Supplemental Online Materials to "Competitive Spatial Pricing for Urban Parking Systems: Network Structures and Asymmetric Information" by Yuguang Wu, Qiao-chu He, and Xin Wang

A Extensions

A.1 Capacitated garages with multiple periods

Suppose that we model garage capacity explicitly and consider a planning horizon of multiple periods, for instance, the entire morning rush hour until the garages are filled up. We index time periods by $t = 1, 2, ..., T$. The baseline demand $\alpha^{[t]}$ as well as the random shock $\theta_1^{[t]}$ are both changing inter-temporally. The random shocks $\{\theta_i^{[t]}\}$'s are independent and identically distributed, with identical variance σ^2, and correlation $\rho^{[t]} = \rho, \forall t$. The underlying economy is stationary such that $\beta^{[t]} = \beta$ and $\hat{\beta}^{[t]} = \hat{\beta}, \forall t$. Garages are symmetric with the same capacity W. In this case, the garages' equilibrium is characterized by the following best-response functions:

$$\max_{p_1^{[t]}, t=1,2,...,T} \pi_1 = \sum_{t=1}^{T} d_1^{[t]} \left(p_1^{[t]} - c \right), \tag{25}$$

$$\max_{p_2^{[t]}, t=1,2,...,T} \pi_2 = \sum_{t=1}^{T} d_2^{[t]} \left(p_2^{[t]} - c \right), \tag{26}$$

$$\sum_{t=1}^{T} d_1^{[t]} \leq W_1, \sum_{t=1}^{T} d_2^{[t]} \leq W_2, \tag{27}$$

wherein

$$\begin{aligned} d_1^{[t]} &= \alpha^{[t]} - \hat{\beta} p_1^{[t]} + \beta p_2^{[t]} + \theta_1^{[t]}, \\ d_2^{[t]} &= \alpha^{[t]} + \beta p_1^{[t]} - \hat{\beta} p_2^{[t]} + \theta_2^{[t]}. \end{aligned} \tag{28}$$

We summarize the results for this extension in the following proposition.

Proposition 5. *The following results for the symmetric capacitated model over multiple periods hold:*

1. *When* $c > \frac{2\hat{\beta}-\beta}{\hat{\beta}\left(\beta-\hat{\beta}\right)} \cdot \frac{W}{T} - \frac{\hat{\beta}}{\hat{\beta}\left(\beta-\hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T} \alpha^{[t]}}{T}$, *the capacity constraints are not binding, the parking rates in equilibrium remain the same as in the static incapacitated model:*

$$p_i^{[t]} = \frac{\alpha^{[t]} + \hat{\beta} c}{2\hat{\beta} - \beta} + \frac{\theta_i^{[t]}}{2\hat{\beta} - \beta\rho}, \tag{29}$$

$\forall t = 1, 2, ..., T$ *and* $i = 1, 2$.

2. *Otherwise, the parking rates in equilibrium are given by*

$$\lim_{T\to\infty} p_i^{[t]} \to \frac{\alpha^{[t]}}{2\hat{\beta}-\beta} + \frac{W/T}{\left(\beta-\hat{\beta}\right)} - \frac{\hat{\beta}}{\left(2\hat{\beta}-\beta\right)\left(\beta-\hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T}\alpha^{[t]}}{T} + \frac{\theta_i^{[t]}}{2\hat{\beta}-\beta\rho}, \quad (30)$$

almost surely, $\forall t = 1, 2, ..., T$ *and* $i = 1, 2$.

3. *When* $W_1 \neq W_2$*, an one-period snapshot is equivalent to an incapacitated static model with heterogeneous cost* c_1 *and* c_2*, wherein* c_1 *and* $c_2 \geq c$.

4. *Compared with the static incapacitated version, while the baseline rates suffer from a constant downward distortion, the response to a private signal remains the same, and thus, the incentives for information sharing remains the same.*

5. *When* $\frac{\beta}{\hat{\beta}} \in (0, 1]$*, the parking rates as well as the aggregate payoff over* T *periods (*$\lim_{T\to\infty} \mathbb{E}\pi$*) decrease in total capacity* W*, but increase in average market potentials* $\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$*. (The converse is true when* $\frac{\beta}{\hat{\beta}} \in (1, 2)$*.)*

From this proposition, as a robustness check, we are assured that the results from a single-period snap-shot extend naturally toward multiple-periods. The proof is via a dual approach wherein we use Lagrangian multipliers to calculate the shadow price of limited capacity. When $c > \frac{2\hat{\beta}-\beta}{\hat{\beta}\left(\beta-\hat{\beta}\right)} \cdot \frac{W}{T} - \frac{\hat{\beta}}{\hat{\beta}\left(\beta-\hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$, the cost of selling one unit capacity is less costly than its shadow price, and thus capacity constraint is not binding. In the same vein, when $W_1 \neq W_2$, an one-period snapshot is equivalent to an incapacitated static model with heterogeneous cost c_1 and c_2, wherein c_1 and $c_2 \geq c$. The additional costs $c_1 - c$ and $c_2 - c$ capture the shadow price for limited capacity.

In fact, if we extend the restriction of elasticities to $\beta < 2\hat{\beta}$ (allow the cross-price elasticity to be greater than price elasticity), the capacity W (or average market potentials $\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$) poses opposite effect toward parking rates/payoffs when $\beta > \hat{\beta}$. In fact, let $\kappa = W/T - \frac{\beta}{\left(2\beta-\hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$ (which is increasing in total capacity W and decreasing in average baseline rates $\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$) capture the extra capacity which satisfies per-period price-sensitive demand. When $\beta > \hat{\beta}$, i.e., price elasticity is more dominant than cross-price elasticity, the inverse price-demand relationship within a garage dominates. Since an higher extra capacity κ satisfies higher price-sensitive demand, this implies a lower parking rate. Therefore, the parking rates (and consequently the aggregate payoff) decrease in the extra capacity κ, and thus increases in W and decreases in $\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}$. Conversely, when $\beta > \hat{\beta}$, i.e., the strategic complement between two garages dominates, and thus an higher capacity κ implies higher parking rates from the other garage.

A.2 Noisy demand forecasting

In the basic model, we assume that private signals are received via a noiseless information channel. We find this to be a harmless assumption by examining real-time parking demand data, as private demand forecasts tend to be fairly accurate using historical data (with error rates around 5%). Nevertheless, we generalize the basic model in this section to incorporate noisy demand forecast. Suppose that there are two symmetric garages, $\beta_{11} = \beta_{22} = \hat{\beta}$, $\beta_{12} = \beta_{12} = -\beta$, and $c_1 = c_2 = c$:

$$\begin{aligned} d_1 &= \alpha - \hat{\beta} p_1 + \beta p_2 + \theta_1, \\ d_2 &= \alpha + \beta p_1 - \hat{\beta} p_2 + \theta_2, \end{aligned} \tag{31}$$

We further assume that $(\theta_1, \theta_2)^\top$ are drawn from symmetric bivariate normal distribution $\mathcal{N}\left(0, \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$. As a standard stability constraint, we require $0 \leq \beta < \hat{\beta}$. In addition, the garages cannot accurately forecast their demand. Instead, each garage observe a noisy signal $x_i = \theta_i + \epsilon_i$, where $\epsilon_i \sim N(0, \gamma^2)$. We assume observation channels are independent, i.e., $\epsilon_1 \perp \epsilon_2$.

Proposition 6. *The following statements are true for the symmetric uncapacitated model with two garages and noisy demand forecasting:*

1. *Baseline parking rate structure remains the same as that with perfect demand forecasting.*
2. *In addition, all comparative statics in Proposition 1 hold, i.e.,* $\frac{\partial B}{\partial \rho} > 0$, $\frac{\partial B}{\partial \sigma} > 0$, $\frac{\partial \mathbb{E}\pi}{\partial \hat{\beta}} < 0$, $\frac{\partial \mathbb{E}\pi}{\partial \beta} > 0$ *and* $\frac{\partial \mathbb{E}\pi}{\partial \rho} > 0$.
3. *Garages respond more aggressively toward private signals when they are more accurate, and the payoff increases, i.e.,* $\frac{\partial B}{\partial \gamma} < 0$ *and* $\frac{\partial \mathbb{E}\pi}{\partial \gamma} < 0$.

The first two statements confirm our structural results in the basic model with perfect demand forecasts, which serve as robustness check with general demand forecasting accuracy. Not surprisingly, garages respond more aggressively toward private signals when they are more accurate, and the payoff increases since the value of information increases. However, information sharing is not always desirable in general, and information sharing being unprofitable is more likely to happen when forecasting noise increases. Note that information sharing is always favorable if the demand correlation is positive, since the knowledge sharing helps garages reduce demand uncertainty and further take advantage of their monopoly power. Fixing forecasting accuracy and price elasticities, there is a negative region of demand correlation such that information sharing is not preferred. Usually, garages tend to positively respond to the other's signal when information is shared, i.e., $B_{12} > 0$, since the competitor's demand surge potentially increases her demand through cross-price elasticity. However, under some environment where demand signals are negatively correlated, they form an equilibrium where garages strongly negatively respond to the shared signal. In the long run, this causes both garages to price lower than the theoretical optimum. This underutilization of the demand on average reduces the expected payoff for both garages.

A.3 Garage coalition model

It is worth noting that multiple garages in an urban area may be controlled by a single entity and thus not independent in the pricing game. In this subsection, we show that our insights on information sharing do not heavily rely on the assumption of independent garages. The model in Section 3.4 can be generalized to a pricing competition among garage coalitions. (A garage coalition is a parking firm that owns several garages in the city.) We derive the pricing equilibrium of the coalition competition.

Consider $\mathcal{K}$ as a set of independent garage coalitions controlling all garages $N = \{1, 2, \cdots, n\}$. Each coalition $K \in \mathcal{K}$ corresponds to a subset of N, and $\mathcal{K}$ is a partition of N. Every coalition has access to a certain set of demand information, decides the prices of her garages, and maximizes her total expected payoff. Let $N(K)$ be the information index set of coalition K. That is, information θ_j can be used for the pricing of garage $i \in K$ if and only if $j \in N(K)$. Given information $\theta_{N(K)}$, every coalition K decides the price vector p_K to maximize her own expected total payoff

$$\mathbb{E}_\theta \left[\Pi(K) \middle| \theta_{N(K)} \right] = \mathbb{E}_\theta \left[\sum_{i \in K} \pi_i \middle| \theta_{N(K)} \right] = \mathbb{E}_\theta \left[\sum_{i \in K} d_i p_i \middle| \theta_{N(K)} \right]. \tag{32}$$

Proposition 7. *Suppose coalition K observes signals $\theta_{N(K)}$. Then, the equilibrium pricing strategy is given by $p(\theta) = A + \boldsymbol{B}\theta$, where coefficients $A \in \mathbb{R}^n$ and $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ are determined by*

$$A = \boldsymbol{Q}^{-1} \left(\alpha + (\boldsymbol{Q} - \beta) c \right), \tag{33}$$

$$\begin{cases} Q_{K\star} \boldsymbol{B} \Sigma_{\star N(K)} - \Sigma_{KN(K)} = \mathbf{0}, \forall K \in \mathcal{K}, \\ B_{Kj} = 0, \forall j \notin N(K), \end{cases} \tag{34}$$

and

$$\boldsymbol{Q} := \beta + diag\left[\left(\beta_{KK}^\top \right)_{K \in \mathcal{K}} \right]. \tag{35}$$

The equilibrium payoff is given by

$$\mathbb{E}_\theta \left[\Pi(K) \right] = \langle A_K - c_K, \beta_{KK} (A_K - c_K) \rangle + \left\langle B_{K\star} \boldsymbol{\Sigma} B_{K\star}^\top, \beta_{KK} \right\rangle, \forall K \in \mathcal{K}. \tag{36}$$

where $\langle \cdot, \cdot \rangle$ is the Frobenius inner product of two vectors/matrices of the same dimension(s).

This proposition extends the pricing equilibrium (Proposition 2) to the case where garages are owned by competing coalitions. Here, Equations (33) & (34) are the extensions of Equations 8 & 9. The definition of $\boldsymbol{Q}$ has slightly changed. In (35), diag $\left[\left(\beta_{KK}^{\top}\right)_{K\in\mathcal{K}}\right]$ is an $n\times n$ block diagonal matrix whose main-diagonal blocks are square matrices $\beta_{KK}^{\top}, \forall K \in \mathcal{K}$. In particular, if every coalition K contains only one garage, this proposition reduces to Proposition 2. In Equation (36), $\left\langle B_{K\star}\boldsymbol{\Sigma}B_{K\star}^{\top}, \beta_{KK}\right\rangle$ corresponds to the information value for coalition K in the coalition model.

The connection between the original model and the coalition model lies in the dimensionality of the pricing decision. The coalition model is high-dimensional: Instead of deciding a single price p_i, each entity decides a set of prices p_K to maximize the total payoff $\sum_{i\in K}\pi_i$. The base model is a special case of a single dimension pricing. Therefore, we start from the base model without coalition formation to capture the main insight related to information sharing, which is robust shown in this extension. Furthermore, we will show in the next example additional insights derived from this coalition version.

Consider a symmetric duopoly setting where each garage coalition owns m garages in the city. The total $2m$ garages are assumed to have a symmetric influence on each other. Specifically, $\beta_{ii} = 1$, $\sigma_{ii} = 1$, and $\sigma_{ij} = \rho$, $\beta_{ij} = -\beta, \forall i \neq j$. (Note that Assumption 1 requires $\frac{1}{2m-1} \leq \rho \leq 1$ and $0 \leq \beta \leq \frac{1}{2m-1}\hat{\beta}$.) Then, under the coalition setting, we can show properties similar to the ones in Proposition 1. In Figure 6, we extend the 4th observation in Proposition 1 to the coalition competition. As the demand signals become more positively correlated, the information value (also the expected payoff since the baseline payoff is independent of ρ) increases. Figure 7 illustrate the additional value gain if information sharing is adopted. This result echoes Proposition 1 in that information sharing always generates positive value even if coalition formation is allowed. It also confirms that information sharing is more beneficial when the cross-elasticity β is higher. The value gain from sharing is non-monotone in the signal correlation. Because information is most useful when demand signals are weakly correlated. In the extreme case when demand signals are perfectly correlated, information sharing has 0 marginal value since competitor's information is already contained in the knowledge of demand correlation.

A.4 Optimal information assignment

Proposition 2 presents the exact equilibrium solution for an arbitrary observation matrix $\boldsymbol{M}$. A natural question one would be curious about is: among a set of possible information structures, which one of them would result in an equilibrium that maximizes the expected payoff of a particular garage (or their total expected payoff). The result for two-garage model in Section 3.3 indicates that information sharing is beneficial to both garages. However, this does not always hold in general.

From the perspective an information service provider, a natural question to ask is who should know what and how much they should know. The insights from answering

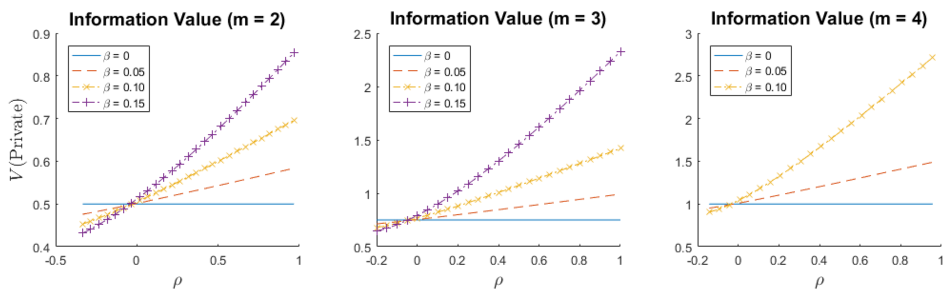

Figure 6: Information value under private information structure increases in the demand correlation ρ

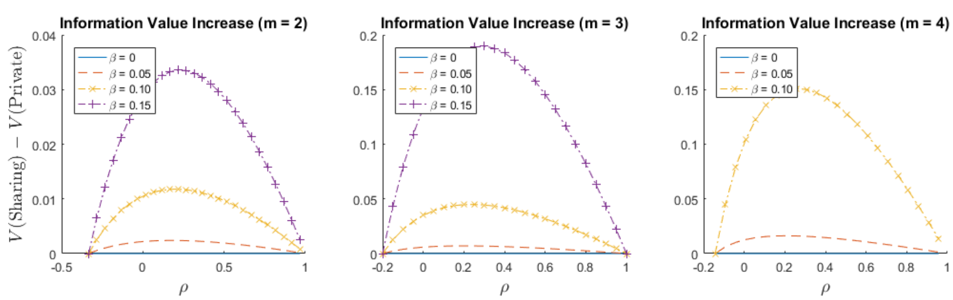

Figure 7: Information value increased from sharing

such questions facilitate the design of information systems. In general, to identify an information structure that maximizes the total information value, we need to solve the following optimization problem

$$\max_{B,M} \left\{ B_{i\star}\Sigma B_{i\star}^{\top} : \boldsymbol{B} \text{ and } \boldsymbol{M} \text{ satisfy (9)} \right\}. \tag{37}$$

This is a mixed integer program with quadratic objective function (convex-maximization). Or generally, we can get rid of binary decision variables M and rewrite it as an Quadratically Constrained Quadratic Program (QCQP)

$$\max_{\boldsymbol{B}\in\mathbb{R}^{n\times n}} \left\{ B_{i\star}\Sigma B_{i\star}^{\top} : (\boldsymbol{QB\Sigma} - \boldsymbol{\Sigma}) * \boldsymbol{B} = \boldsymbol{O} \right\}. \tag{38}$$

Solving the above programs awaits future computational studies, which in itself, is of great interests. In this research, we focus on the strategic aspects of this operational challenge. In Section A.5, we consider a certain type of symmetric information structure — group information sharing. Assume we have one (or multiple) information exchange platform. Every garage can choose to be a member of the platform and share her private signal within the group. In section A.5, we will see that such platform benefits their members and appeals to garages not in the group.

A.5 Information exchange platform

We return to the general model where all parameters can be asymmetric. In this subsection, we discuss the motivation for garages to form an information sharing group. Proposition 8 states that if all garages are using the information exchange platform, then no one has an incentive to quit from the group.

Proposition 8. *All agents joining the group is a Nash equilibrium.*

The equilibrium in Proposition 8 refers to a Nash equilibrium of the group entering strategy. Proposition 8 applies to arbitrary demand correlations as well as cross-elasticities, which is more general than existing literature (Raith, 1996) in this regard, to our best knowledge. It states that given every garage inside the platform, no one can achieve higher profit by unilaterally withdrawing from it. For a two-garage system, Proposition 8 apparently shows that sharing their signal is in the interest of both garages. For a system with more than 2 garages, Proposition 8 ensures that everyone in the group is an equilibrium. The study of the uniqueness and global optimality of such an equilibrium remains for future research.

B Proofs.

In this appendix, we provide detailed proofs of the main results. We use * in the proofs to denote component-wise multiplication of two matrices of identical dimensions.

Proof of Propositions 1 and 2.

Proof. Propositions 1 is a special case of Proposition 2. Here we prove Proposition 2.

Garage i maximizes expected payoff by taking first-order condition:

$$\frac{\partial \pi_i}{\partial p_i} = \left(\alpha_i - \sum_j \beta_{ij} p_j + \theta_i\right) + \beta_{ii}\left(p_i - c_i\right). \tag{39}$$

Garage i observes information θ_{N_i}. Then, her conditional expectation of the entire θ vector is given by

$$\mathbb{E}\left[\theta\left|\theta_{N_i}\right.\right] = \Sigma_{\star N_i}\Sigma_{N_iN_i}^{-1}\theta_{N_i}. \tag{40}$$

Therefore, her anticipation of the pricing vector p is

$$\mathbb{E}\left[p\left|\theta_{N_i}\right.\right] = E\left[A + B\theta\left|\theta_{N_i}\right.\right] = A + B\Sigma_{\star N_i}\Sigma_{N_iN_i}^{-1}\theta_{N_i}. \tag{41}$$

Setting the expected first-order condition to 0, namely $\mathbb{E}\left[\frac{\partial \pi_i}{\partial p_i}\left|\theta_{N_i}\right.\right] = 0$, we obtain

$$\left(\alpha_i - \beta_{i\star}\left(A + B\Sigma_{\star N_i}\Sigma_{N_iN_i}^{-1}\theta_{N_i}\right) + \Sigma_{iN_i}\Sigma_{N_iN_i}^{-1}\theta_{N_i}\right) - \beta_{ii}\left[A + B\Sigma_{\star N_i}\Sigma_{N_iN_i}^{-1}\theta_{N_i} - c\right]_i = 0. \tag{42}$$

Matching the coefficient of θ_{N_i}, we get

$$\left(-Q_{i\star}B\Sigma_{\star N_i} + \Sigma_{iN_i}\right)\Sigma_{N_iN_i}^{-1}\theta_{N_i} - Q_{i\star}A + \alpha_i - [\beta * I]_{i\star}c = 0, \tag{43}$$

where $*$ represents entry-wise multiplication, and $Q = \beta + \beta * I$ is the matrix defined in the proposition. This is a linear equation with respect to θ_{N_i}. And it holds for any θ_{N_i}. Thus, we can get the decision A, B by solving

$$\left\{\begin{array}{l} -Q_{i\star}A + \alpha_i + [\beta * I]_{i\star}c = 0 \\ -Q_{i\star}B\Sigma_{\star N_i} + \Sigma_{iN_i} = 0 \end{array}\right., \forall i \in N. \tag{44}$$

Rewriting the first equation above in vector form, we have

$$A = Q^{-1}\left(\alpha + [\beta * I]\,c\right). \tag{45}$$

Together with the 0 entry constraint, we obtain the system of linear equations which determines our B matrix

$$\left\{\begin{array}{l} Q_{i\star}B\Sigma_{\star N_i} - \Sigma_{iN_i} = 0, \forall i \in N \\ B_{ij} = 0, \forall M_{ij} = 0 \end{array}\right.. \tag{46}$$

Note that, excluding those $B_{ij} = 0$ entries, we have $\sum\limits_{i,j} M_{ij}$ unknowns and $\sum\limits_{i} |N_i| = \sum\limits_{i} M_{ij}$ equations. Thus, there exists at least one solution to this system. In general, the solution is unique.

The expected payoff of garage i conditioned on her observation θ_{N_i} is given by

$$\mathbb{E}\left[\pi_i \left| \theta_{N_i} \right.\right] = E\left[\left(\alpha_i - \beta_{i\star} p + \theta_i\right)\left(p_i - c_i\right)\left| \theta_{N_i} \right.\right] = \left(\alpha_i - \beta_{i\star} E\left[p \left| \theta_{N_i} \right.\right] + \theta_i\right)\left(p_i - c_i\right), \quad (47)$$

which is a convex quadratic function of p_i. Thus, substitute $p = A + B\theta$ which satisfies the first-order condition, we obtain the expected payoff under equilibrium

$$\mathbb{E}\left[\pi_i \left| \theta_{N_i} \right.\right] = \beta_{ii}(p_i - c_i)^2 = \beta_{ii}(A_i + B_{iN_i}\theta_{N_i} - c_i)^2. \quad (48)$$

The expected payoff before the observation of θ_{N_i} is

$$\mathbb{E}\left[\pi_i\right] = \mathbb{E}_{\theta_{N_i}}\left[\beta_{ii}(A_i + B_{iN_i}\theta_{N_i} - c_i)^2\right] = \beta_{ii}\left((A_i - c_i)^2 + B_{iN_i}\Sigma_{N_iN_i}B_{iN_i}^{\top}\right). \quad (49)$$

□

Proof of Lemma 2.

Proof. Rewrite equation (3) in a simpler format,

$$x_{ij} = \frac{l}{2} + \frac{\beta_0}{\lambda}\left(\left(v_i - v_j\right) - \left(p_i - p_j\right)\right). \quad (50)$$

The expectation of v is exogenously given. Thus, $\Delta\mathbb{E}\left[v\right] = 0$. Since $\mathbb{E}\left[\theta\right] = 0$, and followed from Proposition 2,

$$\mathbb{E}\left[p\right] = \mathbb{E}\left[A + \boldsymbol{B}\theta\right] = A.$$

A is independent of the information matrix M as stated in (8). Therefore, $\Delta\mathbb{E}\left[p\right] = 0$, and then $\Delta\mathbb{E}\left[x_{ij}\right] = 0$.

Note that

$$\begin{aligned} C_{ij} &= \mathbb{E}\left[\int_0^{x_{ij}} \left(c_t x + c_w \lambda x^2\right)\lambda dx + \int_0^{l - x_{ij}} \left(c_t x + c_w \lambda x^2\right)\lambda dx\right] \\ &= \lambda\mathbb{E}\left[\frac{1}{2}c_t x_{ij}^2 + \frac{1}{3}c_w\lambda x_{ij}^3 + \frac{1}{2}c_t\left(l - x_{ij}\right)^2 + \frac{1}{3}c_w\lambda\left(l - x_{ij}\right)^3\right] \\ &= \lambda\left(c_t + c_w\lambda l\right)\mathbb{E}\left[x_{ij}^2\right] + \text{constant term}. \end{aligned} \quad (51)$$

Thus, $\Delta C_{ij} = \lambda (c_t + c_w \lambda l) \cdot \Delta \{\mathrm{var}[x_{ij}]\} = \frac{\lambda^2}{2\beta_0} \Delta \{\mathrm{var}[x_{ij}]\}$. For the aggregate cost summed over all links,

$$\begin{aligned}\sum_{ij\in E} \mathrm{var}[x_{ij}] &= \left(\frac{\beta_0}{\lambda}\right)^2 \sum_{ij\in E} \mathrm{var}[(v_i - p_i) - (v_j - p_j)] \\ &= \left(\frac{\beta_0}{\lambda}\right)^2 \sum_{ij\in E} \left(\mathrm{var}[v_i - p_i] + \mathrm{var}[v_j - p_j] - 2\mathrm{cov}(v_i - p_i, v_j - p_j)\right) \\ &= \left(\frac{\beta_0}{\lambda}\right)^2 \langle L, \mathrm{var}[v-p]\rangle .\end{aligned} \tag{52}$$

Hence, $\Delta \left\{ \sum_{ij\in E} C_{ij} \right\} = \frac{\beta_0}{2} \langle L, \mathrm{var}[v-p]\rangle$.

Similarly,

$$\begin{aligned}U_{ij} + C_{ij} &= \mathbb{E}\left[\int_0^{x_{ij}} \lambda dx + \int_0^{l-x_{ij}} (v_j - p_j)\lambda dx\right] \\ &= \lambda \mathbb{E}\left[((v_i - p_i) - (v_j - p_j)) x_{ij}\right] + \text{constant term} \\ &= \frac{\lambda^2}{\beta_0}\mathbb{E}\left[x_{ij}^2\right] + \text{constant term} \\ &= 2C_{ij} + \text{constant term}.\end{aligned} \tag{53}$$

Thus, $\Delta U_{ij} = \Delta C_{ij}$. □

Proof of Proposition 3.

Proof. Both Proposition 3 and 4 are derived from solving (9) for $M = I_n$ and $M = ee^\top$. Then, (11) to solve the information value and (16) to obtain aggregate cost. We omit the algebra for solving B and present the solution directly.

In the private information scenario, $B_{ii} = \frac{3}{16}\beta_0^{-1}, \forall i$. Then,

$$v_i = 2\beta_0 B_{ii}^2 \Sigma_{ii} = \frac{27}{64}\beta_0. \tag{54}$$

$$\langle L, \mathrm{var}[v-p]\rangle = \frac{29}{64} n \cdot \frac{\beta_0}{2}. \tag{55}$$

For the circular model, the B solution is symmetric, i.e., B_{ij} only depends on the distance between i and j but not i or j. Thus, we simply give the solution $B_{1\star}$.

$$B_{1\star} = \begin{cases} \frac{\beta_0^{-1}}{4y_{n/2} - 2y_{n/2-1}} \left[y_{n/2}, y_{n/2-1}, \cdots, y_1, y_0, y_1, \cdots, y_{n/2-1}\right], & \text{if } n \text{ is even}, \\ \frac{\beta_0^{-1}}{4y_{n/2} - 2y_{n/2-1}} \left[y_{n/2}, y_{n/2-1}, \cdots, y_{1/2}, y_{1/2}, \cdots, y_{n/2-1}\right], & \text{if } n \text{ is odd}. \end{cases} \tag{56}$$

Here y_k is a constant defined in Table 2.

Then, by manipulating the hyperbolic functions, the two cases merge to a single analytical format in terms of information value and aggregate cost.

$$v_i = \frac{2}{3\left(y_n - 1\right)} \left(\frac{1}{3}\left(4y_{n-1} + y_n\right) + n - 3 \right) \beta_0 \to \left(2 - \frac{8}{9}\sqrt{3}\right) \beta_0 \approx 0.4604\beta_0 \text{ as } n \to \infty. \tag{57}$$

$$\langle L, \mathrm{var} r \rangle = \frac{2n}{3\left(y_n - 1\right)} \left(\frac{\bar{y}_{n-1} - \bar{y}_n}{\bar{y}_1} + y_n - n \right) \cdot \frac{\beta_0}{2\lambda} \to \frac{2n}{3\sqrt{3}} \cdot \frac{\beta_0}{2} \text{ as } n \to \infty. \tag{58}$$

□

Proof of Proposition 4.

Proof. For private information case, let index 1 denote the center garage. Then,

$$\begin{aligned} B_{11} &= \tfrac{2}{7m-1}\beta_0^{-1}, \\ B_{jj} &= \tfrac{3m-1}{7m-1}\beta_0^{-1}, \forall j \neq 1. \end{aligned} \tag{59}$$

The individual information values are,

$$\begin{aligned} v_1 &= \left(\tfrac{2m}{7m-1}\right)^2 \left(m+1\right)\beta_0, \\ v_j &= 2\left(\tfrac{3m-1}{7m-1}\right)^2 \beta_0, \forall j \neq 1. \end{aligned} \tag{60}$$

The aggregate information value is

$$v_1 + m v_j = \frac{2m\left(11m^2 - 4m + 1\right)}{\left(7m-1\right)^2}\beta_0 \to \frac{22}{49} m\beta_0 \text{ as } m \to \infty. \tag{61}$$

The aggregate cost is

$$\langle L, \mathrm{var}\left[v - p\right] \rangle = \frac{4m^2\left(5m - 3\right)}{\left(7m-1\right)^2} \cdot \frac{\beta_0}{2} \to \frac{20}{49} m \cdot \frac{\beta_0}{2} \text{ as } m \to \infty.$$

For the complete information case, we also list the intermediate and final solutions.

$$B = \left(\frac{1}{3m} \left[\begin{array}{cc} 2 & e^\top \\ e & ee^\top/2 \end{array} \right] + \frac{1}{2} \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right] \right) \beta_0^{-1}. \tag{62}$$

$$\begin{aligned} v_1 &= \tfrac{1}{9}\left(m+1\right)\beta_0, \\ v_j &= \tfrac{13m-5}{36m}\beta_0, \forall i \neq j. \end{aligned} \tag{63}$$

$$v_1 + m v_j = \frac{17m - 1}{36}\beta_0 \to \frac{17}{36} m\beta_0 \text{ as } m \to \infty. \tag{64}$$

$$\langle L, \mathrm{var}\left[v - p\right] \rangle = \frac{13m - 5}{36} \cdot \frac{\beta_0}{2}. \tag{65}$$

□

Proof of Proposition 5.

Proof. We begin by writing a general dual form, using Lagrangian multipliers λ_1 and $\lambda_2 > 0$ to relax the capacity constraints:

$$\begin{aligned}\max_{p_1^{[t]}, t=1,2,...,T} L_1 &= \sum_{t=1}^{T} \left(\alpha^{[t]} - \hat{\beta}^{[t]} p_1^{[t]} + \beta^{[t]} p_2^{[t]} + \theta_1^{[t]}\right)\left(p_1^{[t]} - c\right) \\ &\quad -\lambda_1 \left[W_1 - \sum_{t=1}^{T} \left(\alpha^{[t]} - \hat{\beta}^{[t]} p_1^{[t]} + \beta^{[t]} p_2^{[t]} + \theta_1^{[t]}\right)\right], \end{aligned} \tag{66}$$

$$\begin{aligned}\max_{p_2^{[t]}, t=1,2,...,T} L_2 &= \sum_{t=1}^{T} \left(\alpha^{[t]} - \hat{\beta}^{[t]} p_2^{[t]} + \beta^{[t]} p_1^{[t]} + \theta_2^{[t]}\right)\left(p_2^{[t]} - c\right) \\ &\quad -\lambda_2 \left[W_2 - \sum_{t=1}^{T} \left(\alpha^{[t]} - \hat{\beta}^{[t]} p_2^{[t]} + \beta^{[t]} p_1^{[t]} + \theta_2^{[t]}\right)\right]. \end{aligned} \tag{67}$$

We can decompose these problems by $t = 1, 2, ..., T$, and each sub-problem can by solved by

$$\begin{aligned} p_1^{[t]} &= \frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_1)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_1^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}, \\ p_2^{[t]} &= \frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_2)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_2^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}. \end{aligned} \tag{68}$$

Plug in prices

$$\begin{aligned} &\sum_{t=1}^{T} \alpha^{[t]} - \hat{\beta}^{[t]} \left[\frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_1)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_1^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}\right] \\ &\quad +\beta^{[t]} \left[\frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_2)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_2^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}\right] + \theta_1^{[t]} = W_1, \end{aligned} \tag{69}$$

$$\sum_{t=1}^{T} \alpha^{[t]} - \hat{\beta}^{[t]} \left[\frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_2)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_2^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}\right] \tag{70}$$

$$+\beta^{[t]} \left[\frac{\alpha^{[t]} + \hat{\beta}^{[t]}(c + \lambda_1)}{2\hat{\beta}^{[t]} - \beta^{[t]}} + \frac{\theta_1^{[t]}}{2\hat{\beta}^{[t]} - \beta^{[t]}\rho^{[t]}}\right] + \theta_2^{[t]} = W_2, \tag{71}$$

under stationary conditions: $\hat{\beta}^{[t]} = \hat{\beta}$, $\beta^{[t]} = \beta$, $\rho^{[t]} = \rho$, we have

$$\begin{aligned}
\frac{\hat{\beta}}{2\hat{\beta}-\beta}\sum_{t=1}^{T}\alpha^{[t]}+\frac{T\hat{\beta}\beta}{2\hat{\beta}-\beta}(c+\lambda_2)-\frac{T\hat{\beta}^2}{2\hat{\beta}-\beta}(c+\lambda_1)+\frac{\hat{\beta}+\beta(1-\rho)}{2\hat{\beta}-\beta\rho}\sum_{t=1}^{T}\theta_1^{[t]} &= W_1,\\
\frac{\hat{\beta}}{2\hat{\beta}-\beta}\sum_{t=1}^{T}\alpha^{[t]}+\frac{T\hat{\beta}\beta}{2\hat{\beta}-\beta}(c+\lambda_1)-\frac{T\hat{\beta}^2}{2\hat{\beta}-\beta}(c+\lambda_2)+\frac{\hat{\beta}+\beta(1-\rho)}{2\hat{\beta}-\beta\rho}\sum_{t=1}^{T}\theta_2^{[t]} &= W_2
\end{aligned} \tag{72}$$

When $W_1 = W_2 = W$, and $\lim_{T\to\infty}\frac{\sum_{t=1}^{T}\theta_i^{[t]}}{T} \to E\theta_i^{[t]}$ almost surely, due to Strong Law of Large Numbers, we have

$$\frac{\hat{\beta}}{2\hat{\beta}-\beta}\sum_{t=1}^{T}\alpha^{[t]}+\frac{T\hat{\beta}\left(\beta-\hat{\beta}\right)}{2\hat{\beta}-\beta}(c+\lambda)+\frac{\hat{\beta}+\beta(1-\rho)}{2\hat{\beta}-\beta\rho}\sum_{t=1}^{T}\theta_i^{[t]} = W,$$

$$\lambda = \frac{2\hat{\beta}-\beta}{\hat{\beta}\left(\beta-\hat{\beta}\right)}\cdot\frac{W}{T}-\frac{\left(2\hat{\beta}-\beta\right)\left[\hat{\beta}+\beta(1-\rho)\right]}{\hat{\beta}\left(\beta-\hat{\beta}\right)\left[2\hat{\beta}-\beta\rho\right]}\cdot E\theta_i^{[t]}-\frac{\hat{\beta}}{\hat{\beta}\left(\beta-\hat{\beta}\right)}\cdot\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}-c, \tag{73}$$

whenever this is non-negative. Plug this in the pricing strategies, and $E\theta_i^{[t]}=0$, we obtain

$$\begin{aligned}
p_1^{[t]} &= \frac{\alpha^{[t]}}{2\hat{\beta}-\beta}+\frac{W/T}{\left(\beta-\hat{\beta}\right)}-\frac{\hat{\beta}}{\left(2\hat{\beta}-\beta\right)\left(\beta-\hat{\beta}\right)}\cdot\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}+\frac{\theta_1^{[t]}}{2\hat{\beta}-\beta\rho},\\
p_1^{[t]} &= \frac{\alpha^{[t]}}{2\hat{\beta}-\beta}+\frac{W/T}{\left(\beta-\hat{\beta}\right)}-\frac{\hat{\beta}}{\left(2\hat{\beta}-\beta\right)\left(\beta-\hat{\beta}\right)}\cdot\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}+\frac{\theta_2^{[t]}}{2\hat{\beta}-\beta\rho}.
\end{aligned} \tag{74}$$

When $W_1 \neq W_2$, a similar procedure returns $\lambda_1 \neq \lambda_2$, a one-period snapshot is equivalent to an incapacitated static model with heterogeneous cost $c_1 = c+\lambda_1$, and $c_2 = c+\lambda_2$. For finite T,

$$p_i^{[t]} = \frac{\alpha^{[t]}+\hat{\beta}\left(c+\lambda_i\right)}{2\hat{\beta}-\beta}+\frac{\theta_i^{[t]}}{2\hat{\beta}-\beta\rho}, \tag{75}$$

wherein λ_i will be function of both $\theta_1^{[t]}$ and $\theta_2^{[t]}$, which is inconsistent, and thus we need $\mathbb{E}\left(p_2^{[t]}|\theta_1^{[t]}\right)$ to solve garage 1's maximization problem. This problem is fundamentally more complicated and awaits future research.

Alternatively, when

$$\frac{2\hat{\beta}-\beta}{\hat{\beta}\left(\beta-\hat{\beta}\right)}\cdot\frac{W}{T}-\frac{\left(2\hat{\beta}-\beta\right)\left[\hat{\beta}+\beta(1-\rho)\right]}{\hat{\beta}\left(\beta-\hat{\beta}\right)\left[2\hat{\beta}-\beta\rho\right]}\cdot E\theta_i^{[t]}-\frac{\hat{\beta}}{\hat{\beta}\left(\beta-\hat{\beta}\right)}\cdot\frac{\sum_{t=1}^{T}\alpha^{[t]}}{T}-c<0, \tag{76}$$

we have non-binding capacity constraint ($\lambda = 0$). This case is trivial, with

$$p_i^{[t]} = \frac{\alpha^{[t]} + \hat{\beta}c}{2\hat{\beta} - \beta} + \frac{\theta_i^{[t]}}{2\hat{\beta} - \beta\rho}. \tag{77}$$

Compared with the static incapacitated version:

$$\lim_{T\to\infty} p_i^{[t]} - p_i = \frac{W/T}{\left(\beta - \hat{\beta}\right)} - \frac{\hat{\beta}}{\left(2\hat{\beta} - \beta\right)\left(\beta - \hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T} \alpha^{[t]}}{T} - \frac{\hat{\beta}c}{2\hat{\beta} - \beta} < 0,$$

which means the response to a private signal remains the same, while the baseline rates suffer from a constant downward distortion. The aggregate payoff is

$$\lim_{T\to\infty} \mathbb{E}\pi = \sum_{t=1}^{T} \hat{\beta} \left(\left[\frac{\alpha^{[t]}}{2\hat{\beta} - \beta} - \left(c - \left[\frac{W/T}{\left(\beta - \hat{\beta}\right)} - \frac{\hat{\beta}}{\left(2\hat{\beta} - \beta\right)\left(\beta - \hat{\beta}\right)} \cdot \frac{\sum_{t=1}^{T} \alpha^{[t]}}{T} \right] \right) \right]^2 + \frac{\hat{\beta}\sigma^2}{(2\hat{\beta} - \beta\rho)^2} \right),$$

which is decreasing in W and increasing in $\frac{\sum_{t=1}^{T} \alpha^{[t]}}{T}$ when $\hat{\beta} > \beta$, the converse is true when $\frac{\beta}{2} < \hat{\beta} \leq \beta$. □

Proof of Proposition 6.

Proof. From Proposition 2, we can obtain the equilibrium pricing strategy $p_i = A + Bx_i, \forall i =$1,2, and $\mathbb{E}\pi_1 = \mathbb{E}\pi_2 = \mathbb{E}\pi = \hat{\beta}\left((A - c)^2 + B^2\sigma^2\right)$. Garage i maximize expected payoff by taking first-order condition:

$$\frac{\partial \mathbb{E}\left(\pi_1 | x_1\right)}{\partial p_1} = \alpha - \hat{\beta}p_1 + \beta\mathbb{E}\left(p_2 | x_1\right) + \mathbb{E}\left(\theta_1 | x_1\right) - \hat{\beta}\left(p_1 - c\right), \tag{78}$$

Plug in $p_i = A + Bx_i$, $\mathbb{E}\left(\theta_1 | x_1\right) = \frac{1/\gamma^2}{1/\sigma^2 + 1/\gamma^2} x_1 = \frac{\sigma^2}{\sigma^2 + \gamma^2} x_1$, and

$$\mathbb{E}\left(x_2 | x_1\right) = \mathbb{E}\left(\theta_2 + \epsilon_2 | x_1\right) = \mathbb{E}\left(\theta_2 | x_1\right) = \mathbb{E}\left[\mathbb{E}\left(\theta_2 | \theta_1, x_1\right) | x_1\right] = \rho\mathbb{E}\left(\theta_1 | x_1\right) = \frac{\rho\sigma^2}{\sigma^2 + \gamma^2}. \tag{79}$$

The first-order condition becomes:

$$\alpha - 2\hat{\beta}\left(A + Bx_1\right) + \beta\left[A + B\frac{\rho\sigma^2}{\sigma^2 + \gamma^2} x_1\right] + \frac{\sigma^2}{\sigma^2 + \gamma^2} x_1 + \hat{\beta}c = 0. \tag{80}$$

Matching coefficients:

$$\alpha - 2\hat{\beta}A + \beta A + \hat{\beta}c = 0,$$

$$-2\hat{\beta}B+\beta B\frac{\rho\sigma^2}{\sigma^2+\gamma^2}+\frac{\sigma^2}{\sigma^2+\gamma^2}=0, \tag{81}$$

we have

$$A=\frac{\alpha+\hat{\beta}c}{2\hat{\beta}-\beta}, B=\frac{\sigma^2}{\left(2\hat{\beta}-\beta\rho\right)\sigma^2+2\hat{\beta}\gamma^2}. \tag{82}$$

To summarize:

$$p_i=\frac{\alpha+\hat{\beta}c}{2\hat{\beta}-\beta}+\frac{\sigma^2}{\left(2\hat{\beta}-\beta\rho\right)\sigma^2+2\hat{\beta}\gamma^2}\cdot x_i, \tag{83}$$

$$\begin{aligned}\mathbb{E}\pi &= \hat{\beta}\left[\frac{\alpha+\hat{\beta}c}{2\hat{\beta}-\beta}-c\right]^2+\left[\frac{\sigma^2}{\left(2\hat{\beta}-\beta\rho\right)\sigma^2+2\hat{\beta}\gamma^2}\right]^2\cdot\hat{\beta}\mathbb{E}x_i^2 &(84)\\ &= \hat{\beta}\left[\frac{\alpha-(\hat{\beta}-\beta)c}{2\hat{\beta}-\beta}\right]^2+\frac{\hat{\beta}\sigma^4(\sigma^2+\gamma^2)}{\left[\left(2\hat{\beta}-\beta\rho\right)\sigma^2+2\hat{\beta}\gamma^2\right]^2}. &(85)\end{aligned}$$

It can be checked that $\frac{\partial B}{\partial\rho}>0$, $\frac{\partial B}{\partial\sigma}>0$, $\frac{\partial B}{\partial\gamma}<0$, $\frac{\partial\mathbb{E}\pi}{\partial\hat{\beta}}<0$, $\frac{\partial\mathbb{E}\pi}{\partial\beta}>0$ and $\frac{\partial\mathbb{E}\pi}{\partial\rho}>0$.

Suppose that garages share information. We can obtain the equilibrium pricing strategy $p_1=A+B_1x_1+B_2x_2$, $p_2=A+B_2x_1+B_1x_2$. We have

$$\begin{aligned}\mathbb{E}\left(\theta_1|x_1,x_2\right) &= \mathbb{E}\left[\mathbb{E}\left(\theta_1|x_1,x_2,\theta_2\right)|x_1,x_2\right]\\ &= \mathbb{E}\left[\mathbb{E}\left(\theta_1|x_1,\theta_2\right)|x_1,x_2\right], &(86)\end{aligned}$$

with marginal distribution being $\theta_1|\theta_2\sim N(\rho\theta_2,\sigma^2\left(1-\rho^2\right))$,

$$\mathbb{E}\left(\theta_1|x_1,\theta_2\right)=\frac{\sigma^2\left(1-\rho^2\right)}{\sigma^2\left(1-\rho^2\right)+\gamma^2}\cdot x_1+\frac{\gamma^2\rho\theta_2}{\sigma^2\left(1-\rho^2\right)+\gamma^2}, \tag{87}$$

$$\begin{aligned}\mathbb{E}\left(\theta_1|x_1,x_2\right) &= \frac{\sigma^2\left(1-\rho^2\right)}{\sigma^2\left(1-\rho^2\right)+\gamma^2}\cdot\mathbb{E}\left(x_1|x_1,x_2\right)+\frac{\gamma^2\rho}{\sigma^2\left(1-\rho^2\right)+\gamma^2}\cdot\mathbb{E}\left(\theta_2|x_1,x_2\right)\\ &= \frac{\sigma^2\left(1-\rho^2\right)}{\sigma^2\left(1-\rho^2\right)+\gamma^2}\cdot x_1+\frac{\gamma^2\rho}{\sigma^2\left(1-\rho^2\right)+\gamma^2}\cdot\frac{\sigma^2}{\sigma^2+\gamma^2}\cdot x_2. &(88)\end{aligned}$$

$$E\left(\theta_1\left|x_1,x_2\right.\right)=\frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}\left[\sigma^2\left(1-\rho^2\right)+\gamma^2,\rho\gamma^2\right]\left[\begin{array}{c}x_1\\x_2\end{array}\right].$$

Garage i maximize expected payoff by taking first-order condition:

$$\frac{\partial \mathbb{E}\left(\pi_1|x_1,x_2\right)}{\partial p_1} = \alpha - 2\hat{\beta}(A+B_1x_1+B_2x_2) + \beta\mathbb{E}\left(A+B_2x_1+B_1x_2|x_1,x_2\right) + \mathbb{E}\left(\theta_1|x_1,x_2\right) + \hat{\beta}c. \tag{89}$$

Matching coefficients:

$$\alpha - \left(2\hat{\beta}-\beta\right)A + \hat{\beta}c = 0 \Rightarrow A = \frac{\alpha+\hat{\beta}c}{2\hat{\beta}-\beta}, \tag{90}$$

$$-2\hat{\beta}(B_1x_1+B_2x_2) + \beta\mathbb{E}\left(B_2x_1+B_1x_2|x_1,x_2\right) + \frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}\left[\sigma^2\left(1-\rho^2\right)+\gamma^2, \rho\gamma^2\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = 0$$

$$-2\hat{\beta}B_1 + \beta B_2 + \frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}\cdot\left[\sigma^2\left(1-\rho^2\right)+\gamma^2\right] = 0, \tag{91}$$

$$-2\hat{\beta}B_2 + \beta B_1 + \frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}\cdot\rho\gamma^2 = 0, \tag{92}$$

which gives

$$\begin{aligned} B_1 &= \frac{\gamma^2(2\hat{\beta}+\beta\rho)+2\hat{\beta}(1-\rho^2)\sigma^2}{4\hat{\beta}^2-\beta^2}\frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}, \\ B_2 &= \frac{\gamma^2(2\hat{\beta}\rho+\beta)+\beta(1-\rho^2)\sigma^2}{4\hat{\beta}^2-\beta^2}\frac{\sigma^2}{\left(\sigma^2+\gamma^2\right)^2-\left(\rho\sigma^2\right)^2}. \end{aligned} \tag{93}$$

Since $2\hat{\beta} > \beta$, garages respond positively toward signals.

$$\mathbb{E}\pi = \hat{\beta}\left[\frac{\alpha-(\hat{\beta}-\beta)c}{2\hat{\beta}-\beta}\right]^2 + \hat{\beta}\sigma^2\frac{\sigma^2(\sigma^2+\gamma^2)}{\left[\left(2\hat{\beta}-\beta\rho\right)\sigma^2+2\hat{\beta}\gamma^2\right]^2}. \tag{94}$$

$$\mathbb{E}\pi' = \hat{\beta}\left[\frac{\alpha-(\hat{\beta}-\beta)c}{2\hat{\beta}-\beta}\right]^2 + \hat{\beta}\sigma^2\left(B_1^2+B_2^2+2\rho B_1B_2\right). \tag{95}$$

Recall that $\mathbb{E}\left(\theta_1|x_1,x_2\right)$ has two parts, one associated with forecasting via x_1, the other associated with forecasting indirectly via x_2, since θ_1 and θ_2 is correlated. We can explicitly observe the corresponding information value in the expression of B_1 and B_2. It can be checked that $\mathbb{E}\pi' - \mathbb{E}\pi > 0$ as $\beta/\hat{\beta} \to 2$, i.e., information sharing is desirable when $\beta/\hat{\beta} \to 2$. More comprehensive characterization can be obtained when $\gamma \to 0$, as in Proposition 1. □

Proof of Proposition 7.

Proof. For conciseness, we present the proof of proposition using a two-coalition formulation. The proof naturally extends to the multiple-coalition cases.

We use a sequence of vector-matrix formulations to prove the proposition. To clarify the notations, $\langle \cdot, \cdot \rangle$ is the Frobenius inner product of two vectors/matrices of the same dimension(s); $z_K = [z]_K$ takes the subvector from vector z based on the index set K. Suppose the two coalitions are denoted by index sets K_1 and K_2

$$\Pi(K_1) = \langle d_{K_1}, p_{K_1} - c_{K_1} \rangle = \langle \alpha_{K_1} - \beta_{K_1\star} p + \theta_{K_1}, p_{K_1} - c_{K_1} \rangle .$$

Define $\boldsymbol{Q} := \boldsymbol{\beta} + \begin{bmatrix} \beta_{K_1K_1}^\top & O \\ O & \beta_{K_2K_2}^\top \end{bmatrix}$. Utilizing the expression of $\boldsymbol{Q}$ and the linear pricing $p = A + \boldsymbol{B}\theta$, we have Since $\mathbb{E}\left[\theta | \theta_{N(K_1)}\right] = \Sigma_{\star N(K_1)} \Sigma_{N(K_1)N(K_1)}^{-1} \theta_{N(K_1)}$, we have

$$\begin{aligned} \frac{\partial \Pi(K_1)}{\partial p_{K_1}} &= -\beta_{K_1K_1}^\top (p_{K_1} - c_{K_1}) + \alpha_{K_1} - \beta_{K_1\star} p + \theta_{K_1} \\ &= [-\boldsymbol{Q}p + (\boldsymbol{Q} - \boldsymbol{\beta}) c + \alpha + \theta]_{K_1}, \\ &= [-\boldsymbol{Q}A + (\boldsymbol{Q} - \boldsymbol{\beta}) c + \alpha + (\boldsymbol{I} - \boldsymbol{Q}\boldsymbol{B}) \theta]_{K_1}. \end{aligned}$$

$$\mathbb{E}\left[\left. \frac{\partial \Pi(K_1)}{\partial p_{K_1}} \right| \theta_{N(K_1)} \right] = \left[-\boldsymbol{Q}A + (\boldsymbol{Q} - \boldsymbol{\beta}) c + \alpha + (\boldsymbol{I} - \boldsymbol{Q}\boldsymbol{B}) \Sigma_{\star N(K_1)} \Sigma_{N(K_1)N(K_1)}^{-1} \theta_{N(K_1)} \right]_{K_1}.$$

Under the equilibrium pricing strategy, the R.H.S is 0 for every $\theta_{N(K_1)}$. Therefore, we have

$$\begin{aligned} [-\boldsymbol{Q}A + (\boldsymbol{Q} - \boldsymbol{\beta}) c]_{K_1} &= 0, \\ \Sigma_{K_1 N(K_1)} - Q_{K_1\star} \boldsymbol{B} \Sigma_{\star N(K_1)} &= 0. \end{aligned}$$

The same relations apply to K_2 as well. Thus, we obtain (33) and (34). (The 0 entries in $\boldsymbol{B}$ are enforced by the information structure.)

K_1's expected payoff given the information $\theta_{N(K_1)}$ is

$$\begin{aligned} \mathbb{E}\left[\Pi(K_1) | \theta_{N(K_1)} \right] &= \mathbb{E}\left[\langle \alpha_{K_1} - \beta_{K_1\star} p + \theta_{K_1}, p_{K_1} - c_{K_1} \rangle | \theta_{N(K_1)} \right] \\ &= \left\langle \mathbb{E}\left[\alpha_{K_1} - \beta_{K_1\star} p + \theta_{K_1} | \theta_{N(K_1)} \right], p_{K_1} - c_{K_1} \right\rangle \\ &= \left\langle \beta_{K_1K_1}^\top (p_{K_1} - c_{K_1}), p_{K_1} - c_{K_1} \right\rangle. \end{aligned}$$

The last equality follows from the equilibrium condition

$$\mathbb{E}\left[\left. \frac{\partial \Pi(K_1)}{\partial p_{K_1}} \right| \theta_{N(K_1)} \right] = \mathbb{E}\left[-\beta_{K_1K_1}^\top (p_{K_1} - c_{K_1}) + \alpha_{K_1} - \beta_{K_1\star} p + \theta_{K_1} \right| \left. \theta_{N(K_1)} \right] = 0.$$

Finally, the expected payoff is

$$\begin{aligned}
\mathbb{E}_\theta\left[\Pi\left(K_1\right)\right] &= \mathbb{E}_\theta\left[\Pi\left(K_1\right)\middle|\,\theta_{N(K_1)}\right] \\
&= \mathbb{E}_\theta\left[\left\langle \beta_{K_1K_1}^\top\left(p_{K_1}-c_{K_1}\right), p_{K_1}-c_{K_1}\right\rangle\middle|\,\theta_{N(K_1)}\right] \\
&= \left\langle \beta_{K_1K_1}^\top\left(A_{K_1}-c_{K_1}\right), A_{K_1}-c_{K_1}\right\rangle + \mathbb{E}_\theta\left[\left\langle \beta_{K_1K_1}^\top B_{K_1\star}\theta, B_{K_1\star}\theta\right\rangle\right] \\
&\quad \left\langle \beta_{K_1K_1}^\top\left(A_{K_1}-c_{K_1}\right), A_{K_1}-c_{K_1}\right\rangle + \left\langle B_{K_1\star}\mathbf{\Sigma}B_{K_1\star}^\top, \beta_{K_1K_1}\right\rangle.
\end{aligned}$$

This is equivalent to (36) and we conclude the proof. □ □

Proof of Proposition 8.

Proof. Let $S=N\backslash\{n\}$ be the set group members, and n be the only agent outside the info-sharing group. We need to prove the information value v_n under this structure is less than the value when all agents are in the group.

B_{nn} satisfies

$$\begin{cases} Q_{SS}B_{SS}\Sigma_{SS}+Q_{Sn}B_{nn}\Sigma_{nS}-\Sigma_{SS}=0 \\ Q_{nS}B_{SS}\Sigma_{Sn}+Q_{nn}B_{nn}\Sigma_{nn}-\Sigma_{nn}=0 \end{cases}. \tag{96}$$

Eliminate B_{SS}, we get

$$B_{nn}=-\frac{Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}-\Sigma_{nn}}{Q_{nn}\Sigma_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\Sigma_{nS}\Sigma_{SS}^{-1}\Sigma_{Sn}}. \tag{97}$$

Thus,

$$v_n=\beta_{nn}B_{nn}^2\Sigma_{nn}=\beta_{nn}\left(\frac{Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}-\Sigma_{nn}}{Q_{nn}\Sigma_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\Sigma_{nS}\Sigma_{SS}^{-1}\Sigma_{Sn}}\right)^2\Sigma_{nn}. \tag{98}$$

If all agents are in the group, utilizing the inverse of block matrix, we have

$$\tilde{B}_{n\star}=\left[Q^{-1}\right]_{n\star}=-\frac{1}{Q_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}}\left[Q_{nS}Q_{SS}^{-1},-1\right]. \tag{99}$$

$$\tilde{v}_n=\beta_{nn}\tilde{B}_{n\star}\Sigma\tilde{B}_{n\star}^\top=\frac{Q_{nS}Q_{SS}^{-1}\Sigma_{SS}\left(Q_{nS}Q_{SS}^{-1}\right)^\top-2Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}+\Sigma_{nn}}{\left(Q_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\right)^2}\cdot\beta_{nn} \tag{100}$$

Then, we prove $v_n<\tilde{v}_n$.

$$\begin{aligned}
\frac{v_n}{\beta_{nn}} &= \left(\frac{Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}-\Sigma_{nn}}{Q_{nn}\Sigma_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\Sigma_{nS}\Sigma_{SS}^{-1}\Sigma_{Sn}}\right)^2\Sigma_{nn} = \frac{Q_{nS}Q_{SS}^{-1}\frac{\Sigma_{Sn}\Sigma_{nS}}{\Sigma_{nn}}\left(Q_{nS}Q_{SS}^{-1}\right)^\top-2Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}+\Sigma_{nn}}{\left(Q_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\frac{\Sigma_{nS}\Sigma_{SS}^{-1}\Sigma_{Sn}}{\Sigma_{nn}}\right)^2} \\
&< \frac{Q_{nS}Q_{SS}^{-1}\frac{\Sigma_{Sn}\Sigma_{nS}}{\Sigma_{nn}}\left(Q_{nS}Q_{SS}^{-1}\right)^\top-2Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}+\Sigma_{nn}}{\left(Q_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\right)^2} < \frac{Q_{nS}Q_{SS}^{-1}\Sigma_{SS}\left(Q_{nS}Q_{SS}^{-1}\right)^\top-2Q_{nS}Q_{SS}^{-1}\Sigma_{Sn}+\Sigma_{nn}}{\left(Q_{nn}-Q_{nS}Q_{SS}^{-1}Q_{Sn}\right)^2} = \frac{\tilde{v}_n}{\beta_{nn}}.
\end{aligned} \tag{101}$$

The first inequality follows from $\frac{\Sigma_{nS}\Sigma_{SS}^{-1}\Sigma_{Sn}}{\Sigma_{nn}} < 1$ (since Σ is positive definite) and Q_{nn}, $Q_{nS}Q_{SS}^{-1}Q_{Sn}$, $Q_{nn} - Q_{nS}Q_{SS}^{-1}Q_{Sn} > 0$. The second inequality holds since $\Sigma_{SS} - \frac{\Sigma_{Sn}\Sigma_{nS}}{\Sigma_{nn}}$ is positive definite.

$\square$