Analysis of Process Flexibility Designs under Disruptions

Erfan Mehmanchi, Hoda Bidkhori, Oleg A. Prokopyev
Department of Industrial Engineering, University of Pittsburgh, Pittsburgh, PA, 15261

Supplementary Material: Proofs

Proof of Lemma 2. Based on Problem (3) and Remark 1 under Assumption 2 ($\mathbf{c}^{(p)} = \mathbf{e}$) we have

$$\min_{\boldsymbol{g}\in\mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) = \min_{\mathbf{p},\mathbf{q},\mathbf{t},\boldsymbol{g}} \Big\{ \sum_{i\in I} g_i p_i \mid (3\text{b}) - (3\text{f}), \sum_{i\in I}(1-g_i) = \gamma, g_i \in \{0,1\}, \forall i \in I \Big\} \geqslant 0. \tag{A1}$$

Recall also that $\delta^{k,\ell}(\mathbf{e},\mathcal{D}) = \min_{\mathbf{p},\mathbf{q},\mathbf{t}}\{\sum_{i\in I} p_i \mid (3\text{b}) - (3\text{f})\}$.

If there exists a feasible solution for the set of constraints (3b) to (3f) such that $\sum_{i\in I} p_i \leqslant \gamma$, then clearly $\delta^{k,\ell}(\mathbf{e},\mathcal{D}) \leqslant \gamma$; additionally, for this feasible solution and every plant i such that $p_i = 1$ let $g_i = 0$, then $\min_{\boldsymbol{g}\in\mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) = \big(\delta^{k,\ell}(\mathbf{e},\mathcal{D}) - \gamma\big)^+ = 0$.

Now, suppose that $\sum_{i\in I} p_i > \gamma$ for all feasible solutions of the set of constraints (3b) to (3f). Then since $\sum_{i\in I}(1-g_i) = \gamma$, in any optimal solution of (A1), we have $g_i = 0$ exactly for γ plants that $p_i = 1$. Hence, all the optimal solutions of (A1) satisfy $p_i + g_i \geqslant 1$. Next, we show that $\min_{\boldsymbol{g}\in\mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) = \delta^{k,\ell}(\mathbf{e},\mathcal{D}) - \gamma$.

$$\begin{aligned}
&\min_{\boldsymbol{g}\in\mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g},\mathcal{D}) \\
&= \min_{\mathbf{p},\mathbf{q},\mathbf{t},\boldsymbol{g},\mathbf{y}} \Big\{ \sum_{i\in I} y_i \mid (3\text{b}) - (3\text{f}), \sum_{i\in I}(1-g_i) = \gamma, g_i \in \{0,1\}, \\
&\qquad\qquad y_i \leqslant g_i, y_i \leqslant p_i, y_i \geqslant p_i + g_i - 1, y_i \geqslant 0, \forall i \in I \Big\} && \text{(A2)} \\
&= \min_{\mathbf{p},\mathbf{q},\mathbf{t},\boldsymbol{g}} \Big\{ \sum_{i\in I} (p_i + g_i - 1)^+ \mid (3\text{b}) - (3\text{f}), \sum_{i\in I}(1-g_i) = \gamma, g_i \in \{0,1\}, \forall i \in I \Big\} && \text{(A3)} \\
&= \min_{\mathbf{p},\mathbf{q},\mathbf{t},\boldsymbol{g}} \Big\{ \sum_{i\in I} (p_i + g_i - 1) \mid (3\text{b}) - (3\text{f}), \sum_{i\in I}(1-g_i) = \gamma, g_i \in \{0,1\}, \forall i \in I \Big\} && \text{(A4)} \\
&= \min_{\mathbf{p},\mathbf{q},\mathbf{t}} \Big\{ \sum_{i\in I} p_i - \gamma \mid (3\text{b}) - (3\text{f}) \Big\} = \delta^{k,\ell}(\mathbf{e},\mathcal{D}) - \gamma. && \text{(A5)}
\end{aligned}$$

Equation (A2) holds by linearizing the bilinear terms $g_i p_i$, where $g_i, p_i \in \{0,1\}$ for all $i \in I$, using the standard techniques (see, e.g., Glover and Woolsey 1974) in the objective function of the optimization problem on the right-hand side of (A1). Equation (A3) holds since the optimization problem is in the minimization form; thus, the lower bounds of y are sufficient, i.e., $y_i \geqslant \max\{p_i + g_i - 1, 0\}$ for all $i \in I$. Equation (A4) is correct since $p_i + g_i \geqslant 1$ for any optimal solution. Finally, Equation (A5) holds by substitution of $\sum_{i\in I}(1-g_i) = \gamma$ in the objective function. $\square$

Proof of Lemma 3. To prove Relation (11) and Equations (12) and (13), separately for each one, we first derive $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)$. Then by applying Lemma 2, we obtain the desired result. Recall that the term $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)$ is PCID without any plant disruptions, i.e., the minimum number of plants (under the assumption $\mathbf{c}^{(p)}=\mathbf{e}$) required to create a vertex cover that includes k products after ignoring ℓ arcs.

Proof of Relation (11)***.*** We evaluate $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)$ for any $0\leqslant k\leqslant n$ and $0\leqslant \ell\leqslant n\cdot Q$. Design $\mathcal{LC}_Q$ has $n\cdot Q$ arcs, that exactly $k\cdot Q$ arcs are covered by k products. Among $Q\cdot(n-k)$ uncovered arcs, ℓ arcs are ignored by Equations (3c) and (3d). If $Q\cdot(n-k)\leqslant \ell$, then $n-k-\lfloor\frac{\ell}{Q}\rfloor\leqslant 0$. Thus, $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)=(n-k-\lfloor\frac{\ell}{Q}\rfloor)^{+}=0$.

Otherwise, there remain $Q\cdot(n-k)-\ell$ uncovered arcs. Since each plant can cover Q uncovered arcs at most, we have $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)\geqslant\frac{Q\cdot(n-k)-\ell}{Q}$. Moreover, since $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)\in\mathbb{Z}_+$ we have $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)\geqslant\lceil\frac{Q\cdot(n-k)-\ell}{Q}\rceil=n-k-\lfloor\frac{\ell}{Q}\rfloor$. Therefore, based on Lemma 2, we obtain that $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\ell}(\boldsymbol{g},\mathcal{LC}_Q)\geqslant(n-k-\lfloor\frac{\ell}{Q}\rfloor-\gamma)^{+}$ for any $0\leqslant k\leqslant n$ and $0\leqslant\ell\leqslant n\cdot Q$.

Proof of Equation (12)***.*** For $k=0$, the proof is trivial since by Remark 3 part (i) we know that $\delta^{0,0}(\mathbf{e},\mathcal{LC}_Q)=n$. Then we evaluate $\delta^{k,0}(\mathbf{e},\mathcal{LC}_Q)$ for $1\leqslant k\leqslant n-1$. Under the assumption $\mathbf{c}^{(p)}=\mathbf{e}$ by Remark 3 part (vi) we get $\delta^{k,0}(\mathbf{e},\mathcal{LC}_Q)=\min\limits_{S\subseteq B,|S|=k}|\mathcal{N}(B\setminus S,\mathcal{LC}_Q)|$. Evidently, on the basis of the definition of Q-long chain we have $|\mathcal{N}(V,\mathcal{LC}_Q)|\geqslant\min\{n,|V|+Q-1\}$ for any $V\subseteq B$, $V\neq\emptyset$. Hence, $\delta^{k,0}(\mathbf{e},\mathcal{LC}_Q)=\min\{n,|B\setminus S|+Q-1\}=\min\{n,n-k+Q-1\}$; this minimum value can be obtained by letting S be a set of products with consecutive indices.

Proof of Equation (13)***.*** First, note that by Inequality (11) and Lemma 2 we have

$$\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)\geqslant n-k-\lfloor\frac{\ell}{Q}\rfloor. \tag{A6}$$

Second, if $nQ\leqslant kQ+\ell$, then among nQ arcs of $\mathcal{LC}_Q$, exactly kQ arcs are covered by k products and the remaining arcs all are ignored; as a consequence, $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)=(n-k-\lfloor\frac{\ell}{Q}\rfloor)^{+}=0$. Then we evaluate $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_Q)$ for all $0\leqslant k\leqslant n$ and $(Q-1)^2\leqslant\ell\leqslant Q\cdot n$ such that $nQ>kQ+\ell$ for two cases $Q=2$ and $Q\geqslant 3$ separately as follows.

- Let $Q=2$, then in the following for any $0\leqslant k\leqslant n$ and $1\leqslant\ell\leqslant 2\cdot n$ such that $2n>2k+\ell$, we create a vertex cover necessitating $n-k-\lfloor\frac{\ell}{2}\rfloor>0$ plants. Thus, based on Inequality (A6), the created vertex cover is the minimum and we have $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_2)=n-k-\lfloor\frac{\ell}{2}\rfloor$.

  If $k=0$, then we temporarily put all n plants in the vertex cover. Next, by ignoring every two arcs connected to a plant in $\mathcal{LC}_2$, we can exclude exactly one plant from the vertex cover (in total, $\lfloor\frac{\ell}{2}\rfloor$ plants are excluded). Thus, $\delta^{0,\ell}(\mathbf{e},\mathcal{LC}_2)=n-0-\lfloor\frac{\ell}{2}\rfloor=n-\lfloor\frac{\ell}{2}\rfloor$.

  For $0<k<n$, let $S\subset B$, $|S|=k$ be a set of products with consecutive indices, e.g., $S=\{1,2,\ldots,k\}$. After putting S in the vertex cover, $2n-2k$ uncovered arcs remain. By the selection of S and the

structure of $\mathcal{LC}_2$ exactly two uncovered arcs emanate from $\mathcal{N}(S, \mathcal{LC}_2)$, i.e., arcs with an endpoint in $B \setminus S$. Next, consider two cases where ℓ is either even or odd.

Case 1: if ℓ is even, let us first ignore 2 uncovered arcs of $\mathcal{N}(S, \mathcal{LC}_2)$, and then (if $\ell > 2$) ignore $\ell - 2$ uncovered arcs which are connected to $\frac{\ell-2}{2}$ of plants in $A \setminus \mathcal{N}(S, \mathcal{LC}_2)$. Thus, there remain $2n - 2k - \ell$ uncovered arcs. The number of plants in $A \setminus \mathcal{N}(S, \mathcal{LC}_2)$ that are still connected to two uncovered arcs is $\frac{2n-2k-\ell}{2} = n - k - \frac{\ell}{2} = n - k - \lfloor \frac{\ell}{2} \rfloor$, and we need all of them to create a vertex cover with S. Therefore, by Inequality (A6) we get $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) = n - k - \lfloor \frac{\ell}{2} \rfloor$.

Case 2: if ℓ is odd, let us first ignore uncovered arc(s) of $\mathcal{N}(S, \mathcal{LC}_2)$, and then (if $\ell > 2$) ignore $\ell - 2$ uncovered arcs which are connected to $\lceil \frac{\ell-2}{2} \rceil$ of plants in $A \setminus \mathcal{N}(S, \mathcal{LC}_2)$. After ignoring ℓ arcs, all plants in $A \setminus \mathcal{N}(S, \mathcal{LC}_2)$ except one are connected to either 0 or 2 uncovered arcs and only one plant is connected to a single uncovered arc. Thus, $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) \leqslant \frac{2n-2k-\ell}{2} + 1$. Since $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) \in \mathbb{Z}_+$ we get $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) \leqslant \lfloor \frac{2n-2k-\ell}{2} + 1 \rfloor = n - k - \lfloor \frac{\ell}{2} \rfloor$. As a result, by Inequality (A6) we get $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) = n - k - \lfloor \frac{\ell}{2} \rfloor$.

Therefore, Equation (13) holds true for $Q = 2$ by using Lemma 2.

- Let $Q \geqslant 3$, then for any $0 \leqslant k \leqslant n$ and $(Q-1)^2 \leqslant \ell \leqslant Q \cdot n$ such that $nQ > kQ + \ell$ it suffices to demonstrate that there exist $S \subseteq B,\ |S| = k$ and $E \subseteq \mathcal{LC}_Q,\ |E| = \ell$ such that $|\mathcal{N}(B \setminus S, \mathcal{LC}_Q \setminus E)| = n - k - \lfloor \frac{\ell}{Q} \rfloor$. It implies that $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_Q) = n - k - \lfloor \frac{\ell}{Q} \rfloor$ due to Remark 3 part (vi) and Inequality (A6). Then by applying Lemma 2, we obtain the desired result, that is $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{LC}_Q) = (n - k - \lfloor \frac{\ell}{Q} \rfloor - \gamma)^+$.

  To this end, let S be a set of k products with consecutive indices, e.g., $S = \{1, \dots, k\}$ and $Z := B \setminus S = \{k+1, \dots, n\}$. Clearly, $|S| = k$ and $|Z| = n - k$. Put S in the vertex cover. Hence, all arcs connected to S are covered and all uncovered arcs have an endpoint in products of set Z. We define η_i as the number of uncovered arcs (with an endpoint in Z) connected to plant $i \in A$. It should be noted that $\eta_i > 0$ for all $i \in \mathcal{N}(Z, \mathcal{LC}_Q)$ and $\eta_i = 0$ for all $i \in A \setminus \mathcal{N}(Z, \mathcal{LC}_Q)$. Without excluding E from $\mathcal{LC}_Q$ the set $\mathcal{N}(Z, \mathcal{LC}_Q)$ is required to create a vertex cover along with S. We continue our discussion by considering two cases $|Z| \leqslant Q - 2$ and $|Z| \geqslant Q - 1$, separately.

  Case 1: If $|Z| \leqslant Q - 2$, then $|Z|Q \leqslant (Q-2)Q < (Q-1)^2 \leqslant \ell$. For any $\ell \geqslant (Q-1)^2$ let $E \subseteq \mathcal{LC}_Q$ such that $\{(i,j) \in \mathcal{LC}_Q \mid j \in Z\} \subseteq E$, and $|E| = \ell$. Clearly, $|\{(i,j) \in \mathcal{LC}_Q \mid j \in Z\}| = |Z|Q < |E| = \ell$ and E includes all $|Z|Q$ arcs connected to Z. Thus, $|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = 0$. It should be observed that $\ell > |Z|Q = (n-k)Q$; hence, $(n - k - \lfloor \frac{\ell}{Q} \rfloor)^+ = 0$. Therefore, there exist $S \subseteq B,\ |S| = k$ and $E \subseteq \mathcal{LC}_Q,\ |E| = \ell$ such that $|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = (n - k - \lfloor \frac{\ell}{Q} \rfloor)^+ = 0$.

  Case 2: If $|Z| \geqslant Q - 1$, then define $\tau_t = \big|\{i \in A \mid \eta_i = t\}\big|$, that is the number of plants with $\eta_i = t$, and $x = (|Z| + Q - 1 - n)^+$. By the definition of $\mathcal{LC}_Q$, we observe that either $\eta_i = 0$ or $\eta_i \geqslant x + 1$. Additionally, $\tau_{x+1} = x + 2$, $\tau_t = 2$ for $t \in \{x+2, x+3, \dots, Q-1\}$, and $\tau_Q = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2\big(Q - 1 - (1+x)\big) - (x+2) = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2Q + x + 2$. We first show that,

Fact 1. for any $T \in \{x+1, x+2, \ldots, Q-1\}$,

$$\sum_{t=x+1}^{T} (\tau_t \cdot t) - T = T^2. \tag{A7}$$

We prove Equality (A7) by induction on T. Let $T = x+1$, then since $\tau_{x+1} = x+2$ we get $\sum_{t=x+1}^{x+1}(\tau_t \cdot t) - (x+1) = (x+2)(x+1) - (x+1) = (x+1)^2$. Next, we need to prove that if Equality (A7) holds true for T, then it also holds true for $T+1$. Suppose that Equality (A7) is true for some $T \in \{x+1, x+2, \ldots, Q-2\}$, then by induction hypothesis,

$$\sum_{t=x+1}^{T} (\tau_t \cdot t) - T = T^2. \tag{A8}$$

Additionally, since $\tau_{T+1} = 2$ for $T \in \{x+1, x+2, \ldots, Q-2\}$, we have $\sum_{t=x+1}^{T+1}(\tau_t \cdot t) = \sum_{t=x+1}^{T}(\tau_t \cdot t) + 2(T+1)$; by using this equality and also Equality (A8), starting from the left-hand side of (A7) for $T+1$ we get

$$\sum_{t=x+1}^{T+1} (\tau_t \cdot t) - (T+1) = \sum_{t=x+1}^{T} (\tau_t \cdot t) + T + 1 = T^2 + T + T + 1 = (T+1)^2.$$

Therefore, Equality (A7) is valid for any $T \in \{x+1, x+2, \ldots, Q-1\}$.

Next, recall that $|\mathcal{N}(Z, \mathcal{LC}_Q)|$ plants are required to create a vertex cover along with S. By the definition of Q-long chain design and since Z includes products with consecutive indices, $|\mathcal{N}(Z, \mathcal{LC}_Q)| = \min\{n, |Z| + Q - 1\}$. Next, we continue the discussion for different values of $\ell \geqslant (Q-1)^2$.

- for $\ell = (Q-1)^2$ let $E_0 = \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, 0 < \eta_i \leqslant Q-1\} \setminus \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, \eta_i = Q-1, i \in I_1 \subset I,\ |I_1| = 1\}$. In fact, set E_0 includes uncovered arcs connected to plants $i \in \mathcal{N}(Z, \mathcal{LC}_Q)$ except those with $\eta_i = Q$, and one of two plants with $\eta_i = Q-1$, i.e., E_0 is the set of arcs with an endpoint in Z and connected to $|\mathcal{N}(Z, \mathcal{LC}_Q)| - \tau_Q - 1 = 2Q - x - 3$ plants with the smallest $\eta_i > 0$. Let $T = Q-1$ in Equality (A7). Then, by Fact 1 we have $|E_0| = \sum_{t=x+1}^{Q-1}(\tau_t \cdot t) - (Q-1) = (Q-1)^2 = \ell$. Thus, by excluding E_0 from $\mathcal{LC}_Q$ the number of plants required to create a vertex cover, $|\mathcal{N}(Z, \mathcal{LC}_Q)|$, along with S reduces by $2Q - x - 3$, i.e.,

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = |\mathcal{N}(Z, \mathcal{LC}_Q)| - (2Q - x - 3) = \min\{n, |Z| + Q - 1\} - (2Q - x - 3). \tag{A9}$$

If $|Z| + Q - 1 \geqslant n$, then $x = |Z| + Q - 1 - n$ and from Equality (A9),

$$\begin{aligned} |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| &= n - 2Q + (|Z| + Q - 1 - n) + 3 \\ &= |Z| - (Q-2) = |Z| - \lfloor \frac{(Q-1)^2}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor > 0; \end{aligned}$$

else, $x = 0$ and from Equality (A9),

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = (|Z| + Q - 1) - (2Q - 0 - 3) = |Z| - (Q - 2)$$
$$= |Z| - \lfloor \frac{(Q-1)^2}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor > 0.$$

As a consequence,

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = n - k - \lfloor \frac{\ell}{Q} \rfloor > 0. \tag{A10}$$

Note that the value of η_i for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)$, the remaining required plants for the vertex cover after ignoring arcs in E_0, is $Q-1, Q, Q, \ldots, Q$, i.e.,$\tau_t = 0$ for $t \leqslant Q-2$, $\tau_{Q-1} = 1$ and still $\tau_Q = |\mathcal{N}(Z, \mathcal{LC}_Q)| - 2Q + x + 2$.

- for $(Q-1)^2 < \ell < (Q-1)^2+(Q-1)$ let $E_1 = E_0 \cup \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, (i,j) \notin E_0\}$ such that $|E_1| = \ell$. Excluding E_1 from $\mathcal{LC}_Q$ does not remove any more plants from $\mathcal{N}(Z, \mathcal{LC}_Q)$ than by excluding E_0. Because $|E_1 \setminus E_0| < Q-1$, but $\eta_i = Q-1$ or Q for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)$. Thus, based on (A10), we get

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_1)| = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| = |Z| - \lfloor \frac{(Q-1)^2}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor > 0,$$

for any $(Q-1)^2 < \ell < (Q-1)^2 + (Q-1)$.

- for $\ell = (Q-1)^2+(Q-1)+t \cdot Q = Q(Q-1+t)$, where $t \in \mathbb{Z}_+ \cup \{0\}$. Let $E_2 = \cup_{i \in A}\{(i,j) \in \mathcal{LC}_Q \mid j \in Z, (i,j) \notin E_0, \eta_i = Q-1 \text{ or } \eta_i = Q\} \cup E_0$ such that $|E_2| = \ell$. Subsequently, excluding E_2 from $\mathcal{LC}_Q$ removes $t+1$ plant(s) from $\mathcal{N}(Z, \mathcal{LC}_Q)$ more than E_0, because $E_2 \setminus E_0$ includes arcs with an endpoint in Z and connected to a plant with $\eta_i = Q-1$ and t plants with $\eta_i = Q$. Hence, by (A10), we get

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_2)| = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_0)| - (t+1) = |Z| - \lfloor \frac{(Q-1)^2}{Q} \rfloor - (t+1)$$
$$= |Z| - \lfloor \frac{Q(Q-1+t)}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor, \tag{A11}$$

for $\ell = Q(Q-1+t)$. It should be noted that $\eta_i = Q$ for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_2)$, i.e., for the remaining plants to create the vertex cover after ignoring arcs in E_2.

- for $Q(Q-1+t) < \ell < Q(Q-1+t)+r$, where $t \in \mathbb{Z}_+ \cup \{0\}$ and $1 \leqslant r < Q$. Let $E_3 = E_2 \cup \{(i,j) \in \mathcal{LC}_Q \mid j \in Z, (i,j) \notin E_2\}$ such that $|E_3| = \ell$. It can be clearly seen that excluding E_3 from $\mathcal{LC}_Q$ removes no more plants from $\mathcal{N}(Z, \mathcal{LC}_Q)$ than excluding E_2, because $|E_3 \setminus E_2| = r < Q$, while $\eta_i = Q$ for $i \in \mathcal{N}(Z, \mathcal{LC}_Q \setminus E_2)$. Thus, by (A11) we have

$$|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_3)| = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E_2)| = |Z| - \lfloor \frac{Q(Q-1+t)}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor$$

for any $Q(Q-1+t) < \ell < Q(Q-1+t)+r$.

Therefore, there exist $S \subseteq B$, $|S| = k$ and $E \subseteq \mathcal{LC}_Q$, $|E| = \ell$ in a manner that $|\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = n - k - \lfloor \frac{\ell}{Q} \rfloor$ for $|Z| \geqslant (Q-1)$ and any $\ell \geqslant (Q-1)^2$.

Finally, according to the discussion above for $Q \geqslant 3$, Remark 3 part (vi), and Inequality (A6) we have $\delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_Q) = |\mathcal{N}(Z, \mathcal{LC}_Q \setminus E)| = n - k - \lfloor \frac{\ell}{Q} \rfloor$. Thus, Equation (13) holds true for $Q \geqslant 3$ by using Lemma 2. $\square$

Proof of Lemma 4. We prove Lemma 4 by using a double induction on z and ℓ in the following three steps including the base case, induction over z for $\ell = 1$, and induction over ℓ for fixed z.

Base case. We show that Lemma 4 is true for $z = 1$ and $\ell = 1$, i.e., there exist some $T \subseteq B$, $|T| = 1$ and $E \subseteq \mathcal{D}$, $|E| = 1$ such that $|\mathcal{N}(T, \mathcal{D} \setminus E)| \leqslant 1$. Toward this goal, for any $u \in B$, consider $T = \{u\}$. Note that there exist $a, a' \in A$ such that $|\mathcal{N}(u, \mathcal{D})| = |\{a, a'\}| = 2$. Let $E = \{(a, u)\}$, then $\big|\mathcal{N}\big(T, \mathcal{D} \setminus E\big)\big| = 1$.

Induction over z for $\ell = 1$. We need to prove that if Lemma 4 holds true for $z < |B|$ and $\ell = 1$, then it also holds true for $z + 1$ and $\ell = 1$ (note that $\lfloor \frac{\ell}{2} \rfloor = 0$ for $\ell = 1$). Suppose that Lemma 4 is true for some $z < |B|$ and $\ell = 1$, then by induction hypothesis, there exist sets $T^z \subset B$, $|T^z| = z$, and $E^z = \{(a, b)\} \subset \mathcal{D}$, $|E^z| = 1$ such that $|\mathcal{N}(T^z, \mathcal{D} \setminus E^z)| \leqslant z$. We consider the following two cases:

Case 1: let $E^z = \{(a, b)\} \in \mathcal{D} \cap \{\mathcal{N}(T^z, \mathcal{D}) \times T^z\}$. It should be noted that the vertices in $A \cup B$ over $\mathcal{D}$ form a connected graph, and $T^z \subset B$. Therefore, there exists some $v \in \mathcal{N}(T^z, \mathcal{D} \setminus E^z)$ and $v' \notin \mathcal{N}(T^z, \mathcal{D} \setminus E^z)$ such that (v, u) and (v', u) are arcs for some $u \notin T^z$. Let $T^{z+1} = T^z \cup \{u\}$ and $E^{z+1} = E^z$, subsequently, we get $|\mathcal{N}(T^{z+1}, \mathcal{D} \setminus E^{z+1})| \leqslant z + 1$.

Case 2: let $E^z = \{(a, b)\} \notin \mathcal{D} \cap \{\mathcal{N}(T^z, \mathcal{D}) \times T^z\}$. Since $\mathcal{D}$ is connected, there exists $u \in B \setminus T^z$ such that $\mathcal{N}(u, \mathcal{D}) \cap \mathcal{N}(T^z, \mathcal{D}) \neq \emptyset$. Thus, for $T^{z+1} = T^z \cup \{u\}$ and $E^{z+1} = E^z$ we get $|\mathcal{N}(T^{z+1}, \mathcal{D} \setminus E^{z+1})| \leqslant z+1$.

Induction over ℓ for fixed z. We need to prove that if Lemma 4 holds true for some $1 \leqslant z \leqslant |B|$ and $1 \leqslant \ell < 2n$, then it also holds true for z and $\ell + 1$. Suppose that Lemma 4 is true for some $1 \leqslant z \leqslant |B|$ and $1 \leqslant \ell < 2n$, then by induction hypothesis there exist sets $T^\ell \subseteq B$, $|T^\ell| = z$, and $E^\ell \subset \mathcal{D}$, $|E^\ell| = \ell$ such that $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| \leqslant (z - \lfloor \frac{\ell}{2} \rfloor)^+$. Construct $E^{\ell+1} = E^\ell \cup \{(a, b)\}$ such that $(a, b) \in \mathcal{D} \setminus E^\ell$ and $T^{\ell+1} = T^\ell$; subsequently, we consider the following three cases when $z - \lfloor \frac{\ell}{2} \rfloor > 0$ (the proof is trivial for $(z - \lfloor \frac{\ell}{2} \rfloor)^+ = 0$):

Case 1: if $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| < z - \lfloor \frac{\ell}{2} \rfloor$, then either $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = |\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)|$ or $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = |\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| - 1$. Similarly, either $z - \lfloor \frac{\ell+1}{2} \rfloor = z - \lfloor \frac{\ell}{2} \rfloor$ or $z - \lfloor \frac{\ell+1}{2} \rfloor = z - \lfloor \frac{\ell}{2} \rfloor - 1$. Therefore, we have $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| \leqslant z - \lfloor \frac{\ell+1}{2} \rfloor$.

Case 2: if $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| = z - \lfloor \frac{\ell}{2} \rfloor$ and ℓ is even, then either $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = |\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)|$ or $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = |\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| - 1$, but $z - \lfloor \frac{\ell+1}{2} \rfloor = z - \lfloor \frac{\ell}{2} \rfloor$. As a result, $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| \leqslant z - \lfloor \frac{\ell+1}{2} \rfloor$.

Case 3: if $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| = z - \lfloor \frac{\ell}{2} \rfloor$ and ℓ is odd, i.e., $\ell = 2t - 1,\ t \in \{1, 2, \ldots, n\}$, then $z - \lfloor \frac{\ell+1}{2} \rfloor = z - \lfloor \frac{\ell}{2} \rfloor - 1$. Next, we demonstrate that there exists $a \in \mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)$ which is connected to T^ℓ by only one arc:

Set $\mathcal{N}(T^\ell, \mathcal{D})$ is connected to T^ℓ by $2z$ arcs. Thus, all $i \in \mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)$ cannot be connected to T^ℓ by 2 or more than 2 arcs, i.e., there exists at least a plant $a \in \mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)$ that is connected to T^ℓ by only one arc. Otherwise, the number of arcs connecting T^ℓ to $\mathcal{N}(T^\ell, \mathcal{D})$ would be greater than $2z$ because

$$\begin{aligned} 2\big|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)\big| + \ell &= 2 \cdot \big(z - \lfloor \frac{2t-1}{2} \rfloor\big) + 2t - 1 \\ &= 2 \cdot (z - t + 1) + 2t - 1 > 2z, \qquad\qquad \forall t \in \{1, 2, \ldots, n\}. \end{aligned}$$

Therefore, by ignoring an odd number of arcs connecting T^ℓ to $\mathcal{N}(T^\ell, \mathcal{D})$ such that $|\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| = (z - \lfloor \frac{\ell}{2} \rfloor)^+$, there exists plant $a \in \mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)$ connected to T^ℓ by one arc. Let (a, b) be the arc connecting $b \in T^\ell$ to a and set $E^{\ell+1} = E^\ell \cup \{(a, b)\}$ and $T^{\ell+1} = T^\ell$. Thus, $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = |\mathcal{N}(T^\ell, \mathcal{D} \setminus E^\ell)| - 1$, and we have $|\mathcal{N}(T^{\ell+1}, \mathcal{D} \setminus E^{\ell+1})| = z - \lfloor \frac{\ell+1}{2} \rfloor$. □

Proof of Theorem 2. If we show that $\delta^{k,\ell}(\mathbf{e}, \mathcal{D}) \leqslant \delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2)$ for all $0 \leqslant k \leqslant n$ and $0 \leqslant \ell \leqslant 2n$, then by using Lemma 2 and Theorem 1 it is proved that Theorem 2 holds. If $\ell = 0$, we refer to Simchi-Levi and Wei (2015, Theorem 5). In addition, observe that if $2n \leqslant 2k + \ell$, then $\delta^{k,\ell}(\cdot, \cdot) = 0$. Thus, we only consider the case wherein $2k + \ell < 2n$.

For any $0 \leqslant k \leqslant n$ and $1 \leqslant \ell \leqslant 2n$ such that $2k + \ell < 2n$, it suffices to show that we can find some sets $S \subseteq B, |S| = k$ and $E \subseteq \mathcal{D}, |E| = \ell$ such that $|\mathcal{N}(B \setminus S, \mathcal{D} \setminus E)| \leqslant n - k - \lfloor \frac{\ell}{2} \rfloor$. Then based on Remark 3 part (vi), Equation (13), and Lemma 2,

$$|\mathcal{N}(B \setminus S, \mathcal{D} \setminus E)| = \delta^{k,\ell}(\mathbf{e}, \mathcal{D}) \leqslant \delta^{k,\ell}(\mathbf{e}, \mathcal{LC}_2) = n - k - \lfloor \frac{\ell}{2} \rfloor.$$

Assume that design $\mathcal{D}$ comprises c connected components named $\mathcal{D}_1, \ldots, \mathcal{D}_c$ such that $A_w \subset A$ and $B_w \subset B$, $w \in \{1, 2, \ldots, c\}$, denote the sets of plants and products of the w-th component, respectively. Without loss of generality, let us suppose that $|A_w| - |B_w|$ is nondecreasing with w. Because $\sum_{w=1}^{c}(|A_w| - |B_w|) = 0$, this assumption implies that $\sum_{w=1}^{t} |A_w| \leqslant \sum_{w=1}^{t} |B_w|$ for any $t \leqslant c$.

For any $0 \leqslant k < n$ and $1 \leqslant \ell < 2n$ such that $2k + \ell < 2n$, we have $n - k - \lfloor \frac{\ell}{2} \rfloor > 0$. Let $t_{k\ell}$ denote the largest possible t such that $\sum_{w=1}^{t} |B_w| + \lfloor \frac{\ell-1}{2} \rfloor < n - k$. By our choice of $t_{k\ell}$, we get $t_{k\ell} < c$ and $n - k - \sum_{w=1}^{t_{k\ell}} |B_w| - \lfloor \frac{\ell-1}{2} \rfloor \leqslant |B_{t_{k\ell}+1}|$. Moreover, define $T_0 \subseteq \bigcup_{w=t_{k\ell}+1}^{c} B_w$ with $|T_0| = \lfloor \frac{\ell-1}{2} \rfloor$, and $E_0 = \{(i, j) \in \mathcal{D} \mid j \in T_0\}$; hence, $|E_0| = 2\lfloor \frac{\ell-1}{2} \rfloor$ because $|\mathcal{N}(u, \mathcal{D})| = 2$ for all $u \in B$.

Based on Lemma 4, in the connected component $B_{t_{k\ell}+1}$, we can find some sets T_1 and E_1, where $T_1 \subseteq B_{t_{k\ell}+1}$, $|T_1| = n - k - \sum_{w=1}^{t_{k\ell}} |B_w| - \lfloor \frac{\ell-1}{2} \rfloor$, and $E_1 \subseteq \mathcal{N}(T_1, \mathcal{D}) \times T_1$, $|E_1| = \ell - |E_0| = (\ell - 2\lfloor \frac{\ell-1}{2} \rfloor) \in \{1, 2\}$ such that

$$|\mathcal{N}(T_1,\mathcal{D}\setminus E_1)| \leqslant n-k-\sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell-1}{2}\rfloor - \lfloor\frac{|E_1|}{2}\rfloor$$
$$= n-k-\sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell-1}{2}\rfloor - \lfloor\frac{\ell-2\lfloor\frac{\ell-1}{2}\rfloor}{2}\rfloor = n-k-\sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell}{2}\rfloor. \tag{A12}$$

Next, we select predefined T_0 such that $T_0 \cap T_1 = \emptyset$, this is possible because it can simply be verified that for $T_0 \subseteq \bigcup_{w=t_{k\ell+1}}^{c} B_w$ and $T_1 \subseteq B_{t_{k\ell+1}}$ we have $|T_0|+|T_1| \leqslant |\bigcup_{w=t_{k\ell+1}}^{c} B_w|$; by this selection we also have $E_0 \cap E_1 = \emptyset$. Finally, let $S := \bigcup\limits_{w=t_{k\ell+1}}^{c} B_w \setminus (T_0 \cup T_1)$, and $E = E_0 \cup E_1$; thus, $|E| = \ell$, and by (A12) we get

$$\big|\mathcal{N}(B\setminus S,\mathcal{D}\setminus E)\big| \leqslant \big|\mathcal{N}(T_1,\mathcal{D}\setminus E_1)\big| + \sum_{t=1}^{t_{k\ell}}|A_w|$$
$$\leqslant n-k-\sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell}{2}\rfloor + \sum_{w=1}^{t_{k\ell}}|B_w| = n-k-\lfloor\frac{\ell}{2}\rfloor.$$

We know that $|T_0| = \lfloor\frac{\ell-1}{2}\rfloor$, $S \subset B$ and

$$|S| = \big|\bigcup_{w=t_{k\ell+1}}^{c} B_w \setminus (T_0\cup T_1)\big| = n - \sum_{w=1}^{t_{k\ell}}|B_w| - |T_0| - |T_1|$$
$$= n - \sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell-1}{2}\rfloor - \big(n-k-\sum_{w=1}^{t_{k\ell}}|B_w| - \lfloor\frac{\ell-1}{2}\rfloor\big) = k.$$

Since $|S| = k$, thus $|B\setminus S| = n-k$, and the proof is complete. □

Proof of Theorem 3. Let $\hat{n}$ be the size of the smallest system for which there exists a counter example $\hat{\mathcal{D}}$ to the statement of Theorem 3. For $n=2$, $\mathcal{D}$ is the same as $\mathcal{LC}_2$; thus, we must have $\hat{n} \geqslant 3$. Moreover, we must have $2k+\ell < 2n$; otherwise $\delta^{k,\ell}(\cdot,\cdot) = 0$.

For $\ell = 0$ refer to Simchi-Levi and Wei (2015, Theorem 6). According to Equality (13) and Lemma 2, $\delta^{k,\ell}(\mathbf{e},\mathcal{LC}_2) = n-k-\lfloor\frac{\ell}{2}\rfloor$ for $2n > 2k+\ell$ and $\ell \geqslant 1$. Since $\hat{\mathcal{D}}$ is a counterexample, there exists some $0 \leqslant \hat{k} < \hat{n}$ and $1 \leqslant \hat{\ell} < 2\hat{n}$ such that $\delta^{\hat{k},\hat{\ell}}(\mathbf{e},\hat{\mathcal{D}}) > \hat{n}-\hat{k}-\lfloor\frac{\hat{\ell}}{2}\rfloor > 0$. Moreover, we can find $u \in B$ such that $|\mathcal{N}(u,\hat{\mathcal{D}})| = 1$; otherwise, by the proof of Theorem 2, we have $\delta^{k,\ell}(\mathbf{e},\mathcal{D}) \leqslant \delta^{k,\ell}(\mathbf{e},\mathcal{LC}_2) = n-k-\lfloor\frac{\ell}{2}\rfloor$ for all n, k and ℓ. Let $\{v\} = \mathcal{N}(u,\hat{\mathcal{D}})$. Since $\hat{\mathcal{D}}$ is connected, we have $|\mathcal{N}(v,\hat{\mathcal{D}})| \geqslant 2$.

Next, we define design $\mathcal{D}'$ with the set of plants and products $A\setminus\{v\}$ and $B\setminus\{u\}$, respectively, such that $\mathcal{D}' = \{(v',u') \mid (v',u') \in \hat{\mathcal{D}}, u' \neq u, v' \neq v\}$. Design $\mathcal{D}'$ is not necessarily connected. If $\mathcal{D}'$ has c components, then $|\mathcal{N}(v,\hat{\mathcal{D}})| \geqslant c+1$. By adding $c-1$ arcs to $\mathcal{D}'$ we can make it connected. Define $\mathcal{D}''$ as the arc set that contains $\mathcal{D}'$ and $c-1$ added arcs. Hence, $\mathcal{D}''$ is connected. It should be noted that $\mathcal{D}''$ is defined on a system with size $\hat{n}-1$ and $|\mathcal{D}''| \leqslant 2(\hat{n}-1)$.

Based on the minimality assumption on $\hat{n}$, $\delta^{\hat{k},\hat{\ell}}(\mathbf{e},\mathcal{D}'') \leqslant \hat{n}-\hat{k}-\lfloor\frac{\hat{\ell}}{2}\rfloor-1$. Thus, by Remark 3 part (vi), there exists some $S \subset B \setminus \{u\}$, $E \subset \mathcal{D}''$, $|E| = \hat{\ell}$, and $|S| = \hat{n} - \hat{k} - 1$ such that $|\mathcal{N}(S, \mathcal{D}'' \setminus E)| \leqslant \hat{n} - \hat{k} - \lfloor\frac{\hat{\ell}}{2}\rfloor - 1$. This implies that $S \cup \{u\} \subseteq B$, $|S \cup \{u\}| = \hat{n} - \hat{k}$ and $|\mathcal{N}(S \cup \{u\}, \hat{\mathcal{D}} \setminus E)| \leqslant \hat{n} - \hat{k} - \lfloor\frac{\hat{\ell}}{2}\rfloor$. Hence, by Remark 3 part (vi) we have $\delta^{\hat{k},\hat{\ell}}(\mathbf{e},\hat{\mathcal{D}}) \leqslant \hat{n} - \hat{k} - \lfloor\frac{\hat{\ell}}{2}\rfloor$. This contradicts the assumption that $\delta^{\hat{k},\hat{\ell}}(\mathbf{e},\hat{\mathcal{D}}) > \hat{n} - \hat{k} - \lfloor\frac{\hat{\ell}}{2}\rfloor$. $\square$

Proof of Lemma 5. Let $\{1,\ldots,c\}$ and $z_1,\ldots,z_c$ represent the components of $\mathcal{SC}_Q$ and their sizes, respectively. For each of Relation (14) and Equation (15), we first derive $\delta^{k,\ell}(\mathbf{e},\mathcal{SC}_Q)$, that is PCID without plant disruptions or (under the assumption $\mathbf{c}^{(p)} = \mathbf{e}$) the minimum required number of plants to create a vertex cover along with k products after ignoring ℓ arcs. Then by applying Lemma 2 we obtain the desired results.

Before we proceed, similar to the proof of Lemma 3 part 1, we can demonstrate that

$$n - k - \lfloor\frac{\ell}{Q}\rfloor \leqslant \delta^{k,\ell}(\mathbf{e},\mathcal{SC}_Q), \tag{A13}$$

for any $0 \leqslant k \leqslant n$ and $0 \leqslant \ell \leqslant n \cdot Q$.

***Proof of Relation* (14) *for* $\ell = 0$.** We consider two cases $k = \sum_{i\in I_1} z_i$ for some $I_1 \subseteq \{1,\ldots,c\}$ and $k \neq \sum_{i\in I} z_i$ for any $I \subseteq \{1,\ldots,c\}$, separately, as follows.

Case 1: let $k = \sum_{i\in I_1} z_i$ for some $I_1 \subseteq \{1,\ldots,c\}$; then we put all products of components in I_1 in the vertex cover. Clearly, all arcs of components in I_1 are covered. Thus, we need all plants of components in $\{1,\ldots,c\}\setminus I_1$ to create a vertex cover that is $n-\sum_{i\in I_1} z_i = n-k$. By Inequality (A13) we conclude that, for $\ell = 0$, this constructed vertex cover is the minimum one. Hence, $\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q) = n-k$. Additionally, by Equation (11) for $\mathcal{LC}_Q$ we have $n-k \leqslant \delta^{k,0}(\mathbf{e},\mathcal{LC}_Q)$. Thus,

$$n - k = \delta^{k,0}(\mathbf{e},\mathcal{SC}_Q) \leqslant \delta^{k,0}(\mathbf{e},\mathcal{LC}_Q). \tag{A14}$$

Case 2: let $k \neq \sum_{i\in I} z_i$ for any $I \subseteq \{1,\ldots,c\}$, then let $I_1 \subset \{1,\ldots,c\}$ be the largest subset of components such that $\sum_{i\in I_1} z_i < k$ and $k_1 := \sum_{i\in I_1} z_i$. Thus, there exists component $x \in \{1,\ldots,c\}\setminus I_1$ such that $z_x > k_2$, where $k_2 := k - k_1$. Now, we put all products of components of I_1 and k_2 products of component x into the vertex cover. Since component x is a Q-long chain, by Equation (12) we need $\min\{z_x, z_x - k_2 + Q - 1\}$ plants from component x for the vertex cover. Thus,

$$\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q) \leqslant n - k_1 - z_x + \min\{z_x, z_x - k_2 + Q - 1\} = \min\{n - k_1, n - k + Q - 1\}. \tag{A15}$$

Moreover, by Equation (12) for $\mathcal{LC}_Q$ we have

$$\delta^{k,0}(\mathbf{e},\mathcal{LC}_Q) = \min\{n, n - k + Q - 1\}. \tag{A16}$$

By (A15) and (A16), we get

$$\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q) \leqslant \delta^{k,0}(\mathbf{e}, \mathcal{LC}_Q) = \min\{n, n-k+Q-1\}. \tag{A17}$$

Therefore, by (A13), (A14), (A17) and Lemma 2 for $\ell = 0$ we have

$$(n-k-\gamma)^+ \leqslant \min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,0}(\boldsymbol{g}, \mathcal{SC}_Q) \leqslant \min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,0}(\boldsymbol{g}, \mathcal{LC}_Q).$$

Proof of Relation (15) ***for*** $(Q-1)^2 \leqslant \ell \leqslant n \cdot Q$**.** First, note that by Equation (13), we have $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{LC}_Q) = (n-k-\lfloor \frac{\ell}{Q} \rfloor - \gamma)^+$. Hence, we only need to demonstrate that $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{SC}_Q) = (n-k-\lfloor \frac{\ell}{Q} \rfloor - \gamma)^+$, as well. The proof is trivial if $kQ+\ell \geqslant nQ$ because in this case, $\min_{\boldsymbol{g} \in \mathcal{U}_p} \delta^{k,\ell}(\boldsymbol{g}, \mathcal{D}) = 0$ for any design $\mathcal{D}$; thus, in the following we let $kQ+\ell < nQ$. Next, we consider two cases $k = \sum_{i \in I_1} z_i$ for some $I_1 \subseteq \{1, \ldots, c\}$ and $k \neq \sum_{i \in I} z_i$ for any $I \subseteq \{1, \ldots, c\}$, separately, as follows.

Case 1: let $k = \sum_{i \in I_1} z_i$ for some $I_1 \subseteq \{1, \ldots, c\}$; then we put all products of components in I_1 in the vertex cover. Moreover, let I_2 denote the largest subset in $\{1, \ldots, c\} \setminus I_1$ such that $Q \sum_{i \in I_2} z_i < \ell$; then ignore all arcs in I_2. Thus, all arcs of components in $I_1 \cup I_2$ are either covered or ignored. Define $\ell_y = \ell - Q \sum_{i \in I_2} z_i$. Since $Q \sum_{i \in I_2} z_i < \ell$, we have $\ell_y > 0$. Hence, there exists component $y \in \{1, \ldots, c\} \setminus (I_1 \cup I_2)$ such that $z_y Q \geqslant \ell_y$. The minimum number of plants required from component y is $z_y - \lfloor \frac{\ell_y}{Q} \rfloor$. Because, no products of component y are included in the vertex cover and by ignoring each batch of Q arcs from y we can exclude only one plant of y from the vertex cover. Therefore, the total number of required plants is

$$\begin{aligned} & n - \sum_{i \in I_1} z_i - \sum_{i \in I_2} z_i - \lfloor \frac{\ell_y}{Q} \rfloor = \\ & n - k - \sum_{i \in I_2} z_i - \lfloor \frac{\ell - Q \sum_{i \in I_2} z_i}{Q} \rfloor = n - k - \lfloor \frac{\ell}{Q} \rfloor. \end{aligned}$$

By Inequality (A13), we conclude that the aforementioned constructed vertex cover is the minimum one, and we get $\delta^{k,\ell}(\mathbf{e}, \mathcal{SC}_Q) = n - k - \lfloor \frac{\ell}{Q} \rfloor$.

Case 2: let $k \neq \sum_{i \in I} z_i$ for any $I \subseteq \{1, \ldots, c\}$, then let $I_1 \subset \{1, \ldots, c\}$ be the largest subset of components such that $\sum_{i \in I_1} z_i < k$ and define $k_1 := \sum_{i \in I_1} z_i$. Next, we create a vertex cover in the following manner. Put all products of components in I_1 into the vertex cover (all arcs in I_1 are covered by k_1 products). There exists component $x \in \{1, \ldots, c\} \setminus I_1$ such that $z_x > k_2$, $k_2 := k - k_1$. We also put k_2 products of x (with consecutive indices) in the vertex cover. Observe that $Q(z_x - k_2)$ arcs of competent x remain uncovered. Then consider the two following cases where either $\ell \leqslant Q(z_x - k_2)$ or $\ell > Q(z_x - k_2)$.

If $\ell \leqslant Q(z_x - k_2)$, then since component x is a Q-long chain and $(Q-1)^2 \leqslant \ell$, by Equation (13), the number of plants required from x to put into the vertex cover is $z_x - k_2 - \lfloor \frac{\ell}{Q} \rfloor$. In addition, we need all plants in components $i \in \{1, \ldots, c\} \setminus (I_1 \cup \{x\})$ for the vertex cover. Thus, we need

$(n-k_1-z_x)+(z_x-k_2-\lfloor\frac{\ell}{Q}\rfloor) = n-k-\lfloor\frac{\ell}{Q}\rfloor$ plants along with the specified products to create a vertex cover. By Inequality (A13) we arrive at a minimum vertex cover. Therefore, $\delta^{k,\ell}(\mathbf{e},\mathcal{SC}_Q) = n-k-\lfloor\frac{\ell}{Q}\rfloor$.

If $\ell > Q(z_x-k_2)$, then ignore all uncovered arcs of x. Let I_2 denote the largest subset in $\{1,\ldots,c\}\setminus(I_1\cup\{x\})$ such that $Q\sum_{i\in I_2} z_i < \ell - Q(z_x-k_2)$. If so, we can ignore all arcs in I_2. Clearly, all arcs of components in $I_1\cup I_2\cup\{x\}$ are either covered or ignored.

Next, define $\ell_y := \ell - Q\sum_{i\in I_2} z_i - Q(z_x-k_2)$. We observe that $\ell_y > 0$ since $Q\sum_{i\in I_2} z_i < \ell - Q(z_x-k_2)$. Thus, there exists component $y\in\{1,\ldots,c\}\setminus(I_1\cup I_2\cup\{x\})$ such that $z_yQ \geqslant \ell_y$. It should be noted that no product of component y is in the vertex cover. Moreover, by ignoring each batch of Q arcs connected to a plant only one plant of y is excluded from the vertex cover. Thus, the required number of plants from component y in the vertex cover is $z_y - \lfloor\frac{\ell_y}{Q}\rfloor$. Therefore, by (A13) the minimum number of plants required to create a vertex cover is

$$\begin{aligned}\delta^{k,\ell}(\mathbf{e},\mathcal{SC}_Q) &= n - \sum_{i\in I_1} z_i - \sum_{i\in I_2} z_i - z_x - z_y + (z_y - \lfloor\frac{\ell_y}{Q}\rfloor)\\ &= n - k_1 - \sum_{i\in I_2} z_i - z_x - \lfloor\frac{\ell - Q\sum_{i\in I_2} z_i - Q(z_x-k_2)}{Q}\rfloor = n-k-\lfloor\frac{\ell}{Q}\rfloor.\end{aligned}$$

Therefore, based on Equation (13) and applying Lemma 2 for $\delta^{k,\ell}(\mathbf{e},\mathcal{SC}_Q)$ discussed above, we conclude that Equality (13) holds for any $0\leqslant k\leqslant n$ and $(Q-1)^2\leqslant \ell\leqslant n\cdot Q$. □

Proof of Proposition 3. If $\alpha=0$, then by Assumption 1 we have $\ell^\star=\alpha=0$ in Equation (10). Moreover, from Inequality (14) we have $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,0}(\boldsymbol{g},\mathcal{SC}_Q)\leqslant\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,0}(\boldsymbol{g},\mathcal{LC}_Q)$ at any $0\leqslant k\leqslant n$. Therefore, $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q)\leqslant R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q)$ by Theorem 1.

Similarly, if $\alpha\geqslant(Q-1)^2$, then $\ell^\star=\alpha\geqslant(Q-1)^2$. Furthermore, $\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{SC}_Q) = \min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{LC}_Q)$ by Equality (15). Consequently, from Theorem 1 we obtain $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q) = R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q)$ for any $(Q-1)^2\leqslant\alpha\leqslant n\cdot Q$. □

Proof of Proposition 5. Under Assumption 1, we conclude that $\ell^\star=\alpha$ in Equation (10) for any design $\mathcal{D}$. Moreover, for $\alpha\geqslant(Q-1)^2$ based on Equation (15), we have

$$\min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{SC}_Q^{(1)}) = \min_{\boldsymbol{g}\in\mathcal{U}_p}\delta^{k,\alpha}(\boldsymbol{g},\mathcal{SC}_Q^{(2)}) \qquad \forall k\in\{1,\ldots,n\};$$

the latter equation leads to $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q^{(1)}) = R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{SC}_Q^{(2)})$ by Theorem 1. Next, we ascertain the relationship between the performances without any disruptions, i.e., $R(\mathcal{U}_d,\mathcal{SC}_Q^{(1)})$ and $R(\mathcal{U}_d,\mathcal{SC}_Q^{(2)})$. To this end, if we demonstrate that $\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q^{(1)})\leqslant\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q^{(2)})$ holds true for any $0\leqslant k\leqslant n$, then we get $R(\mathcal{U}_d,\mathcal{SC}_Q^{(1)})\leqslant R(\mathcal{U}_d,\mathcal{SC}_Q^{(2)})$ according to Lemma 2 and Theorem 1.

Recall that $\delta^{k,0}(\mathbf{e},\mathcal{D})$ is the minimum number of plants that is required to create a vertex cover along with k products on design $\mathcal{D}$. Notably, based on Inequality (14) and Lemma 2 we have $\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q^{(t)})\geqslant n-k$, $t\in\{1,2\}$. It is clear that $\delta^{k,0}(\mathbf{e},\mathcal{SC}_Q^{(1)}) = \delta^{k,0}(\mathbf{e},\mathcal{SC}_Q^{(2)})$ for $k\in\{0,n\}$. Additionally, let

$z_1^{(1)}, \ldots, z_{c_{(1)}}^{(1)}$ and $z_1^{(2)}, \ldots, z_{c_{(2)}}^{(2)}$, denote the component sizes of $\mathcal{SC}_Q^{(1)}$ and $\mathcal{SC}_Q^{(2)}$, respectively. In order to evaluate $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(t)})$, $t \in \{1,2\}$ for $1 \leqslant k < n$, we consider the following four disjoint cases:

Case 1: there exist some $I_1 \subset \{1, \ldots, c_{(1)}\}$ and $I_2 \subset \{1, \ldots, c_{(2)}\}$, such that $\sum_{i \in I_1} z_i^{(1)} = \sum_{i \in I_2} z_i^{(2)} = k$. Thus, in $\mathcal{SC}_Q^{(t)}$, $t \in \{1,2\}$, in addition to k products of components in I_t, we need $n-k$ plants of components in $\{1, \ldots, c_{(t)}\} \backslash I_t$ to create a vertex cover. This is the minimum vertex cover for a particular k since by Inequality (14), we have $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(t)}) \geqslant n-k$, $t \in \{1,2\}$. Therefore, $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) = \delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(2)}) = n-k$.

Case 2: there exists $I_1 \subset \{1, \ldots, c_{(1)}\}$ such that $\sum_{i \in I_1} z_i^{(1)} = k$; however, there does not exist any $I_2 \subset \{1, \ldots, c_{(2)}\}$ such that $\sum_{i \in I_2} z_i^{(2)} = k$. Thus, by the argument given in above, we have $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) = n-k$. By Inequality (14) for any k, we have $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(2)}) \geqslant n-k$. Therefore, $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) \leqslant \delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(2)})$.

Case 3: the cases where there exists no $I_1 \subset \{1, \ldots, c_{(1)}\}$ such that $k = \sum_{i \in I_1} z_i^{(1)}$, but there is $I_2 \subset \{1, \ldots, c_{(2)}\}$ such that $k = \sum_{i \in I_2} z_i^{(2)}$, are not considered. These cases are impossible due to the fact that the components of $\mathcal{SC}_Q^{(1)}$ are decomposition of the components of $\mathcal{SC}_Q^{(2)}$.

Case 4: there does not exist any $I_1 \subset \{1, \ldots, c_{(1)}\}$ such that $k = \sum_{i \in I_1} z_i^{(1)}$. Likewise, there is no $I_2 \subset \{1, \ldots, c_{(2)}\}$ such that $k = \sum_{i \in I_2} z_i^{(2)}$.

Fact 1. Consider an arbitrary $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ with the set of components $\{1, \ldots, c\}$ and component sizes $\{z_1, \ldots, z_c\}$. For any given k, let $k_i \leqslant k$ represent the number of products of component $i \in \{1, \ldots, c\}$ in a vertex cover. Then we define J as the subset of components that have all of their products within the vertex cover ($k_i = z_i$ for all $i \in J$), and $k_J := \sum_{i \in J} z_i = \sum_{i \in J} k_i$. By the definition of J, it is clear that we have $0 \leqslant k_i < z_i$ for all $i \in \{1, \ldots, c\} \setminus J$. It should also be noted that we must have $k_J + \sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k$. By the assumption there does not exist any $I \subset \{1, \ldots, c\}$ such that $k = \sum_{i \in I} z_i$; it implies that $k_J < k$ and $\sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k - k_J > 0$.

We demonstrate that for any given k, a minimum vertex cover for $\mathcal{SC}_Q$ in $\{\mathcal{SC}_Q\}$ can be obtained when k_J has the maximum possible value and $\sum_{i \in \{1,\ldots,c\} \setminus J} \mathbb{1}_{\{0<k_i<z_i\}} = 1$; i.e., exactly one component has at least one ($0 < k_i$), but not all of its products ($k_i < z_i$) within the vertex cover.

It is observed that all arcs of the components in J are covered by k_J products. On the other hand, since each component $i \in \{1, \ldots, c\} \setminus J$ is a Q-long chain, by Equation (12) we require $\min\{z_i, z_i - k_i + Q - 1\}$ of its plants along with its k_i products to cover all its arcs. Thus, the total number of plants that are required to create a vertex cover is

$$
\begin{aligned}
&(n - k_J) - \sum_{i \in \{1,\ldots,c\} \setminus J} \big(z_i - \min\{z_i, z_i - k_i + Q - 1\}\big) \\
&= n - k_J + \sum_{i \in \{1,\ldots,c\} \setminus J} \min\{0, -k_i + Q - 1\}. \qquad \text{(A18)}
\end{aligned}
$$

To obtain a minimum vertex cover we need to minimize the right-hand side of (A18) that is strictly decreasing in k_J and is non-increasing in k_i for each $i \in \{1, \ldots, c\} \setminus J$. In particular, the unit increase of k_J and k_i decreases the right-hand side of (A18) by exactly 1 unit and at the most 1 unit, respectively. Hence, we set k_J at its maximum possible value.

Recall that $\sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k - k_J > 0$. The minimum quantum of the right-hand side of (A18) is obtained when along with the largest value of k_J, for exactly one component, say $t \in \{1, \ldots, c\} \setminus J$, $0 < k_t < z_t$, i.e., $k_t = k - k_J$ and $k_i = 0$ for all $i \in \{1, \ldots, c\} \setminus (J \cup \{t\})$. To elaborate it further, it should be observed that $\min\{0, -k_i + Q - 1\} = -k_i + \min\{k_i, Q - 1\}$. Then we get

$$\begin{aligned}\sum_{i \in \{1,\ldots,c\} \setminus J} (-k_i + \min\{k_i, Q-1\}) &= -(k - k_J) + \sum_{i \in \{1,\ldots,c\} \setminus J} \min\{k_i, Q-1\} \\ &\geqslant -k_t + \min\{k_t, Q-1\},\end{aligned}$$

or equivalently $\sum_{i \in \{1,\ldots,c\} \setminus J} \min\{k_i, Q-1\} \geqslant \min\{k_t, Q-1\}$ from the last inequality. This is true because if $k_i \leqslant Q - 1$ for all $i \in \{1, \ldots, c\} \setminus J$, then $\sum_{i \in \{1,\ldots,c\} \setminus J} \min\{k_i, Q-1\} = \sum_{i \in \{1,\ldots,c\} \setminus J} k_i = k - k_J \geqslant \min\{k_t, Q-1\}$. On the other hand, if $k_i > Q - 1$ for $i \in I$ where $I \subseteq \{1, \ldots, c\} \setminus J$, then $|I| \cdot (Q-1) + \sum_{i \in \{1,\ldots,c\} \setminus (J \cup I)} \min\{k_i, Q-1\} \geqslant \min\{k_t, Q-1\}$. It should be noted that there may exist multiple minimum vertex covers; we can obtain one of them in this manner.

Now, by using Fact 1, we create a minimum vertex cover in $\mathcal{SC}_Q^{(1)}$. Let J_1 be the largest subset of $\{1, \ldots, c_{(1)}\}$ such that $\sum_{i \in J_1} z_i^{(1)} \leqslant k$ and $k_{J_1} := \sum_{i \in J_1} z_i^{(1)}$. Next, select component $x \in \{1, \ldots, c_{(1)}\} \setminus J_1$ and $k_x := k - k_{J_1}$. Put all products of J_1 and k_x products of x within the vertex cover. By (A18), we can see that the minimum number of required plants is $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) = n - k_{J_1} + \min\{0, -k_x + Q - 1\} = \min\{n - k_{J_1}, n - k + Q - 1\}$.

Similarly, in $\mathcal{SC}_Q^{(2)}$ let J_2 denote the largest subset of $\{1, \ldots, c_{(2)}\}$ such that $\sum_{i \in J_2} z_i^{(2)} \leqslant k$ and $k_{J_2} := \sum_{i \in J_2} z_i^{(2)}$. Then we can select component $y \in \{1, \ldots, c_{(2)}\} \setminus J_2$ and $k_y := k - k_{J_2}$. By (A18), the minimum number of required plants is $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) = n - k_{J_2} + \min\{0, -k_y + Q - 1\} = \min\{n - k_{J_2}, n - k + Q - 1\}$.

It should be noted that $k_{J_1} \geqslant k_{J_2}$ because by assumption, the components of $\mathcal{SC}_Q^{(1)}$ are decomposition of $\mathcal{SC}_Q^{(2)}$ components. Thus, $\min\{n - k_{J_1}, n - k + Q - 1\} \leqslant \min\{n - k_{J_2}, n - k + Q - 1\}$. As a result, $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) \leqslant \delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(2)})$.

Therefore base on the discussion provided above we have $\delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(1)}) \leqslant \delta^{k,0}(\mathbf{e}, \mathcal{SC}_Q^{(2)})$, for all $0 \leqslant k \leqslant n$, and by using Lemma 2 and Theorem 1, we get $R(\mathcal{U}_d, \mathcal{SC}_Q^{(1)}) \leqslant R(\mathcal{U}_d, \mathcal{SC}_Q^{(2)})$. Recalling that $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(1)}) = R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q^{(2)})$, the proof is complete. □

Proof of Proposition 6. For general design $\mathcal{D}$, let ($k^{\circ}_{\mathcal{D}}$,$\ell^{\circ}_{\mathcal{D}}$,$\mathbf{d}^{\circ}_{\mathcal{D}}$) and ($k^{\star}_{\mathcal{D}}$,$\mathbf{d}^{\star}_{\mathcal{D}}$) denote optimal solutions of the optimization problems (10) and (19) – which represent the worst-case performances with and without disruptions – respectively. Since there are no arc disruptions ($\alpha = 0$), by Assumption 1, we have $\ell^{\circ}_{\mathcal{D}} = 0$. Next, we consider four disjoint cases on $k^{\star}_{\mathcal{SC}_Q}$ and $k^{\star}_{\mathcal{LC}_Q} \in \{0, \ldots, n\}$ including: $k^{\star}_{\mathcal{SC}_Q}, k^{\star}_{\mathcal{LC}_Q} < n$; $k^{\star}_{\mathcal{SC}_Q} < k^{\star}_{\mathcal{LC}_Q} = n$; $k^{\star}_{\mathcal{SC}_Q} = k^{\star}_{\mathcal{LC}_Q} = n$; and $k^{\star}_{\mathcal{LC}_Q} < k^{\star}_{\mathcal{SC}_Q} = n$.

- Let $k^{\star}_{\mathcal{SC}_Q} < n$ and $k^{\star}_{\mathcal{LC}_Q} < n$. Any vertex cover of general design $\mathcal{D}$ can be represented by sets $S \subseteq B$ and $\mathcal{N}(B \setminus S, \mathcal{D})$. In addition, let $S^{\star} \subset B$, $|S^{\star}| = k^{\star}_{\mathcal{SC}_Q} < n$ and $\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q) > 0$ denote a minimum vertex cover of $\mathcal{SC}_Q$ without disruptions. Hence, by Remark 3 part (vi), we have $\delta^{k^{\star}_{\mathcal{SC}_Q},0}(\mathbf{e}, \mathcal{SC}_Q) = |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)|$ and by Equation (19), we get

$$R(\mathcal{U}_d, \mathcal{SC}_Q) = |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)| + \sum_{j=1}^{k^{\star}_{\mathcal{SC}_Q}} \min{}^j(\mathbf{d}^{\star}_{\mathcal{SC}_Q}) = |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)| + \min_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^{k^{\star}_{\mathcal{SC}_Q}} d_j. \tag{A19}$$

  The second equality in (A19) follows by fixing $k^{\star}_{\mathcal{SC}_Q}$ in Equation (19). A single plant disruption makes the capacity of a plant zero. We claim that sets $S^{\star}$ and $\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)$ are also a minimum vertex cover of $\mathcal{SC}_Q$ with a plant disruption. If our claim holds true, then $|S^{\star}| = k^{\star}_{\mathcal{SC}_Q} = k^{\circ}_{\mathcal{SC}_Q}$; as a consequence, by Equation (10) we have

$$\begin{aligned} R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q) &= (\delta^{k^{\star}_{\mathcal{SC}_Q},0}(\mathbf{e}, \mathcal{SC}_Q) - 1)^{+} + \sum_{j=1}^{k^{\star}_{\mathcal{SC}_Q}} \min{}^j(\mathbf{d}^{\circ}_{\mathcal{SC}_Q}) \\ &= |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)| - 1 + \min_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^{k^{\star}_{\mathcal{SC}_Q}} d_j. \end{aligned} \tag{A20}$$

  The second equality in (A20) is obtained by fixing $k^{\star}_{\mathcal{SC}_Q}$ in Equation (10), and due to the fact that $\delta^{k^{\star}_{\mathcal{SC}_Q},0}(\mathbf{e}, \mathcal{SC}_Q) = |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)| > 0$. Thus, by considering (A19) and (A20), we get $Fr(\mathcal{SC}_Q) = R(\mathcal{U}_d, \mathcal{SC}_Q) - R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q) = 1$.

  We prove the validity of our claim by contradiction. Suppose $S^{\star} \cup \mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)$ is the minimum vertex cover of $\mathcal{SC}_Q$ without the disruption, but it is not the minimum vertex cover of $\mathcal{SC}_Q$ with the disruption, and sets $\bar{S} \subseteq B$, $|\bar{S}| = k^{\circ}_{\mathcal{SC}_Q}$, and $\mathcal{N}(B \setminus \bar{S}, \mathcal{SC}_Q)$ corresponds to the minimum vertex cover of $\mathcal{SC}_Q$ with the disruption. In this case,

$$R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q) = \Big(|\mathcal{N}(B \setminus \bar{S}, \mathcal{SC}_Q)| - 1\Big)^{+} + \min_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^{k^{\circ}_{\mathcal{SC}_Q}} d_j < |\mathcal{N}(B \setminus S^{\star}, \mathcal{SC}_Q)| - 1 + \min_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^{k^{\star}_{\mathcal{SC}_Q}} d_j.$$

  As a result, by (A19)

$$|\mathcal{N}(B \setminus \bar{S}, \mathcal{SC}_Q)| + \min_{\mathbf{d} \in \mathcal{U}_d} \sum_{j=1}^{k^{\circ}_{\mathcal{SC}_Q}} d_j < R(\mathcal{U}_d, \mathcal{SC}_Q). \tag{A21}$$

Both $S^\star \cup \mathcal{N}(B \backslash S^\star, \mathcal{SC}_Q)$ and $\bar{S} \cup \mathcal{N}(B \backslash \bar{S}, \mathcal{SC}_Q)$ represent vertex covers for $\mathcal{SC}_Q$. Thus, Relation (A21) contradicts the minimality of vertex cover $S^\star \cup \mathcal{N}(B \setminus S^\star, \mathcal{SC}_Q)$ for $\mathcal{SC}_Q$ without disruption.

Indeed, the proof above holds for any design $\mathcal{D}$ (i.e., if $k^\star_{\mathcal{D}} < n$, then $Fr(\mathcal{D}) = 1$) since we did not exploit the structure of $\mathcal{SC}_Q$. Therefore, $Fr(\mathcal{LC}_Q) = Fr(\mathcal{SC}_Q) = 1$ when $k^\star_{\mathcal{SC}_Q} < n$ and $k^\star_{\mathcal{LC}_Q} < n$.

- Let $k^\star_{\mathcal{SC}_Q} < k^\star_{\mathcal{LC}_Q} = n$, then by the discussion above, we have $Fr(\mathcal{SC}_Q) = 1$ because $k^\star_{\mathcal{SC}_Q} < n$. In addition, by Equation (19) we have $R(\mathcal{U}_d, \mathcal{LC}_Q) = \sum_{j=1}^n d^\star_{j,\mathcal{LC}_Q}$. Next, we evaluate $Fr(\mathcal{LC}_Q)$ to prove that $Fr(\mathcal{LC}_Q) \leqslant 1$. Toward this goal, let B_1 denote the set of products whose demands are larger than 1 in $\mathbf{d}^\circ_{\mathcal{LC}_Q}$, i.e., $B_1 = \{j \in B \mid d^\circ_{j,\mathcal{LC}_Q} > 1\}$ and $A_1 := \mathcal{N}(B_1, \mathcal{LC}_Q)$. Clearly, $|B_1| \leqslant |A_1|$ and set $(B \setminus B_1) \cup A_1$ is a vertex cover for $\mathcal{LC}_Q$.

Fact 1. The inequality

$$|A_1| + \sum_{j=1}^{n-|B_1|} \min{}^j(\mathbf{d}^\circ_{\mathcal{LC}_Q}) \geqslant \sum_{j=1}^{n} d^\star_{j,\mathcal{LC}_Q}, \tag{A22}$$

is true since $(B \setminus B_1) \cup A_1$ creates a vertex cover in $\mathcal{LC}_Q$ that corresponds to the feasible solution $(k = n - |B_1|, \mathbf{d}^\circ_{\mathcal{LC}_Q})$ with the objective function value $|A_1| + \sum_{j=1}^{n-|B_1|} \min^j(\mathbf{d}^\circ_{\mathcal{LC}_Q})$ for Equation (19). If (A22) does not hold true, then for this feasible solution we have $|A_1| + \sum_{j=1}^{n-|B_1|} \min^j(\mathbf{d}^\circ_{\mathcal{LC}_Q}) < \sum_{j=1}^{n} d^\star_{j,\mathcal{LC}_Q} = R(\mathcal{U}_d, \mathcal{LC}_Q)$, that contradicts the optimality of $(k^\star_{\mathcal{LC}_Q} = n, \mathbf{d}^\star_{\mathcal{LC}_Q})$.

Fact 2. By Equation (10), we have $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) = \left(\delta^{k^\circ_{\mathcal{LC}_Q},0}(\mathbf{e}, \mathcal{LC}_Q) - 1\right)^+ + \sum_{j=1}^{k^\circ_{\mathcal{LC}_Q}} \min^j(\mathbf{d}^\circ_{\mathcal{LC}_Q})$. For any $k \in \{0, \dots, n\}$, we define $G(k) = G_1(k) + G_2(k)$, where $G_1(k) = \left(\delta^{k,0}(\mathbf{e}, \mathcal{LC}_Q) - 1\right)^+$, and $G_2(k) = \sum_{j=1}^{k} \min^j(\mathbf{d}^\circ_{\mathcal{LC}_Q})$. Observe that $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) = \min_{0 \leqslant k \leqslant n} G(k)$.

Evidently, $G_1(k)$ and $G_2(k)$ are non-increasing and non-decreasing functions of k, respectively. Moreover, each value of k corresponds to a vertex cover that is associated with $G(k)$ with the total capacity $G_1(k)$ and the total demand $G_2(k)$, respectively. Note that $G_1(k)$ is minimized when products with consecutive indices are selected in the vertex cover; hence, without loss of generality, for each k we suppose that the vertex cover includes product set $S = \{1, 2, \dots, k\}$ and plant set $\mathcal{N}(B \setminus S, \mathcal{LC}_Q)$.

Next, we demonstrate that the value of any local minimum of $G(k)$ is the same as $G(\bar{k})$ at one of the points $\bar{k} \in \{0, n - |B_1|, n\}$. As a result, it is sufficient to compute $G(k)$ only at $k \in \{0, n - |B_1|, n\}$ and consider the minimum one as $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q)$.

Specifically, define k' as the largest value of k such that $G_1(k) = n - 1$ for all $k \leqslant k'$. If $k' \leqslant n - |B_1| - 1$, then we show that

(a) $G(n - |B_1|) < G(n - |B_1| + 1) < \ldots < G(n - 1)$

(b) $G(k') \geqslant \ldots \geqslant G(n-|B_1|-2) \geqslant G(n-|B_1|-1)$

(c) $G(0) \leqslant G(1) \leqslant \ldots \leqslant G(k')$

Hence, $k=0$ is a local minimum of $G(k)$. Moreover, if $G(n-|B_1|-1) \geqslant G(n-|B_1|)$, then $k=n-|B_1|$ is a also local minimum.

In case $k' \geqslant n-|B_1|$, we only need to demonstrate that

(d) $G(0) \leqslant G(1) \leqslant \ldots \leqslant G(k') < \ldots < G(n-1)$

Thus, $k=0$ is a local minimum of $G(k)$. It should be noted that for $k=n$, if $G(n-1) \geqslant G(n)$, then $k=n$ is also a local minimum. In the following, we prove Relations (a) to (d).

Relation (a) holds true for any $k \in \{n-|B_1|+1,\ldots,n-1\}$ because $G_1(k-1)-G_1(k)=1$, and $G_2(k-1)-G_2(k) = -\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) < -1$; therefore, $G(k-1)-G(k)<0$. To elaborate further, utilizing the structure of $\mathcal{LC}_Q$, product k $(k>k')$ has a different neighbor plant from product $k-1$; i.e., $G_1(k-1)-G_1(k) = |\mathcal{N}(k,\mathcal{LC}_Q) \setminus \mathcal{N}(k-1,\mathcal{LC}_Q)| = 1$. Furthermore, by the definition of set B_1 we have $\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) > 1$ for $n-|B_1| < k \leqslant n$. As a consequence, $G_2(k-1)-G_2(k) < -1$.

Relation (b) holds true for any $k \in \{k',\ldots,n-|B_1|\}$ because $G_1(k-1)-G_1(k)=1$ and $G_2(k-1)-G_2(k) = -\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) \geqslant -1$; therefore $G(k-1)-G(k) \geqslant 0$. By the structure of $\mathcal{LC}_Q$, product k $(k>k')$ has one different neighbor plant from product $k-1$, i.e., $G_1(k-1)-G_1(k) = |\mathcal{N}(k,\mathcal{LC}_Q) \setminus \mathcal{N}(k-1,\mathcal{LC}_Q)| = 1$. Furthermore, since $\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) \leqslant 1$ for $k \leqslant n-|B_1|$, we have $G_2(k-1)-G_2(k) = -\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) \geqslant -1$.

Relation (c) holds true for any $k \in \{0,\ldots,k'\}$ because $G_1(k-1)-G_1(k) = 0$ and $G_2(k-1)-G_2(k) = -\min^k(\mathbf{d}^\circ_{\mathcal{LC}_Q}) \leqslant 0$; therefore, $G(k-1)-G(k) \leqslant 0$. We get $G_1(k-1)-G_1(k) = 0$ because $G_1(k) = n-1$ for all $k \leqslant k'$. Since there are no negative demands, it follows that $G_2(k-1)-G_2(k) \leqslant 0$.

Relation (d) holds true since Relations (a) and (c) are true. If $k' \geqslant n-|B_1|$, we have $G(k-1) < G(k)$ for $k \in \{k'+1,\ldots,n-1\}$ by Relation (a) and $G(k-1) \leqslant G(k)$ for $k \in \{0,\ldots,k'\}$ by Relation (c).

It should be noted that $G(k)$ may have local minimums other than set $\{0, n-|B_1|, n\}$. However, on the basis of Fact 2, any local minimum takes the value of $G(k)$ at one of the points $\{0, n-|B_1|, n\}$. By considering Fact 2 and since $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q) = \min_{0\leqslant k\leqslant n} G(k)$, in the following we evaluate $Fr(\mathcal{LC}_Q)$ only for $k^\circ_{\mathcal{LC}_Q} \in \{0, n-|B_1|, n\}$. Recall that it is supposed $k^\star_{\mathcal{LC}_Q} = n$.

Let $k^\circ_{\mathcal{LC}_Q} = n$, then we have $k^\star_{\mathcal{LC}_Q} = k^\circ_{\mathcal{LC}_Q} = n$. Using Equations (10) and (19), we obtain $R(\mathcal{U}_d,\mathcal{LC}_Q) = \sum_{j=1}^n d^\star_{j,\mathcal{LC}_Q}$ and $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q) = \sum_{j=1}^n d^\circ_{j,\mathcal{LC}_Q}$, respectively. Thus, $R(\mathcal{U}_d,\mathcal{LC}_Q) = R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q)$ because it is clear that $R(\mathcal{U}_d,\mathcal{U}_p,\mathcal{U}_a,\mathcal{LC}_Q) \leqslant R(\mathcal{U}_d,\mathcal{LC}_Q) \leqslant \sum_{j=1}^n d^\circ_{j,\mathcal{LC}_Q}$ and since $(k^\star_{\mathcal{LC}_Q} = n, \mathbf{d}^\circ_{j,\mathcal{LC}_Q})$ is a feasible solution for (19). Hence, $Fr(\mathcal{LC}_Q) = 0$. Recall that $Fr(\mathcal{SC}_Q) = 1$ because $k^\star_{\mathcal{LC}_Q} < n$. Therefore, $Fr(\mathcal{LC}_Q) \leqslant Fr(\mathcal{SC}_Q) = 1$.

Let $k^{\circ}_{\mathcal{LC}_Q} = n-|B_1|$, then the corresponding minimum vertex cover for $k^{\circ}_{\mathcal{LC}_Q} = n-|B_1|$ is $A_1 \cup (B \setminus B_1)$. By Equation (10) we have $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) = |A_1| - 1 + \sum_{j=1}^{n-|B_1|} \min^j(\mathbf{d}^{\circ}_{\mathcal{LC}_Q})$. According to the definition of fragility coupled with Relation (A22) in Fact 1, we get

$$Fr(\mathcal{LC}_Q) = R(\mathcal{U}_d, \mathcal{LC}_Q) - R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{LC}_Q} - \big(|A_1| - 1 + \sum_{j=1}^{n-|B_1|} \min^j(\mathbf{d}^{\circ}_{\mathcal{LC}_Q})\big) \leqslant 1.$$

Therefore, $Fr(\mathcal{LC}_Q) \leqslant Fr(\mathcal{SC}_Q) = 1$.

Let $k^{\circ}_{\mathcal{LC}_Q} = 0$, then $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q) = n - 1$ by Equation (10). Importantly, $R(\mathcal{U}_d, \mathcal{LC}_Q) = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{LC}_Q} \leqslant n$ because $k^{\star}_{\mathcal{LC}_Q} = n$; otherwise, one can select all plants as the vertex cover and as a result $R(\mathcal{U}_d, \mathcal{LC}_Q) = n$. Hence,

$$Fr(\mathcal{LC}_Q) = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{LC}_Q} - (n-1) \leqslant 1.$$

Therefore, $Fr(\mathcal{LC}_Q) \leqslant Fr(\mathcal{SC}_Q) = 1$ when $k^{\star}_{\mathcal{SC}_Q} < k^{\star}_{\mathcal{LC}_Q} = n$.

- Let $k^{\star}_{\mathcal{SC}_Q} = k^{\star}_{\mathcal{LC}_Q} = n$, then $R(\mathcal{U}_d, \mathcal{SC}_Q) = R(\mathcal{U}_d, \mathcal{LC}_Q) = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q} = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{LC}_Q}$. On the other hand, based on (16) we have $R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{SC}_Q) \leqslant R(\mathcal{U}_d, \mathcal{U}_p, \mathcal{U}_a, \mathcal{LC}_Q)$. Therefore, $Fr(\mathcal{LC}_Q) \leqslant Fr(\mathcal{SC}_Q)$.

- Let $k^{\star}_{\mathcal{LC}_Q} < k^{\star}_{\mathcal{SC}_Q} = n$. Since $k^{\star}_{\mathcal{SC}_Q} = n$, based on Equation (19), we get $R(\mathcal{U}_d, \mathcal{SC}_Q) = \sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q}$. Then by considering Inequality (14) and Theorem 1, we have $R(\mathcal{U}_d, \mathcal{SC}_Q) \leqslant R(\mathcal{U}_d, \mathcal{LC}_Q)$. Hence, $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q} \leqslant R(\mathcal{U}_d, \mathcal{LC}_Q)$. Additionally, note that $(k = n, \mathbf{d}^{\star}_{\mathcal{SC}_Q})$ is a feasible solution of (19) for $\mathcal{LC}_Q$ with the objective function value of $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q}$.

  If $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q} < R(\mathcal{U}_d, \mathcal{LC}_Q)$, then we have a feasible solution $(k = n, \mathbf{d}^{\star}_{\mathcal{SC}_Q})$ with the objective function value $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q}$ being less than the optimal value $R(\mathcal{U}_d, \mathcal{LC}_Q)$; hence, $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q} < R(\mathcal{U}_d, \mathcal{LC}_Q)$ cannot occur. If $\sum_{j=1}^{n} d^{\star}_{j,\mathcal{SC}_Q} = R(\mathcal{U}_d, \mathcal{LC}_Q)$, then $(k = n, \mathbf{d}^{\star}_{\mathcal{SC}_Q})$ is an optimal solution for (19). Thus, $k^{\star}_{\mathcal{LC}_Q} = k^{\star}_{\mathcal{SC}_Q} = n$ and refer to the corresponding discussion above for this case.

Therefore, on the basis of the discussions above, Proposition 6 is proved. □