Online Supplement for "Two-Stage Facility Location Problems with Restricted Recourse" by E. Koca, N. Noyan, and H. Yaman

A Additional Notes on Conditional Value-at-Risk

It is well-known that there exists an optimal solution of (2), where η is equal to the α-quantile, also known as the value-at-risk at confidence level α, denoted by $\mathrm{VaR}_\alpha(\Xi)$. Accordingly, the w variables provide the excess values with respect to the threshold of $\mathrm{VaR}_\alpha(\Xi)$. Without loss of generality assume that $\xi^1 \leq \xi^1 \leq \cdots \leq \xi^{|S|}$, and let $s^* = \min\{s \in S \ : \ \sum_{s=1}^{s^*} p^s \geq \alpha\}$. Then, there exists an optimal solution, where $\eta = \mathrm{VaR}_\alpha(\Xi) = \xi^{s^*}$ and the optimal objective function value is calculated as

$$\xi^{s^*} + \frac{1}{1-\alpha}\sum_{s^*+1}^{|S|} p^s(\xi^s - \xi^{s^*}) = \frac{1}{1-\alpha}((1-\alpha-\sum_{s^*+1}^{|S|} p^s)\xi^{s^*} + \sum_{s^*+1}^{|S|} p^s\xi^s).$$

This expression shows that CVaR of a random variable corresponds to a weighted sum of the least favorable outcomes that are larger than or equal to the α-quantile of that random variable. This interpretation can be easily derived from the following knapsack type representation of CVaR, which is equivalent to the linear programming dual of (2):

$$\mathrm{CVaR}_\alpha(\Xi) = \max\left\{\frac{1}{1-\alpha}\sum_{s\in S} \upsilon^s\xi^s \ : \ \sum_{s\in S}\upsilon^s = 1-\alpha, \quad 0 \leq \upsilon^s \leq p^s \ \ \forall\, s \in S\right\}.$$

B Illustrative Examples

Example 1. *Consider a setting with four demand nodes and two open facilities, and suppose that the pre-allocation decisions, and the allocation decisions along with the demand values under a particular scenario $s \in S$ are given as in Table 4. By definition of $\boldsymbol{\zeta}^{(1)}$ and*

Table 4: Illustrative example for calculating the dispersion measures of our particular choice.

Demand node (i)	x_{i1}	x_{i2}	y^s_{i1}	y^s_{i2}	d^s_i	$\zeta^{(2)}_{si1} = d^s_i\lvert y^s_{i1} - x_{i1}\rvert$	$\zeta^{(2)}_{si2} = d^s_i\lvert y^s_{i2} - x_{i2}\rvert$	$\zeta^{(1)}_{si} = \zeta^{(2)}_{si1} + \zeta^{(2)}_{si2}$
1	0.5	0.5	1	0	120	60	60	120
2	0.5	0.5	0.3	0.7	200	40	40	80
3	0.5	0.5	0	1	300	150	150	300
4	0.5	0.5	0.2	0.8	400	120	120	240

$\boldsymbol{\zeta}^{(2)}$, we obtain $\boldsymbol{\zeta}^{(1)}_s = (120, 80, 300, 240)^\top$ and $\boldsymbol{\zeta}^{(2)}_s = (60, 40, 150, 120, 60, 40, 150, 120)^\top$. Setting $\bar{\alpha} = 0.50$ we specify $k' = \lfloor |I|(1-\bar{\alpha})\rfloor = 2$ and $k = \lfloor |I||J|(1-\bar{\alpha})\rfloor = 4$, and then averaging the largest two (resp., four) components of $\boldsymbol{\zeta}^{(1)}_s$ (resp., $\boldsymbol{\zeta}^{(2)}_s$) provides us with $\|\boldsymbol{\zeta}^{(1)}_s\|_{\mathrm{CVaR}_{0.5}} = (300+240)/2 = 270$ and $\|\boldsymbol{\zeta}^{(2)}_s\|_{\mathrm{CVaR}_{0.5}} = (150+150+120+120)/4 = 135$.

Example 2. *Let us consider a simple setting with a single demand point (node 1) having a deterministic demand of 100 and two candidate locations for the facilities with a fixed setup cost of 1000. Suppose that the unit cost of allocating the demand to facility j is a random variable with two equally likely outcomes: $c^s_{1j} = 1$ if $j = s$, and $c^s_{1j} = 40$ otherwise, for $j,\ s = 1, 2$. It is easy to see that, for the underlying unrestricted problem, there exists an optimal solution where each demand point is fully assigned to the closest open facility under each scenario; we refer to this structure as the "single-sourcing" property. Accordingly, for our simple problem instance, the optimal solution of the underlying unrestricted model is to locate a facility at both candidate locations, and allocate the demand to the facilities at node 1 and node 2 under scenarios 1 and 2, respectively. The associated expected total cost is calculated as* $1000*2+(0.5*100*1+0.5*100*1) = 2100$. *Here, our main concern is the nervousness caused by the significant difference in the scenario-dependent allocation decisions; each demand is fully served by two different facilities under two scenarios. We cannot know in advance which scenario will occur in the future, and therefore, it is not straightforward to take preparedness actions for such very different potential allocation plans. One possible option is to make preparedness decisions based on the expected demand to be satisfied by each facility (50 for each facility in our example). However, even in this case, there is fifty percent deviation in allocation decisions under each scenario, leading to a nervousness problem in the system.*

Following our modeling approach, we introduce the scenario-independent pre-allocation decisions (x_{11} and x_{12}) and opt for limiting the deviations of the scenario-dependent allocation vectors ($(y^s_{11}, y^s_{12}),\ s = 1, 2$) from the pre-determined counterpart ((x_{11}, x_{12})). To this end, suppose that we use the $|I|$-dimensional deviation vector $\boldsymbol{\zeta}^{(1)}$, whose single component takes the value of $\zeta^{(1)}_{s1} = d_1(|y^s_{11} - x_{11}| + |y^s_{12} - x_{12}|)$ under scenario $s = 1, 2$. For this one-dimensional vector, setting $\alpha = 0.90$, the corresponding constraint (6) simply becomes $\mathrm{CVaR}_{0.9}(\zeta^{(1)}_1(\varpi)) \leq \epsilon$. The threshold value ϵ depends on the deviation tolerance of the decision maker: if it is very small, a scenario-independent solution might be observed, on the other hand, if it is sufficiently large, the optimal solution of the unrestricted model might be obtained. For example, we obtain the following solutions for this problem instance under different threshold values:

- *For a sufficiently small value of ϵ: the optimal solution is to locate only one of the facilities, and assign the demand point to that facility under each scenario. The associated expected total cost is calculated as* $1000+0.5*100*1+0.5*100*40 = 3050$. *In this case, we have the single-sourcing structure for the decisions in both stages.*

- *For a sufficiently large value of ϵ: the optimal solution coincides with that of the underlying unrestricted model (discussed above) with the following pre-determined allocation decisions $x_{11} = x_{12} = 0.5$. The associated expected total cost is equal to 2100.*

In this case, we have the single-sourcing structure only for the second-stage decisions.

- *For the ϵ values between the previously listed two limiting ones, we in general do not have the single-sourcing structure in any stage. The optimal solution suggests to open both facilities with the following additional allocation decisions: $x_{11} = 0.5 - a$, $x_{12} = 0.5 + a$, $y^1_{11} = 2a$, $y^2_{12} = 1 - 2a$ for $0 \leq a \leq 0.50$. The value of the variable a depends on the threshold parameter ϵ. The associated expected total cost would be between 2100 and 3050. Note that this particular form of the solution is just for our specific problem instance, and it cannot be generalized.*

C Omitted Theorem and Proofs

C.1 Proof of Theorem 1

Proof. It is easy to see that if $d^s_i = 0$ or $v^s_i + \bar{\eta}^s = 0$, then the solution $(\mathbf{y}^s_i, \mathbf{r}^s_i)$ where $y^s_{ij} = x_{ij}$ and $r^s_{ij} = 0$ for all $j \in J$ is optimal to the primal subproblem. It is also easy to prove that the dual solutions given in 1a and 1b are optimal for the dual subproblem as the objective function values of these dual solutions are equal to the primal objective function values.

For the second case, we first prove that the defined solution $(\mathbf{y}^s_i, \mathbf{r}^s_i)$ is feasible for the primal subproblem. It is easy to check that the constraints (11b) and (11c) are satisfied.

Now consider the case 2a. Clearly, $\mathbf{y}^s_i$ is nonnegative. In addition, $r^s_{i,\pi(1)} = d^s_i(1 - x_{i,\pi(1)})$ and $r^s_{ij} = d^s_i x_{ij}$ for all $j \in J \setminus \{\pi(1)\}$. Then, by the specified condition of $(v^s_i + \bar{\eta}^s)/2d^s_i \geq \sum_{j \in J' \setminus \{\pi(1)\}} x_{ij}$, we have $\sum_{j \in J} r^s_{ij} = d^s_i(1 - x_{i,\pi(1)}) + \sum_{j \in J' \setminus \{\pi(1)\}} d^s_i x_{ij} = 2d^s_i \sum_{j \in J' \setminus \{\pi(1)\}} x_{ij} \leq v^s_i + \bar{\eta}^s$, and the constructed solution $(\mathbf{y}^s_i, \mathbf{r}^s_i)$ is feasible. It is easy to verify the feasibility of the dual solution. It is also easy to see that the objective function values of the primal and dual solutions coincide, proving that they are optimal to their corresponding problems.

Finally, consider the case 2b. By the choice of j', we know that $\sum_{j \in J' : \pi^-(j) \geq \pi^-(j')} x_{ij} \geq (v^s_i + \bar{\eta}^s)/2d^s_i$. Then it is easy to verify that $\mathbf{y}^s_i$ is nonnegative. Now we compute $\mathbf{r}^s_i$: $r^s_{i,\pi(1)} = (v^s_i + \bar{\eta}^s)/2$, $r_{i,j'} = (v^s_i + \bar{\eta}^s)/2 - d^s_i \sum_{j \in J' : \pi^-(j) > \pi^-(j')} x_{ij}$, $r^s_{ij} = d^s_i x_{ij}$ for all $j \in J'$ with $\pi^-(j) > \pi^-(j')$, and $r_{ij} = 0$ otherwise. Hence, we have $\sum_{j \in J} r^s_{ij} = (v^s_i + \bar{\eta}^s)/2 + (v^s_i + \bar{\eta}^s)/2 - d^s_i \sum_{j \in J' : \pi^-(j) > \pi^-(j')} x_{ij} + \sum_{j \in J' : \pi^-(j) > \pi^-(j')} d^s_i x_{ij} = v^s_i + \bar{\eta}^s$.

Now we prove that the defined dual solution is feasible. It is easy to verify that the constraints (12c) and the nonnegativity restrictions are satisfied. We will show that the dual solution also satisfies the constraints (12b). For $j \in J \setminus J'$, if $(c^s_{i,\pi(1)} + c^s_{ij'})/2 \geq c^s_{ij}$, then $\gamma^s_i - \beta^s_{ij} = d^s_i(c^s_{i,\pi(1)} + c^s_{ij'})/2 - d^s_i((c^s_{i,\pi(1)} + c^s_{ij'})/2 - c^s_{ij}) = d^s_i c^s_{ij}$. This is equal to $d^s_i(c^s_{ij} + \lambda^s_{ij} - \rho^s_{ij})$ since $\lambda^s_{ij} = \rho^s_{ij} = 0$. If $(c^s_{i,\pi(1)} + c^s_{ij'})/2 < c^s_{ij}$, then $\beta^s_{ij} = 0$ and

$\gamma_i^s - \beta_{ij}^s = d_i^s(c_{i,\pi(1)}^s + c_{ij'}^s)/2 < d_i^s c_{ij}^s$. In both cases, $\gamma_i^s - \beta_{ij}^s \leq d_i^s c_{ij}^s$ and this is equal to $d_i^s(c_{ij}^s + \lambda_{ij}^s - \rho_{ij}^s)$ since $\lambda_{ij}^s = \rho_{ij}^s = 0$.

For $j \in J'$ with $\pi^-(j) < \pi^-(j')$, $\gamma_i^s - \beta_{ij}^s = d_i^s(c_{i,\pi(1)}^s + c_{ij'}^s)/2$ and $c_{ij}^s + \lambda_{ij}^s - \rho_{ij}^s = c_{ij}^s + (c_{ij'}^s - c_{ij}^s)/2 - (c_{ij}^s - c_{i,\pi(1)}^s)/2 = (c_{i,\pi(1)}^s + c_{ij'}^s)/2$. Now it is easy to see that $\gamma_i^s - \beta_{ij}^s \leq d_i^s(c_{ij}^s + \lambda_{ij}^s - \rho_{ij}^s)$.

Finally, for $j \in J'$ with $\pi^-(j) \geq \pi^-(j')$, using the relation $c_{ij'}^s \leq c_{ij}^s$, we have

$$\gamma_i^s - \beta_{ij}^s = d_i^s \frac{c_{i,\pi(1)}^s + c_{ij'}^s}{2} = d_i^s \left(\frac{c_{i,\pi(1)}^s + c_{ij'}^s + c_{ij}^s - c_{ij}^s}{2} \right) \leq d_i^s \left(\frac{c_{i,\pi(1)}^s + c_{ij}^s + c_{ij}^s - c_{ij'}^s}{2} \right).$$

The last quantity is equal to $d_i^s(c_{ij}^s + \lambda_{ij}^s - \rho_{ij}^s)$. This shows that the dual solution satisfies the constraints (12b) and proves its feasibility.

Now we compute the objective function value of the dual solution:

$$\begin{aligned}
&- (v_i^s + \bar{\eta}^s)\, \psi_i^s + \gamma_i^s - \sum_{j \in J} z_j \beta_{ij}^s + \sum_{j \in J} d_i^s x_{ij} \left(\rho_{ij}^s - \lambda_{ij}^s \right) \\
&= - (v_i^s + \bar{\eta}^s) \frac{c_{ij'}^s - c_{i,\pi(1)}^s}{2} + d_i^s \frac{c_{ij'}^s + c_{i,\pi(1)}^s}{2} \\
&\qquad + \sum_{j \in J' \,:\, \pi^-(j) < \pi^-(j')} d_i^s \left(c_{ij}^s - \frac{c_{ij'}^s + c_{i,\pi(1)}^s}{2} \right) x_{ij} + \sum_{j \in J' \,:\, \pi^-(j) \geq \pi^-(j')} d_i^s \frac{c_{ij'}^s - c_{i,\pi(1)}^s}{2} x_{ij} \\
&= d_i^s \frac{c_{i,\pi(1)}^s}{2} \left(x_{i,\pi(1)} - \sum_{j \in J' \,:\, \pi^-(j) > 1} x_{ij} + \frac{v_i^s + \bar{\eta}^s}{d_i^s} + 1 \right) + \sum_{\substack{j \in J': \\ 1 < \pi^-(j) < \pi^-(j')}} d_i^s c_{ij}^s x_{ij} \\
&\qquad + d_i^s \frac{c_{i,j'}^s}{2} \left(- \sum_{j \in J' \,:\, \pi^-(j) < \pi^-(j')} x_{ij} + \sum_{j \in J' \,:\, \pi^-(j) \geq \pi^-(j')} x_{ij} - \frac{v_i^s + \bar{\eta}^s}{d_i^s} + 1 \right) \\
&= d_i^s c_{i,\pi(1)}^s y_{i,\pi(1)}^s + \sum_{j \in J' \,:\, 1 < \pi^-(j) < \pi^-(j')} d_i^s c_{ij}^s y_{ij}^s + d_i^s c_{i,j'}^s y_{ij'}^s = \sum_{j \in J'} d_i^s c_{ij}^s y_{ij}^s.
\end{aligned}$$

The above chain of equations shows that objective function values of the dual and primal solutions coincide, which proves that both solutions are optimal to their corresponding problems. □

C.2 Theorem 2 and Its Proof

For the case of **CFSDP**, the primal subproblem for demand point $i \in I$ and scenario $s \in S$ takes the following form

$$Q_i^s\left(\mathbf{z}, \mathbf{x}, \mathbf{w}, \mathbf{v}, \eta, \bar{\boldsymbol{\eta}}\right) = \min \quad \sum_{j \in J} c_{ij}^s d_i^s y_{ij}^s \tag{14a}$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{j \in J} y_{ij}^s = 1, & & & (\gamma_i^s) \quad & (14b)\\
& y_{ij}^s \leq z_j, & & \forall\, j \in J & (-\beta_{ij}^s) \quad & (14c)\\
& d_i^s y_{ij}^s \leq d_i^s x_{ij} + v_{ij}^s + \bar{\eta}^s, & & \forall\, j \in J & (-\lambda_{ij}^s) \quad & (14d)\\
& d_i^s y_{ij}^s \geq d_i^s x_{ij} - v_{ij}^s - \bar{\eta}^s, & & \forall\, j \in J & (\rho_{ij}^s) \quad & (14e)\\
& \mathbf{y}_i^s \in \mathbb{R}_+^{|J|}. & & & & (14f)
\end{aligned}
$$

Before stating the counterpart of Theorem 1, we first present the dual formulation of (14):

$$
\begin{aligned}
\max \quad & \gamma_i^s - \sum_{j \in J} z_j \beta_{ij}^s + \sum_{j \in J} \left(d_i^s x_{ij} \left(\rho_{ij}^s - \lambda_{ij}^s \right) - \left(v_{ij}^s + \eta^s \right) \left(\rho_{ij}^s + \lambda_{ij}^s \right) \right) & & (15a)\\
\text{s.t.} \quad & \gamma_i^s - \beta_{ij}^s \leq d_i^s \left(c_{ij}^s + \lambda_{ij}^s - \rho_{ij}^s \right), & \forall\, j \in J \quad & (15b)\\
& \boldsymbol{\beta}_i^s, \boldsymbol{\lambda}_i^s, \boldsymbol{\rho}_i^s \geq 0. & & (15c)
\end{aligned}
$$

This dual subproblem provides the optimality cut taking the following form for $s \in S$ and $i \in I$:

$$
\theta_i^s \geq \gamma_i^s - \sum_{j \in J} \beta_{ij}^s z_j + \sum_{j \in J} d_i^s (\rho_{ij}^s - \lambda_{ij}^s) x_{ij}^s - \sum_{j \in J} (\lambda_{ij}^s + \rho_{ij}^s)(v_{ij}^s + \bar{\eta}^s).
$$

Note that one can also use an aggregated version of these optimality cuts in an implementation of the corresponding Benders decomposition algorithm.

Theorem 2. *Consider the primal subproblem (14) along with its dual (15) for a particular pair of scenario $s \in S$ and demand point $i \in I$.*

1. *If $d_i^s = 0$ or $v_{ij}^s + \bar{\eta}^s = 0$ for all $j \in J$, then the solution $(\mathbf{y}_i^s)$ such that $y_{ij}^s = x_{ij}$ for all $j \in J$ is optimal to the primal subproblem (14).*

   (a) *If $d_i^s = 0$, then the solution such that all dual variables equal to zero is an optimal solution to the dual subproblem (15).*

   (b) *If $v_{ij}^s + \bar{\eta}^s = 0$ for all $j \in J$, let $\bar{c}_{ij}^s$ be the minimum unit allocation cost for i among the locations with $x_{ij} > 0$, i.e., $\bar{c}_{ij}^s = \min_{j \in J: x_{ij} > 0} c_{ij}^s$. Then, an optimal dual solution is given by*
   $\gamma_i^s = \bar{c}_{ij}^s d_i^s$,
   $\beta_{ij}^s = \left(\gamma_i^s - c_{ij}^s d_i^s \right)^+$ *for $j \in J$ with $z_j = 0$,* $\beta_{ij}^s = 0$ *for $j \in J$ with $z_j = 1$,*
   $\rho_{ij}^s = \left(c_{ij}^s - \bar{c}_{ij}^s \right)^+$ *for $j \in J$ with $x_{ij} > 0$,* $\rho_{ij}^s = 0$ *for $j \in J$ with $x_{ij} = 0$,*
   $\lambda_{ij}^s = 0$ *for $j \in J$.*

2. *If $d_i^s > 0$ and $v_{ij}^s + \bar{\eta}^s > 0$ for some $j \in J$, then J', $\pi(t)$ and $\pi^-(j)$ are defined as in Theorem 1. We find $j' \in J'$ with smallest $\pi^-(j')$ such that*

$$\frac{1}{d_i^s}(d_i^s x_{ij'} - \bar{\eta}^s - v_{ij}^s)^+ \leq 1 - \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} \frac{1}{d_i^s}(d_i^s x_{ij} + \bar{\eta}^s + v_{ij}^s) - \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} \frac{1}{d_i^s}(d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s)^+ \leq \frac{1}{d_i^s}(d_i^s x_{ij'} + \bar{\eta}^s + v_{ij}^s).$$

- *The solution $\mathbf{y}_i^s$ with the following structure is an optimal solution to the primal subproblem:*

$$y_{ij}^s = \begin{cases} \frac{1}{d_i^s}(d_i^s x_{ij} + \bar{\eta}^s + v_{ij}^s), & \text{if } j \in J' \text{ and } \pi^-(j) < \pi^-(j'), \\ \frac{1}{d_i^s}(d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s)^+, & \text{if } j \in J' \text{ and } \pi^-(j) > \pi^-(j'), \\ 1 - \sum\limits_{j \in J': \pi^-(j) < \pi^-(j')} \frac{1}{d_i^s}(d_i^s x_{ij} + \bar{\eta}^s + v_{ij}^s) & \\ \qquad - \sum_{j \in J': \pi^-(j) > \pi^-(j')} \frac{1}{d_i^s}(d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s)^+, & \text{if } j = j', \\ 0, & \text{if } j \in J \setminus J', \end{cases}$$

  for all $j \in J$.

- *The dual solution $(\gamma_i^s, \boldsymbol{\beta}_i^s, \boldsymbol{\lambda}_i^s, \boldsymbol{\rho}_i^s)$ with the following structure is optimal to the dual subproblem:*

$$\begin{aligned} &\gamma_i^s = d_i^s c_{ij'}^s, \\ &\lambda_{ij}^s = \rho_{ij}^s = 0 \text{ and } \beta_{ij}^s = d_i^s (c_{ij'}^s - c_{ij}^s)^+ \text{ for } j \in J \setminus J', \\ &\lambda_{ij}^s = (c_{ij'}^s - c_{ij}^s)^+ \text{ and } \beta_{ij}^s = 0, \text{ for } j \in J', \\ &\rho_{ij}^s = (c_{ij}^s - c_{ij'}^s)^+ \text{ for } j \in J' \text{ with } d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s \geq 0, \\ &\rho_{ij}^s = 0 \text{ for } j \in J' \text{ with } d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s < 0. \end{aligned}$$

Proof. Since it is easy to see that the primal and dual solutions defined in case 1 are optimal for their corresponding problems, we skip that part.

For the case 2, the defined primal and dual and solutions are clearly feasible for their corresponding problems. The objective function value of the dual solution is given by

$$d_i^s c_{ij'}^s + \sum_{j \in J'} (-(d_i^s x_{ij} + \bar{\eta}^s + v_{ij}^s)(c_{ij'}^s - c_{ij}^s)^+ + (d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s)^+ (c_{ij}^s - c_{ij'}^s)^+)$$

$$
\begin{aligned}
&= d_i^s c_{ij'}^s + \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} -(d_i^s x_{ij} + \bar{\eta}^s + v_{ij}^s)(c_{ij'}^s - c_{ij}^s) + \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} (d_i^s x_{ij} - \bar{\eta}^s - v_{ij}^s)^+ (c_{ij}^s - c_{ij'}^s) \\
&= d_i^s c_{ij'}^s + \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} -(c_{ij'}^s - c_{ij}^s) d_i^s y_{ij}^s + \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} (c_{ij}^s - c_{ij'}^s) d_i^s y_{ij}^s \\
&= c_{ij'}^s d_i^s \left(1 - \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} y_{ij}^s - \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} y_{ij}^s \right) + \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} c_{ij}^s d_i^s y_{ij}^s + \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} c_{ij}^s d_i^s y_{ij}^s \\
&= c_{ij'}^s d_i^s y_{ij'}^s + \sum_{\substack{j \in J': \\ \pi^-(j) < \pi^-(j')}} c_{ij}^s d_i^s y_{ij}^s + \sum_{\substack{j \in J': \\ \pi^-(j) > \pi^-(j')}} c_{ij}^s d_i^s y_{ij}^s = \sum_{j \in J'} c_{ij}^s d_i^s y_{ij}^s.
\end{aligned}
$$

Since the objective function values of the dual and primal solutions are equal to each other, these solutions are optimal for their corresponding problems. □

D Additional Numerical Results

D.1 For Section 4.1

Table 5: Fixed costs in different problem instances.

Instance Type	Fixed Cost (f)	Instance Type	Fixed Cost (f)
cap71	7,500	cap73	17,500
cap72	12,500	cap74	25,000

D.2 For Section 4.2

In this section, we investigate the impact of our modeling approach on the multi/single-sourcing structure of the optimal solutions. First recall the single-sourcing property of the underlying unrestricted model (see Example 1), i.e., there exists an optimal solution where each demand point is fully assigned to the closest open facility under each scenario—as this is the case for the deterministic UFL problem. For this unrestricted model, we consider an auxiliary pre-allocation decision vector $\mathbf{x}$ specified as $\mathbf{x} = \mathbf{y}^s$ for all $s \in S$. We perform an analysis to quantify and demonstrate the deviations from such a single-sourcing plan—an extended version of the discussion in Example 2. To this end, we follow two types of calculations to quantify both the deviations from the single-sourcing property in the first stage (by focusing on the scenario-independent $\mathbf{x}$ decision vectors) and in the second stage (by focusing on the scenario-dependent $\mathbf{y}$ vectors). We consider the problem instances of

type cap71 with 25 demand points and 100 scenarios for varying values of $\bar{\alpha}$ and ϵ, and particularly, solve **CSTDP** for the following parameters settings: $\bar{\alpha} = 0.5, 0.6, 0.7, 0.8, 0.9$ when $b = 0.005$, and $\bar{\alpha} = 0.9$ when $b = 0.01, 0.05$. For all the instances, except those with $\bar{\alpha} = 0.9$ and $b = 0.05$ (seven facilities are opened), a total of eight facilities are opened. To quantify the first-stage single-sourcing level, for each instance, we determine the percentage of the nodes whose demand is pre-assigned to at most τ many facilities: $G_1 = |\{i \in I : |\{j \in J : x_{ij} > 0\}| \leq \tau\}|/|I|$ for $\tau = 1 \ldots, 8$. We can clearly say that a higher value of G_1 corresponds to a higher level of first-stage single-sourcing. The values of the G_1 metric (under varying values of τ) associated with the optimal solutions of **CSTDP** are presented in Figure 1. Single-sourcing in the first-stage is attained for $\bar{\alpha} = 0.5$ and $\bar{\alpha} = 0.6$, while the deviations from the first-stage single-sourcing is observed for larger $\bar{\alpha}$ values. For $\bar{\alpha} = 0.7$ and $\bar{\alpha} = 0.8$, all demand points are pre-allocated to at most four and five facilities, respectively, while for $\bar{\alpha} = 0.9$, there are some nodes whose demand is satisfied from seven different facilities. Similarly, for smaller values of b, we can observe a lower level of single-sourcing. For example, when $b = 0.05$, each node is assigned to at most four facilities, whereas for $b = 0.01$ the maximum number of facilities that serve the same demand point is six. Thus, for larger $\bar{\alpha}$ and smaller b values, the level of deviation from the first-stage single-sourcing is generally higher.

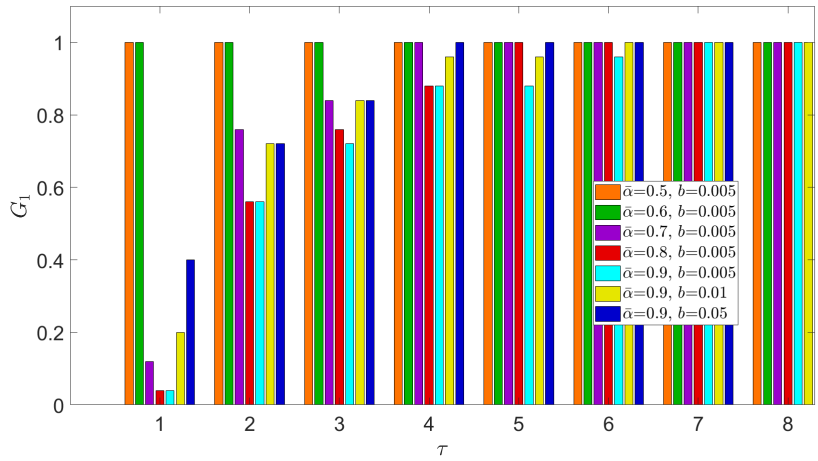

Figure 1: Results on the multi/single-sourcing structure in the first-stage for varying $\bar{\alpha}$ and ϵ.

We follow a similar approach to analyse the second-stage multi/single-sourcing structure under each scenario. We focus on a random variable G_2 representing the percentage of the nodes whose demand is allocated to at most τ many facilities. We calculate the realization of G_2 under each scenario $s \in S$, i.e., $G_2^s = |\{i \in I : |\{j \in J : y_{ij}^s > 0\}| \leq \tau\}|/|I|$ along with its expectation $\bar{G}_2 = \sum_{s \in S} G_2^s/|S|$. In line with an expectation-based view, we can

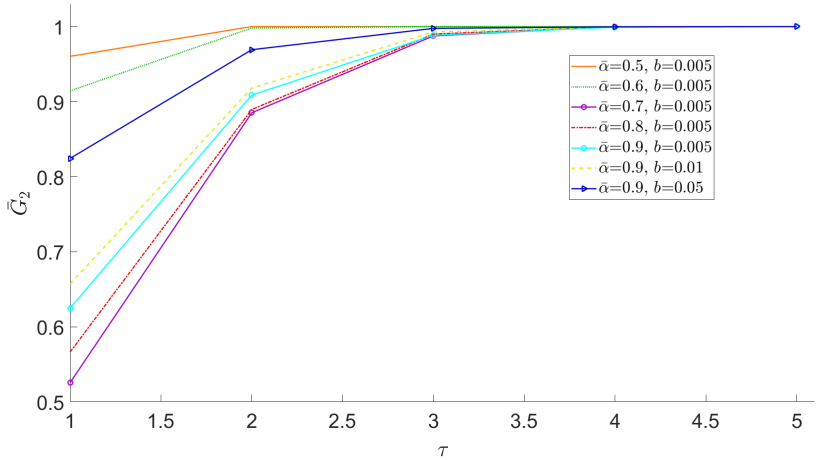

Figure 2: Results on the multi/single-sourcing structure in the second-stage for varying $\bar{\alpha}$ and ϵ.

say that a higher value of $\bar{G}_2$ on average, in general, indicates a higher level of second-stage single-sourcing. Figure 2 presents the values of the $\bar{G}_2$ metric (under varying values of τ) associated with the optimal solutions of **CSTDP**. We observe a similar behavior as in the first-stage case: for smaller $\bar{\alpha}$ values **CSTDP** becomes less restrictive, and most of the nodes are allocated to a single facility. Observe that for $\bar{\alpha} = 0.5$, the $\bar{G}_2$ takes a value larger than 0.95, and all demand points are assigned to at most two facilities. This setting has the highest single-sourcing level according to our expectation-based view. Even if there is no monotone structure, the deviations from the single-sourcing becomes more pronounced as $\bar{\alpha}$ increases. Similarly, the level of second-stage single-sourcing increases as b gets larger. Consequently, Figures 1 and 2 indicate that the right tail of the G_1 and $\bar{G}_2$ functions in general tends to shift to the left for less restrictive cases (larger b and smaller $\bar{\alpha}$), which implies that a larger number of demand points has a smaller number of facility sources.

D.3 For Section 4.3

Table 6 is the counterpart of Table 3 for 50 demand points. On the other hand, Table 7 reports the worst and the best solution times and the final gaps over 10 instances for each setting considered in Tables 3 and 6.

We perform a similar experiment for **CFSDP** and present the results for 25 demand points in Table 8. First we observe that it is harder to solve **CFSDP** than **CSTDP** for the same problem sizes: DEF can be solved to optimality only for 100-300 scenarios, and terminates with 100% optimality gap for larger instances. As it can be clearly observed from Table 8, the first type of decomposition approach (OPT1) perfoms better than the second

Table 6: Results for **CSTDP** with 50 demand points.

		DEF			**OPT1**					**OPT2 - CPLEX**					**OPT2 - ALG**				
$\|S\|$	cap	cpu (sec)	fgap (%)	# opt (# feas)	cpu (sec)	fgap (%)	# opt (# feas)	node #	cut #	cpu (sec)	fgap (%)	# opt (# feas)	node #	cut #	cpu (sec)	fgap (%)	# opt	node #	cut #
100	71	875	0	**10(10)**	2,888	2.33	9(10)	302	30,342	244	**0**	**10(10)**	44	40,377	**119**	**0**	**10(10)**	42	28,803
	72	1,770	0	**10(10)**	4,682	1.58	4(9)	754	39,055	276	**0**	**10(10)**	79	38,609	**170**	**0**	**10(10)**	75	31,964
	73	2,759	0	**10(10)**	4,075	3.06	2(9)	624	41,093	446	**0**	**10(10)**	133	44,439	**308**	**0**	**10(10)**	108	37,183
	74	3,201	0	**10(10)**	4,182	5.05	7(9)	599	32,964	308	**0**	**10(10)**	67	36,707	**197**	**0**	**10(10)**	85	34,837
200	71	4,490	0	**10(10)**	2,593	0.92	5(10)	273	44,561	2,088	**0**	**10(10)**	120	116,389	**666**	**0**	**10(10)**	91	68,663
	72	5,896	6.56	4(10)	6,289	1.96	2(9)	546	51,564	4,026	**0**	**10(10)**	173	126,522	**1,988**	**0**	**10(10)**	177	99,010
	73	6,689	19.3	1(10)	5,356	2.64	1(8)	523	50,811	4,055	2.28	9(10)	189	114,728	**3,198**	**0**	**10(10)**	229	110,379
	74	7,201	62.54	2(10)	3,864	3.19	2(8)	415	50,071	3,170	**0**	**10(10)**	156	102,646	**1,566**	**0**	**10(10)**	141	90,076
300	71	TL	72.66	0(10)	5,336	1.59	4(10)	274	62,520	4,345	4.34	7(10)	119	198,533	**1,833**	**0**	**10(10)**	85	119,238
	72	TL	100	0(10)	TL	**2.38**	0(10)	307	64,541	4,840	4.41	3(10)	129	189,985	**2,928**	2.89	**6(10)**	129	148,930
	73	TL	100	0(10)	TL	**3.2**	0(9)	352	64,335	**4,677**	4.88	**4(10)**	139	164,725	4,837	4.64	**4(10)**	146	162,922
	74	TL	100	0(10)	5,215	**3.46**	4(10)	424	59,416	3,433	4.76	**9(10)**	116	131,797	**3,217**	5.94	6(10)	104	142,418
400	71	TL	100	0(10)	5,199	2.2	3(10)	193	67,867	5,457	5.33	4(10)	104	272,923	**2,073**	0	**10(10)**	86	135,554
	72	TL	100	0(10)	TL	**3.37**	0(10)	175	72,979	**4,051**	5.61	1(10)	92	220,534	4,222	3.85	**6(10)**	129	177,505
	73	TL	100	0(10)	TL	4.28	0(7)	193	66,525	TL	**3.32**	0(10)	92	196,987	**2,973**	4.95	**1(10)**	100	191,144
	74	TL	100	0(10)	6,378	3.69	1(10)	303	69,659	**3,492**	**2.98**	1(10)	99	160,119	3,629	5.75	**2(10)**	91	181,965
500	71	TL	100	0(10)	6,950	2.09	1(10)	143	73,457	4,841	5.63	3(8)	98	319,968	**4,153**	**0**	**10(10)**	101	185,333
	72	TL	100	0(10)	TL	**3.98**	0(10)	140	71,939	TL	7.31	0(8)	80	274,160	**5,746**	4.26	**3(9)**	102	221,757
	73	TL	100	0(10)	TL	**4.59**	0(9)	140	75,548	TL	6.47	0(8)	80	257,120	TL	7.34	**0(10)**	83	251,347
	74	TL	100	0(10)	TL	**3.64**	0(10)	211	71,468	5,637	4.51	**3(9)**	84	194,805	**3,323**	4.7	**1(10)**	78	221,965
	TOTAL			57(200)			45(187)					114(193)					139(199)		

Table 7: Minimum and maximum values of solution times and final gaps for **CSTDP**

			DEF		**OPT1**		**OPT2-CPLEX**		**OPT2-ALG**	
$\|I\|$	$\|S\|$	cap	min cpu (fgap)	max cpu (fgap)	min cpu (fgap)	max cpu (fgap)	min cpu (fgap)	max cpu (fgap)	min cpu (fgap)	max cpu (fgap)
25	100	71	68	235	63	262	31	47	**9**	**19**
		72	64	153	70	138	26	41	**11**	**20**
		73	88	304	47	107	21	43	**9**	**17**
		74	60	125	17	39	14	30	**5**	**13**
	300	71	671	2,681	323	1,000	223	406	**56**	**233**
		72	755	1,840	473	1,898	247	400	**178**	**350**
		73	1,493	4,826	194	893	170	**317**	**106**	475
		74	524	1,201	68	**138**	83	160	**33**	176
	500	71	3,225	6,498	553	1,394	690	1,970	**314**	**765**
		72	2,315	5,516	788	1,384	625	1,943	**472**	**1,017**
		73	6,214	(4.38)	**308**	(4.99)	452	**1,692**	363	1,855
		74	1,824	5,148	**116**	**443**	171	507	156	500
	800	71	7,200	(19.95)	1,201	**2,265**	1,932	3,185	**885**	5,750
		72	7,201	(0.04)	1,134	4,266	1,230	3,783	**1,019**	**2,993**
		73	(0.08)	(5.02)	746	**2,263**	898	3,627	**859**	4,648
		74	4,484	(0.05)	**172**	**661**	383	1,051	274	888
	1000	71	(0.05)	(19.77)	**1,741**	**5,024**	2,264	(0.38)	1,917	(2.49)
		72	(0.03)	(9.78)	**1,111**	**4,303**	2,446	(7.99)	2,024	(0.52)
		73	(0.13)	(5.12)	**1,119**	**3,400**	2,306	5,191	1,790	(11.53)
		74	7,170	(0.06)	**324**	2,244	658	**1,463**	431	1,449
50	100	71	375	1,547	668	(2.33)	69	485	**25**	**416**
		72	1,123	2,335	1,827	(1.99)	87	625	**50**	**473**
		73	1,341	4,767	3,469	(6.83)	215	835	**89**	**602**
		74	1,346	5,262	1,315	(5.32)	79	655	**38**	**535**
	200	71	3,079	6,198	1,383	(1.52)	432	3,374	**206**	**1,325**
		72	3,917	(24.48)	5,636	(3.21)	1,381	**6,451**	**449**	6,623
		73	6,689	(100)	5,356	(3.63)	2,192	**(2.28)**	**735**	5,745
		74	7,200	(100)	2,696	(5.36)	1,753	4,859	**615**	**3,461**
	300	71	(0.06)	(100)	4,618	(1.84)	1,017	(4.83)	**508**	**3,022**
		72	(100)	(100)	(0.82)	**(3.64)**	4,023	(4.86)	**989**	(4.28)
		73	(100)	(100)	(1.96)	**(4.9)**	3,141	(7.63)	**1,967**	(7.0)
		74	(100)	(100)	3,956	(6.09)	1,016	**(4.76)**	**638**	(8.95)
	400	71	(100)	(100)	3,311	(3.29)	4,278	(7.84)	**1,107**	**2,901**
		72	(100)	(100)	(1.72)	**(4.36)**	4,051	(10.61)	**2,211**	(7.64)
		73	(100)	(100)	(3.28)	**(4.96)**	(0.2)	(5.58)	**2,973**	(10.3)
		74	(100)	(100)	6,378	(5.88)	3,492	**(5.42)**	**2,742**	(9.51)
	500	71	(100)	(100)	6,950	(2.7)	3,824	(12.44)	**1,499**	**6,156**
		72	(100)	(100)	(2.36)	**(5.21)**	(3.92)	(9.76)	**3,425**	(9.95)
		73	(100)	(100)	**(3.67)**	**(5.49)**	(4.53)	(9.97)	(4.8)	(11.36)
		74	(100)	(100)	(0.05)	**(5.45)**	4,958	(6.2)	**3,323**	(10.31)

type of decomposition methods (OPT2-CPLEX and OPT2-ALG). Also, using the algorithm given in Theorem 2 does not improve the solution times. Note that in our second type of decomposition algorithms, the Benders master problem $\mathbf{RMP}^{(2)}$ of **CFSDP** involves a larger number of scenario-dependent decision variables compared to that of **CSTDP**, and this may be the main cause of this difference. Thus, keeping the complicating constraints in the relaxed master problem does not help us to improve the solution times for **CFSDP**. Actually, for a larger number of scenarios, even a feasible solution could not be found within the time limit by the second type of decomposition algorithms (see the Column "# feas"). Since OPT1 is clearly the best solution method for **CFSDP**, in Table 9, we report the worst and the best solution times and the final gaps over 10 instances only for OPT1. It can be observed from Tables 8 and 9 that its performance is the best for the cap73 and cap74 instances (all the instances are solved to optimality within the time limit) where the fixed costs are relatively larger.

Table 8: Results for **CFSDP** with 25 demand points.

		DEF			**OPT1**					**OPT2 - CPLEX**					**OPT2 - ALG**				
$\|S\|$	cap	cpu (sec)	fgap (%)	# opt (# feas)	cpu (sec)	fgap (%)	# opt (# feas)	node #	cut #	cpu (sec)	fgap (%)	# opt (# feas)	node #	cut #	cpu (sec)	fgap (%)	# opt	node #	cut #
100	71	988	0	**10(10)**	**384**	0	**10(10)**	249	20,804	1,272	10.92	9(10)	76	30,694	4,604	13.69	5(10)	90	57,814
	72	502	0	**10(10)**	**153**	0	**10(10)**	233	10,076	994	8.31	9(10)	61	29,344	4,851	9.50	6(10)	94	56,681
	73	945	0	**10(10)**	**98**	0	**10(10)**	146	7,813	1,308	5.90	9(10)	57	28,176	3,747	6.01	6(9)	106	49,965
	74	256	0	**10(10)**	**39**	0	**10(10)**	47	4,641	1,239	0.00	**10(10)**	34	24,239	1,977	3.53	8(10)	59	35,426
300	71	TL	27.28	0(10)	**1,908**	0	**10(10)**	222	63,904	6,857	2.39	1(6)	93	96,738	-	-	0(0)	-	-
	72	7,064	38.75	2(10)	**1,013**	0	**10(10)**	269	40,062	5,377	2.87	4(6)	91	82,774	-	-	0(0)	-	-
	73	TL	40.97	0(10)	**316**	0	**10(10)**	180	20,212	6,106	1.30	5(6)	71	64,456	-	-	0(0)	-	-
	74	TL	70.15	0(10)	**126**	0	**10(10)**	47	12,390	5,002	6.43	5(6)	42	71,288	6,280	12.53	1(3)	50	110,160
500	71	TL	100	0(10)	**3,825**	0	**10(10)**	209	101,554	-	-	0(0)	-	-	-	-	0(0)	-	-
	72	TL	100	0(10)	**2,696**	0	**10(10)**	247	72,844	-	-	0(0)	-	-	-	-	0(0)	-	-
	73	TL	100	0(10)	**961**	0	**10(10)**	178	40,275	-	-	0(0)	-	-	-	-	0(0)	-	-
	74	TL	100	0(10)	**209**	0	**10(10)**	54	19,032	-	-	0(0)	-	-	-	-	0(0)	-	-
800	71	TL	100	0(10)	**5,710**	3.74	**5(10)**	181	135,583	-	-	0(0)	-	-	-	-	0(0)	-	-
	72	TL	100	0(10)	**4,249**	4.91	**7(10)**	223	105,083	-	-	0(0)	-	-	-	-	0(0)	-	-
	73	TL	100	0(10)	**1,282**	0.00	**10(10)**	156	55,648	-	-	0(0)	-	-	-	-	0(0)	-	-
	74	TL	100	0(10)	**494**	0.00	**10(10)**	55	34,180	-	-	0(0)	-	-	-	-	0(0)	-	-
1000	71	TL	100	0(10)	**4,806**	3.93	**3(10)**	188	139,045	-	-	0(0)	-	-	-	-	0(0)	-	-
	72	TL	100	0(10)	**3,444**	6.06	**6(10)**	182	112,978	-	-	0(0)	-	-	-	-	0(0)	-	-
	73	TL	100	0(10)	**2,392**	0	**10(10)**	172	81,858	-	-	0(0)	-	-	-	-	0(0)	-	-
	74	TL	100	0(10)	**749**	0	**10(10)**	58	45,968	-	-	0(0)	-	-	-	-	0(0)	-	-
	TOTAL			42(200)			181(200)					52(64)					26(42)		

Table 9: Minimum and maximum values of solution times and final gaps for OPT1—**CFSDP**.

$\|S\|$	cap	**min cpu (min fgap)**	**max cpu (max fgap)**
100	71	125	1,380
	72	49	427
	73	35	289
	74	15	104
300	71	841	3,725
	72	398	2,372
	73	116	662
	74	34	277
500	71	1,346	6,961
	72	640	6,883
	73	239	2,235
	74	110	386

$\|S\|$	cap	**min cpu (min fgap)**	**max cpu (max fgap)**
800	71	4,869	(4.88)
	72	2,399	(5.65)
	73	485	2,887
	74	224	1,039
1000	71	3,479	(7.95)
	72	2,155	(8.77)
	73	795	5,405
	74	193	1,483