On-line Supplement for “An analytical investigation of alternative batching policies for remanufacturing under stochastic demands and returns” by “Yi Zhang, Elif Akçalı, and Sıla Çetinkaya"

Proof of Properties 1 to 5

Properties 1 to 5 can be proved using the following assumptions/observations:

- $Y_i \sim \exp(r)$, $Z_i \sim \text{Gamma}\,(i,r)$, $W(t) \sim \text{Poisson}(rt)$, $X_i \sim \exp(a)$, $S_i \sim \text{Gamma}\,(i,a)$, and $N(t) \sim \text{Poisson}(at)$.
- X_1 and S_{Q_D} are stopping times[§] for $\{N(t), t > 0\}$.
- Y_1 and Z_{Q_R} are stopping times for $\{W(t), t > 0\}$.

Proof of Property 6

Since $N(t) - W(t) \sim \text{Normal}((a-r)t, \sqrt{(a+r)t})$, we have

$$\lim_{t\to\infty} \frac{E\,[B(t)]}{t} = \lim_{t\to\infty} \frac{E\left[(N(t)-W(t))^+\right]}{t} \approx \lim_{t\to\infty} \frac{\int_0^\infty \frac{z}{\sqrt{2\pi}\sqrt{(a+r)t}} e^{-\frac{(z-(a-r)t)^2}{2(a+r)t}}\,dz}{t}.$$

Letting $v = \frac{z-(a-r)t}{\sqrt{(a+r)t}}$ in the right hand side of the above expression, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_0^\infty \frac{z}{\sqrt{2\pi}\sqrt{(a+r)t}} e^{-\frac{(z-(a-r)t)^2}{2(a+r)t}}\,dz}{t} &= \lim_{t\to\infty} \frac{\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^\infty \frac{\sqrt{(a+r)t}v+(a-r)t}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv}{t} \\
&= \lim_{t\to\infty} \frac{\sqrt{(a+r)t}\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^\infty \frac{v}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv}{t} + \lim_{t\to\infty} \frac{(a-r)t\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv}{t} \\
&= \lim_{t\to\infty} \frac{\sqrt{(a+r)}\int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^\infty \frac{v}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv}{\sqrt{t}} + \lim_{t\to\infty} (a-r) \int_{-\frac{(a-r)\sqrt{t}}{\sqrt{a+r}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv \\
&= 0 \cdot \int_{-\infty}^\infty \frac{v}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv + (a-r) \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv \\
&= a-r.
\end{aligned}$$

Proof of Observation 1

1. The proof is straightforward using (13), (30) and (35), and, hence, it is omitted.

[§] A random variable, e.g., X_1, is a stopping time with respect to the process $\{N(t), t > 0\}$ if for every $t \geq 0$, the event $[X_1 \leq t]$ is determined by the process up to time t ([23], Page 504).

2. Recall (18), (20), (24), and (26). Now, observe that, in order to show $E\left[CL\left(\hat{T}_R\right)\right] > E\left[CL\left(\hat{T}_D\right)\right]$, it is sufficient to show that $B - A > 0$. That is,

$$\begin{aligned}
B - A &= \frac{2(wa+hr) + 2r^2K + a(w+h)}{r^2(wa+hr)} - \frac{2(wa+hr) + 2a^2K - a(w+h)}{a^2(wa+hr)} \\
&= \frac{2a^2(wa+hr) + 2a^2r^2K + a^3(w+h)}{a^2r^2(wa+hr)} - \frac{2r^2(wa+hr) + 2a^2r^2K - ar^2(w+h)}{a^2r^2(wa+hr)} \\
&= \frac{2(a^2-r^2)(wa+hr) + a(a^2+r^2)(w+h)}{a^2r^2(wa+hr)}.
\end{aligned}$$

Recalling $r < a$, we conclude that $B - A > 0$.

3. The proof is straightforward using (13) and (24), and, hence, it is omitted.

4. Recall (13) and (18) and observe that in order to show $E\left[CL\left(\hat{T}_F\right)\right] < E\left[CL\left(\hat{T}_D\right)\right]$, we need to verify

$$\sqrt{\frac{2}{a^2} + \frac{2K - \frac{w+h}{a}}{wa+hr}} - \frac{1}{a} + \frac{1}{a} > \sqrt{\frac{2K}{wa+hr}} \Rightarrow \frac{2}{a^2} + \frac{2K}{wa+hr} - \frac{w+h}{a(wa+hr)} > \frac{2K}{wa+hr} \Rightarrow \frac{wa+h(2r-a)}{a^2(wa+hr)} > 0.$$

Rearranging the terms of the last inequality above, it then follows that if $\frac{r}{a} > \frac{1}{2}\left(1 - \frac{w}{h}\right)$ then $E\left[CL\left(\hat{T}_F\right)\right] < E\left[CL\left(\hat{T}_D\right)\right]$.

5. Recall that, by assumption, $r < a$. Considering this assumption along with $w > h$, the condition $\frac{r}{a} > \frac{1}{2}\left(1 - \frac{w}{h}\right)$ in Part 4 of Observation 1 is immediately satisfied. Hence, the result is an immediate consequence of Parts 1–4 of Observation 1.

6. It follows from the proof of Part 5 of Observation 1 that if $w < h$ then the condition $\frac{r}{a} > \frac{1}{2}\left(1 - \frac{w}{h}\right)$ in Part 4 of Observation 1 may or may not be satisfied. However, it is straightforward to show that if $w < h$ and $\frac{a}{2} < r < a$ then we still have $\frac{r}{a} > \frac{1}{2}\left(1 - \frac{w}{h}\right)$ so that the result follows as an immediate consequence of Part 5 of Observation 1.

7. The result is an immediate consequence of the assumption that $r < a$ and Parts 1–6 of Observation 1.

Proof of Observation 2

Let us recall the closed-form expressions of $E\left[CL\left(\hat{T}_F\right)\right]$, $E\left[CL\left(\hat{T}_D\right)\right]$, $E\left[CL\left(\hat{T}_R\right)\right]$, $E\left[CL\left(\hat{Q}_D\right)\right]$, and $E\left[CL\left(\hat{Q}_R\right)\right]$ given by (13), (18), (24), (30), and (35), respectively, along with the definitions of A and B given by (20) and (26), respectively. Now, using (20) and (26) and recalling $r < a$, one can easily verify that

$$\frac{\partial A}{\partial w} = \frac{-2a^2K + h(a-r)}{a^3(wa+hr)^2} < 0 \text{ if } \frac{K}{h} > \frac{1}{2a^2}(a-r); \tag{60}$$

$$\frac{\partial A}{\partial h} \frac{-2arK - w(a-r)}{a^3(wa+hr)^2} < 0; \tag{61}$$

$$\frac{\partial B}{\partial w} = \frac{-2r^2K - ha(a-r)}{ar^6(wa+hr)^2} < 0; \tag{62}$$

$$\frac{\partial B}{\partial h} = \frac{-2r^3K + wa^2(a-r)}{r^6(wa+hr)^2} < 0 \text{ if } \frac{K}{w} > \frac{a^2}{2r^3}(a-r). \tag{63}$$

The results in Table 3 can now be verified in a straightforward fashion by utilizing the closed-form expressions (13), (18), (24), (30), and (35) along with the results (60)–(63) above.

Proof of Observation 3

The proof builds on the argument that if one can identify the conditions under which I_n (the number of used-items in inventory at the end of remanufacturing cycle n, which we also refer as the supply overage quantity) can be safely omitted in cost and policy parameter computations then the proposed approach offers solid approximations. Hence, in proving the following specific parts, we examine the cases either the expected supply overage quantity is negligible (i.e., $E[I_n] < 1$) or a supply overage is highly unlikely (i.e., $P(R_n \geq D_n)$). More specifically, parts 1, 2, 3, and 4 of Observation 3 rely on expressions (10), (15), (21), and (27), while parts 5, 6, and 7 rely on Properties 1, 4, and 5.

1. Recalling (10), we are interested in the parametric setting where

$$E\left[I_n\right] \leq \frac{1+\frac{r}{a}}{2(1-\frac{r}{a})} < 1.$$

Rearranging the terms of the above inequality, we have the condition $r/a < 1/3$.

2., 3., 4. The proofs are straightforward, and, hence they are omitted.

5. Recall from Property 1 that $D_n \sim \text{Poisson}(aT_F)$, $R_n \sim \text{Poisson}(rT_F)$, $E[D_n] = \text{Var}(D_n) = aT_F$, and $E[R_n] = \text{Var}(R_n) = rT_F$. Considering that a Poisson random variable with a large arrival rate can be effectively approximated with a Normal random variable (see [3], Page 40), we let $D_n \sim \text{Normal}(aT_F, \sqrt{aT_F})$ and $R_n \sim \text{Normal}(rT_F, \sqrt{rT_F})$. Now, recalling the well-known property of a Normal random variable which implies that about 99.7% of its possible values lie within three standard deviations of the mean, we argue that $P(R_n \geq D_n) \approx 0$ when the difference between $E[D_n]$ and $E[R_n]$ exceeds three times the sum of $\sqrt{\text{Var}(D_n)}$ and $\sqrt{\text{Var}(R_n)}$, as illustrated in Figure 4. That is, if

$$aT_F - rT_F \geq 3\left(\sqrt{aT_F} + \sqrt{rT_F}\right)$$

then $P(R_n \geq D_n) \approx 0$. Rearranging the terms of the above inequality completes the proof.

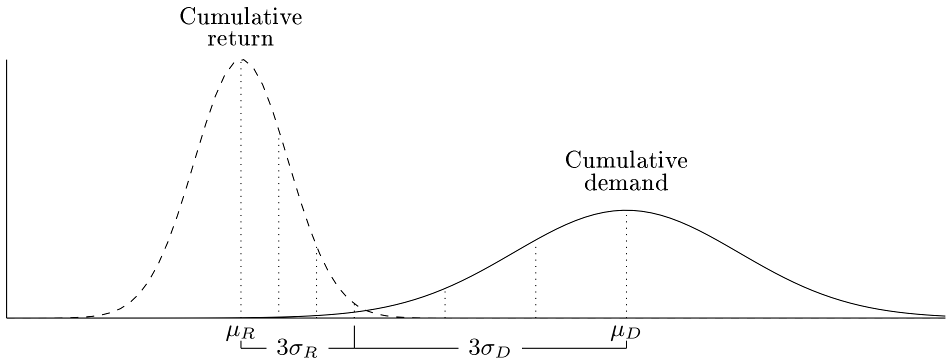

Figure 4: Normal distribution approximations for cumulative demand and cumulative return where μ_D and σ_D represent the mean and standard deviation of the cumulative demand distribution, respectively, and μ_R and σ_R represent the mean and standard deviation of the cumulative return distribution, respectively.

6. Recalling Property 2 and relying on the idea introduced in proof of Part 5 above, we consider the difference between Q_D and $E[R_n]$. That is, if

$$Q_D - \frac{rQ_D}{a} > 3\sqrt{\frac{rQ_D}{a} + \frac{r^2Q_D}{a^2}}$$

then $P(R_n \geq Q_D) \approx 0$. Rearranging the terms of the above inequality completes the proof.

7. The proof is similar to the proofs of Parts 5 and 6, and, hence, it is omitted.