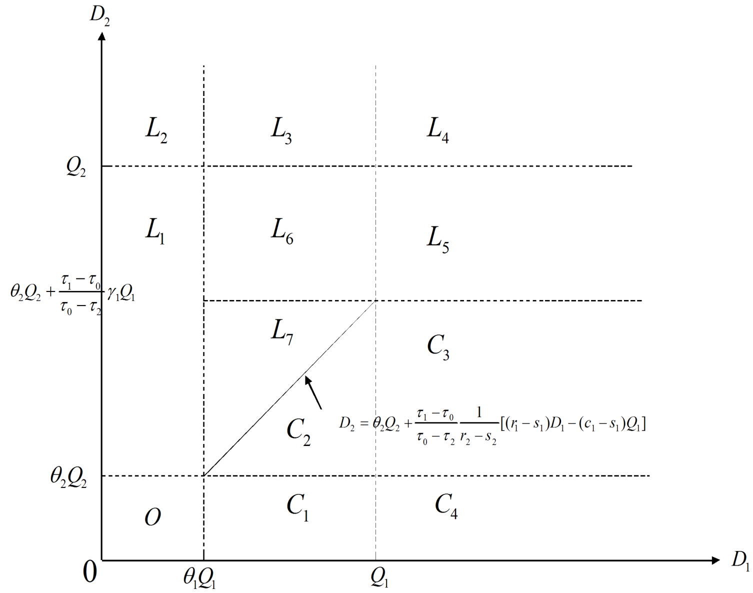

Figure 1 Demand Realization Regions under the L Condition.

1.2. Definition of Δs in Figure 2

$\Delta(O) = Pr\{D_1 < \theta_1 Q_1, D_2 < \theta_2 Q_2\}$

$\Delta(C_1) = Pr\{\theta_1 Q_1 < D_1 < \theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2, D_2 < \theta_2 Q_2\}$

$\Delta(C_2) = Pr\{\theta_1 Q_1 < D_1 < \theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2, D_2 < \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2}\frac{1}{r_2 - s_2}[(r_1 - s_1)D_1 - (c_1 - s_1)Q_1]\}$

$\Delta(C_3) = Pr\{\theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2 < D_1 < Q_1, D_2 > Q_2\}$

$\Delta(C_4) = Pr\{D_1 > Q_1, D_2 > Q_2\}$

$\Delta(C_5) = Pr\{D_1 > Q_1, \theta_2 Q_2 < D_2 < Q_2\}$

$\Delta(C_6) = Pr\{\theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2 < D_1 < Q_1, \theta_2 Q_2 < D_2 < Q_2\}$

$\Delta(C_7) = Pr\{\theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2 < D_1 < Q_1, D_2 < \theta_2 Q_2\}$

$\Delta(C_8) = Pr\{D_1 > Q_1, D_2 < \theta_2 Q_2\}$

$\Delta(L_1) = Pr\{D_1 < \theta_1 Q_1, \theta_2 Q_2 < D_2 < Q_2\}$

$\Delta(L_2) = Pr\{D_1 < \theta_1 Q_1, D_2 > Q_2\}$

$\Delta(L_3) = Pr\{\theta_1 Q_1 < D_1 < \theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2, D_2 > Q_2\}$

$\Delta(L_4) = Pr\{\theta_1 Q_1 < D_1 < \theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2, D_2 > \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2}\frac{1}{r_2 - s_2}[(r_1 - s_1)D_1 - (c_1 - s_1)Q_1]\}$

1.3. Derivation of the First-Order Conditions Under Conditions L, C, and E

Case 1. Q *satisfies the L condition* Define $\gamma_i = (r_i - c_i)/(r_j - s_j)$, $i \neq j$ and note that $\theta_i = (c_i - s_i)/(r_i - s_i)$. Figure 1 shows different demand realization regions in which ex post either event L

(in regions L_1-L_7) or event C (in regions C_1-C_4) occurs. Specifically, we note that when $D_i < \theta_i Q_i$, $i = 1, 2$, neither subsidiary makes a profit, so no tax incurs in region O.

The partial derivatives of $P^C(\mathbf{Q})$ with respect to Q_1, provided that $\mathbf{Q}$ satisfies ex ante condition L, are as given by

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} = & -(c_1 - s_1)\{\Delta(O) + \Delta(L_1) + \Delta(L_2)\} - (1-\tau_1)(c_1 - s_1)[\Delta(C_1) + \Delta(C_2)] \\ & - (1-\tau_0)(c_1 - s_1)[\Delta(L_3) + \Delta(L_6) + \Delta(L_7)] + (1-\tau_1)(r_1 - c_1)[\Delta(C_3) + \Delta(C_4)] \\ & + (1-\tau_0)(r_1 - c_1)[\Delta(L_4) + \Delta(L_5)].\end{aligned} \tag{1}$$

After collapsing terms, (1) can be rewritten as

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} = & (1-\tau_1)[(r_1 - c_1)Pr\{D_1 > Q_1\} - (c_1 - s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}] \\ & - (c_1 - s_1)Pr\{D_1 \le \theta_1 Q_1\} + (\tau_1 - \tau_0)\Big\{(r_1 - c_1)[\Delta(L_4) + \Delta(L_5)] \\ & - (c_1 - s_1)[\Delta(L_3) + \Delta(L_6) + \Delta(L_7)]\Big\}.\end{aligned} \tag{2}$$

Equation (2) can be interpreted as follows: the first and second lines are the marginal profitability with respect to an increase of Q_1 for the subsidiary S_1 under its own after-local-tax profit maximization problem discussed earlier in Section 3 (see (10)). The third line represents the marginal benefits of the global firm due to tax cross-crediting across the two subsidiaries.

With a similar analysis, we can also derive the partial derivative of $P^C(\mathbf{Q})$ with respect to Q_2 as follows:

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} = & (1-\tau_2)[(r_2 - c_2)Pr\{D_2 > Q_2\} - (c_2 - s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}] \\ & - (c_2 - s_2)Pr\{D_2 \le \theta_2 Q_2\} + (\tau_2 - \tau_0)\Big\{(r_2 - c_2)[\Delta(L_2) + \Delta(L_3) + \Delta(L_4)] \\ & - (c_2 - s_2)[\Delta(L_1) + \Delta(L_5) + \Delta(L_6) + \Delta(L_7)]\Big\}.\end{aligned} \tag{3}$$

The interpretation for (3) is similar to that for (2).

Case 2. $\mathbf{Q}$ *satisfies the C condition*

We now turn to the situation in which a given sourcing decision $\mathbf{Q}$ satisfies ex ante condition C. Figure 2 illustrates different demand realization regions. Similar to Figure 1, in region O, no tax incurs; in regions L_1-L_4, we have event L ex post; and in regions C_1-C_7, we have event C ex post.

Following a similar analysis as in Case 1, we can show that when $\mathbf{Q}$ satisfies condition C, the partial derivatives are as follows:

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} = & (1-\tau_1)[(r_1 - c_1)Pr\{D_1 > Q_1\} - (c_1 - s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}] \\ & - (c_1 - s_1)Pr\{D_1 \le \theta_1 Q_1\} - (\tau_1 - \tau_0)(c_1 - s_1)[\Delta(L_3) + \Delta(L_4)],\end{aligned} \tag{4}$$

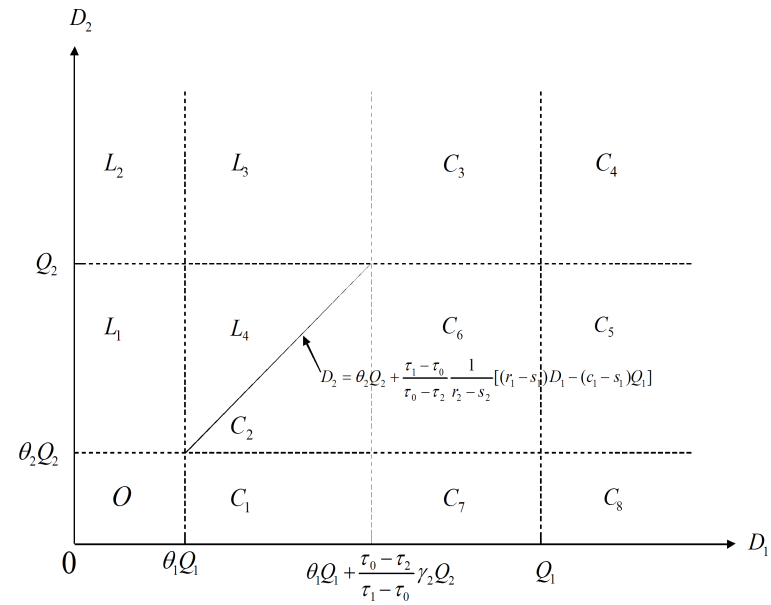

Figure 2 Demand Realization Regions under the C Condition

and

$$\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} = (1-\tau_2)[(r_2-c_2)Pr\{D_2 > Q_2\} - (c_2-s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}] - (c_2-s_2)Pr\{D_2 \le \theta_2 Q_2\}$$
$$- (\tau_0-\tau_2)\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)] - (c_2-s_2)[\Delta(L_1)+\Delta(L_4)]\}. \tag{5}$$

Case 3. Q *satisfies the ex ante E condition*

Under this condition, $\theta_1 Q_1 + \frac{\tau_0-\tau_2}{\tau_1-\tau_0}\gamma_2 Q_2 = Q_1$ and $\theta_2 Q_2 + \frac{\tau_1-\tau_0}{\tau_0-\tau_2}\gamma_1 Q_1 = Q_2$. As a result, Figures 1 and 2 become identical. Thus, if condition E holds, $\mathbf{Q^C}$ satisfies the following equations:

$$\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} + \frac{(\tau_1-\tau_0)(r_1-c_1)}{(\tau_0-\tau_2)(r_2-c_2)}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} = 0, \tag{6}$$
$$(\tau_1-\tau_0)(r_1-c_1)Q_1 - (\tau_0-\tau_2)(r_2-c_2)Q_2 = 0, \tag{7}$$

where $\frac{\partial P^C(\mathbf{Q})}{\partial Q_1}$ and $\frac{\partial P^C(\mathbf{Q})}{\partial Q_2}$ are given by (4) and (5), respectively.

1.4. A single firm's optimal quantity with tax consideration

For any given sourcing quantity Q_i and realized demand D_i, let A_i be S_i's pretax profits. We have

$$A_i(Q_i) = r_i \min\{Q_i, D_i\} + s_i(Q_i-D_i)^+ - c_i Q_i = (r_i-c_i)Q_i - (r_i-s_i)(Q_i-D_i)^+. \tag{8}$$

At the end of each year (period), if S_i is profitable, its profits are taxed on the internal managerial books at the managerial tax rate τ. However, if a loss incurs, no tax will be levied. The after-tax profits or losses corresponding to $A_i(Q_i)$ is therefore given by

$$\Pi_i(\tau, Q_i) = A_i(Q_i) - \tau A_i(Q_i)^+ = (1-\tau)A_i(Q_i) + \tau A_i(Q_i)^-$$

where $A_i(Q_i)^- = min(0, A_i(Q_i))$ and $A_i(Q_i)^+ = max(0, A_i(Q_i))$. The subsidiary S_i's objective is to choose an optimal sourcing quantity $Q_i(\tau)$ which solves the following maximization problem:

$$P_i(\tau) \equiv \max_{Q_i}\{E_{D_i}(\Pi_i(\tau, Q_i))\}. \tag{9}$$

It is easy to verify that $A_i(Q_i)$, $-A_i(Q_i)^+$ and $A_i(Q_i)^-$ are all concave in Q_i. Thus, $\Pi_i(\tau, Q_i)$ and $E_{D_i}\{\Pi_i(\tau, Q_i)\}$ are also concave in Q_i. The partial derivative of the expected after-tax profits for subsidiary S_i with respect to Q_i as follows

$$\begin{aligned}\frac{\partial E_{D_i}(\Pi_i(\tau, Q_i))}{\partial Q_i} &= (1-\tau)[(r_i - c_i)Pr\{D_i \geq Q_i\} - (c_i - s_i)Pr\{\theta_i Q_i < D_i < Q_i\}] \\ &\quad - (c_i - s_i)Pr\{D_i < \theta_i Q_i\},\end{aligned} \tag{10}$$

where Pr represents probability.

Setting (10) to zero yields

$$Pr\{D_i < Q_i(\tau)\} = \frac{r_i - c_i}{r_i - s_i} - \frac{\tau(c_i - s_i)Pr\{D_i < \theta_i Q_i(\tau)\}}{(1-\tau)(r_i - s_i)}. \tag{11}$$

Note that when $\tau = 0$, $Q_i(0)$ is the quantity chosen by a traditional newsvendor without tax consideration. From (11), we gain some insights on the optimal policy under the after-tax objective which are summarized in the following proposition.

Single Firm Proposition: *(i) The after-tax objective causes the firm to produce less (i.e., $Q_i(\tau) \leq Q_i(0)$) than the optimal sourcing quantity under the pretax objective; (ii) The optimal sourcing quantity $Q_i(\tau)$ satisfies (5) in the main body of this paper.*

2. Proof of the Main Results

Proof of Proposition 1 Let $g_1(\mathbf{Q}) = \Pi_1(\tau_0, Q_1) + \Pi_2(\tau_0, Q_2)$ and $g_2(\mathbf{Q}) = \Pi_1(\tau_1, Q_1) + \Pi_2(\tau_2, Q_2)$. It is straightforward to see that $E[\Pi_i(\tau, Q_i)]$ is strictly concave in Q_i under the assumption that $f_i(\cdot)$ is strictly positive. The sum of strictly concave functions is also strictly concave. Hence, $E[g_1(\mathbf{Q})]$ is strictly concave. By definition, we need to show $P^C(t\mathbf{Q_1} + (1-t)\mathbf{Q_2}) > tP^C(\mathbf{Q_1}) + (1-t)P^C(\mathbf{Q_2})$ for the strict concavity of $P^C(\cdot)$ for $0 < t < 1$.

$$\begin{aligned}&P^C(t\mathbf{Q_1} + (1-t)\mathbf{Q_2}) = E[min[g_1(t\mathbf{Q_1} + (1-t)\mathbf{Q_2}), g_2(t\mathbf{Q_1} + (1-t)\mathbf{Q_2})] \\ &\geq min[E[g_1(t\mathbf{Q_1} + (1-t)\mathbf{Q_2})], E[g_2(t\mathbf{Q_1} + (1-t)\mathbf{Q_2})]] \\ &> min[tE[g_1(\mathbf{Q_1})] + (1-t)E[g_1(\mathbf{Q_2})], tE[g_2(\mathbf{Q_1})] + (1-t)E[g_1(\mathbf{Q_2})]] \\ &= tminE[g_1(\mathbf{Q_1}), g_2(\mathbf{Q_1})] + (1-t)minE[g_1(\mathbf{Q_2}), g_2(\mathbf{Q_2})] = tP^C(\mathbf{Q_1}) + (1-t)P^C(\mathbf{Q_2}).\end{aligned}$$

The first inequality holds due to Jensen's inequality. The second strict inequality holds because of the strict concavity of $g_1(\cdot)$ and $g_2(\cdot)$. $\square$

Proof of Proposition 2 Proposition 2 is derived by Proposition 1 and setting the first-order conditions given by equations (2) to (7) to zero.

The next result will be used for the proof of Proposition 3.

LEMMA 1. *For all $Q_i \le Q_i(0)$,*

$$(r_i - c_i)Pr\{D_i > Q_i\} - (c_i - s_i)Pr\{\theta_i Q_i < D_i < Q_i\} \ge 0. \tag{12}$$

Proof of Lemma 1. This result follows directly from the marginal after-tax profit of a subsidiary:

$$\begin{aligned}\frac{\partial E_{D_i}(\Pi_i(\tau, Q_i))}{\partial Q_i} &= (1-\tau)[(r_i - c_i)Pr\{D_i \ge Q_i\} - (c_i - s_i)Pr\{\theta_i Q_i < D_i < Q_i\}] \\ &\quad - (c_i - s_i)Pr\{D_i < \theta_i Q_i\}.\end{aligned} \tag{13}$$

Since $E_{D_i}[\Pi_i(0, Q_i)]$ is concave, for all $Q_i \le Q_i(0)$, $\partial E_{D_i}[\Pi_i(0, Q_i)]/\partial Q_i \ge 0$, namely,

$$(r_i - c_i)Pr\{D_i \ge Q_i\} - (c_i - s_i)Pr\{\theta_i Q_i < D_i < Q_i\} \ge (c_i - s_i)Pr\{D_i < \theta_i Q_i\} \ge 0.$$

Proof of Proposition 4 For Part (i), because of the concavity of $P^C(\mathbf{Q})$, it suffices to show at $Q_i = Q_i(\tau_0)$, $\partial P^C(\mathbf{Q})/\partial Q_i \le 0$ for any Q_j. We can derive the marginal profit corresponding to each of the regions in Figure 1 and show that the partial derivatives are given as below:

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} &= -(c_1 - s_1)[\Delta(O) + \Delta(L_1) + \Delta(L_2)] - (1-\tau_1)(c_1 - s_1)[\Delta(C_1) + \Delta(C_2)] \\ &\quad - (1-\tau_0)(c_1 - s_1)[\Delta(L_3) + \Delta(L_6) + \Delta(L_7)] + (1-\tau_1)(r_1 - c_1)[\Delta(C_3) + \Delta(C_4)] \\ &\quad + (1-\tau_0)(r_1 - c_1)[\Delta(L_4) + \Delta(L_5)].\end{aligned} \tag{14}$$

After collapsing terms, it becomes

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q}))}{\partial Q_1} &= (1-\tau_0)[(r_1 - c_1) - (r_1 - s_1)Pr\{D_1 < Q_1\}] - \tau_0(c_1 - s_1)Pr\{D_1 < \theta_1 Q_1\} \\ &\quad - (\tau_1 - \tau_0)\{(r_1 - c_1)[\Delta(C_3) + \Delta(C_4)] - (c_1 - s_1)[\Delta(C_1) + \Delta(C_2)]\}.\end{aligned} \tag{15}$$

At $Q_1 = Q_1(\tau_0)$, the first line of (15) vanishes. Moreover, since D_1 and D_2 are independent of each other,

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q}))}{\partial Q_1} =& (r_1 - c_1)[\Delta(C_3) + \Delta(C_4)] - (c_1 - s_1)[\Delta(C_1) + \Delta(C_2)] \\ &\le -(\tau_1 - \tau_0)\{(r_1 - c_1)[\Delta(C_3) + \Delta(C_4)] - (c_1 - s_1)[\Delta(C_1) + \Delta(C_2) + \Delta(L_7)]\} \\ &= Pr\{D_2 < \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2}\gamma_1 Q_1\}[(r_1 - c_1)Pr\{D_1 > Q_1\} - (c_1 - s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}].\end{aligned}$$

By Lemma 1, at $Q_1 = Q_1(\tau_0)$,

$$(r_1 - c_1)Pr\{D_1 > Q_1\} - (c_1 - s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\} \ge 0.$$

Therefore, at $Q_1 = Q_1(\tau_0)$, $\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} \le 0$. Thus, the concavity of $P^C(\mathbf{Q})$ and Proposition 1(ii) yield $Q_1^C \le Q_1(\tau_0) \le Q_1(0)$.

Similarly, using Figure 1,

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} &= -(c_2 - s_2)[\Delta(O) + \Delta(C_1) + \Delta(C_4)] - (1-\tau_0)(c_2 - s_2)[\Delta(L_1) + \Delta(L_5) + \Delta(L_6) + \Delta(L_7)] \\ &\quad - (1-\tau_2)(c_2 - s_2)[\Delta(C_2) + \Delta(C_3)] + (1-\tau_0)(r_2 - c_2)[\Delta(L_2) + \Delta(L_3) + \Delta(L_4)].\end{aligned} \tag{16}$$

After collapsing terms, it becomes

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} &= (1-\tau_0)[(r_2 - c_2) - (r_2 - s_2)Pr\{D_2 < Q_2\}] - \tau_0(c_2 - s_2)Pr\{D_2 < \theta_2 Q_2\} \\ &\quad - (\tau_0 - \tau_2)(c_2 - s_2)[\Delta(C_2) + \Delta(C_3)].\end{aligned} \tag{17}$$

The first line of (17) vanishes at $Q_2 = Q_2(\tau_0)$, so

$$\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} = -(\tau_0 - \tau_2)(c_2 - s_2)[\Delta(C_2) + \Delta(C_3)] \le 0.$$

Hence, the concavity of $P^C(\mathbf{Q})$ and Proposition 1(ii) yield $Q_2^C \le Q_2(\tau_0) \le Q_2(\tau_2) \le Q_2(0)$.

For part (ii), we derive the marginal profit corresponding to each of the regions in Figure 2 and show that the partial derivatives of $P^C(\mathbf{Q})$ with respect to Q_1 is as follows:

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} &= -(c_1 - s_1)[\Delta(O) + \Delta(L_1) + \Delta(L_2)] - (1-\tau_0)(c_1 - s_1)[\Delta(L_3) + \Delta(L_4)] \\ &\quad - (1-\tau_1)(c_1 - s_1)[\Delta(C_1) + \Delta(C_2) + \Delta(C_3) + \Delta(C_6) + \Delta(C_7))] \\ &\quad + (1-\tau_1)(r_1 - c_1)[\Delta(C_4) + \Delta(C_5) + \Delta(C_8)].\end{aligned} \tag{18}$$

After collapsing terms,

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} &= (1-\tau_1)[(r_1 - c_1) - (r_1 - s_1)Pr\{D_1 \le Q_1\}] - \tau_1(c_1 - s_1)Pr\{D_1 \le \theta_1 Q_1\} \\ &\quad - (\tau_1 - \tau_0)(c_1 - s_1)[\Delta(L_3) + \Delta(L_4)].\end{aligned} \tag{19}$$

At $Q_1 = Q_1(\tau_1)$, the first line of (19) vanishes, so

$$\frac{\partial P^C(\mathbf{Q})}{\partial Q_1} = -(\tau_1 - \tau_0)(c_1 - s_1)[\Delta(L_1) + \Delta(L_2)] \le 0.$$

Therefore, the concavity of $P^C(\mathbf{Q})$ and Proposition 1(ii) yield $Q_1^C \le Q_1(\tau_1) \le Q_1(\tau_0) \le Q_1(0)$.

Similarly, using Figure 2 and the definitions of its areas, the global firm's marginal profit with respect to Q_2 is as follows:

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} &= -(c_2 - s_2)[\Delta(O) + \Delta(C_1) + \Delta(C_7) + \Delta(C_8)] - (1-\tau_2)(c_2 - s_2)[\Delta(C_2) + \Delta(C_5) + \Delta(C_6)] \\ &\quad + (1-\tau_2)(r_2 - c_2)[\Delta(C_3) + \Delta(C_4)] - (1-\tau_0)(c_2 - s_2)[\Delta(L_1) + \Delta(L_4)] \\ &\quad + (1-\tau_0)(r_2 - c_2)[\Delta(L_2) + \Delta(L_3)].\end{aligned} \tag{20}$$

After collapsing terms,

$$\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} = (1-\tau_2)[(r_2-c_2)-(r_2-s_2)Pr\{D_2 \le Q_2\}] - \tau_2(c_2-s_2)Pr\{D_2 \le \theta_2 Q_2\} \\ -(\tau_0-\tau_2)\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)]-(c_2-s_2)[\Delta(L_1)+\Delta(L_4)]\}. \tag{21}$$

At $Q_2 = Q_2(\tau_2)$,

$$\begin{aligned}\frac{\partial P^C(\mathbf{Q})}{\partial Q_2} &= -(\tau_0-\tau_2)\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)]-(c_2-s_2)[\Delta(L_1)+\Delta(L_4)]\} \\ &\le -(\tau_0-\tau_2)\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)]-(c_2-s_2)[\Delta(L_1)+\Delta(L_4)+\Delta(C_2)]\} \\ &= -(\tau_0-\tau_2)Pr\{D_1 < \theta_1 Q_1 + \frac{\tau_0-\tau_2}{\tau_1-\tau_0}\gamma_2 Q_2\}[(r_2-c_2)Pr\{D_2 > Q_2\} \\ &\quad -(c_2-s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}] \le 0.\end{aligned} \tag{22}$$

because of Lemma 1. Hence, the concavity of $P^C(\mathbf{Q})$ and Proposition 1(ii) yield $Q_2^C \le Q_2(\tau_2) \le Q_2(0)$.

Proof of Corollary 1 Under all three conditions, $dQ_{2j}^C/d\tau_0$ must be decreasing in τ_0 as subsidiary S_2 is subject to home tax rate τ_0 although under certain demand realizations, the excess tax liability may be partially offset by the tax credit generated from subsidiary S_1. The strict proof is shown below. From (15),

$$\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_1 \partial \tau_0} = -(c_1-s_1)[\Delta(L_2)+\Delta(L_4)+(\tau_1-\tau_0)\frac{d(\Delta(L_3)+\Delta(L_4))}{d\tau_0}] \le 0.$$

As shown in Figure 2, as τ_0 increase, the line $Q_1 = \theta_1 Q_1 + \frac{(\tau_1-\tau_0)}{(\tau_0-\tau_2)}\gamma_2 Q_2$ shifts to the right as τ_0 increases. Thus, $\Delta(L_3)$ and $\Delta(L_4)$ increase.

$$\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_1 \partial Q_2} = -(\tau_1-\tau_0)(c_1-s_1)\frac{d(\Delta(L_3)+\Delta(L_4))}{dQ_2} \ge 0,$$

and

$$\begin{aligned}\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_2 \partial \tau_0} &= -\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)]-(c_2-s_2)[\Delta(L_1)+\Delta(L_4)]\} \\ &\quad -(\tau_0-\tau_2)\{(r_2-c_2)\frac{d(\Delta(L_2)+\Delta(L_3))}{d\tau_0}]-(c_2-s_2)\frac{d(\Delta(L_1)+\Delta(L_4)}{d\tau_0}\} \le 0.\end{aligned}$$

From Figure 2, ΔL_2 and ΔL_1 do not change as τ_0 increases. Moreover,

$$\begin{aligned}&(r_2-c_2)\frac{d\Delta L_3}{d\tau_0} - (c_2-s_2)\frac{d\Delta(L_4)}{d\tau_0} \\ &\ge ((r_2-c_2)Pr\{D_2 > Q_2\} - (c_2-s_2)Pr\{Pr\{\theta_2 Q_2 \le D_2 \le Q_2\} f(\theta_1 Q_1 + \frac{\tau_0-\tau_2}{\tau_1-\tau_0}\gamma_2 Q_2)\frac{\tau_1-\tau_2}{(\tau_1-\tau_0)^2} \ge 0.\end{aligned}$$

Hence, $\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_2 \partial \tau_0} \le 0$. The concavity of $P^C(\cdot)$ yields $dQ_{2C}^C/d\tau_0 \le 0$. Above three inequalities and the concavity of $P^C(\cdot)$ implies $dQ_{iC}^C/d\tau_0 < 0$, $i = 1, 2$.

The proof of $dQ^C_{1L}/d\tau_0 \le 0$ and $dQ^C_{2L}/d\tau_0 \le 0$ can be shown similarly. Under condition L,

$$\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_1 \partial Q_2} = -(\tau_1 - \tau_0)\{(r_1 - c_1)\frac{d[\Delta(C_3) + \Delta(C_4)]}{dQ_2} - (c_1 - s_1)\frac{d[\Delta(C_1) + \Delta(C_2)]}{dQ_2}\}$$

$$= \theta_2 f(\theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2}\gamma_1 Q_1)(\tau_1 - \tau_0)\{(r_1 - c_1)Pr\{D_1 > Q_1\} - (c_1 - s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}/2\} \ge 0.$$

Similarly, $\frac{\partial P^C(\mathbf{Q})}{\partial Q_i \partial \tau_0} \le 0$.

Under condition E, the line $D_2 = \theta_2 Q_2 + \frac{\tau_1 - \tau_0}{\tau_0 - \tau_2}\gamma_1 Q_1$ merges with $D_2 = Q_2$ in Figure 1 and the line $D_1 = \theta_1 Q_1 + \frac{\tau_0 - \tau_2}{\tau_1 - \tau_0}\gamma_2 Q_2$ merges with $D_1 = Q_1$. As a result, Figures 1 and 2 become identical. Additionally, tax liability exactly equals tax credit when both subsidiaries sell up inventory, i.e., $(\tau_1 - \tau_0)(r_1 - c_1)Q_1 = (\tau_0 - \tau_2)(r_2 - c_2)Q_2$.

Here is a simpler proof of the monotonicity of $\mathbf{Q^C}$ with respect to τ_0. Under condition L or C, $\mathbf{Q^C}$ satisfies the first-order conditions. $dQ^C_i/d\tau_0 \le 0$ yields directly from the strict concavity of $P^C(\mathbf{Q})$ and $\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_i \partial \tau_0} \le 0$, $\frac{\partial^2 P^C(\mathbf{Q})}{\partial Q_1 \partial Q_2} \ge 0$ and the Envelop Theorem. Under condition E, $u = 0$. Complete differentiating the equality with respect to τ_0 yields

$$-(r_1 - c_1)Q^C_{1E} - (r_2 - c_2)Q^C_{2E} = -(\tau_1 - \tau_0)(r_1 - c_1)\frac{dQ^C_{1E}}{d\tau_0} + (\tau_0 - \tau_2)(r_2 - c_2)\frac{dQ^C_{2E}}{d\tau_0}.$$

Since $(\tau_0 - \tau_2)|\frac{dQ^C_{2E}}{d\tau_0}| \le Q^C_{2E}$ and $Q^C_{1E} \ge -(\tau_1 - \tau_0)\frac{dQ^C_{1E}}{d\tau_0}$, for the equation above to hold, $\frac{dQ^C_{1E}}{d\tau_0} \ge 0$ must hold. □

Proof of Corollary 2 Let t be the Lagrange multiplier, the new objective function can be rewritten as

$$L(\mathbf{Q}; \tau_0) = P^C(\mathbf{Q}; \tau_0) + t[u - (\tau_1 - \tau_0)(r_1 - c_1)Q_1 + (\tau_0 - \tau_2)(r_2 - c_2)Q_2].$$

The optimal solution satisfies

$$\frac{\partial P^C(\mathbf{Q}; \tau_0)}{\partial Q_1} = t(\tau_1 - \tau_0)(r_1 - c_1) \qquad \frac{\partial P^C(\mathbf{Q}; \tau_0)}{\partial Q_2} = -t(\tau_0 - \tau_2)(r_2 - c_2),$$
$$ut = 0 \qquad u = (\tau_1 - \tau_0)(r_1 - c_1)Q_1 - (\tau_0 - \tau_2)(r_2 - c_2)Q_2.$$

note that at $\tau_0 = \tau_2$, $u > 0$ for $\mathbf{Q} \neq 0$, i.e., C condition holds and at $\tau_0 = \tau_1$, $u < 0$, i.e., the **ex ante** condition L holds. As shown below, as τ_0 increases within the range of $[\tau_2, \tau_1]$, u decreases.

$$\frac{du}{d\tau_0} = -(r_1 - c_1)Q^C_1 - (r_2 - c_2)Q^C_2 + (\tau_1 - \tau_0)(r_1 - c_1)dQ^C_1/d\tau_0 - (\tau_0 - \tau_2(r_2 - c_2)dQ^C_2/d\tau_0.$$

Note that $(r_i - c_i)Q_i$ is subsidiary i's maximum profit with excess demand. The first-order impact of τ_0 on tax credit (liability) must dominate the absolute value of the second order effect; i.e., $Q^C_1 \ge (\tau_1 - \tau_0)|dQ^C_1/d\tau_0|$ and $Q^C_2 \ge (\tau_0 - \tau_2)(r_2 - c_2)|dQ^C_2/d\tau_0|$ because the probability of cross-crediting is strictly less than 1 and at the two extreme points (i.e., $\tau_0 = \tau_2, \tau_1$, $Q^C_i > 0$). Hence, $\frac{du}{d\tau_0} \le 0$. Consequently, there must exist two threshold values, $\tau_2 \le \hat{\tau}_0 \le \tilde{\tau}_0 \le \tau_1$ such that condition C holds for $\tau_0 \in [\tau_2, \hat{\tau}_0)$; E condition holds for $\tau_0 \in [\hat{\tau}_0, \tilde{\tau}_0]$; and for $\tau \in (\tilde{\tau}_0, \tau_1]$, L condition holds.

□

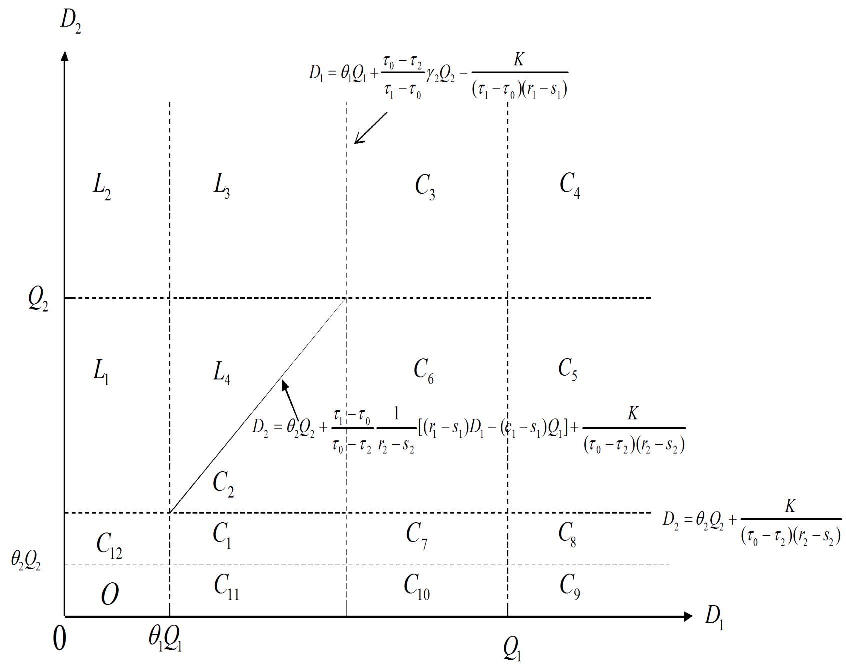

Figure 3 Demand Realization Regions

Proof of Proposition 5 Part (i) holds because, $Q_i(0)$ deviates further away from the optimal quantity Q_i^C than $\mathbf{Q}(\mathcal{D}_\mathbf{h})$ and $\mathbf{Q}(\mathcal{D}_\mathbf{l})$, respectively and the concavity of $P^C(\mathbf{Q})$. First, from definition (**??**), $P(\tau_0;\mathbf{Q})$ is non-increasing and continuous in τ_0 for $\tau_0 \in (\tau_2, \tau_1)$ and any $\mathbf{Q}$. Hence, $P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{h})), P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{l}))$ and $P(\tau_0;\mathbf{Q}^\mathbf{C})$ are all non-increasing in τ_0. At $\tau_0 = \tau_2$, $\mathbf{Q}^\mathbf{C} = \mathbf{Q}(\mathcal{D}_\mathbf{l})$, and $\mathcal{D}_l$ is suboptimal for all $\tau_0 > \tau_2$, so at $\tau_0 = \tau_2$, $P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{l})) = P(\mathbf{Q}^\mathbf{C}) > P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{h}))$. Similarly, at $\tau_0 = \tau_1, \mathbf{Q}^\mathbf{C} = \mathbf{Q}(\mathcal{D}_\mathbf{h})$; for $\tau_0 \in (\tau_2, \tau_1)$, $\mathbf{Q}(\mathcal{D}_\mathbf{h}))$ is suboptimal. Hence, $P(\tau_0;\mathbf{Q}^\mathbf{C}) = P(\tau_0, \mathbf{Q}(\mathcal{D}_\mathbf{l})) > P(\tau_0, \mathbf{Q}(\mathcal{D}_\mathbf{h}))$.

To show (ii), we next establish that $P(\tau_0;\mathbf{Q})$ is Lipschitz continuous in τ_0. From (**??**), for any $\tau_0^1, \tau_0^2 \in (\tau_2, \tau_1)$ with $\tau_0^1 \tau_0 2$,

$$\left| \frac{P(\tau_0^1;\mathbf{Q}) - P(\tau_0^2;\mathbf{Q})}{\tau_0^1 - \tau_0^2} \right| \leq (A_1^+ + A_2^+) \leq \sum_1^2 (r_i - c_i) Q_i.$$

The monotonicity and continuity of $P(\tau_0;\mathbf{Q})$ in τ_0 guarantee there exists a $\tilde{\tau}_0 \in (\tau_2, \tau_1)$, for $\tau_0 \in (\tau_2, \tilde{\tau}_0)$, $P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{l}))) > P(\tau_0;\mathbf{Q}(\mathcal{D}_\mathbf{h})))$ and the opposite holds for $\tau_0 \in (\tilde{\tau}_0, \tau_1)$. □

Proof of Proposition 6 This proposition is a direct result of Proposition 1.4 in Appendix 1.4. □

Proof of Propositions 7 and 8 The proof is embedded in the main body of the paper. □

3. Marginal Profit for the Extensions

3.1. FTC Carry-Forward

Under the revised ex ante L condition, i.e.,

$$(\tau_0 - \tau_2)(r_2 - c_2)Q_2 - (\tau_1 - \tau_0)(r_1 - c_1)Q_1 > K,$$

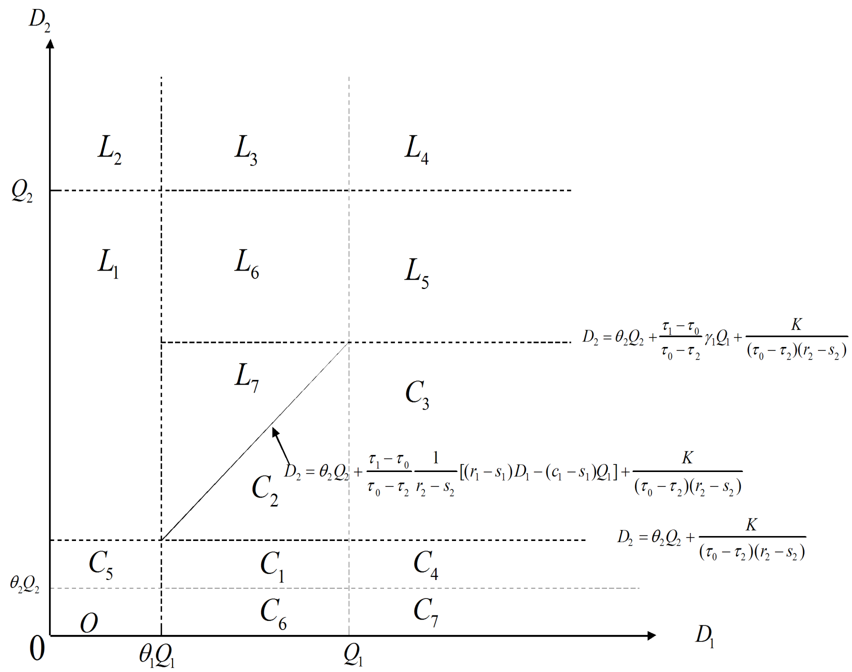

Figure 4 Demand Realization Regions

the demand realization space can be partitioned as in Figure 4. Following a similar analysis as in Section 4, the MNF's marginal expected profits with respect to Q_i are

$$\begin{aligned}\frac{\partial P_1^C(\mathbf{Q};K)}{\partial Q_1} &= (1-\tau_0)\{(r_1-c_1)Pr\{D_1>Q_1\}-(c_1-s_1)Pr\{\theta_1 Q_1<D_1<Q_1\}\} \\ &\quad -(c_1-s_1)Pr\{D_1<\theta_1 Q_1\} \\ &\quad -(\tau_1-\tau_0)\{(r_1-c_1)[\Delta(C_3)+\Delta(C_4)+\Delta(C_7)]-(c_1-s_1)[\Delta(C_1) \\ &\quad +\Delta(C_2)+\Delta(C_6)]\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{\partial P_1^C(\mathbf{Q};K)}{\partial Q_2} &= (1-\tau_0)\{(r_2-c_2)Pr\{D_2>Q_2\}-(c_2-s_2)Pr\{\theta_2 Q_2<D_2<Q_2\} \\ &\quad -(c_2-s_2)Pr\{D_2<\theta_2 Q_2\} \\ &\quad -(\tau_0-\tau_2)(c_2-s_2)[\Delta(C_1)+\Delta(C_2)+\Delta(C_3)+\Delta(C_4)+\Delta(C_5)].\end{aligned} \tag{24}$$

Under the revised ex ante condition C,

$$(\tau_1-\tau_0)(r_1-c_1)Q_1+K>(\tau_0-\tau_2)(r_2-c_2)Q_2,$$

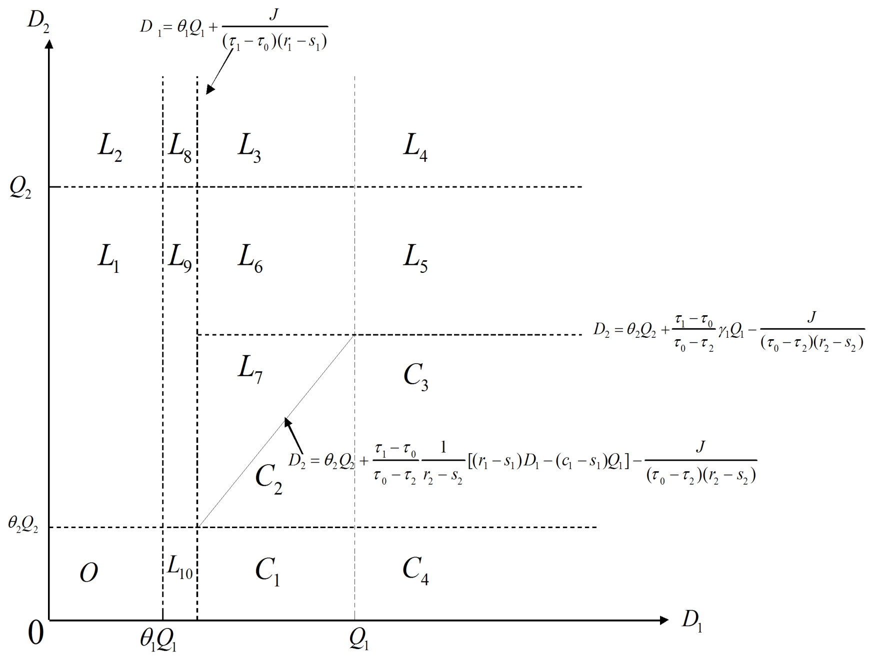

Figure 5 Demand Realization Regions

the demand realization space can be partitioned as in Figure 3. After a few transformations, the MNF's marginal profits can be written as

$$\begin{aligned}\frac{\partial P_1^C(\mathbf{Q};K)}{\partial Q_1} &= (1-\tau_1)\{(r_1-c_1)Pr\{D_1 > Q_1\} - (c_1-s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}\} \\ &\quad - (c_1-s_1)Pr\{D_1 < \theta_1 Q_1\} \\ &\quad - (\tau_1-\tau_0)(c_1-s_1)[\Delta(L_3)+\Delta(L_4)]\end{aligned} \tag{25}$$

and

$$\begin{aligned}\frac{\partial P_1^C(\mathbf{Q};K)}{\partial Q_2} &= (1-\tau_2)\{(r_2-c_2)Pr\{D_2 > Q_2\} - (c_2-s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}\} \\ &\quad - (c_2-s_2)Pr\{D_2 < \theta_2 Q_2\} \\ &\quad - (\tau_0-\tau_2)\{(r_2-c_2)[\Delta(L_2)+\Delta(L_3)] - (c_2-s_2)[\Delta(L_1)+\Delta(L_4)]\}.\end{aligned} \tag{26}$$

3.2. FTC Carry-Back

Under the revised **ex ante** L condition

$$(\tau_0-\tau_2)(r_2-c_2)Q_2 + J > (\tau_1-\tau_0)(r_1-c_1)Q_1,$$

the demand realization space can be described by Figure 5. Using Figure 5, the MNF's marginal expected profit with respect to Q_1 is

$$\begin{aligned}\frac{\partial P_2^C(\mathbf{Q};J)}{\partial Q_1} &= (1-\tau_0)[(r_1-c_1)Pr\{D_1 > Q_1\} - (c_1-s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}] \\ &\quad - (c_1-s_1)Pr\{D_1 < \theta_1 Q_1\} \\ &\quad - (\tau_1-\tau_0)[(r_1-c_1)[\Delta(C_3)+\Delta(C_4)] - (c_1-s_1)[\Delta(C_1)+\Delta(C_2)],\end{aligned} \tag{27}$$

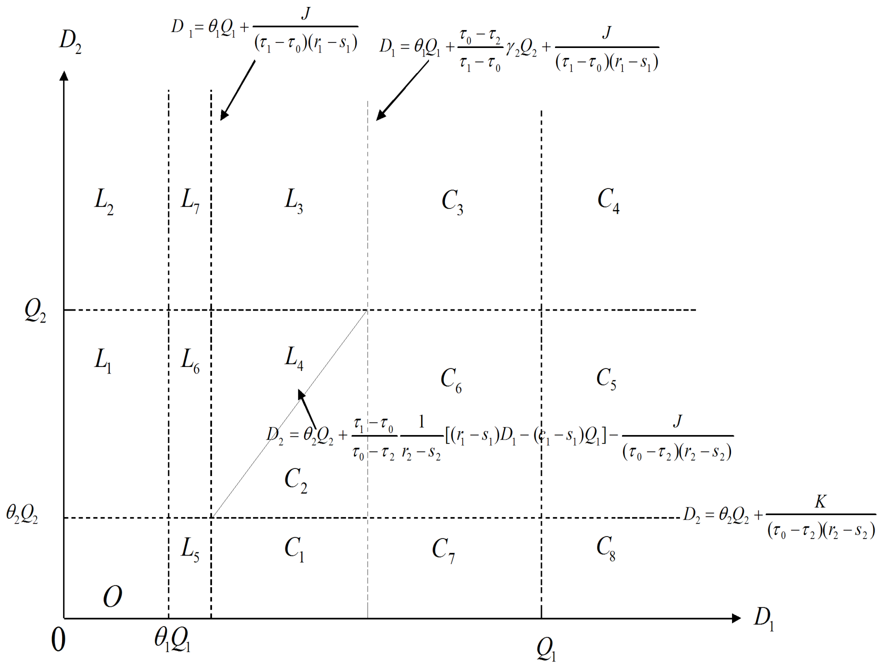

Figure 6 Demand Realization Regions

and that with respect to Q_2 is

$$\begin{aligned}\frac{\partial P_2^C(\mathbf{Q};J)}{\partial Q_2} = &(1-\tau_0)[(r_2-c_2)Pr\{D_2 > Q_2\} - (c_2-s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}\\ &-(c_2-s_2)Pr\{D_2 < \theta_2 Q_2\}\\ &-(\tau_0-\tau_2)(c_2-s_2)[\Delta(C_2)+\Delta(C_3)].\end{aligned} \tag{28}$$

Under the revised **ex ante** condition C

$$(\tau_0-\tau_2)(r_2-c_2)Q_2 + J < (\tau_1-\tau_0)(r_1-c_1)Q_1,$$

the demand space can be partitioned as in Figure 6. The MNF's marginal profits are as below:

$$\begin{aligned}\frac{\partial P_2^C(\mathbf{Q};J)}{\partial Q_1} = &(1-\tau_1)[(r_1-c_1)Pr\{D_1 > Q_1\} - (c_1-s_1)Pr\{\theta_1 Q_1 < D_1 < Q_1\}]\\ &-(c_1-s_1)Pr\{D_1 < \theta_1 Q_1\}\\ &+(\tau_1-\tau_0)[-(c_1-s_1)[\Delta(L_3)+\Delta(L_4)+\Delta(L_5)+\Delta(L_6)+\Delta(L_7)]\end{aligned} \tag{29}$$

and

$$\begin{aligned}\frac{\partial P_2^C(\mathbf{Q};J)}{\partial Q_2} = &(1-\tau_2)[(r_2-c_2)Pr\{D_2 > Q_2\} - (c_2-s_2)Pr\{\theta_2 Q_2 < D_2 < Q_2\}\\ &-(c_2-s_2)Pr\{D_2 < \theta_2 Q_2\}\\ &-(\tau_0-\tau_2)[(r_2-c_2)[\Delta(L_2)+\Delta(L_3)+\Delta(L_7)]\\ &-(c_2-s_2)[\Delta(L_1)+\Delta(L_4)+\Delta(L_6)].\end{aligned} \tag{30}$$

3.3. Loss Carry-Forward

The demand spaces partitions under the revised **ex ante** L and C conditions are shown in Figures 7 and 8, respectively. By comparing Figures 7 and 8 with Figures 1 and 2, respectively, it is clear that four of the boundary lines have shifted upward (or to the right) by a constant, $T_2/(r_2 + s_2)$. As a consequence, the marginal profits of the MNF will have the identical expressions as in Section 4, although the boundaries for some of regions have be adjusted by a constant. Hence, we omit the equations for the MNF's marginal profits here for brevity.

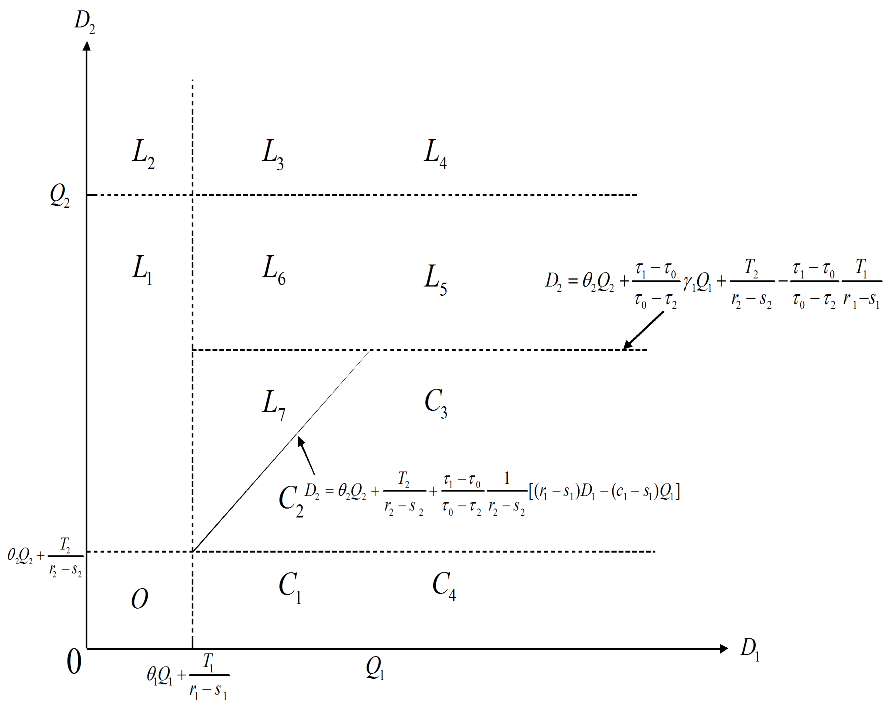

Figure 7 Demand Realization Regions

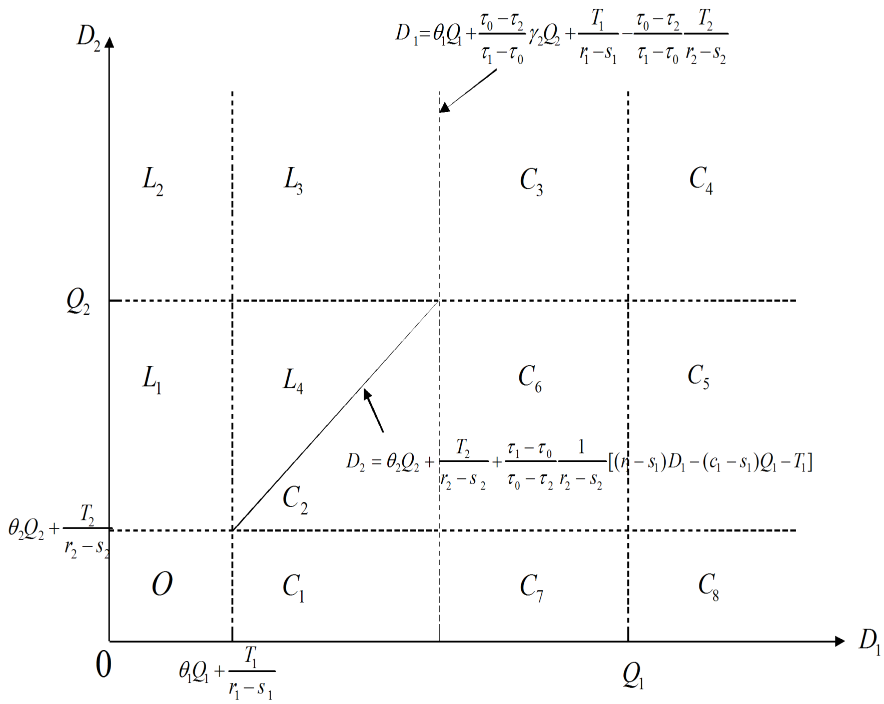

Figure 8 Demand Realization Regions

3.4. Loss Carry-Back

The global firm's after-tax profits with loss carry-back can be expressed as:

$$\Pi_4^C(\mathbf{Q};Y) = \Pi^C(\mathbf{Q}) + \tau_0 \min\{-A_2^-, Y\},$$

where $\Pi^C(\mathbf{Q})$, defined in Section 3, is the expected after-tax profit without loss carry-back consideration. Since the last term in of Π_4^C is independent of S_1's decision. Moreover, the tax cross-averaging effect and tax refund will not occur simultaneously. Let $P_4^C(\mathbf{Q}) \equiv E_{\mathbf{D}}\Pi_4^C(\mathbf{Q};Y))$. We have the following partial derivatives:

$$\frac{\partial P_4^C(\mathbf{Q};Y)}{\partial Q_1} = \frac{\partial P^C(\mathbf{Q})}{\partial Q_1},$$
$$\frac{\partial P_4^C(\mathbf{Q};Y)}{\partial Q_2} = \frac{\partial P^C(\mathbf{Q})}{\partial Q_2} + \tau_0(c_2 - s_2)Pr\{\theta_2 Q_2 > D_2 > \theta_2 Q_2 - \frac{Y}{r_2 - s_2}\}.$$