Supplemental Online Materials to "Optimal Inventory Management with Buy-One-Give-One (BOGO) Promotion" by Soeun Park, Woonghee Tim Huh and Byung Cho Kim

Proof of Proposition 1. Part (a) follows directly from $s < c_0$ in (3.4), and part (b) from (3.7).

For part (c), the form of $\tilde{y}^*$ given in (3.13) follows directly from (3.12). Since $\tilde{G}(x)$ in (3.10) is independent of c_0 and the ratio $(p - \tilde{c} - c_0)/(p - \tilde{c} - s)$ is decreasing in c_0, we obtain that $\tilde{y}^*$ is decreasing in c_0. The lower bound for $\tilde{y}^*$ follows from (3.11). The upper bounds for $\tilde{y}^*$ follow from (a) and (3.11). □

Proof of Proposition 2. The equivalence result is immediate from observing that while each unit sold generates the profit of p, it also incurs the cash donation cost of $\beta\tilde{c}$. Then the expressions for $\hat{\pi}(y, \hat{D})$ and $\hat{y}^*$ follow from (2.1) and (2.3), respectively. □

Proof of Prosition 3. For any $y \geq 0$, recall from Proposition 2 that the profit function with cash donation is given by:

$$\hat{\pi}(y, D) \quad = \quad -c_0 y + p \cdot \min\{D, y\} - \beta\tilde{c} \cdot \min\{D, y\} + s \cdot \max\{y - D, 0\}.$$

Also, recall from (3.7) that the profit function with BOGO is given by:

$$\tilde{\pi}(y, D) \quad = \quad -c_0 y + p \cdot \min\{D, y\} - \tilde{c} \cdot Z + s \cdot \max\{y - 2D, 0\}$$

where $Z = \max\{\min\{D, y\} - \max\{y - D, 0\}, 0\}$.

Since Z is nonnegative and $\max\{y - 2D, 0\} \leq \max\{y - D, 0\}$, it follows from the nonnegativity of $\tilde{c}$ and s that

$$\hat{\pi}(y, D) + \beta\tilde{c} \cdot \min\{D, y\} \quad \geq \quad \tilde{\pi}(y, D) \ .$$

It shows that if $\beta = 0$ holds, BOGO yields a lower profit for any fixed y provided that BOGO has demand distribution as cash donation. This result is intuitive since BOGO introduces additional commitment to supply (for the give-away units), which is costly. With positive β, BOGO can be better than cash donation, but the difference is at most $\beta\tilde{c} \cdot \min\{D, y\}$, i.e., $\hat{\pi}(y, D) - \tilde{\pi}(y, D) \geq -\beta\tilde{c} \cdot \min\{D, y\}$. Thus, we obtain $\hat{\pi}(y, D) - \tilde{\pi}(y, D) \geq -\beta\tilde{c} \cdot D$.

We proceed to upper-bound the difference in profits:

$$\hat{\pi}(y,D) - \tilde{\pi}(y,D) = \tilde{c}\cdot Z + s\cdot(\max\{y-D,0\} - \max\{y-2D,0\}) - \beta\tilde{c}\cdot\min\{D,y\}.$$

We have

$$\max\{y-D,0\} - \max\{y-2D,0\} = \begin{cases} 0 & \text{if } 0 \le y \le D \\ y-D & \text{if } D \le y \le 2D \\ D & \text{if } y \ge 2D. \end{cases}$$

Recall from (3.6)

$$Z = \begin{cases} y & \text{if } 0 \le y \le D \\ 2D-y & \text{if } D \le y \le 2D \\ 0 & \text{if } y \ge 2D. \end{cases}$$

Thus, it follows that

$$\begin{aligned} \hat{\pi}(y,D) - \tilde{\pi}(y,D) &= \tilde{c}\cdot Z + s\cdot\{\max\{y-D,0\} - \max\{y-2D,0\}\} - \beta\tilde{c}\cdot\min\{D,y\} \\ &= \begin{cases} (1-\beta)\tilde{c}\cdot y & \text{if } 0 \le y \le D \\ -(\tilde{c}-s)\cdot(y-D) + (1-\beta)\tilde{c}\cdot D & \text{if } D \le y \le 2D \\ (s-\beta\tilde{c})\cdot D & \text{if } y \ge 2D. \end{cases} \\ &\le (1-\beta)\tilde{c}\cdot D\ , \end{aligned}$$

where the last inequality holds from our assumption $\tilde{c} > s$ from (3.4). This bound $(1-\beta)\tilde{c}\cdot D$ makes sense because one could (suboptimally) manage BOGO by purchasing all of its give-away quantities after observing the demand, at the cost of $\tilde{c}$ per unit, whereas the cash donation model has $\beta\tilde{c}$ donated per unit sold. This establishes the first part of the required result.

The first part implies, for every $y \ge 0$,

$$-\beta\tilde{c}\cdot E[D] \le E_D[\hat{\pi}(y,D)] - E_D[\tilde{\pi}(y,D)] \le (1-\beta)\tilde{c}\cdot E[D].$$

Let $\hat{y}^*$ be the value of y maximizing $E_D[\hat{\pi}(y,D)]$, and let $\tilde{y}^*$ be the value of y maximizing

$E_D[\tilde{\pi}(y, D)]$. Then,

$$\begin{aligned}
\max_y E_D[\hat{\pi}(y, D)] - \max_y E_D[\tilde{\pi}(y, D)] &= E_D[\hat{\pi}(\hat{y}^*, D)] - E_D[\tilde{\pi}(\tilde{y}^*, D)] \\
&\leq E_D[\hat{\pi}(\hat{y}^*, D)] - E_D[\tilde{\pi}(\hat{y}^*, D)] \\
&\leq (1-\beta)\tilde{c} \cdot E[D].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\max_y E_D[\hat{\pi}(y, D)] - \max_y E_D[\tilde{\pi}(y, D)] &= E_D[\hat{\pi}(\hat{y}^*, D)] - E_D[\tilde{\pi}(\tilde{y}^*, D)] \\
&\geq E_D[\hat{\pi}(\tilde{y}^*, D)] - E_D[\tilde{\pi}(\tilde{y}^*, D)] \\
&\geq -\beta\tilde{c} \cdot E[D].
\end{aligned}$$

These inequalities complete the proof. □

Proof of Proposition 4. Suppose $\beta \geq 1$. Then, $(p - \tilde{c} - c_0)/(p - \tilde{c} - s) \leq (p - \beta\tilde{c} - c_0)/(p - \beta\tilde{c} - s)$. Since (3.11) implies $\tilde{G}^{-1}(\cdot) \geq \hat{F}^{-1}(\cdot)$, it follows from (4.14) that $\hat{y}^* \leq \tilde{y}^*$ holds. We proceed by assuming $\beta < 1$.

Recall that $E_{\tilde{D}}[\tilde{\pi}(y, \tilde{D})]$ is concave in y, and achieves its maximum at $\tilde{y}^*$. Thus, $\tilde{y}^* > \hat{y}^*$ holds if, from (3.12),

$$0 < \left. \frac{dE_{\tilde{D}}[\tilde{\pi}(y, \tilde{D})]}{dy} \right|_{y=\hat{y}^*} = -(p - \tilde{c} - s) \cdot \tilde{G}(\hat{y}^*) + p - \tilde{c} - c_0 \ .$$

The above inequality is equivalent to $\tilde{G}(\hat{y}^*) < (p - \tilde{c} - c_0)/(p - \tilde{c} - s)$.

By using a similar argument, it can be shown that if $\tilde{G}(\hat{y}^*) > (p - \tilde{c} - c_0)/(p - \tilde{c} - s)$, then $\tilde{y}^* < \hat{y}^*$ holds. Furthermore, if $\tilde{G}(\hat{y}^*) = (p - \tilde{c} - c_0)/(p - \tilde{c} - s)$, then (3.12) shows

$$\left. \frac{dE_{\tilde{D}}[\tilde{\pi}(y, \tilde{D})]}{dy} \right|_{y=\hat{y}^*} = -(p - \tilde{c} - s) \cdot \tilde{G}(\hat{y}^*) + p - \tilde{c} - c_0 = 0 \ ,$$

which implies $\tilde{y}^* = \hat{y}^*$. This proves the required result. □

Proof of Corollary 1. For this proof, we consider the sign of $\frac{dE_{\tilde{D}}[\tilde{\pi}(y, \tilde{D})]}{dy}$ at $y = \hat{y}^*$. If the derivative is nonnegative, then $\tilde{y}^* \geq \hat{y}^*$ holds; if the derivative is nonpositive, then $\tilde{y}^* \leq \hat{y}^*$

holds. From (3.12) and (3.10),

$$
\begin{aligned}
& \frac{dE_{\tilde{D}}[\tilde{\pi}(y,\tilde{D})]}{dy}\Big|_{y=\hat{y}^*} \\
= \;& -(p-\tilde{c}-s)\cdot\tilde{G}(\hat{y}^*)+p-\tilde{c}-c_0 \\
= \;& -(p-\tilde{c}-s)\cdot\left[\frac{\tilde{c}-s}{p-\tilde{c}-s}\cdot\tilde{F}(\hat{y}^*/2)+\frac{p-2\tilde{c}}{p-\tilde{c}-s}\cdot\tilde{F}(\hat{y}^*)\right]+p-\tilde{c}-c_0 \\
= \;& -(\tilde{c}-s)\cdot\tilde{F}(\hat{y}^*/2)-(p-2\tilde{c})\cdot\tilde{F}(\hat{y}^*)+p-\tilde{c}-c_0\ .
\end{aligned}
$$

Recall $\hat{y}^*=\hat{F}^{-1}\left(\frac{p-\beta\tilde{c}-c_0}{p-\beta\tilde{c}-s}\right)$ is given in Proposition 2, and $\tilde{F}(\cdot)=\hat{F}(\cdot)$. Since $\tilde{F}(\hat{y}^*/2)=0$ holds by assumption, we have

$$
\begin{aligned}
& \frac{dE_{\tilde{D}}[\tilde{\pi}(y,\tilde{D})]}{dy}\Big|_{y=\hat{y}^*} \\
= \;& -(p-2\tilde{c})\cdot\frac{p-\beta\tilde{c}-c_0}{p-\beta\tilde{c}-s}+p-\tilde{c}-c_0 \\
= \;& \frac{1}{p-\beta\tilde{c}-s}\cdot[-(p-2\tilde{c})(p-\beta\tilde{c}-c_0)+(p-\beta\tilde{c}-s)(p-\tilde{c}-c_0)] \\
= \;& \frac{-p^2+2\tilde{c}p+c_0p-2c_0\tilde{c}+p^2-\tilde{c}p-c_0p-ps+\tilde{c}s+c_0s}{p-\beta\tilde{c}-s}+\frac{(p-2\tilde{c})\beta\tilde{c}-\beta\tilde{c}(p-\tilde{c}-c_0)}{p-\beta\tilde{c}-s} \\
= \;& \frac{\tilde{c}p-2c_0\tilde{c}-ps+\tilde{c}s+c_0s+(c_0-\tilde{c})\beta\tilde{c}}{p-\beta\tilde{c}-s}\ .
\end{aligned}
$$

Above, the denominator $p-\beta\tilde{c}-s$ is always positive for the cash donation model to yield a positive profit. The numerator is increasing in p since $\tilde{c}-s>0$. Thus, there exists τ_p such that if $p\le\tau_p$, then the above expression is at most 0, i.e., $\tilde{y}^*\le\hat{y}^*$ holds, and that if $p\ge\tau_p$, then the above expression is at least 0, i.e., $\tilde{y}^*\ge\hat{y}^*$ holds. This proves (a).

Similarly, the numerator is decreasing in c_0 if $0\le\beta<\frac{2\tilde{c}-s}{\tilde{c}}$ since $-2\tilde{c}+s+\beta\tilde{c}<0$, and an argument analogous to part (a) proves (b). Finally, the numerator is decreasing in s since $-p+\tilde{c}+c_0<0$. This proves (c). $\square$

Proof of Proposition 5. Recall that $\tilde{F}$ is the cumulative distribution function of $\tilde{D}\sim\alpha\hat{D}$. In this proof, it is convenient to use subscript α to explicitly denote the dependency on α. In other words, let $\tilde{F}_\alpha$ denote the cumulative distribution function of $\alpha\hat{D}$. Then, $\tilde{F}_1$ is identical to $\hat{F}$, i.e., $\tilde{F}_1$ is the cumulative distribution function of $\hat{D}$. For any $\alpha>0$ and $x\ge0$, we have

$$\tilde{F}_\alpha(\alpha x)=\tilde{F}_1(x).$$

Moreover, from (3.10),

$$\begin{aligned}\tilde{G}_\alpha(\alpha x) &= \frac{\tilde{c}-s}{p-\tilde{c}-s}\cdot \tilde{F}_\alpha(\alpha x/2) + \frac{p-2\tilde{c}}{p-\tilde{c}-s}\cdot \tilde{F}_\alpha(\alpha x) \\ &= \frac{\tilde{c}-s}{p-\tilde{c}-s}\cdot \tilde{F}_1(x/2) + \frac{p-2\tilde{c}}{p-\tilde{c}-s}\cdot \tilde{F}_1(x) \\ &= \tilde{G}_1(x).\end{aligned}$$

This implies, for any $\rho \in [0,1]$,

$$\tilde{G}_\alpha^{-1}(\rho) = \alpha \cdot \tilde{G}_1^{-1}(\rho).$$

Then, it follows from (3.13) that the optimal quantity $\tilde{y}_\alpha^*$ satisfies

$$\tilde{y}_\alpha^* = \alpha \cdot \tilde{y}_1^*$$

where $\tilde{y}_1^*$ is given by $\tilde{y}_1^* = \tilde{G}^{-1}\left(\frac{p-\tilde{c}-c_0}{p-\tilde{c}-s}\right)$. Thus, $\tilde{y}_\alpha^*$ is increasing in α in a linear manner. This proves the first part of (b).

From (3.7), the optimal profit under BOGO is given by

$$\tilde{\pi}_\alpha^* = \tilde{\pi}_\alpha(\tilde{y}_\alpha^*, \tilde{D}) = -c_0\tilde{y}_\alpha^* + p\cdot \min\{\tilde{D}, \tilde{y}_\alpha^*\} - \tilde{c}\cdot Z + s\cdot \max\{y - 2\tilde{D}, 0\}$$

where $Z = \max\{\min\{\tilde{D}, \tilde{y}_\alpha^*\} - \max\{\tilde{y}_\alpha^* - \tilde{D}, 0\}, 0\}$. Since $\tilde{D} \sim \alpha\hat{D}$ is linear in α and $\tilde{y}_\alpha^*$ is also linear in α, each of $\min\{\tilde{D}, \tilde{y}_\alpha^*\}$, $\max\{y - 2\tilde{D}, 0\}$ and Z is linear in α. Thus, the optimal profit given above is linear in α, i.e., $\tilde{\pi}_\alpha^* = \alpha \cdot \tilde{\pi}_1^*$. This proves the first part of (a).

Now, since $\tilde{\pi}_\alpha^* = \alpha \cdot \tilde{\pi}_1^*$ holds from the first part of (a), there exists τ_α^π such that $\alpha \le \tau_\alpha^\pi$ implies $\tilde{\pi}_\alpha^* \le \hat{\pi}^*$, and $\alpha \ge \tau_\alpha^\pi$ implies $\tilde{\pi}_\alpha^* \ge \hat{\pi}^*$. This proves the second part of (a). The second part of (b) follows easily from part (a) since $\tilde{y}_\alpha^* = \alpha\tilde{y}_1^*$ grows linearly in α while the optimal quantity under cash donation, $\hat{y}^*$, is a nonegative constant. □

Proof of Proposition 6. Part (a) is already established before the statement of the proposition.

Note that the $\tilde{c}$ parameter appears in the BOGO model, not the newsvendor model. It is convenient to denote the optimal profit under the BOGO model by $\tilde{\pi}_{\alpha,\tilde{c}}^*$. Recall from Proposition 1 that $\tilde{\pi}^*$ is decreasing in $\tilde{c}$. Since π^* is independent of $\tilde{c}$, it follows that $\tilde{\pi}^*/\pi^*$ is decreasing in $\tilde{c}$. This shows that $\tilde{\pi}^*/\pi^*$ decreases in $\tilde{c}$, proving (b).

Proposition 3 shows that if $\alpha = 1$, then the optimal profit with BOGO is bounded above by the optimal newsvendor profit, i.e., $\tilde{\pi}_{1,\tilde{c}}^* \le \pi^*$. Thus, for any $\alpha \le 1$ and any $\tilde{c} > s$, Proposition 5 implies $\tilde{\pi}_{\alpha,\tilde{c}}^* \le \tilde{\pi}_{1,\tilde{c}}^* \le \pi^*$.

Since $\lim_{\tilde{c}\to s} \tilde{\pi}^*_{\alpha,\tilde{c}} = \lim_{\tilde{c}\to s} \alpha \cdot \tilde{\pi}^*_{1,\tilde{c}}$ from Proposition 5, there exists $\tau_\alpha > 1$ such that $\lim_{\tilde{c}\to s} \tilde{\pi}^*_{\alpha,\tilde{c}} \geq \tilde{\pi}^*$ if $\alpha \geq \tau_\alpha$ and $\lim_{\tilde{c}\to s} \tilde{\pi}^*_{\alpha,\tilde{c}} \leq \tilde{\pi}^*$ if $\alpha \leq \tau_\alpha$. Since for any α, $\tilde{\pi}^*_{\alpha,\tilde{c}}$ decreases in $\tilde{c} > s$ from Proposition 1, this shows that the threshold $\tau^\pi_{\tilde{c}}$ should be as low as possible, and equal to s for any $\alpha \leq \tau_\alpha$.

For any given $\alpha \geq \tau_\alpha$, since $\lim_{\tilde{c}\to s} \tilde{\pi}^*_{\alpha,\tilde{c}} \geq \tilde{\pi}^*$ holds, the value of $\tau^\pi_{\tilde{c}}$ is chosen such that it is the value of $\tilde{c}$ satisfying

$$\pi^* \;=\; \tilde{\pi}^*_{\alpha,\tau^\pi_{\tilde{c}}} \;=\; \alpha \cdot \tilde{\pi}^*_{1,\tau^\pi_{\tilde{c}}} \;,$$

where the second equality follows from Proposition 5. Since π^* is a constant independent of α, this implies that as α increases, $\tilde{\pi}^*_{1,\tau^\pi_{\tilde{c}}}$ must decrease. Thus, since $\tilde{\pi}^*_{\alpha,\tilde{c}}$ is decreasing in $\tilde{c}$, the value chosen for $\tilde{c}$, namely $\tau^\pi_{\tilde{c}}$, must increase in α. This completes the proof of part (c). $\square$