CHAPTER 30

Polyhedral Combinatorics

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Contents

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1. Introduction

Polyhedral combinatorics studies combinatorial problems with the help of polyhedra. Let us first give a simple, illustrative example. Let $G=(V,E)$ be a graph, and let $c\colon E \to \mathbb{R}_+$ be a weight function on the edges of G. Suppose we want to find a matching M in G with "weight"

$$c(M)=\sum_{e\in M} c(e) \tag{1.1}$$

as large as possible, Thus we want to "solve"

$$\max\{c(M) \mid M \text{ matching in } G\}\,. \tag{1.2}$$

Denote for any matching M, the incidence vector of M in $\mathbb{R}^E$ by χ^M, i.e., $\chi^M(e):=1$ if $e\in M$ and $:=0$ if $e\not\in M$. Considering the weight function $c\colon E\to\mathbb{R}$ as a *vector* in $\mathbb{R}^E$, we can write problem (1.2) as

$$\max\{c^{\mathrm{T}}\chi^M \mid M \text{ matching in } G\}\,. \tag{1.3}$$

This amounts to maximizing a linear function over a finite set of vectors. Hence we can equally well maximize over the *convex hull* of these vectors:

$$\max\{c^{\mathrm{T}}x \mid x\in \operatorname{conv}\{\chi^M \mid M \text{ matching in } G\}\}\,. \tag{1.4}$$

The set

$$\operatorname{conv}\{\chi^M \mid M \text{ matching in } G\} \tag{1.5}$$

is a polytope in $\mathbb{R}^E_+$, called the *matching polytope* of G. It follows that there exist a matrix A and a vector b such that

$$\operatorname{conv}\{\chi^M \mid M \text{ matching in } G\}=\{x\in\mathbb{R}^E \mid x\geqslant 0,\ Ax\leqslant b\}\,. \tag{1.6}$$

Then problem (1.4) is equal to

$$\max\{c^{\mathrm{T}}x \mid x\geqslant 0,\ Ax\leqslant b\}\,. \tag{1.7}$$

In this way we have formulated the original combinatorial problem (1.2) as a *linear programming problem*. This enables us to apply linear programming methods to study the original problem.

The problem at this point is, however, how to find the matrix A and the vector b. We know that A and b exist, but we must know them in order to apply linear programming methods.

If G is bipartite, it turns out that the matching polytope of G is equal to the set of all vectors $x\in\mathbb{R}^E$ satisfying

$$\begin{aligned} &x(e)\geqslant 0\,, \quad e\in E \\ &\sum_{e\ni v} x(e)\leqslant 1\,, \quad v\in V\,. \end{aligned} \tag{1.8}$$

That is, for A we can take the $V \times E$ incidence matrix of G and for b the all-one vector $\mathbf{1}$ in $\mathbb{R}^V$.

It is not difficult to show that the matching polytope for bipartite graphs is indeed completely determined by (1.8). First note that the matching polytope is contained in the polytope defined by (1.8), since χ^M satisfies (1.8) for each matching M. To see the converse inclusion, we note that if G is bipartite, then the matrix A is *totally unimodular*, i.e., each square submatrix of A has determinant belonging to $\{0, +1, -1\}$. This may be seen to imply that the vertices of the polytope determined by (1.8) are *integral* vectors, i.e., they belong to $\mathbb{Z}^E$. Now each integral vector satisfying (1.8) must trivially be equal to χ^M for some matching M. Hence the polytope determined by (1.8) is equal to the matching polytope of G.

For each *nonbipartite* graph, the matching polytope is not completely determined by (1.8). Indeed, if C is an odd circuit in G, then the vector $x \in \mathbb{R}^E$ defined by $x(e) = \frac{1}{2}$ if $e \in C$ and 0 if $e \notin C$, satisfies (1.8) but does not belong to the matching polytope.

In fact, it is a pioneering theorem in polyhedral combinatorics due to J. Edmonds that gives a complete description of the inequalities needed to describe the matching polytope for arbitrary graphs.

When we have formulated the matching problem as LP problem (1.7), we can apply LP techniques to study the matching problem. Thus we can find a maximum weighted matching in a bipartite graph, e.g., with the simplex method. Moreover, the Duality Theorem of Linear Programming gives

$$\begin{aligned}&\max\{c(M) \mid M \text{ matching in } G\} = \max\{c^{\mathrm{T}}x \mid x \geq 0,\ Ax \leq \mathbf{1}\}\\ &\quad = \min\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0,\ y^{\mathrm{T}}A \geq c^{\mathrm{T}}\}\ . \end{aligned} \tag{1.9}$$

In the special case of G bipartite and c being the all-one vector in $\mathbb{R}^E$, we can derive from this the *König–Egerváry Theorem*. The left-most expression in (1.9) is equal to the maximum size of a matching. The minimum can be seen to be attained by an integral vector y, again by the total unimodularity of A. This y is a $\{0, 1\}$-vector in $\mathbb{R}^V$, and is the incidence vector of some subset W of V intersecting every edge of G. Thus (1.9) implies that the maximum size of a matching is equal to the minimum size of a set of vertices intersecting all edges of G.

As an extension, one can derive the *Tutte–Berge Formula* from the inequality system given by Edmonds for arbitrary graphs.

Bipartite matching forms an easy example in polyhedral combinatorics. We now discuss the central idea of polyhedral combinatorics – taking convex hulls – in a more general framework.

Let $\mathcal{F}$ be a collection of subsets of a finite set S, let $c: S \rightarrow \mathbb{R}$, and suppose we are interested in

$$\max\left\{\sum_{s \in U} c(s) \mid U \in \mathcal{F}\right\}. \tag{1.10}$$

(For example, S is the set of edges of a graph, and $\mathcal{F}$ is the collection of matchings, in which case (1.10) is the maximum "weight" of a matching.) Usually, $\mathcal{F}$ is too large to evaluate each set U in $\mathcal{F}$ in order to determine the maximum (1.10). (For example, the collection of matchings is exponentially large in the size of the graph.) Now (1.10) is equal to

$$\max\{c^{\mathrm{T}}\chi^{U} \mid U \in \mathcal{F}\}\,, \tag{1.11}$$

where χ^{U} denotes the *incidence vector* of U in $\mathbb{R}^{S}$, i.e., $\chi^{U}(s) = 1$ if $s \in U$ and 0 otherwise. [Here we identify functions $c: S \rightarrow \mathbb{R}$ with vectors in the linear space $\mathbb{R}^{S}$, and accordingly we shall sometimes denote $c(s)$ by c_s.] Since (1.11) means maximizing a linear function over a finite set of vectors, we could equally well maximize over the convex hull of these vectors:

$$\max\{c^{\mathrm{T}}x \mid x \in \operatorname{conv}\{\chi^{U} \mid U \in \mathcal{F}\}\}\,. \tag{1.12}$$

Since this convex hull is a polytope, there exist a matrix A and a column vector b such that

$$\operatorname{conv}\{\chi^{U} \mid U \in \mathcal{F}\} = \{x \in \mathbb{R}^{S} \mid Ax \leqslant b\}\,. \tag{1.13}$$

Hence (1.12) is equal to

$$\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}\,. \tag{1.14}$$

Thus we have formulated the original combinatorial problem as a linear programming problem, and we can appeal to linear programming methods to study the combinatorial problem.

In order to determine the maximum (1.10) algorithmically, we could use LP algorithms like the simplex method or the primal–dual method. Sometimes, with the ellipsoid method the polynomial-time solvability of (1.10) can be shown. Moreover, by the Duality Theorem of Linear Programming, problem (1.14), and hence problem (1.10), is equal to

$$\min\{y^{\mathrm{T}}b \mid y \geqslant 0,\, y^{\mathrm{T}}A = c^{\mathrm{T}}\}\,, \tag{1.15}$$

which gives us a min–max equation for the combinatorial maximum. Often this provides us with a "good characterization" [i.e., problem (1.10) belongs to NP $\cap$ co-NP], and it enables us to carry out a "sensitivity analysis" of the combinatorial problem, etc.

However, in order to apply LP techniques, we should be able to find matrix A and vector b satisfying (1.13). This is one of the main theoretical problems in polyhedral combinatorics.

Often, one first "guesses" a system $Ax \leqslant b$, and next, one tries to prove that $Ax \leqslant b$ forms a complete description of the polytope. Sometimes, like in bipartite matching, this can be shown with the help of the total unimodularity of A. However, in general A is not totally unimodular, and one has to try more complicated techniques to show that $Ax \leqslant b$ completely describes the polytope. In this survey, we mention the techniques of "total dual integrality", "blocking polyhedra", "anti-blocking polyhedra", and "cutting planes".

In several cases, the guessed system $Ax \leqslant b$ turns out not to be a complete description, but just gives an approximation of the polytope. This can still be useful, since in that case the linear programming problem $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}$ gives a (hopefully good) upper bound for the combinatorial maximum. This can be very useful in a so-called *branch-and-bound* algorithm for the combinatorial problem.

Historically, applying LP techniques to combinatorial problems came along with the introduction of linear programming in the 1940s and 1950s. Dantzig, Ford, Fulkerson, Hoffman, Johnson and Kruskal studied problems like the transportation, flow, and assignment problems, which can be reduced to linear programming (by the total unimodularity of the constraint matrix), and the traveling salesman problem, using a rudimentary version of a cutting plane technique (extended by Gomory to general integer linear programming).

The field of polyhedral combinatorics was extended and deepened considerably by the work of Edmonds in the 1960s and 1970s. He characterized basic polytopes like the matching polytope, the arborescence polytope, and the matroid intersection polytope; he introduced (with Giles) the important concept of total dual integrality; and he advocated the link between polyhedra, min–max relations, good characterizations, and polynomial-time solvability. Fulkerson designed the clarifying framework of blocking and anti-blocking polyhedra, enabling the deduction of one polyhedral characterization or min–max relation from another.

In this chapter we describe the basic techniques in polyhedral combinatorics, and we derive as illustrations polyhedral characterizations for some concrete combinatorial problems. First, in sections 2 and 3, we give some background information on polyhedra and linear programming methods.

For background and related literature we refer to Grötschel et al. (1988), Grötschel and Padberg (1985), Grünbaum (1967), Lovász (1977, 1979), Pulleyblank (1983), Schrijver (1983b, 1986), and Stoer and Witzgall (1970).

2. Background information on polyhedra

For an in-depth survey on polyhedra (focusing on the combinatorial properties) we refer the reader to chapter 18. In this section, we give a brief review on polyhedra, covering those parts of polyhedral theory required for the present chapter.

A set $P \subseteq \mathbb{R}^n$ is called a *polyhedron* if there exist a matrix A and a column vector b such that

$$P = \{x \mid Ax \leqslant b\} . \tag{2.1}$$

If (2.1) holds, we say that $Ax \leqslant b$ *determines* P. A set $P \subseteq \mathbb{R}^n$ is called a *polytope* if there exist $x_1, \ldots, x_t$ in $\mathbb{R}^n$ such that $P = \operatorname{conv}\{x_1, \ldots, x_t\}$. The following theorem is intuitively clear, but is not trivial to prove, and is usually attributed to Minkowski (1896), Steinitz (1916), and Weyl (1935).

Finite Basis Theorem for Polytopes 2.2. *A set P is a polytope if and only if P is a bounded polyhedron.*

Motzkin, in 1936, extended this to:

Decomposition Theorem for Polyhedra 2.3. *$P \subseteq \mathbb{R}^n$ is a polyhedron if and only if there exist $x_1, \ldots, x_t, y_1, \ldots, y_s \in \mathbb{R}^n$ such that*

$$P = \{\lambda_1 x_1 + \cdots + \lambda_t x_t + \mu_1 y_1 + \cdots + \mu_s y_s \mid \lambda_1, \ldots, \lambda_t, \mu_1, \ldots, \mu_s \geq 0;\\ \lambda_1 + \cdots + \lambda_t = 1\}\,.$$

Now let $P = \{x \mid Ax \leq b\}$ be a nonempty polyhedron, where A has order $m \times n$. If $c \in \mathbb{R}^n$ with $c \neq 0$ and $\delta = \max\{c^{\mathrm{T}}x \mid x \in P\}$, then the set $\{x \mid c^{\mathrm{T}}x = \delta\}$ is called a *supporting hyperplane* of P. A subset F of P is called a *face* of P if $F = P$ or if $F = P \cap H$ for some supporting hyperplane H of P. Clearly, a face of P is a polyhedron again. It can be shown that for any face F of P there exists a subsystem $A'x \leq b'$ of $Ax \leq b$ such that $F = \{x \in P \mid A'x = b'\}$. Hence P has only finitely many faces. They are ordered by inclusion. *Minimal faces* are the faces minimal with respect to inclusion. The following theorem is due to Hoffman and Kruskal (1956).

Theorem 2.4. *A set F is a minimal face of P if and only if $\emptyset \neq F \subseteq P$ and*

$$F = \{x \mid A'x = b'\}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$.

All minimal faces have the same dimension, viz. n-rank(A). If this is 0, minimal faces correspond to vertices: a *vertex* of P is an element of P which is not a convex combination of two other elements of P. Only if rank$(A) = n$, does P have vertices, and then those vertices are exactly the minimal faces. Hence:

Theorem 2.5. *Vector z in P is a vertex of P if and only if $A'z = b'$ for some subsystem $A'x \leq b'$ of $Ax \leq b$, with A' nonsingular of order n.*

The matrix A' (or subsystem $A'x \leq b'$) is sometimes called a *basis* for z. Generally, such a basis is not unique. P is called *pointed* if it has vertices. A polytope is always pointed, and is the convex hull of its vertices.

Two vertices x and y of P are *adjacent* if conv$\{x, y\}$ is a face of P. It can be shown that if P is a polytope, then two vertices x and y are adjacent if and only if the vector $\frac{1}{2}(x + y)$ is not a convex combination of other vertices of P. Moreover, one can show:

Theorem 2.6. *Vertices z' and z'' of the polyhedron P are adjacent if and only if z' and z'' have bases $A'x \leqslant b'$ and $A''x \leqslant b''$, respectively, so that they have exactly $n-1$ constraints in common.*

The polyhedron P gives rise to a graph, whose nodes are the vertices of P, two of them being adjacent in the graph if and only if they are adjacent on P. The *diameter* of P is the diameter of this graph. The following conjecture is due to W. M. Hirsch (cf. Dantzig 1963).

Hirsch's Conjecture 2.7. A polytope in $\mathbb{R}^n$ determined by m inequalities has diameter at most $m-n$.

This conjecture is related to the number of iterations in the simplex method (see section 3). See also Klee and Walkup (1967) and Larman (1970). [The Hirsch conjecture was proved by Naddef (1989) for polytopes all of whose vertices are $\{0,1\}$ – vectors.]

A *facet* of P is an inclusion-wise maximal face F of P with $F \neq P$. A face F of P is a facet if and only if $\dim(F) = \dim(P) - 1$. An inequality $c^{\mathrm{T}}x \leqslant \delta$ is called a *facet-inducing inequality* if $P \subseteq \{x \mid c^{\mathrm{T}}x \leqslant \delta\}$ and $P \cap \{x \mid c^{\mathrm{T}}x = \delta\}$ is a facet of P.

Suppose $Ax \leqslant b$ is an *irredundant* (or *minimal*) system determining P, i.e., no inequality in $Ax \leqslant b$ is implied by the other. Let $A^+x \leqslant b^+$ be those inequalities $a^{\mathrm{T}}x \leqslant \beta$ from $Ax \leqslant b$ for which $a^{\mathrm{T}}z < \beta$ for at least one z in P. Then each inequality in $A^+x \leqslant b^+$ is a facet-inducing inequality. Moreover, this defines a one-to-one relation between facets and inequalities in $A^+x \leqslant b^+$. If P is full-dimensional, then the irredundant system $Ax \leqslant b$ is unique up to multiplication of inequalities by positive scalars. The following characterization holds.

Theorem 2.8. *If $P = \{x \mid Ax \leqslant b\}$ is full-dimensional, then $Ax \leqslant b$ is irredundant if and only if for each pair $a_i^{\mathrm{T}}x \leqslant b_i$ and $a_j^{\mathrm{T}}x \leqslant b_j$ of constraints from $Ax \leqslant b$ there is a vector x' in P satisfying $a_i^{\mathrm{T}}x' = b_i$ and $a_j^{\mathrm{T}}x'' < b_j$.*

The polyhedron P is called *rational* if we can take A and b in (2.1) rational-valued (and hence we can take them integer-valued). P is rational if and only if the vectors $x_1, \ldots, x_t$, and $y_1, \ldots, y_s$ in Theorem 2.3 can be taken to be rational. P is called *integral* if we can take $x_1, \ldots, x_t$, and $y_1, \ldots, y_s$ in Theorem 2.3 integer-valued. Hence P is integral if and only if P is the convex hull of the integer vectors in P or, equivalently, if and only if every minimal face of P contains integer vectors.

3. Background information on linear programming

Linear programming, abbreviated by LP, studies the problem of maximizing or minimizing a linear function $c^{\mathrm{T}}x$ over a polyhedron P. Examples of such a problem are:

$$\begin{aligned}
&\text{(i)} && \max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}\,, \\
&\text{(ii)} && \max\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax \leqslant b\}\,, \\
&\text{(iii)} && \max\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax = b\}\,, \\
&\text{(iv)} && \min\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax \geqslant b\}\,.
\end{aligned} \tag{3.1}$$

It can be shown, for each of the problems (i)–(iv), that if the set involved is a polyhedron with vertices [for (ii)–(iv) this follows if it is nonempty], and if the optimum value is finite, then it is attained by a vertex of the polyhedron.

Each of the optima (3.1) is equal to the optimum value in some other LP problem, called the *dual problem.*

Duality Theorem of Linear Programming 3.2. *Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then*

$$\begin{aligned}
&\text{(i)} && \max\{c^{\mathrm{T}}x \mid Ax \leqslant b\} &&= \min\{y^{\mathrm{T}}b \mid y \geqslant 0,\, y^{\mathrm{T}}A = c^{\mathrm{T}}\}\,; \\
&\text{(ii)} && \max\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax \leqslant b\} &&= \min\{y^{\mathrm{T}}b \mid y \geqslant 0,\, y^{\mathrm{T}}A \geqslant c^{\mathrm{T}}\}\,; \\
&\text{(iii)} && \max\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax = b\} &&= \min\{y^{\mathrm{T}}b \mid y^{\mathrm{T}}A \geqslant c^{\mathrm{T}}\}\,; \\
&\text{(iv)} && \min\{c^{\mathrm{T}}x \mid x \geqslant 0,\, Ax \geqslant b\} &&= \max\{y^{\mathrm{T}}b \mid y \geqslant 0,\, y^{\mathrm{T}}A \leqslant c^{\mathrm{T}}\}\,;
\end{aligned} \tag{3.3}$$

provided that these sets are nonempty.

It is not difficult to derive this from:

Farkas's Lemma 3.4. *Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. Then $Ax = b$ has a solution $x \geqslant 0$ if and only if $y^{\mathrm{T}}b \geqslant 0$ holds for each vector $y \in \mathbb{R}^m$ with $y^{\mathrm{T}}A \geqslant 0$.*

The principle of *complementary slackness* says: let x and y satisfy $Ax \leqslant b$, $y \geqslant 0$, $y^{\mathrm{T}}A = c^{\mathrm{T}}$ then x and y are optimum solutions in Theorem 3.2(i) if and only if $y_i = 0$ or $a_i^{\mathrm{T}}x = b_i$ for each $i = 1, \ldots, m$ (where $a_i^{\mathrm{T}}x = b_i$ denotes the ith line in the system $Ax = b$). Similar statements hold for Theorem 3.2(ii)–(iv).

We now describe briefly three of the methods for solving LP problems. The first two methods, the famous simplex method and the primal–dual method, can be considered also, when applied to combinatorial problems, as a guideline to deriving a "combinatorial" algorithm from a polyhedral characterization. The third method, the ellipsoid method, is more of theoretical value: it is a tool sometimes used to derive the polynomial-time solvability of a combinatorial problem.

3.1. The simplex method

The simplex method, due to Dantzig (1951a), is the method used most often for linear programming. Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Suppose we wish to solve $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}$, where the polyhedron $P := \{x \mid Ax \leqslant b\}$ is a polyhedron with vertices, i.e., $\operatorname{rank}(A) = n$.

The idea of the simplex method is to make a trip, going from a vertex to a better adjacent vertex, until an optimal vertex is reached. By Theorem 2.5, vertices can be described by bases, while by Theorem 2.6 adjacency can be described by bases differing in exactly one constraint. Thus the process can be described by a series

$$A_0x \leqslant b_0,\ A_1x \leqslant b_1,\ A_2x \leqslant b_2, \ldots \tag{3.5}$$

of bases, where each $x_k := A_k^{-1}b_k$ is a vertex of P, where $A_{k+1}x \leqslant b_{k+1}$ differs by one constraint from $A_kx \leqslant b_k$, and where $c^{\mathrm{T}}x_{k+1} \geqslant c^{\mathrm{T}}x_k$.

The series can be found as follows. Suppose $A_kx \leqslant b_k$ has been found. If $c^{\mathrm{T}}A_k^{-1} \geqslant 0$, then x_k is an optimal solution of $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}$, since for each x satisfying $Ax \leqslant b$ one has $A_kx \leqslant b_k$ and hence $c^{\mathrm{T}}x = (c^{\mathrm{T}}A_k^{-1})A_kx \leqslant (c^{\mathrm{T}}A_k^{-1})b_k = c^{\mathrm{T}}x_k$.

If $c^{\mathrm{T}}A_k^{-1} \not\geqslant 0$, choose an index i so that $(c^{\mathrm{T}}A_k^{-1})_i < 0$, and let $z := -A_k^{-1}e_i$ (where e_i denotes the ith unit basis vector in $\mathbb{R}^n$). Note that for $\lambda \geqslant 0$, $x_k + \lambda z$ traverses an edge or ray of P (i.e., face of dimension 1), or it is outside of P for all $\lambda > 0$. Moreover, $c^{\mathrm{T}}z = -c^{\mathrm{T}}A_k^{-1}e_i > 0$. Now if $Az \leqslant 0$, then $x_k + \lambda z \in P$ for all $\lambda \geqslant 0$, whence $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\} = \infty$. If $Az \not\leqslant 0$, let λ_0 be the largest λ such that $x_k + \lambda z$ belongs to P, i.e.,

$$\lambda_0 := \min\left\{\frac{b_j - a_j^{\mathrm{T}}x_k}{a_j^{\mathrm{T}}z} \;\middle|\; j = 1, \ldots, m,\ a_j^{\mathrm{T}}z > 0\right\}. \tag{3.6}$$

Choose an index j attaining this minimum. Replacing the ith inequality in $A_kx \leqslant b_k$ by inequality $a_j^{\mathrm{T}}x \leqslant b_j$ then gives us the next system $A_{k+1}x \leqslant b_{k+1}$.

Note that $x_{k+1} = x_k + \lambda_0 z$, implying that if $x_{k+1} \neq x_k$ then $c^{\mathrm{T}}x_{k+1} > c^{\mathrm{T}}x_k$. Clearly, the above process stops if $c^{\mathrm{T}}x_{k+1} > c^{\mathrm{T}}x_k$ for each k (since P has only finitely many vertices). This is the case if each vertex has exactly one basis – the *nondegenerate* case. However, in general it can happen that $x_{k+1} = x_k$ for certain k. Several "pivot selection rules", prescribing the choice of i and j above, have been found which could be proved to yield termination of the simplex method. No one of these rules could be proved to give a polynomial-time method – in fact, most of them could be shown to require an exponential number of iterations in the worst case.

The number of iterations in the simplex method is related to the diameter of the underlying polyhedron P. Suppose P is a polytope. If there is a pivot selection rule such that for each $c \in \mathbb{R}^n$ the problem $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}$ can be solved within t iterations of the simplex method (starting with an arbitrary first basis $A_0x \leqslant b_0$ corresponding to a vertex), then clearly P has diameter at most t. However, as Padberg and Rao (1974) showed, the "traveling-salesman polytopes" (see section 10) form a class of polytopes of diameter at most 2, while maximizing a linear function over these polytopes is NP-complete.

A main problem seems that we do not have a better criterion for adjacency than Theorem 2.6. Note that a vertex of P can be adjacent to an exponential number of vertices (in the sizes of A and b), whereas for any basis A' there are at most $n(m-n)$ bases differing from A' in exactly one row. In the degenerate case, there can be several bases corresponding to one and the same vertex. Just this phenomenon shows up frequently in polytopes occurring in combinatorial optimization, and one of the main objectives is to find pivoting rules preventing us going through many bases corresponding to the same vertex (cf. Cunningham 1979).

3.2. *Primal–dual method*

As a generalization of similar methods for network flow and transportation problems, Dantzig et al. (1956) designed the "primal–dual method" for LP. The general idea is as follows. Starting with a dual feasible solution y, the method searches for a primal feasible solution x satisfying the complementary slackness condition with respect to y. If such a primal feasible solution is found, x and y form a pair of optimal (primal and dual) solutions. If no such primal solution is found, the method prescribes a modification of y, after which we start anew.

The problem now is how to find a primal feasible solution x satisfying the complementary slackness condition, and how to modify the dual solution y if no such primal solution is found. For general LP problems this problem can be seen to amount to another LP problem, generally simpler than the original LP problem. To solve the simpler problem we could use any LP method, e.g., the simplex method. In many combinatorial applications, however, this simpler LP problem is a simpler combinatorial optimization problem, for which direct methods are available (see Papadimitriou and Steiglitz 1982). Thus, if we can describe a combinatorial optimization problem as a linear program, the primal–dual method gives us a scheme for reducing one combinatorial problem to an easier combinatorial problem.

We shall now describe the primal–dual method more precisely. Suppose we wish to solve the LP problem.

$$\min\{c^{\mathrm{T}}x \mid x \geq 0, Ax = b\}\ , \tag{3.7}$$

where A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. The dual problem is

$$\max\{y^{\mathrm{T}}b \mid y^{\mathrm{T}}A \leq c^{\mathrm{T}}\}\ . \tag{3.8}$$

The primal–dual method consists of repeating the following *primal–dual iteration*. Suppose we have a feasible solution y_0 for problem (3.8). Let A' be the submatrix of A consisting of those columns a_j of A for which $y_0^{\mathrm{T}}a_j = c_j$. To find a feasible primal solution for which the complementary slackness condition holds, solve the *restricted linear program*

$$\min\{\lambda \mid x', \lambda \geq 0; A'x' + b\lambda = b\} = \max\{y^{\mathrm{T}}b \mid y^{\mathrm{T}}A' \leq 0, y^{\mathrm{T}}b \leq 1\}\ . \tag{3.9}$$

If the optimum value is 0, let x'_0, λ be an optimum solution for the minimum. So $x'_0 \geq 0$, $A'x'_0 = b$, and $\lambda = 0$. Hence by adding zero-components, we obtain a vector $x_0 \geq 0$ such that $Ax_0 = b$ and $(x_0)_j = 0$ if $y_0^{\mathrm{T}} a_j < c_j$. By complementary slackness, it follows that x_0 and y_0 are optimum solutions for problems (3.7) and (3.8). If the optimum value in problem (3.9) is positive, it is 1. Let u be an optimum solution for the maximum. Let θ be the largest real number satisfying

$$(y_0 + \theta u)^{\mathrm{T}} A \leq c^{\mathrm{T}} . \tag{3.10}$$

(Note that $\theta > 0$.) Reset $y_0 := y_0 + \theta u$, and start the iteration anew.

This describes the primal–dual method. It reduces problem (3.7) to (3.9), which is often an easier problem, consisting only of testing feasibility of: $x' \geq 0$, $A'x' = b$.

The primal–dual method can equally be considered as a *gradient method*. Suppose we wish to solve problem (3.8), and we have a feasible solution y_0. This y_0 is not optimal if and only if we can find a vector u such that $u^{\mathrm{T}} b > 0$ and u is a *feasible direction* in y_0 [i.e., $(y_0 + \theta u)^{\mathrm{T}} A \leq c^{\mathrm{T}}$ for some $\theta > 0$]. If we let A' consist of those columns of A in which $y_0^{\mathrm{T}} A \leq c^{\mathrm{T}}$ has equality, then u is a feasible direction if and only if $u^{\mathrm{T}} A' \leq 0$. So u can be found by solving the right-hand side of problem (3.9).

Application 3.11 (*Maximum flow*). Let $D = (V, A)$ be a directed graph, let r, $s \in V$, and let a "capacity" function $c: A \to \mathbb{Q}_+$ be given. The *maximum flow problem* is to find the maximum amount of flow from r to s, subject to c:

$$\begin{aligned} \text{maximize} \quad & \sum_{a \in \delta^+(r)} x(a) - \sum_{a \in \delta^-(r)} x(a) \\ \text{subject to} \quad & \sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = 0 , \quad v \in V,\ v \neq r, s , \\ & 0 \leq x(a) \leq c(a) , \quad a \in A . \end{aligned} \tag{3.12}$$

If we have a feasible solution x_0, we have to find a feasible direction in x_0, i.e., a function $u: A \to \mathbb{R}$ satisfying

$$\begin{aligned} & \sum_{a \in \delta^+(r)} u(a) - \sum_{a \in \delta^-(r)} u(a) > 0 , \\ & \sum_{a \in \delta^+(v)} u(a) - \sum_{a \in \delta^-(v)} u(a) = 0 , \quad v \in V,\ v \neq r, s , \\ & u(a) \geq 0 , \quad a \in A,\ x_0(a) = 0 , \\ & u(a) \leq 0 , \quad a \in A,\ x_0(a) = c(a) . \end{aligned} \tag{3.13}$$

One easily checks that this problem is equivalent to the problem of finding an undirected path from r to s in $D=(V,A)$ so that for any arc a in the path,

$$\begin{aligned}&\text{if } x_0(a)=0, \quad \text{then arc } a \text{ is traversed forward,}\\ &\text{if } x_0(a)=c(a), \text{ then arc } a \text{ is traversed backward,}\\ &\text{if } 0<x_0(a)<c(a), \text{ then arc } a \text{ is traversed forward or backward.}\end{aligned} \tag{3.14}$$

If we have found such a path, we find u as in (3.13) (by taking $u(a)=+1$ or -1 if a occurs in the path forward or backward, respectively, and $u(a)=0$ if a does not occur in the path). Taking the highest θ for which $x_0+\theta u$ is feasible in problem (3.12) gives us the next feasible solution. The path is called a *flow-augmenting path*, since the new solution has a higher objective value than the old. Iterating this process we finally get an optimum flow. This is exactly Ford and Fulkerson's algorithm (1957) for finding a maximum flow, which is therefore an example of a primal–dual method. [Dinits (1970) and Edmonds and Karp (1972) showed that a version of this algorithm is a polynomial-time method.]

3.3. The ellipsoid method

The ellipsoid method, developed by Shor (1970a,b, 1977) and Yudin and Nemirovskiĭ (1976/1977, 1977) for nonlinear programming, was shown by Khachiyan (1979) to solve linear programming in polynomial time. Very roughly speaking, it works as follows.

Suppose we wish to solve the LP problem

$$\max\{c^{\mathrm{T}}x \mid Ax\leqslant b\}\,, \tag{3.15}$$

where $A\in\mathbb{Q}^{m\times n}$, $b\in\mathbb{Q}^m$, and $c\in\mathbb{Q}^n$. Let us assume that the polyhedron $P:=\{x\mid Ax\leqslant b\}$ is bounded. Then it is not difficult to calculate a number R such that $P\subseteq\{x\in\mathbb{R}^n \mid \|x\|\leqslant R\}$. We construct a sequence of ellipsoids E_0, E_1, $E_2,\ldots$, each containing the optimum solutions of problem (3.15). First, $E_0:=\{x\in\mathbb{R}^n \mid \|x\|\leqslant R\}$. Suppose ellipsoid E_t has been found. Let z be its center.

If $Az\leqslant b$ does not hold, let $a_k^{\mathrm{T}}x\leqslant b_k$ be an inequality in $Ax\leqslant b$ violated by z. Next let E_{t+1} be the ellipsoid of smallest volume satisfying $E_{t+1}\supseteq E_t\cap\{x\mid a_k^{\mathrm{T}}x\leqslant a_k^{\mathrm{T}}z\}$. If $Az\leqslant b$ does hold, let E_{t+1} be the ellipsoid of smallest volume satisfying $E_{t+1}\supseteq E_t\cap\{x\mid c^{\mathrm{T}}x\geqslant c^{\mathrm{T}}z\}$.

One can prove that these ellipsoids of smallest volume are unique, and that the parameters determining E_{t+1} can be expressed straightforwardly in those determining E_t and in a_k, respectively c. Moreover, $\mathrm{vol}(E_{t+1})<\mathrm{e}^{-1/3n}\cdot\mathrm{vol}(E_t)$. Hence the volumes of the successive ellipsoids decrease exponentially fast. Since the optimum solutions of problem (3.15) belong to each E_t, we may hope that the centers of the ellipsoids converge to an optimum solution of problem (3.15).

To make this description more precise, an important problem to be solved is that ellipsoids with very small volume can still have a large diameter [so that the centers of the ellipsoids can remain far from any optimum solution of problem (3.15)]. Another, technical, problem is that the unique smallest elipsoid is usually determined by irrational parameters, so that if we work in rational arithmetic we must allow approximations of the successive ellipsoids. These problems can be overcome, and a polynomially bounded running time can be proved.

It was observed by Grötschel et al. (1981), Karp and Papadimitriou (1982) and Padberg and Rao (1980) that in applying the ellipsoid method, it is not necessary that the system $Ax \leqslant b$ be explicitly given. It suffices to have a "subroutine" to decide whether or not a given vector z belongs to the feasible region of problem (3.15), and to find a separating hyperplane in case z is not feasible. This is especially useful for linear programs coming from combinatorial optimization problems, where the number of inequalities can be exponentially large (in the size of the underlying data-structure), but can yet be tested in polynomial time.

This leads to the following result (Grötschel et al. 1981). Suppose we are given, for each graph $G=(V, E)$, a collection $\mathscr{F}_G$ of subsets of E. For example,

(i) $\mathscr{F}_G$ is the collection of matchings in G;
(ii) $\mathscr{F}_G$ is the collection of spanning trees in G; (3.16)
(iii) $\mathscr{F}_G$ is the collection of Hamiltonian circuits in G.

With any class ($\mathscr{F}_G \mid G$ graph), we can associate the following problem.

Optimization Problem 3.17. Given a graph $G=(V, E)$ and $c \in \mathbb{Q}^E$, find $F \in \mathscr{F}_G$ maximizing $\sum_{e \in F} c_e$.

So if ($\mathscr{F}_G \mid G$ graph) is as in (i), (ii), and (iii) above, Problem 3.17 amounts to the problems of finding a maximum weighted matching, a maximum weighted spanning tree, and a maximum weighted Hamiltonian circuit (the traveling salesman problem), respectively.

The optimization problem is called *solvable in polynomial time*, or *polynomially solvable*, if it is solvable by an algorithm whose running time is bounded by a polynomial in the input size of Problem 3.17, which is $|V| + |E| +$ size(c). Here size$(c) := \sum_{e \in E}$ size(c_e), where the size of a rational number p/q is $\log_2((|p| + 1) + \log_2(|q|)$. So size$(c)$ is about the space needed to specify c in binary notation.

Define also the following problem for any fixed class ($\mathscr{F}_G \mid G$ graph).

Separation Problem 3.18. Given a graph $G=(V, E)$ and $x \in \mathbb{Q}^E$, determine whether or not x belongs to $\operatorname{conv}\{\chi^F \mid F \in \mathscr{F}_G\}$, and if not, find a separating hyperplane.

Theorem 3.19. *For any fixed class* ($\mathscr{F}_G \mid G$ *graph*), *the Optimization Problem* 3.17 *is polynomially solvable if and only if the Separation Problem* 3.18 *is polynomially solvable.*

The theorem implies that with respect to the question of polynomial-time solvability, the polyhedral combinatorics approach described in section 1 (i.e., studying the convex hull) is, implicitly or explicitly, unavoidable: a combinatorial optimization problem is polynomially solvable if and only if the corresponding convex hulls can be described decently, in the sense of the polynomial-time solvability of the separation problem. This can be stated also in the negative: if a combinatorial optimization problem is not polynomially solvable (perhaps the traveling salesman problem), then the corresponding polytopes have no such decent description.

The ellipsoid method does not give a practical method, so Theorem 3.19 is more of theoretical value. In some cases, with Theorem 3.19 the polynomial solvability of a combinatorial optimization problem was proved, and that then formed a motivation for finding a practical polynomial-time algorithm for the problem.

One drawback of the ellipsoid method is that the number of ellipsoids to be evaluated depends on the size of the objective vector c. This does not conflict with the definition of polynomial solvability, but is not very attractive in practice. It would be preferable for the size of c only to influence the sizes of the numbers occurring when we perform the algorithm, but not the number of arithmetic operations to be performed. An algorithm for Optimization Problem 3.17 is called *strongly polynomial* if it consists of a number of arithmetic operations, bounded by a polynomial in $|V| + |E|$, on numbers of size bounded by a polynomial in $|V| + |E| + \text{size}(c)$. Such an algorithm is obviously polynomial-time.

Interestingly, Frank and Tardos (1985) showed, with the help of the "basis reduction method" (Lenstra et al. 1982):

Theorem 3.20. *For any fixed class* ($\mathcal{F}_G \mid G$ graph), *if there exists a polynomial-time algorithm for Optimization Problem* 3.17, *then there exists a strongly polynomial algorithm for it.*

At the moment of writing, it is not yet clear whether this result leads to practical algorithms.

Finally we note that it is not necessary to restrict $\mathcal{F}_G$ to collections of subsets of the edge set E. For instance, similar results hold if we consider collections $\mathcal{F}_G$ of subsets of the vertex set V. Moreover, we can consider classes $(\mathcal{F}_G \mid G \in \mathcal{G})$, where $\mathcal{G}$ is a subcollection of the set of all graphs. Similarly, we can consider classes ($\mathcal{F}_D \mid D$ directed graph), ($\mathcal{F}_H \mid H$ hypergraph), ($\mathcal{F}_M \mid M$ matroid), and so on.

More on the ellipsoid method can be found in Grötschel et al. (1988).

We finally mention the method of Karmarkar (1984) for linear programming; this appears to be competitive with the simplex method, but its impact on polyhedral combinatorics is not yet clear at the moment of writing.

4. Total unimodularity

A matrix is called *totally unimodular* if each subdeterminant belongs to $\{0, +1, -1\}$. In particular, each entry of a totally unimodular matrix belongs to $\{0, +1,$

$-1\}$. The importance of total unimodularity for polyhedral combinatorics comes from the following theorem (Hoffman and Kruskal 1956).

Theorem 4.1. *Let A be a totally unimodular $m \times n$ matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron $P := \{x \mid Ax \leq b\}$ is integral.*

Proof. Let $F = \{x \mid A'x = b'\}$ be a minimal face of P, where $A'x \leq b'$ is a subsystem of $Ax \leq b$. Without loss of generality, $A' = [A_1\ A_2]$, with A_1 nonsingular. Then A_1^{-1} is an integral matrix (as $\det A_1 = \pm 1$), and hence the vector

$$x := \begin{pmatrix} A_1^{-1}b' \\ 0 \end{pmatrix} \tag{4.2}$$

is an integral vector in F. $\square$

In fact, Hoffman and Kruskal showed that an integral $m \times n$ matrix A is totally unimodular if and only if for each $b \in \mathbb{Z}^m$, each vertex of the polyhedron $\{x \in \mathbb{R}^n \mid x \geq 0, Ax \leq b\}$ is integral.

We mention a strengthening of Theorem 4.1 due to Baum and Trotter (1977). A polyhedron P in $\mathbb{R}^n$ is said to have the *integer decomposition property* if for each $k \in \mathbb{N}$ and for each integral vector z in kP $(=\{kx \mid x \in P\})$, there exist integral vectors $x_1, \ldots, x_k$ in P so that $z = x_1 + \cdots + x_k$. It is not difficult to see that each polyhedron with the integer decomposition property is integral.

Theorem 4.3. *Let A be a totally unimodular $m \times n$ matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron $P := \{x \mid Ax \leq b\}$ has the integer decomposition property.*

Proof. Let $k \in \mathbb{N}$ and $z \in kP \cap \mathbb{Z}^n$. By induction on k we show that $z = x_1 + \cdots + x_k$ for integral vectors $x_1, \ldots, x_k$ in P. By Theorem 4.1, there exists an integral vector, say x_k, in the polyhedron $\{x \mid Ax \leq b, -Ax \leq (k-1)b - Az\}$ [since (i) the constraint matrix $\left[\begin{smallmatrix} A \\ -A \end{smallmatrix}\right]$ is totally unimodular, (ii) the right-hand-side vector $\left(\begin{smallmatrix} b \\ (k-1)b - Az \end{smallmatrix}\right)$ is integral, and (iii) the polyhedron is nonempty, as it contains $k^{-1}z$]. Then $z - x_k \in (k-1)P$, whence by induction $z - x_k = x_1 + \cdots + x_{k-1}$ for integral vectors $x_1, \ldots, x_{k-1}$ in P. $\square$

The following theorem collects together several other characterizations of total unimodularity.

Theorem 4.4. *Let A be a matrix with entries* 0, +1, *and* −1. *Then the following characterizations are equivalent*:

(i) *A is totally unimodular, i.e., each square submatrix of A has determinant in* $\{0, +1, -1\}$;

(ii) *each collection of columns of A can be split into two parts so that the sum of the columns in one part, minus the sum of the columns in the other part, is a vector with entries* 0, +1, *and* −1 *only*;

(iii) *each nonsingular submatrix of A has a row with an odd number of nonzero components*;

(iv) *the sum of the entries in any square submatrix of A with even row and column sums, is divisible by four*;

(v) *no square submatrix of A has determinant* $+2$ *or* -2.

Characterization (ii) is due to Ghouila-Houri (1962), (iii) and (iv) to Camion (1965), and (v) to R. E. Gomory (cf. Camion 1965).

There are several further characterizations of total unimodularity. By far the deepest is due to Seymour (1980) (see chapter 10). For an efficient algorithm to test total unimodularity, see Truemper (1982). See also Truemper (1990).

4.1. Application: bipartite graphs

It is not difficult to see that the $V \times E$ incidence matrix A of a bipartite graph $G = (V, E)$ is totally unimodular: any square submatrix B of A either has a column with at most one 1 (in which case $\det B \in \{0, \pm 1\}$ by induction), or has two 1's in each column (in which case $\det B = 0$ by the bipartiteness of G). In fact, the incidence matrix of a graph G is totally unimodular if and only if G is bipartite.

The total unimodularity of the incidence matrix of a bipartite graph has several consequences, some of which we will describe now.

Definition 4.5. The *matching polytope* of a graph $G = (V, E)$ is the polytope $\mathrm{conv}\{\chi^M \mid M \text{ matching}\}$ in $\mathbb{R}^E$. Theorem 4.1 directly implies that the matching polytope of a bipartite graph G is equal to the set of all vectors x in $\mathbb{R}^E$ satisfying

$$\begin{aligned} &\text{(i)} \quad x_e \geq 0\,, \quad e \in E\,, \\ &\text{(ii)} \quad \sum_{e \ni v} x_e \leq 1\,, \quad v \in V \end{aligned} \tag{4.6}$$

[since the polyhedron determined by (4.6) is integral].

Clearly, the matching polytope of $G = (V, E)$ has dimension $|E|$. Each inequality in (4.6) is facet-determining, except if G has a vertex of degree at most 1. It is not difficult to see that the incidence vectors χ^M, $\chi^{M'}$ of two matchings M, M' are adjacent on the matching polytope iff $M \,\Delta\, M'$ is a path or circuit, where Δ denotes symmetric difference. Hence, the matching polytope of G has diameter at most $\nu(G)$. (This paragraph holds also for nonbipartite graphs.)

The above characterization of the matching polytope for bipartite graphs implies that for any bipartite graph $G = (V, E)$ and any "weight" function $c: E \to \mathbb{R}_+$:

$$\text{maximum weight of a matching} = \max\{c^{\mathrm{T}}x \mid x \geq 0,\, Ax \leq \mathbf{1}\}\,, \tag{4.7}$$

where A is the incidence matrix of A, $\mathbf{1}$ denotes an all-one column vector, and where the *weight* of a set is the sum of the weights of its elements. In particular,

$$\nu(G) = \max\{\mathbf{1}^{\mathrm{T}}x \mid x \geq 0, Ax \leq \mathbf{1}\} . \tag{4.8}$$

Definition 4.9. The *node-cover polytope* of a graph $G = (V, E)$ is the polytope $\operatorname{conv}\{\chi^N \mid N \text{ node cover}\}$ in $\mathbb{R}^V$. Again, Theorem 4.1 implies that, if G is bipartite, the node-cover polytope of G is equal to the set of all vectors y in $\mathbb{R}^V$ satisfying

$$\begin{aligned} &\text{(i)} \quad 0 \leq y_v \leq 1 , \quad v \in V , \\ &\text{(ii)} \quad y_v + y_w \geq 1 , \quad \{v, w\} \in E . \end{aligned} \tag{4.10}$$

It follows that for any weight function $w: V \to \mathbb{R}_+$:

$$\text{minimum weight of a node cover} = \min\{w^{\mathrm{T}}y \mid y \geq 0, y^{\mathrm{T}}A \geq \mathbf{1}\} , \tag{4.11}$$

where A again is the $V \times E$ incidence matrix of G. In particular,

$$\tau(G) = \min\{\mathbf{1}^{\mathrm{T}}y \mid y \geq 0, y^{\mathrm{T}}A \geq \mathbf{1}\} . \tag{4.12}$$

Now, by linear programming duality, we know that problems (4.8) and (4.12) are equal, i.e., we have *König's Matching Theorem*: $\nu(G) = \tau(G)$ for bipartite G.

By Theorem 4.3, the matching polytope P of G has the integer decomposition property. This has the following consequence. Let $k := \Delta(G)$ (the maximum degree of G). Then $(1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^E$ belongs to kP, and hence is the sum of k integer vectors in P. Each of these vectors being the incidence vector of a matching, it follows that E can be partitioned into k matchings. So we have *König's Edge-Coloring Theorem*: the edge-coloring number $\gamma(G)$ of a bipartite graph G is equal to its maximum degree.

We briefly mention some more examples of the consequences of Theorems 4.1 and 4.3 to bipartite graphs.

Definition 4.13. The *perfect matching polytope* of a graph $G = (V, E)$ is the polytope $\operatorname{conv}\{\chi^M \mid M \text{ perfect matching}\}$ in $\mathbb{R}^E$. It is a face of the matching polytope of G. For bipartite graphs, by (4.6), the perfect matching polytope is determined by

$$\begin{aligned} &\text{(i)} \quad x_e \geq 1 , \quad e \in E , \\ &\text{(ii)} \quad \sum_{e \ni v} x_e = 1 , \quad v \in V . \end{aligned} \tag{4.14}$$

This is equivalent to a theorem of Birkhoff (1946): each doubly stochastic matrix is a convex combination of permutation matrices.

One easily checks that the incidence vectors χ^M, $\chi^{M'}$ of two perfect matchings M, M' are adjacent on the perfect matching polytope if and only if $M \Delta M'$ is a circuit (cf. Balinski and Russakoff 1974). So the perfect matching polytope has diameter at most $\frac{1}{2}|V|$. The dimension of the perfect matching polytope of a bipartite graph is equal to $|E'| - |V| + 1$, where $E' := \bigcup M \setminus (\bigcap M)$, where the union and intersection both range over all perfect matchings (see Lovász and Plummer 1986).

Definition 4.15. The *assignment polytope* of order n is the perfect matching polytope of $K_{n,n}$. Equivalently, it is the polytope in $\mathbb{R}^{n \times n}$ of all matrices $(x_{ij})_{i,j=1}^{n}$ satisfying

$$\begin{aligned} &\text{(i)} \quad x_{ij} \geq 0\,, \quad i, j = 1, \ldots, n\,, \\ &\text{(ii)} \quad \sum_{i=1}^{n} x_{ij} = 1\,, \quad j = 1, \ldots, n\,, \\ &\text{(iii)} \quad \sum_{j=1}^{n} x_{ij} = 1\,, \quad i = 1, \ldots, n\,. \end{aligned} \tag{4.16}$$

(Such matrices are called *doubly stochastic*.)

Balinski and Russakoff (1974) studied assignment polytopes, proving inter alia that they have diameter 2 (if $n \geq 4$). See also Balinski (1985), Bertsekas (1981), Goldfarb (1985), Hung (1983), Padberg and Rao (1974), and Roohy-Laleh (1981).

Definition 4.17. The *stable-set polytope* of a graph $G = (V, E)$ is the polytope $\text{conv}\{\chi^C \mid C \text{ stable set}\}$ in $\mathbb{R}^V$. By Theorem 4.1, for bipartite G, it is determined by

$$\begin{aligned} &\text{(i)} \quad 0 \leq y_v \leq 1\,, \quad v \in V\,, \\ &\text{(ii)} \quad y_v + y_w \leq 1\,, \quad \{v, w\} \in E\,. \end{aligned} \tag{4.18}$$

So if A is the $V \times E$ incidence matrix of the bipartite graph G, and $w: V \to \mathbb{R}_+$ is a "weight" function, then

$$\text{maximum weight of a stable set} = \max\{w^{\mathrm{T}} y \mid y \geq 0,\ y^{\mathrm{T}} A \leq \mathbf{1}^{\mathrm{T}}\}\,. \tag{4.19}$$

In particular:

$$\alpha(G) = \max\{\mathbf{1}^{\mathrm{T}} y \mid y \geq 0,\ y^{\mathrm{T}} A \leq \mathbf{1}^{\mathrm{T}}\}\,. \tag{4.20}$$

Definition 4.21. The *edge-cover polytope* of a graph $G = (V, E)$ is the polytope $\text{conv}\{\chi^F \mid F \text{ edge cover}\}$ in $\mathbb{R}^E$. By Theorem 4.1, for bipartite G, it is determined by

$$\begin{aligned} &\text{(i)} \quad 0 \leq x_e \leq 1\,, \quad e \in E\,, \\ &\text{(ii)} \quad \sum_{e \ni v} x_e \geq 1\,, \quad v \in V\,, \end{aligned} \tag{4.22}$$

assuming G has no isolated vertices. Hurkens (1991) characterized adjacency on the edge-cover polytope, and showed that its diameter is $|E| - \rho(G)$.

From (4.22) it follows that for any "weight" function $w: E \to \mathbb{R}_+$,

$$\text{minimum weight of an edge cover} = \min\{w^{\mathrm{T}}x \mid x \geq 0, Ax \geq \mathbf{1}\}\,. \tag{4.23}$$

In particular:

$$\rho(G) = \min\{\mathbf{1}^{\mathrm{T}}x \mid x \geq 0, Ax \geq \mathbf{1}\}\,. \tag{4.24}$$

By linear programming duality, (4.20) and (4.24) are equal, and hence we have *König's Covering Theorem*: $\alpha(G) = \rho(G)$ for bipartite G.

By Theorem 4.3, the edge-cover polytope of a bipartite graph has the integer decomposition property, implying a result of Gupta (1967): the maximum number of pairwise disjoint edge covers in a bipartite graph is equal to its minimum degree.

Let A be the incidence matrix of the bipartite graph $G = (V, E)$, let $w \in \mathbb{Z}^E$, $b \in \mathbb{Z}^V$, and consider the linear programs in the following duality equations:

$$\begin{aligned} &\text{(i)} \quad \max\{w^{\mathrm{T}}x \mid x \geq 0, Ax \leq b\} = \min\{y^{\mathrm{T}}b \mid y \geq 0, y^{\mathrm{T}}A \geq w^{\mathrm{T}}\}\,, \\ &\text{(ii)} \quad \min\{w^{\mathrm{T}}x \mid x \geq 0, Ax \geq b\} = \max\{y^{\mathrm{T}}b \mid y \geq 0, y^{\mathrm{T}}A \leq w^{\mathrm{T}}\}\,. \end{aligned} \tag{4.25}$$

By Theorem 4.1, these programs have integer optimum solutions. The special case $b = \mathbf{1}$ is equivalent to the following min–max relations of Egerváry (1931):

(i) the maximum weight of a matching is equal to the minimum value of $\sum_{v \in V} y_v$, where $y: V \to \mathbb{Z}_+$ such that $y_u + y_v \geq w_e$ $\forall e = \{u, v\} \in E$; (4.26)

(ii) The minimum weight of an edge cover is equal to the maximum value of $\sum_{v \in V} y_v$, where $y: V \to \mathbb{Z}_+$ such that $y_u + y_v \leq w_e$ $\forall e = \{u, v\} \in E$.

Definition 4.27. The *transportation polytope* for $a \in \mathbb{R}^m_+$, $b \in \mathbb{R}^n_+$ is the set of all vectors $(x_{ij} \mid i = 1, \ldots, m,\ j = 1, \ldots, n)$ in $\mathbb{R}^{m \times n}$ satisfying

$$\begin{aligned} &\text{(i)} \quad x_{ij} \geq 0\,, \quad i = 1, \ldots, m,\ j = 1, \ldots, n\,, \\ &\text{(ii)} \quad \sum_{j=1}^{n} x_{ij} = a_i\,, \quad i = 1, \ldots, m\,, \\ &\text{(iii)} \quad \sum_{i=1}^{n} x_{ij} = b_j\,, \quad j = 1, \ldots, n\,. \end{aligned} \tag{4.28}$$

It is related to the *Hitchcock–Koopmans transportation problem* (Hitchcock 1941, Koopmans 1948). Klee and Witzgall (1968) studied transportation polytopes, showing that x satisfying (4.28) is a vertex iff $\{\{p_i, q_j\} \mid x_{ij} > 0\}$ contains no circuits (where $p_1, \ldots, p_m, q_1, \ldots, q_n$ are vertices). Moreover, the dimension is $(m-1)\ (n-1)$ if a and b are positive (if the polytope is nonempty, i.e., if $\sum_i a_i = \sum_j b_j$). Bolker (1972) and Balinski (1974) showed the Hirsch Conjecture for some classes of transportation polytopes. Bolker (1972) and Ahrens (1981) studied the number of vertices of transportation polytopes.

Definition 4.29. Related is the *dual transportation polyhedron*, which is, for fixed $c \in \mathbb{R}^{m \times n}$, defined as the set of all vectors $(u; v)$ in $\mathbb{R}^m \times \mathbb{R}^n$ satisfying

$$u_1 = 0, \qquad u_i + v_j \geq c_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n. \tag{4.30}$$

It is not difficult to see that the dimension is $m + n - 1$, and that $(u; v)$ satisfying (4.30) is a vertex iff $\{\{p_i, q_j\} \mid u_i + v_j = c_{ij}\}$ is a connected graph on vertex set $\{p_1, \ldots, p_m, q_1, \ldots, q_n\}$. Balinski (1984) showed that the diameter of (4.30) is at most $(m-1)(n-1)$, thus proving the Hirsch Conjecture for this class of polyhedra. Balinski and Russakoff (1984) made a further study of dual transportation polyhedra, characterizing vertices and higher-dimensional faces by means of partitions. See also Balinski (1983), Ikura and Nemhauser (1983), and Zhu (1963).

4.2. Application: directed graphs

Total unimodularity also implies several results for flows and circulations in directed graphs. Let M be the $V \times A$ incidence matrix of a digraph $D = (V, A)$. Then M is totally unimodular. Again this can be shown by induction: let B be a square submatrix of M. If B has a column with at most one nonzero, then $\det B \in \{0, \pm 1\}$ by induction. If each column of B contains a $+1$ and a -1, then $\det B = 0$.

There are the following consequences.

Definition 4.31. Let $D = (V, A)$ be a digraph, let $r, s \in V$, and let $c \in \mathbb{R}_+^A$ be a "capacity" function. Then the *r–s-flow polytope* is the set of all vectors x in $\mathbb{R}^A$ satisfying

$$\begin{aligned} &\text{(i)} \quad 0 \leq x_a \leq c_a, \quad a \in A, \\ &\text{(ii)} \quad \sum_{a \in \delta^-(v)} x_a = \sum_{a \in \delta^+(v)} x_a, \quad v \in V, \; v \neq r, s. \end{aligned} \tag{4.32}$$

Any vector x satisfying (4.32) is called an *r–s-flow* (*under* c). By the total unimodularity of the incidence matrix of D, if c is integral, then the r–s-flow polytope has integral vertices. Hence, if c is integral, the maximum *value* $(:= \sum_{a \in \delta^+(r)} x_a - \sum_{a \in \delta^-(r)} x_a)$ of an r–s-flow under c is attained by an integral vector (Dantzig 1951b).

4.33 (*Max-Flow Min-Cut Theorem*). By LP duality, the maximum value of an r–s-flow under c is equal to the minimum value of $\sum_{a \in A} y_a c_a$, where $y \in \mathbb{R}_+^A$ is such that there exists a vector z in $\mathbb{R}^V$ satisfying

$$\begin{aligned} &\text{(i)} \quad y_a - z_v + z_w \geq 0, \quad a = (v, w) \in A, \\ &\text{(ii)} \quad z_r = 1, \qquad z_s = 0. \end{aligned} \tag{4.34}$$

Again, by the total unimodularity of the incidence matrix of D, we may take the minimizing y, z to be integral. Let $W := \{v \in V \mid z_v \geq 1\}$. Then for $a = (v, w) \in$

$\delta^+(W)$ we have $y_a \geq z_v - z_w \geq 1$, and hence

$$\sum_{a\in A} y_a c_a \geq \sum_{a\in\delta^-(W)} y_a c_a \geq \sum_{a\in\delta^-(W)} c_a . \tag{4.35}$$

So the maximum flow value is not less than the capacity of cut $\delta^+(W)$. Since it also cannot be larger, we have Ford and Fulkerson's Max-Flow Min-Cut Theorem.

Definition 4.36. Given digraph $D = (V, A)$ and $r, s \in V$, the *shortest-path polytope* is the convex hull of all incidence vectors χ^P of subsets P of A, being a disjoint union of an r–s-path and some directed circuits. By the total unimodularity of the incidence matrix of D, this polytope is equal to the set of all vectors $x \in \mathbb{R}^A$ satisfying

$$\begin{aligned} &\text{(i)} \quad 0 \leq x_a \leq 1\,, \quad a \in A\,,\\ &\text{(ii)} \quad \sum_{a\in\delta^+(v)} x_a = \sum_{a\in\delta^-(v)} x_a\,, \quad v \in V,\ v \neq r, s\,,\\ &\text{(iii)} \quad \sum_{a\in\delta^+(r)} x_a - \sum_{a\in\delta^-(r)} x_a = 1\,. \end{aligned} \tag{4.37}$$

So it is the intersection of an r–s-flow polytope with the hyperplane determined by (iii). Saigal (1969) showed that the Hirsch Conjecture holds for the class of shortest-path polytopes.

Definition 4.38. For digraph $D = (V, A)$ and l, $u \in \mathbb{R}^A$, the *circulation polytope* is the set of all *circulations* between l and u, i.e., vectors $x \in \mathbb{R}^A$ satisfying

$$\begin{aligned} &\text{(i)} \quad l_a \leq x_a \leq u_a\,, \quad a \in A\,,\\ &\text{(ii)} \quad Mx = 0\,, \end{aligned} \tag{4.39}$$

where M is the incidence matrix of D. By the total unimodularity of M, if l and u are integral, then the circulation polytope is integral. So if l and u are integral, and there exists a circulation, there exists an integral circulation. Similarly, a minimum-cost circulation can be taken to be integral.

By Farkas's Lemma, the circulation polytope is nonempty iff there are no vectors z, $w \in \mathbb{R}^A$, $y \in \mathbb{R}^V$ satisfying

$$\begin{aligned} &\text{(i)} \quad z, w \geq 0\,,\\ &\text{(ii)} \quad z - w + M^{\mathrm{T}} y = 0\,,\\ &\text{(iii)} \quad u^{\mathrm{T}} z - l^{\mathrm{T}} w < 0\,. \end{aligned} \tag{4.40}$$

Suppose now $l \leq u$, and (4.40) has a solution. Then there is also a solution satisfying $\mathbf{0} \leq y \leq \mathbf{1}$, and hence, by the total unimodularity of M, there is a solution z, w, y with y a $\{0, 1\}$-vector. We may assume that $z_a w_a = 0$ for each arc

a. Then, for $W := \{v \in V \mid y_v = 1\}$,

$$\sum_{a \in \delta^-(W)} u_a - \sum_{a \in \delta^+(W)} l_a = u^{\mathrm{T}} z - l^{\mathrm{T}} w < 0 . \tag{4.41}$$

Thus we have Hoffman's *Circulation Theorem* (Hoffman 1960): there exists a circulation x satisfying $l \leqslant x \leqslant u$ iff $l \leqslant u$ and there is no subset W of V with $\sum_{a \in \delta^-(W)} u_a < \sum_{a \in \delta^+(W)} l_a$.

4.42. More generally, for $l, u \in \mathbb{R}^A$ and $b', b'' \in \mathbb{R}^V$, the polyhedron

$$\{x \in \mathbb{R}^A \mid l \leqslant x \leqslant u, b' \leqslant Mx \leqslant b''\} \tag{4.43}$$

is integral, if l, u, b' and b'' are integral. Moreover, the total unimodularity of M yields a characterization of the nonemptiness of the polyhedron (4.43), extending Hoffman's Circulation Theorem.

It is not difficult to see that (4.43) is an affine transformation of the polytope of vectors $(x'; x''; y'; y'')$ in $\mathbb{R}^A \times \mathbb{R}^A \times \mathbb{R}^V \times \mathbb{R}^V$ satisfying

$$\begin{aligned}
&x'_a \geqslant 0 , \quad x''_a \geqslant 0 , \qquad a \in A , \\
&y'_v \geqslant 0 , \quad y''_v \geqslant 0 , \qquad v \in V , \\
&\sum_{a \in \delta^+(v)} x'_a + \sum_{a \in \delta^-(v)} x''_a + y'_v = b''_v + \sum_{a \in \delta^-(v)} u_a - \sum_{a \in \delta^+(v)} l_a , \quad v \in V , \\
&x'_a + x''_a = u_a - l_a , \quad a \in A , \\
&y'_v + y''_v = b''_v - b'_v , \quad v \in V
\end{aligned} \tag{4.44}$$

(the transformation is given by $x_a := x'_a + l_a$). Thus (4.43) is transformed into a face of the transportation polytope (4.27). In this way, several results for (4.43) can be derived from results for transportation polytopes.

Let $D = (V, A)$ be a directed graph, and let $T \subseteq A$ be a spanning tree in D. Consider the $T \times (A \backslash T)$ matrix N defined, for $a \in T$ and $a' = (v, w) \in A \backslash T$, by:

$$N_{a',a} := \begin{cases} 0 & \text{if } a \text{ does not occur in the } v\text{–}w \text{ path in } T , \\ -1 & \text{if } a \text{ occurs forward in the } v\text{–}w \text{ path in } T , \\ +1 & \text{if } a \text{ occurs backward in the } v\text{–}w \text{ path in } T . \end{cases} \tag{4.45}$$

Then N is totally unimodular, as can be seen with the help of Ghouila-Houri's characterization (4.4) (ii). A vector $x = \binom{x'}{x''}$ in $\mathbb{R}^{A \backslash T} \times \mathbb{R}^T$ satisfies $Mx = 0$ (where M is the incidence matrix of D) if and only if $x'' = Nx'$. Thus (4.39) can be reformulated as

$$\begin{aligned}
&l_a \leqslant x'_a \leqslant u_a , \quad a \in A \backslash T , \\
&l_a \leqslant (Nx')_a \leqslant u_a , \quad a \in T .
\end{aligned} \tag{4.46}$$

By the total unimodularity of N, the polytope determined by (4.46) has integer vertices, if all l_a and u_a are integer.

A special case is formed by the $\{0, 1\}$-matrices with the *consecutive ones property*: in each column, the 1's form an interval (fixing some ordering of the rows, as usual). This special case arises when T is a directed path, and each arc in $A\backslash T$ forms a directed circuit with some subpath in T.

For related results, see also Hoffman (1960, 1979).

5. Total dual integrality

Total dual integrality appears to be a powerful technique in deriving min–max relations and the integrality of polyhedra. It is based on the following result, shown, implicitly or explicitly, by Gomory (1963), Lehman (1965), Fulkerson (1971), Chvátal (1973a), Hoffman (1974) and Lovász (1976) for pointed polyhedra, and by Edmonds and Giles (1977) for general polyhedra.

Theorem 5.1. *A rational polyhedron P is integral if and only if each rational supporting hyperplane of P contains integral vectors.*

Proof. Since the intersection of a supporting hyperplane with P is a face of P, necessity of the condition is trivial. To prove sufficiency, suppose that each rational supporting hyperplane of P contains integral vectors. Let $P = \{x \mid Ax \leqslant b\}$, with A and b integral. Let $F = \{x \mid A'x = b'\}$ be a minimal face of P, where $A'x \leqslant b'$ is a subsystem of $Ax \leqslant b$. If F does not contain any integral vector, there exists a vector y such that $c^{\mathrm{T}} := y^{\mathrm{T}}A'$ is an integral vector, while $\delta := y^{\mathrm{T}}b'$ is not an integer (this follows, e.g., from Hermite's Normal Form Theorem). We may suppose that all entries in y are nonnegative (we may replace each entry y_i of y by $y_i - \lfloor y_i \rfloor$). Now $H := \{x \mid c^{\mathrm{T}}x = \delta\}$ is a supporting hyperplane of P, not containing any integral vector. $\square$

Note that the special case where P is pointed can be shown without appealing to Hermite's Theorem: if x^* is a nonintegral vertex of P, w.l.o.g. $x_1^* \not\in \mathbb{Z}$. There exist supporting hyperplanes $H = \{x \mid c^{\mathrm{T}}x = c^{\mathrm{T}}x^*\}$ and $\tilde{H} = \{x \mid \tilde{c}^{\mathrm{T}}x = \tilde{c}^{\mathrm{T}}x^*\}$ touching P in x^* such that c and $\tilde{c}$ are integral and such that $c^{\mathrm{T}} - \tilde{c}^{\mathrm{T}} = (1, 0, \ldots, 0)$. If both H and $\tilde{H}$ contain integral vectors, we know $c^{\mathrm{T}}x^* \in \mathbb{Z}$ and $\tilde{c}^{\mathrm{T}}x^* \in \mathbb{Z}$. However, $(c - \tilde{c})^{\mathrm{T}}x^* = x_1^* \not\in \mathbb{Z}$.

Theorem (5.1) can be applied as follows. Consider the LP problem

$$\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}, \tag{5.2}$$

for rational matrix A and rational vectors b, c.

Corollary 5.3. *The following are equivalent*:

(i) *the maximum value in* (5.2) *is an integer for each integral vector c for which the maximum is finite*;

(ii) *the maximum* (5.2) *is attained by an integral optimum solution for each rational vector c for which the maximum is finite*;

(iii) *the polyhedron* $\{x \mid Ax \leq b\}$ *is integral.*

Now consider the LP-duality equation

$$\max\{c^{\mathrm{T}}x \mid Ax \leq b\} = \min\{y^{\mathrm{T}}b \mid y \geq 0,\ y^{\mathrm{T}}A = c^{\mathrm{T}}\}\,. \tag{5.4}$$

Clearly, we may derive that the maximum value is an integer if we know that the minimum has an integral optimum solution and b is integral. This motivated Edmonds and Giles (1977) to define a system $Ax \leq b$ of linear inequalities to be *totally dual integral* (*TDI*) if for each integral vector c, the minimum in (5.4) is attained by an integral optimum solution. Then we have the following consequence.

Corollary 5.5. *Let* $Ax \leq b$ *by a system of linear inequalities, with* A *rational and* b *integral. If* $Ax \leq b$ *is TDI* (*i.e., the minimum in* (5.4) *is attained by an integral optimum solution* y, *for each integral vector* c *for which the minimum is finite*), *then* $\{x \mid Ax \leq b\}$ *is integral* (*i.e., the maximum in* (5.4) *is attained by an integral optimum solution* x, *for each* c *for which the maximum is finite*).

Note that the notion of total dual integrality is *not* symmetric in objective function c and right-hand-side vector b. Indeed, the implication in Corollary 5.5 cannot be reversed: the system $x_1 \geq 0$, $x_1 + 2x_2 \geq 0$ determines an integral polyhedron in $\mathbb{R}^2$, while it is not TDI. However, Giles and Pulleyblank (1979) showed that if P is an integral polyhedron, then $P = \{x \mid Ax \leq b\}$ for some TDI-system $Ax \leq b$ with b integral. In Schrijver (1981) it is shown that if P is moreover full-dimensional, then there is a unique minimal TDI-system determining P with A and b integral (minimal under deleting inequalities).

Related to total dual integrality is the notion of *Hilbert basis*: This is a collection $\{a_1, \ldots, a_m\}$ of vectors with the property that if an integer vector x is a nonnegative linear combination of the vectors $a_1, \ldots, a_m$, then it is an *integer* nonnegative linear combination of them.

The relation to total dual integrality is as follows. Let $Ax \leq b$ be a system of linear inequalities, and set $P := \{x \mid Ax \leq b\}$. If $a^{\mathrm{T}}x \leq \beta$ is an inequality from $Ax \leq b$ and F is a face of P, we say a is *tight* in F if $a^{\mathrm{T}}x = \beta$ for all x in F. Now $Ax \leq b$ is TDI if and only if for each face F of P, the rows of A that are tight in A form a Hilbert basis.

It was shown by Cook et al. (1986a) that if $\{a_1, \ldots, a_m\}$ is a Hilbert basis consisting of integer vectors in $\mathbb{R}^n$, then any integer vector x that is a nonnegative linear combination of $a_1, \ldots, a_m$ is in fact an integer nonnegative linear combination of at most $2n - 1$ of these vectors.

As a consequence one has that if $Ax \leq b$ is TDI (in n variables, say), and A is integral, then for any $c \in \mathbb{Z}^n$, $\min\{y^{\mathrm{T}}b \mid y \geq 0,\ y^{\mathrm{T}}A = c^{\mathrm{T}}\}$ is attained by an integer vector y with at most $2n - 1$ nonzero components (if the minimum is finite).

For more on total dual integrality, see Cook (1983, 1986), Edmonds and Giles (1984), and Cook et al. (1984).

We now consider some combinatorial applications of total dual integrality.

Application 5.6 (*Arborescences*). Let $D = (V, A)$ be a directed graph, and let r be a fixed vertex of D. An *r-arborescence* is a set A' of $|V| - 1$ arcs forming a spanning tree such that each vertex $v \neq r$ is entered by exactly one arc in A'. So for each vertex v there is a unique directed path in A' from r to v. An *r-cut* is an arc set of the form $\delta^-(U)$, for some nonempty subset U of $V \setminus \{r\}$. As usual $\delta^-(U)$ denotes the set of arcs entering U.

It is not difficult to see that r-arborescences are the inclusion-wise minimal sets of arcs intersecting r-cuts. Conversely, the inclusion-wise minimal r-cuts are the inclusion-wise minimal sets of arcs intersecting all r-arborescences.

Fulkerson (1974) showed:

Fulkerson's Optimum Arborescence Theorem 5.7. *For any "length" function $l: A \to \mathbb{Z}_+$, the minimum length of an r-arborescence is equal to the maximum number t of r-cuts $C_1, \ldots, C_t$ (repetition allowed) so that no arc a is in more than $l(a)$ of these cuts.*

This result can be formulated in polyhedral terms as follows. Let C be the matrix whose rows are the incidence vectors of all r-cuts. So the columns of C are indexed by A, and the rows by the collection $\mathcal{H} := \{U \mid \emptyset \neq U \subseteq V \setminus \{r\}\}$. Then Theorem 5.7 is equivalent to both optima in the LP-duality equation

$$\min\{l^{\mathrm{T}}x \mid x \geq 0, Cx \geq \mathbf{1}\} = \max\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0, y^{\mathrm{T}}C \leq l^{\mathrm{T}}\} \tag{5.8}$$

having integral optimum solutions, for each $l \in \mathbb{Z}_+^A$. So in order to show the theorem, by (5.5) it suffices to show that the maximum in (5.8) has an integral optimum solution, for each $l: A \to \mathbb{Z}$, i.e., that the system $x \geq 0$, $Cx \geq \mathbf{1}$ is TDI. This can be proved as follows (Edmonds and Giles 1977).

Proof of Theorem 5.7. Note that the matrix C is generally not totally unimodular. However, in order to prove that the maximum (5.8) has an integer optimum solution, it suffices to show that there exists a "basis" that is totally unimodular and that attains the maximum. That is, it is enough to find a totally unimodular submatrix C' of C (consisting of rows of C) such that

$$\max\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0, y^{\mathrm{T}}C \leq l^{\mathrm{T}}\} = \max\{z^{\mathrm{T}}\mathbf{1} \mid z \geq 0, z^{\mathrm{T}}C' \leq l^{\mathrm{T}}\}\,. \tag{5.9}$$

Since the second maximum is attained by an integer optimum solution z (by the total unimodularity of C'), extending z by 0's in the appropriate positions gives an integer optimum solution y for the first maximum.

How can we find such a C'? The key observation is the following. Call a subcollection $\mathcal{F}$ of $\mathcal{H}$ *laminar* if for all T, $U \in \mathcal{F}$ one has $T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$. Then, if C' is the matrix consisting of the rows of C with index in some laminar family $\mathcal{F}$, C' is totally unimodular.

This last fact can be derived with Ghouila-Houri's characterization (4.4) (ii). Choose a set of rows of C', i.e., choose a subcollection $\mathcal{G}$ of $\mathcal{F}$. Define, for each U in $\mathcal{G}$, the "height" $h(U)$ of U as the number of sets T in $\mathcal{G}$ with $T \supseteq U$. Now split $\mathcal{G}$ into $\mathcal{G}_{\text{odd}}$ and $\mathcal{G}_{\text{even}}$, according as $h(U)$ is odd or even. One easily derives from the laminarity of $\mathcal{G}$ that for any arc a of D, the number of sets in $\mathcal{G}_{\text{odd}}$ entered by a, and the number of sets in $\mathcal{G}_{\text{even}}$ entered by a, differ by at most 1. Therefore, we can split the rows corresponding to $\mathcal{G}$ into two classes fulfilling Ghouila-Houri's criterion. So C' is totally unimodular.

So it suffices to find a laminar subcollection $\mathcal{F}$ or $\mathcal{H}$ so that the corresponding matrix C' satisfies (5.9). This can be done as follows. We may assume that all components of l are nonnegative. (If some component is negative, the maximum in (5.8) is infinite.) Choose a vector y that attains the maximum in (5.8), and for which

$$\sum_{U \in \mathcal{H}} y_U \cdot |U| \cdot |V \setminus U| \tag{5.10}$$

is as small as possible. Such a vector y exists by compactness arguments.

Define

$$\mathcal{F} := \{U \mid y_U > 0\} . \tag{5.11}$$

Then $\mathcal{F}$ is laminar. To see this, suppose there are $T, U \in \mathcal{F}$ with $T \not\subseteq U \not\subseteq T$ and $T \cap U \neq \emptyset$. Let $\varepsilon := \min\{y_T, y_U\} > 0$. Next reset:

$$\begin{aligned} y_T &:= y_T - \varepsilon , & y_{T\cap U} &:= y_{T\cap U} + \varepsilon , \\ y_U &:= y_U - \varepsilon , & y_{T\cup U} &:= y_{T\cup U} + \varepsilon , \end{aligned} \tag{5.12}$$

while y does not change in the other coordinates. By this resetting, $y^{\mathrm{T}}C$ does not increase in any coordinate (since $\varepsilon \cdot \chi^{\delta^-(T)} + \varepsilon \cdot \chi^{\delta^-(U)} \geq \varepsilon \cdot \chi^{\delta^-(T\cap U)} + \varepsilon \cdot \chi^{\delta^-(T\cup U)}$), while $y^{\mathrm{T}}\mathbf{1}$ does not change. However, the sum (5.10) did decrease, contradicting the minimality of (5.10). This shows that $\mathcal{F}$ is laminar.

We finally show that (5.9) holds. The inequality $\leq$ is trivial, since C' is a submatrix of C. The inequality $\geq$ follows from the fact that the vector y above attains the second maximum in (5.9), while y has 0's in the positions corresponding to rows of C not in C'. □

A direct consequence is that the *r-arborescence polytope* of $D = (V, A)$ (being the convex hull of the incidence vectors of r-arborescences) is determined by

$$\begin{aligned} &0 \leq x_a \leq 1 , \quad a \in A , \\ &\sum_{a \in \delta^-(U)} x_a \geq 1 , \quad \emptyset \neq U \subseteq V \setminus \{r\} . \end{aligned} \tag{5.13}$$

This is a result of Edmonds (1967). It follows, with the ellipsoid method, that a minimum-length r-arborescence can be found in polynomial time if and only if we can test (5.13) in polynomial time. This last is indeed possible: given $x \in \mathbb{Q}^A$, we first test if $0 \leq x_a \leq 1$ for each arc a; if $x_a < 0$ or $x_a > 1$ for some a, we have a separating hyperplane. Otherwise, consider x as a capacity function on the arcs of D, and find an r-cut C of minimum capacity (with an adaptation of Ford and Fulkerson's algorithm): if C has capacity at least 1, then (5.13) is satisfied; otherwise, C yields a hyperplane separating x from the polyhedron determined by (5.13).

For a characterization of the facets of the r-arborescence polytope, see Held and Karp (1970) and Giles (1975, 1978).

One similarly shows that for any directed graph $D = (V, A)$, the following system, in $x \in \mathbb{R}^A$, is TDI:

$$\begin{aligned} & x_a \geq 0 , \quad a \in A , \\ & \sum_{a \in \delta^-(U)} x_a \geq 1 , \quad \emptyset \neq U \subseteq V,\ \delta^+(U) = \emptyset , \end{aligned} \tag{5.14}$$

which is a result of Lucchesi and Younger (1978). It is equivalent to:

Lucchesi–Younger Theorem 5.15. *The minimum size of a directed-cut covering in a digraph $D = (V, A)$ is equal to the maximum number of pairwise disjoint directed cuts.*

Here a *directed cut* is a set of arcs of the form $\delta^-(U)$ with $\emptyset \neq U \neq V$, $\delta^+(U) = \emptyset$. A *directed-cut covering* is a set of arcs intersecting each directed cut, or equivalently, a set of arcs whose contraction makes the digraph strongly connected.

Note that the Lucchesi–Younger Theorem is of a self-refining nature: it implies that for any "length" function $l: A \to \mathbb{Z}_+$, the minimum length of a directed-cut covering is equal to the maximum number t of directed cuts $C_1, \ldots, C_t$ (repetition allowed), so that no arc a is in more than $l(a)$ of these cuts. [To derive this from Theorem 5.15, replace each arc a by a directed path of length $l(a)$.] In this weighted form, the Lucchesi–Younger Theorem is easily seen to be equivalent to the total dual integrality of (5.14).

Application 5.16 (*Polymatroid intersection*). Let S be a finite set. A function $f: \mathcal{P}(S) \to \mathbb{R}$ is called *submodular* if

$$f(T) + f(U) \geq f(T \cap U) + f(T \cup U) \quad \text{for all } T, U \subseteq S . \tag{5.17}$$

There are several examples of submodular functions. For example, the rank function of any matroid is submodular (see chapters 9 and 11).

Let f_1, f_2 be two submodular functions on S, and consider the following system in the variable $x \in \mathbb{R}^S$:

$$\begin{aligned} &\text{(i)} \quad x_s \geq 0 , \quad s \in S , \\ &\text{(ii)} \quad \sum_{s \in U} x_s \leq f_1(U) , \quad U \subseteq S , \\ &\text{(iii)} \quad \sum_{s \in U} x_s \leq f_2(U) , \quad U \subseteq S . \end{aligned} \tag{5.18}$$

Edmonds (1970, 1979) proved:

Theorem 5.19. *System* (5.18) *is TDI.*

Proof. The proof is similar to that of Theorem 5.7. Let $c: S \to \mathbb{Z}$, and consider the LP problem dual to maximizing $c^{\mathrm{T}}x$ over (5.18):

$$\min\Big\{\sum_{U\subseteq S} y_U f_1(U) + \sum_{U\subseteq S} z_U f_2(U) \,\Big|\, y, z \in \mathbb{R}_+^{\mathcal{P}(S)};\ \sum_{U\subseteq S}(y_U + z_U)\chi^U \geq c\Big\}. \tag{5.20}$$

We show that this minimum has an integral optimum solution, by a version of the "uncrossing" technique. Let y, z attain this minimum, so that

$$\sum_{U\subseteq S}(y_U + z_U)\cdot|U|\cdot|S\setminus U| \tag{5.21}$$

is as small as possible. Let

$$\mathcal{F} := \{U \subseteq S \mid y_U > 0\}. \tag{5.22}$$

We show that $\mathcal{F}$ forms a chain with respect to inclusion. Suppose not. Let T, $U \in \mathcal{F}$ with $T \not\subseteq U \not\subseteq T$. Let $\varepsilon := \min\{y_T, y_U\} > 0$. Next reset as in (5.12). Again, the modified y forms, with the original z, an optimum solution of (5.20) [since $\chi^T + \chi^U = \chi^{T\cap U} + \chi^{T\cup U}$ and $f_1(T) + f_1(U) \geq f_1(T\cap U) + f_1(T\cup U)$]. However, (5.21) did decrease, contradicting its minimality. This shows that $\mathcal{F}$ forms a chain. Similarly,

$$\mathcal{G} := \{U \subseteq S \mid z_U > 0\} \tag{5.23}$$

forms a chain.

Now (5.20) is equal to

$$\min\Big\{\sum_{U\in\mathcal{F}} y_U f_1(U) + \sum_{U\in\mathcal{G}} z_U f_2(U) \,\Big|\, y \in \mathbb{R}_+^{\mathcal{F}},\ z \in \mathbb{R}_+^{\mathcal{G}}; \\ \sum_{U\in\mathcal{F}} y_U\chi^U + \sum_{U\in\mathcal{G}} z_U\chi^U \geq c\Big\} \tag{5.24}$$

since y, z attain (5.20), using (5.22) and (5.23).

The constraint matrix in (5.24) is totally unimodular, as can be derived easily with Ghouila-Houri's criterion (4.4) (ii). Hence (5.24) has an integral optimum solution y, z. By extending y, z with 0-components, we obtain an integral optimum solution of (5.20). □

This result has several corollaries, as we shall see. If f_1 and f_2 are integer-valued submodular functions, then the total dual integrality of (5.18) implies that (5.18) determines an integral polyhedron. In particular, let f_1 and f_2 be the rank functions of two matroids $(S, \mathcal{I}_1)$ and $(S, \mathcal{I}_2)$. Then the following result of Edmonds (1970) follows.

Corollary 5.25. *The polytope* $\operatorname{conv}\{\chi^I \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$ *is determined by* (5.18).

Proof. Note that an integral vector satisfies (5.18) iff it is equal to χ^I for some I in $\mathcal{I}_1 \cap \mathcal{I}_2$. □

A special case is that if we have one matroid $(S, \mathcal{I})$, with rank function, say, f, then its *independence polytope* ($=\operatorname{conv}.\{\chi^I \mid I \in \mathcal{I}\}$) is determined by $x_s \geqslant 0$, $s \in S$; $\sum_{s\in U} x_s \leqslant f(U)$, $U \subseteq S$ (Edmonds 1971). So Corollary 5.25 concerns the intersection of two independence polytopes. The facets of independence polytopes, and of the intersection of two of them, are described by Giles (1975). Hausmann and Korte (1978) characterized adjacency on the independence polytope. See also Edmonds (1979) and Cunningham (1984).

Another direct consequence for matroids is:

Edmonds' Matroid Intersection Theorem 5.26. *The maximum size of a common independent set of two matroids* $(S, \mathcal{I}_1)$ *and* $(S, \mathcal{I}_2)$ *is equal to* $\min_{U\subseteq S} (f_1(U) + f_2(S\setminus U))$, *where* f_1 *and* f_2 *are the rank functions of these matroids.*

Proof. By Corollary 5.25, the maximum size of a common independent set is equal to $\max\{\mathbf{1}^{\mathrm{T}}x \mid x$ satisfies (5.18)$\}$, and hence, by the total dual integrality of (5.18), to

$$\min\left\{\sum_{U\subseteq S} (y_U f_1(U) + z_U f_2(U)) \mid y, z \in \mathbb{Z}_+^{\mathcal{P}(S)};\ \sum_{U\subseteq S} (y_U + z_U)\chi^U \geqslant \mathbf{1}\right\}.$$

It is not difficult (using the nonnegativity, the monotonicity and the submodularity of f_1 and f_2) to derive that this last minimum is equal to the minimum in Theorem 5.26. □

For more consequences of Theorem 5.19, we refer to chapter 11.

The proofs of Theorems 5.7 and 5.19 given above are examples of a general proof technique for total dual integrality studied by Edmonds and Giles (1977). First show that there exists an optimum dual solution whose nonzero components correspond to a "nice" colelction of sets (e.g., laminar, a chain, "cross-free"). Next prove that such nice collections yield a restricted linear program with totally unimodular constraint matrix. Finally, appeal to Hoffman and Kruskal's Theorem to deduce the existence of an integral optimum dual solution for the restricted, and hence for the original, problem.

We now illustrate how total dual integrality helps in showing one of the pioneering successes of polyhedral combinatorics, the characterization of the matching polytope by Edmonds (1965). For the basic theory on matchings we refer to chapter 3.

Definition 5.27. The *matching polytope* of an undirected graph $G = (V, E)$ is the polytope $\operatorname{conv}\{\chi^M \mid M \text{ matching}\}$ in $\mathbb{R}^E$. Edmonds showed that this polytope is equal to the set of all vectors x in $\mathbb{R}^E$ satisfying

$$\begin{aligned} &\text{(i)} \quad x_e \geq 0\,, \quad e \in E\,, \\ &\text{(ii)} \quad \sum_{e \ni v} x_e \leq 1\,, \quad v \in V\,, \\ &\text{(iii)} \quad \sum_{e \subseteq U} x_e \leq \lfloor \tfrac{1}{2}|U| \rfloor\,, \quad U \subseteq V\,. \end{aligned} \tag{5.28}$$

Since the integral vectors satisfying (5.28) are exactly the incidence vectors χ^M of matchings M, it suffices to show that (5.28) determines an integral polyhedron. In fact, Cunningham and Marsh (1978) showed:

Theorem 5.29. *System* (5.28) *is TDI.*

This implies that for each $w\colon E \to \mathbb{Z}$, both optima in the LP-duality equation

$$\begin{aligned} &\max\{w^{\mathrm{T}}x \mid x \text{ satisfies (5.28)}\} \\ &= \min\Big\{\sum_{v \in V} y_v + \sum_{U \subseteq V} z_U \lfloor \tfrac{1}{2}|U| \rfloor \;\Big|\; y \in \mathbb{R}^V_+,\, z \in \mathbb{R}^{\mathcal{P}(V)}_+; \\ &\qquad \forall e \in E\colon \sum_{v \in e} y_v + \sum_{U \supseteq e} z_U \geq w_e \Big\} \end{aligned} \tag{5.30}$$

are attained by integral optimum solutions. It means: for each undirected graph $G = (V, E)$ and for each "weight" function $w\colon E \to \mathbb{Z}$

$$\begin{aligned} &\max\{w(M) \mid M \text{ matching}\} \\ &= \min\Big\{\sum_{v \in V} y_v + \sum_{U \subseteq V} z_U \lfloor \tfrac{1}{2}|U| \rfloor \;\Big|\; y \in \mathbb{Z}^V_+,\, z \in \mathbb{Z}^{\mathcal{P}(V)}_+; \\ &\qquad \forall e \in E\colon \sum_{v \in e} y_v + \sum_{U \supseteq e} z_U \geq w_e \Big\} \end{aligned} \tag{5.31}$$

Here $w(E') := \sum_{e \in E'} w_e$ for any subset E' of E. [Note that (5.31) contains the Tutte–Berge formula as special case, by taking $w = \mathbf{1}$.]

Proof of Theorem 5.29. We may assume that w is nonnegative, since replacing any negative component of w by 0 does not change any optimum in (5.31).

For any w, let ν_w denote the left-hand term in (5.31). It suffices to show that ν_w is not less than the right-hand term in (5.31) (since $\leq$ is trivial). Suppose (5.31) does not hold, and suppose we have chosen $G = (V, E)$ and $w\colon E \to \mathbb{Z}_+$ so that $|V| + |E| + w(E)$ is as small as possible. Then G is connected (otherwise, one of the components of G will form a smaller counterexample) and $w_e \geq 1$ for each edge e (otherwise we could delete e). Now there are two cases.

Case 1. There exists a vertex v covered by every maximum–weighted matching. In this case, let $w' \in \mathbb{Z}_+^E$ arise from w by decreasing the weights of edges incident to v by 1. Then $\nu_{w'} = \nu_w - 1$. Since $w'(E) < w(E)$, (5.31) holds for w'. Increasing component y_v of the optimal y for w' by 1, shows (5.31) for w.

Case 2. No vertex is covered by every maximum-weighted matching. Now let w' arise from w by decreasing all weights by 1. We show that $\nu_w \geq \nu_{w'} + \lfloor \frac{1}{2}|V| \rfloor$. This will imply (5.31) for w: since $w'(E) < w(E)$, (5.31) holds for w'. Increasing component z_V of the optimal z for w' by 1, shows (5.31) for w.

Assume $\nu_w < \nu_{w'} + \lfloor \frac{1}{2}|V| \rfloor$, and let M be a matching with $\nu_{w'} = w'(M)$, such that $w(M)$ is as large as possible. Then M leaves at least two vertices in V uncovered, since otherwise $w(M) = w'(M) + \lfloor \frac{1}{2}|V| \rfloor$, implying $\nu_w \geq w(M) = w'(M) + \lfloor \frac{1}{2}|V| \rfloor = \nu_{w'} + \lfloor \frac{1}{2}|V| \rfloor$.

Let u and v be not covered by M, and suppose we have chosen M, u and v so that the distance $d(u, v)$ in G is as small as possible. Then $d(u, v) > 1$, since otherwise augmenting M by $\{u, v\}$ would increase $w(M)$. Let t be an internal vertex of a shortest path between u and v. Let M' be a matching with $w(M') = \nu_w$ not covering t.

Now $M \,\Delta\, M'$ is a disjoint union of paths and circuits. Let P be the set of edges of the component of $M \,\Delta\, M'$ containing t. Then P forms a path starting in t and not covering both u and v (as t, u and v each have degree at most 1 in $M \,\Delta\, M'$). Say P does not cover u. Now the symmetric difference $M \,\Delta\, P$ is a matching with $|M \,\Delta\, P| \leq |M|$, and therefore

$$\begin{aligned} w'(M \,\Delta\, P) - w'(M) &= w(M \,\Delta\, P) - |M \,\Delta\, P| - w(M) + |M| \\ &\geq w(M \,\Delta\, P) - w(M) = w(M') - w(M' \,\Delta\, P) \geq 0\,. \end{aligned} \tag{5.32}$$

Hence $\nu_{w'} = w'(M \,\Delta\, P)$ and $w(M \,\Delta\, P) \geq w(M)$. However, $M \,\Delta\, P$ does not cover t and u, and $d(u, t) < d(u, v)$, contradicting our choice of M, u, and v. □

So (5.28) is TDI. A consequence is the following fundamental result of Edmonds (1965).

Edmonds' Matching Polyhedron Theorem 5.33. *The matching polytope of a graph is equal to the polyhedron determined by* (5.28).

In fact, Edmonds found Theorem 5.33 as a by-product of a polynomial-time algorithm for finding a maximum-weighted matching. In turn, with the ellipsoid method, Padberg and Rao (1982) showed that Theorem 5.33 yields a polynomial-time algorithm finding a maximum-weighted matching, see (5.37) below.

5.34. A consequence of Theorem 5.33 is a characterization of the *perfect matching polytope* of a graph $G = (V, E)$, which is the polytope conv$\{\chi^M \mid M$ perfect matching$\}$ in $\mathbb{R}^E$. This polytope clearly is a face of the matching polytope of G (or is empty), viz. the intersection of the matching polytope with the (supporting) hyperplane $\{x \in \mathbb{R}^E \mid \sum_{e \in E} x_e = \frac{1}{2}|V|\}$. It follows that the perfect matching polytope is determined by the following inequalities:

$$
\begin{aligned}
&\text{(i)} && x_e \geqslant 0\,, \quad e \in E\,, \\
&\text{(ii)} && \sum_{e \ni v} x_e = 1\,, \quad v \in V\,, \\
&\text{(iii)} && \sum_{e \in \delta(U)} x_e \geqslant 1\,, \quad U \subseteq V,\ |U| \text{ odd}\,.
\end{aligned}
\tag{5.35}
$$

(Note that (ii) and (iii) imply (5.28) (iii).)

From the description (5.35) of the perfect matching polytope one can derive with the ellipsoid method a polynomial-time algorithm for finding a maximum-weighted perfect matching (and through this a maximum-weighted matching). It amounts to showing that it can be tested in polynomial time whether a vector x satisfies (5.35). Padberg and Rao (1982) showed that this can be done as follows.

For a given $x \in \mathbb{Q}^E$ we must test if x satisfies (5.35). The inequalities in (i) and (ii) can be checked one by one. If one of them is not satisfied, it gives us a separating hyperplane. So we may assume that (i) and (ii) are satisfied. If $|V|$ is odd, then clearly (iii) is not satisfied for $U := V$. So we may assume that $|V|$ is even. We cannot check the constraints in (iii) one by one in polynomial time, simply because there are exponentially many of them. Yet, there is a polynomial-time method of checking time. First, note that from Ford and Fulkerson's max-flow min-cut algorithm we can easily derive a polynomial-time algorithm having the following as input and output:

Input: Subset W of V.
Output: Subset T of V such that $W \cap T \neq \emptyset \neq W \backslash T$ and such that $x(\delta(T))$ is as small as possible. (5.36)

Here $x(E') := \sum_{e \in E'} x_e$ for any subset E' of E. We next describe recursively an algorithm with the following input and output specification:

Input: Subset W of V with $|W|$ even.
Output: Subset U of V such that $|W \cap U|$ is odd and such that $x(\delta(U))$ is as small as possible. (5.37)

First, we find with algorithm (5.36) a subset T of V with $W \cap T \neq \emptyset \neq W \backslash T$ and with $x(\delta(T))$ minimal. If $|W \cap T|$ is odd, we are done. If $|W \cap T|$ is even, call, recursively, the algorithm (5.37) for the inputs $W \cap T$ and $W \cap \bar{T}$, respectively, where $\bar{T} := V \backslash T$. Let it yield a subset U' of V such that $|W \cap T \cap U'|$ is odd and $x(\delta(U'))$ is minimal, and a subset U'' of V such that $|W \cap \bar{T} \cap U''|$ is odd and $x(\delta(U''))$ is minimal. Without loss of generality, $W \cap \bar{T} \not\subseteq U'$ (otherwise replace U' by $V \backslash U'$), and $W \cap T \not\subseteq U''$ (otherwise replace U'' by $V \backslash U''$).

We claim that if $x(\delta(T \cap U')) < x(\delta(\bar{T} \cap U''))$, then $U := T \cap U'$ is output of (5.37) for input W, and otherwise $U := \bar{T} \cap U''$. To see that this output is justified suppose to the contrary that there exists a subset Y of V such that $|W \cap Y|$ is odd, and $x(\delta(Y)) < x(\delta(T \cap U'))$ and $x(\delta(Y)) < x(\delta(\bar{T} \cap U''))$. Then either $|W \cap Y \cap T|$ is odd or $|W \cap Y \cap \bar{T}|$ is odd.

Case 1. $|W \cap Y \cap T|$ is odd. Then $x(\delta(Y)) \geq x(\delta(U'))$, since U' is output of (5.37) for input $W \cap T$. Moreover, $x(\delta(T \cup U') \geq x(\delta(T))$, since T is output of (5.36) for input W, and since $W \cap (T \cup U') \neq \emptyset \neq W \backslash (T \cup U')$. Therefore, we have the following contradiction:

$$\begin{aligned} x(\delta(Y)) &\geq x(\delta(U')) \geq x(\delta(T \cap U')) + x(\delta(T \cup U')) - x(\delta(T)) \\ &\geq x(\delta(T \cap U')) > x(\delta(Y)) \end{aligned} \tag{5.38}$$

[the second inequality follows since $x(\delta(A)) + x(\delta(B)) \geq x(\delta(A \cap B)) + x(\delta(A \cup B))$ for all $A, B \subseteq V$].

Case 2. $|W \cap Y \cap \bar{T}|$ is odd: similar.

Given the polynomial speed of the algorithm for (5.36), it is not difficult to see that the algorithm described for (5.37) is also polynomial-time. As a consequence, we can test (5.35) (iii) in polynomial time.

Further notes on TDI: for a deep characterization of certain TDI systems, see Seymour (1977). For an application of TDI to non-optimizational combinatorics (viz. Nash–Williams Orientation Theorem), see Frank (1980), and Frank and Tardos (1984).

6. Blocking polyhedra

Another useful technique in polyhedral combinatorics is a variant of the classical polarity in Euclidean space, viz. the blocking relation between polyhedra. It was introduced by Fulkerson (1970a, 1971), who noticed its importance to combinatorics and optimization. Often, with the theory of blocking polyhedra, one polyhedral characterization (or min–max relation) can be derived from another, and conversely.

The basic idea is the following result. Let $c_1, \ldots, c_m, d_1, \ldots, d_t \in \mathbb{R}^n_+$ satisfy

$$\operatorname{conv}\{c_1, \ldots, c_m\} + \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid d_j^{\mathrm{T}} x \geq 1 \text{ for } j = 1, \ldots, t\}\,. \tag{6.1}$$

Then the same holds after interchanging the c_i and d_j:

$$\operatorname{conv}\{d_1, \ldots, d_t\} + \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid c_i^{\mathrm{T}} x \geq 1 \text{ for } i = 1, \ldots, m\}. \tag{6.2}$$

In a sense, in (6.2) the ideas of "vertex" and "facet" are interchanged as compared with (6.1). The proof is a simple application of Farkas's Lemma.

Theorem 6.3. *For any* $c_1, \ldots, c_m, d_1, \ldots, d_t \in \mathbb{R}^n_+$, (6.1) *holds if and only if* (6.2) *holds.*

Proof. Suppose (6.1) holds. Then $\subseteq$ in (6.2) is direct, since $c_i^{\mathrm{T}} d_j \geq 1$ for all i, j, as the c_i belong to the right-hand side in (6.1), and since $c \geq 0$.

To show $\supseteq$ in (6.2), suppose $x \not\in \operatorname{conv}\{d_1, \ldots, d_t\} + \mathbb{R}^n_+$. Then there exists a separating hyperplane, i.e., there is a vector y such that

$$y^{\mathrm{T}}x > \min\{y^{\mathrm{T}}z \mid z \in \mathrm{conv}\{d_1, \ldots, d_t\} + \mathbb{R}_+^n\} . \tag{6.4}$$

We may assume $t \geq 1$ [since if $t = 0$, then (6.1) gives that $0 \in \{c_1, \ldots, c_m\}$, and therefore x does not belong to the right-hand side of (6.2)]. By scaling y, we can assume that the minimum in (6.4) is 1. Therefore, y belongs to the right-hand side of (6.1), and therefore to the left-hand side. So $y \geq \lambda_1 c_1 + \cdots + \lambda_m c_m$ for certain $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. Since $y^{\mathrm{T}}x < 1$, it follows that $c_i^{\mathrm{T}}x < 1$ for at least one i. Hence x does not belong to the right-hand side of (6.2).

This shows (6.1) $\Rightarrow$ (6.2). The reverse implication follows by symmetry. □

This theorem has the following consequences. For any $X \subseteq \mathbb{R}^n$, we define the *blocker* $B(X)$ of X by:

$$B(X) := \{x \in \mathbb{R}_+^n \mid y^{\mathrm{T}}x \geq 1 \text{ for each } y \text{ in } X\} . \tag{6.5}$$

Clearly, for $c_1, \ldots, c_m \in \mathbb{R}_+^n$, if P is the polyhedron

$$P := \mathrm{conv}\{c_1, \ldots, c_m\} + \mathbb{R}_+^n , \tag{6.6}$$

then

$$B(P) = \{x \in \mathbb{R}_+^n \mid c_i^{\mathrm{T}}x \geq 1 \text{ for } i = 1, \ldots, m\} . \tag{6.7}$$

So $B(P)$ is also a polyhedron, called the *blocking polyhedron* of P. If $R = B(P)$, the pair P, R is called a *blocking pair* of polyhedra. By the following direct corollary of Theorem 6.3, this is a symmetric relation.

Corollary 6.8. *For any polyhedron of type* (6.6), $B(B(P)) = P$.

So both (6.1) and (6.2) are equivalent to:

the pair $\mathrm{conv}\{c_1, \ldots, c_m\} + \mathbb{R}_+^n$ and $\mathrm{conv}\{d_1, \ldots, d_t\} + \mathbb{R}_+^n$ forms a blocking pair of polyhedra. (6.9)

The following corollary shows the equivalence of certain min–max relations.

Corollary 6.10. *Let* $c_1, \ldots, c_m, d_1, \ldots, d_t \in \mathbb{R}_+^n$. *Then the following are equivalent*:

(i) *for each* $l \in \mathbb{R}^n$: $\min\{l^{\mathrm{T}}c_1, \ldots, l^{\mathrm{T}}c_m\}$

$$= \max\Big\{\lambda_1 + \cdots + \lambda_t \mid \lambda_1, \ldots, \lambda_t \in \mathbb{R}_+; \sum_j \lambda_j d_j \leq l\Big\} ; \tag{6.11}$$

(ii) *for each* $w \in \mathbb{R}_+^n$: $\min\{w^{\mathrm{T}}d_1, \ldots, w^{\mathrm{T}}d_t\}$

$$= \max\left\{\mu_1 + \cdots + \mu_m \mid \mu_1, \ldots, \mu_m \in \mathbb{R}_+ ; \sum_i \mu_i c_i \leq w\right\} . \tag{6.12}$$

Proof. By LP duality, the maximum in (6.11) is equal to min $\{l^{\mathrm{T}}x \mid x \in \mathbb{R}^n_+$ $d_j^{\mathrm{T}}x \geq 1$ for $j = 1, \ldots, t\}$. Hence, (6.11) is equivalent to (6.1). Similarly, (6.12) i equivalent to (6.2). Therefore, Theorem 6.3 implies Corollary 6.10. □

Note that by continuity, in (6.11) we may restrict l to rational, and hence t integral vectors, without changing the condition. Similarly for (6.12). This i sometimes useful when showing one of them by induction.

A symmetric characterization of the blocking relation is the "length-widt inequality" given by Lehman (1965):

Lehman's Length-Width inequality 6.13. *Let* $c_1, \ldots, c_m, d_1, \ldots, d_t \in \mathbb{R}^n_+$. *The* (6.1) [*equivalently* (6.2), (6.11), *or* (6.12)] *holds if and only if*

$$\begin{aligned} &\text{(i)} \quad d_j^{\mathrm{T}}c_i \geq 1 \textit{ for all } i = 1, \ldots, m \textit{ and } j = 1, \ldots, t; \\ &\text{(ii)} \quad \min\{l^{\mathrm{T}}c_1, \ldots, l^{\mathrm{T}}c_m\} \cdot \min\{w^{\mathrm{T}}d_1, \ldots, w^{\mathrm{T}}d_t\} \leq l^{\mathrm{T}}w \textit{ for all } l, w \in \mathbb{Z}^n_+ . \end{aligned} \tag{6.14}$$

Proof. Suppose (6.14) holds. We derive (6.11). Let $l \in \mathbb{R}^n_+$. By LP duality, th maximum in (6.11) is equal to $\min\{l^{\mathrm{T}}x \mid x \in \mathbb{R}^n_+ ;\ d_j^{\mathrm{T}}x \geq 1$ for $j = 1, \ldots, t\}$. L this minimum be attained by vector w. Then by (6.14)

$$l^{\mathrm{T}}w \geq (\min_i l^{\mathrm{T}}c_i)(\min_j w^{\mathrm{T}}d_j) \geq \min_i l^{\mathrm{T}}c_i \geq l^{\mathrm{T}}w .$$

So the minimum in (6.11) is equal to $l^{\mathrm{T}}w$.

Next, suppose (6.1) holds. Then (6.11) and (6.12) hold. Now (6.14) (i) follov by taking $l = d_j$ in (6.11). To show (6.14) (ii), let $\lambda_1, \ldots, \lambda_t, \mu_1, \ldots, \mu_m$ atta the maxima in (6.11) and (6.12). Then

$$\begin{aligned}\Big(\sum_j \lambda_j\Big)\Big(\sum_i \mu_i\Big) &= \sum_j \sum_i \lambda_j \mu_i \leq \sum_j \sum_i \lambda_j \mu_i d_j^{\mathrm{T}} c_i = \Big(\sum_j \lambda_j d_j\Big)^{\mathrm{T}}\Big(\sum_i \mu_i c_i\Big) \\ &\leq l^{\mathrm{T}}w .\end{aligned}$$

This implies (6.14) (ii). □

It follows from the ellipsoid method that if $c_1, \ldots, c_m, d_1, \ldots, d_t \in \mathbb{R}^n_+$ satis (6.1) [equivalently, (6.2), (6.11), or (6.12)], then

> for each $l \in \mathbb{R}^n_+$: $\min\{l^{\mathrm{T}}c_1, \ldots, l^{\mathrm{T}}c_m\}$ can be found in polynomial tin if and only if
> for each $w \in \mathbb{R}^n_+$: $\min\{w^{\mathrm{T}}d_1, \ldots, w^{\mathrm{T}}d_t\}$ can be found in polynom time. (6.1

This is particularly interesting if t or m is exponentially large (cf. the applications below).

For more on blocking (and anti-blocking) polyhedra, see Aráoz (1973), Aráoz et al. (1983), Bland (1978), Griffin (1977), Griffin et al. (1982), Huang and Trotter (1980), and Johnson (1978).

Application 6.18 (*Shortest paths and network flows*). The theory of blocking polyhedra yields an illustrative short proof of the Max-Flow-Min-Cut Theorem. Let $D = (V, A)$ be a directed graph, and let $r, s \in V$. Let $c_1, \ldots, c_m \in \mathbb{R}_+^A$ be the incidence vectors of the r–s-paths in D. Similarly, let $d_1, \ldots, d_t \in \mathbb{R}_+^A$ be the incidence vectors of the r–s-cuts.

Considering a given function $l: A \to \mathbb{Z}_+$ as a "length" function, one easily verifies: the minimum length of an r–s-path is equal to the maximum number of r–s-cuts (repetitition allowed) so that no arc a is in more than $l(a)$ of these cuts. [Indeed, the inequality min $\geq$ max is easy. To see the reverse inequality, let p be the minimum length of an r–s-path. For $i = 1, \ldots, p$, let

$$V_i := \{v \in V \mid \text{the shortest } r\text{–}v\text{-path has length at least } i\}.$$

Then $\delta^-(V_1), \ldots, \delta^-(V_p)$ are r–s-cuts as required.] This implies (6.11). Hence (6.12) holds, which is equivalent to the Max-Flow Min-Cut Theorem: the maximum amount of r–s-flow subject to a capacity function w is equal to the minimum capacity of an r–s-cut. (Note that $\sum_i \mu_i c_i$ is an r–s-flow.) In fact, there exists an integral optimum flow if the capacities are integer, but this fact does not seem to follow from the theory of blocking polyhedra.

The above implies that the polyhedra $\text{conv}\{c_1, \ldots, c_m\} + \mathbb{R}_+^A$ and $\text{conv}\{d_1, \ldots, d_t\} + \mathbb{R}_+^A$ form a blocking pair of polyhedra. By (6.17), the polynomial-time solvability of the minimum-capacitated cut problem is equivalent to that of the shortest-path problem; note that this latter problem is much easier.

Application 6.19 (*r-arborescence*). Let $D = (V, A)$ be a digraph and let $r \in V$. Let $c_1, \ldots, c_m$ be the incidence vectors of r-arborescences, and let $d_1, \ldots, d_t$ be the incidence vectors of r-cuts (cf. Application 5.6).

From (5.13) we know that (6.1) holds. Therefore, by Theorem 6.3, also (6.2) holds. It means that for any "capacity" function $w \in \mathbb{R}_+^A$, the minimum capacity of an r-cut is equal to the maximum value of $\mu_1 + \cdots + \mu_k$ where $\mu_1, \ldots, \mu_k \geq 0$ are such that there exist r-arborescences $T_1, \ldots, T_k$ with the property that for each arc a, the sum of the μ_j for which $a \in T_j$ is at most c_a.

Hence the convex hull of the incidence vectors of sets containing an r-cut as a subset, is determined by the system (in $x \in \mathbb{R}^A$)

$$\begin{aligned} &\text{(i)} && 0 \leq x_a \leq 1\,, \quad a \in A\,, \\ &\text{(ii)} && \sum_{a \in T} x_a \geq 1\,, \quad T \ r\text{-arborescence}\,. \end{aligned} \tag{6.20}$$

Edmonds (1973) in fact showed that (6.20) is TDI (again, this does not seem to follow from the theory of blocking polyhedra). It is equivalent to: the minimum size of an r-cut is equal to the maximum number of pairwise disjoint r-arborescences.

The theory of blocking polyhedra can also be applied to directed cuts and directed-cut covers (cf. Theorem 5.15). Again it follows that the convex hull of incidence vectors of sets containing a directed cut as a subset, is determined by (6.20), with "r-arborescence" replaced by "directed-cut cover". However, in this case the system is not TDI (cf. Schrijver 1980b, 1982, 1983a).

Similar arguments apply to T-joins and T-cuts.

7. Anti-blocking polyhedra

The theory of anti-blocking polyhedra, due to Fulkerson (1971, 1972), is to a large extent parallel to that of blocking polyhedra, and arises mostly by reversing inequality signs and by interchanging "min" and "max". We here restrict ourselves to listing results analogous to those given in section 6, the proofs being similar.

Let $c_1, \dots, c_m, \ d_1, \dots, d_t \in \mathbb{R}^n_+$ be such that $\dim(\langle c_1, \dots, c_m \rangle) = \dim(\langle d_1, \dots, d_t \rangle) = n$. Then the following are equivalent:

$$(\operatorname{conv}\{c_1, \dots, c_m\} + \mathbb{R}^n_-) \cap \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid d_j^{\mathrm{T}} x \leq 1 \text{ for } j = 1, \dots, t\}\,, \tag{7.1}$$

$$(\operatorname{conv}\{d_1, \dots, d_t\} + \mathbb{R}^n_-) \cap \mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ \mid c_i^{\mathrm{T}} x \leq 1 \text{ for } i = 1, \dots, m\}\,. \tag{7.2}$$

Define for any subset X of $\mathbb{R}^n$ the *anti-blocker* $A(X)$ of X by:

$$A(X) := \{x \in \mathbb{R}^n_+ \mid y^{\mathrm{T}} x \leq 1 \text{ for each } y \text{ in } X\}\,.$$

Clearly, if

$$P := (\operatorname{conv}\{c_1, \dots, c_m\} + \mathbb{R}^n_-) \cap \mathbb{R}^n_+\,, \tag{7.4}$$

then

$$A(P) = \{x \in \mathbb{R}^n_+ \mid c_i^{\mathrm{T}} x \leq 1 \text{ for } i = 1, \dots, m\}\,. \tag{7.5}$$

$A(P)$ is called the *anti-blocking polyhedron* of P. If $R = A(P)$, the pair P, R is called an *anti-blocking pair* of polyhedra. Again, this is a symmetric relation:

$$\text{For any polyhedron } P \text{ of type (7.4), } A(A(P)) = P\,. \tag{7.6}$$

Each of the following are equivalent among themselves and to (7.1) and (7.2):

(a) The pair $(\operatorname{conv}\{c_1, \dots, c_m\} + \mathbb{R}^n_-) \cap \mathbb{R}^n_+$ and $(\operatorname{conv}\{d_1, \dots, d_t\} + \mathbb{R}^n_-)$

$\cap \mathbb{R}^n_+$ forms an anti-blocking pair of polyhedra; (7.7)

(b) For each $l \in \mathbb{R}^n_+$: $\max\{l^{\mathrm{T}}c_1, \ldots, l^{\mathrm{T}}c_m\}$

$$= \min\left\{\lambda_1 + \cdots + \lambda_t \mid \lambda_1, \ldots, \lambda_t \in \mathbb{R}_+; \sum_j \lambda_j d_j \geq l\right\}; \tag{7.8}$$

(c) For each $w \in \mathbb{R}^n_+$: $\max\{w^{\mathrm{T}}d_1, \ldots, w^{\mathrm{T}}d_t\}$

$$= \min\left\{\mu_1 + \cdots + \mu_m \mid \mu_1, \ldots, \mu_m \in \mathbb{R}_+; \sum_i \mu_i c_i \geq w\right\}; \tag{7.9}$$

(d) (i) $d_j^{\mathrm{T}}c_i \leq 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, t$,
(ii) $\max\{l^{\mathrm{T}}c_1, \ldots, l^{\mathrm{T}}c_m\} \cdot \max\{w^{\mathrm{T}}d_1, \ldots, w^{\mathrm{T}}d_t\} \geq l^{\mathrm{T}}w$ for all $l, w \in \mathbb{Z}^n_+$. (7.10)

This last characterization is again due to Lehman (1965).

Application 7.11 (*Perfect graphs*). The theory of anti-blocking polyhedra yields a proof of Lovász's Perfect Graph Theorem (cf. chapter 4). This line of proof was developed by Fulkerson (1970b, 1972, 1973), Lovász (1972), and Chvátal (1975).

Define for any graph $G = (V, E)$, the *stable-set polytope* $\mathrm{STAB}(G)$ of G as the convex hull of the incidence vectors of stable sets in G. Clearly, any vector x in the stable-set polytope satisfies

$$\begin{aligned} &\text{(i)} \quad x_v \geq 0, \quad v \in V, \\ &\text{(ii)} \quad \sum_{v \in K} x_v \leq 1, \quad K \subseteq V,\ K \text{ clique}, \end{aligned} \tag{7.12}$$

since the incidence vector of any stable set satisfies (7.12). Note that the polytope determined by (7.12) is exactly $A(\mathrm{STAB}(\bar{G}))$. The circuit on five vertices shows that generally $A(\mathrm{STAB}(\bar{G}))$ can be larger than $\mathrm{STAB}(G)$. Chvátal (1975) showed that $\mathrm{STAB}(G)$ is exactly determined by (7.12) if and only if G is perfect. Anti-blocking then yields the Perfect Graph Theorem.

First observe the following. Let $Ax \leq \mathbf{1}$ denote the inequality system (7.12) (ii). So the rows of A are the incidence vectors of cliques. By definition, G is perfect if and only if the (dual) linear programs

$$\max\{w^{\mathrm{T}}x \mid x \geq 0, Ax \leq \mathbf{1}\} = \min\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0, y^{\mathrm{T}}A \geq w^{\mathrm{T}}\} \tag{7.13}$$

have integral optimum solutions, for each $\{0, 1\}$-vector w.

Chvátal's Theorem 7.14. *G is perfect if and only if its stable-set polytope is determined by* (7.12).

Proof. (I) First suppose G is perfect. For $w: V \to \mathbb{Z}_+$, let α_w denote the maximum weight of a stable set. To prove that the stable-set polytope is determined by

(7.12), it suffices to show that

$$\alpha_w = \max\{w^{\mathrm{T}}x \mid x \geq 0,\ Ax \leq \mathbf{1}\} \tag{7.15}$$

for each $w: V \to \mathbb{Z}_+$. This will be done by induction on $\sum_{v \in V} w_v$.

If w is a $\{0,1\}$-vector, then (7.15) follows from the remark on (7.13). So we may assume that $w_u \geq 2$ for some vertex u. Let $e_u = 1$ and $e_v = 0$ if $v \neq u$. Replacing w by $w - e$ in (7.13) and (7.15) gives, by induction, a vector $y \geq 0$ so that $y^{\mathrm{T}}A \geq (w-e)^{\mathrm{T}}$ and $y^{\mathrm{T}}\mathbf{1} = \alpha_{w-e}$. Since $(w-e)_u \geq 1$, there is a clique K with $y_K > 0$ and $u \in K$. We may assume that $\chi^K \leq w - e$. Denote $a := \chi^K$.

Then $\alpha_{w-a} < \alpha_w$. For suppose $\alpha_{w-a} = \alpha_w$. Let S be any stable set with $\sum_{v \in S}(w-a)_v = \alpha_{w-a}$. Since $\alpha_{w-a} = \alpha_w$, $K \cap S = \emptyset$. On the other hand, since $w - a \leq w - e \leq w$, we know that $\sum_{v \in K}(w-e)_v = \alpha_{w-e}$ and hence, by complementary slackness, $|K \cap S| = 1$, which is a contradiction.

Therefore,

$$\begin{aligned} \alpha_w &= 1 + \alpha_{w-a} = 1 + \max\{(w-a)^{\mathrm{T}}x \mid x \geq 0,\ Ax \leq \mathbf{1}\} \\ &\geq \max\{w^{\mathrm{T}}x \mid x \geq 0,\ Ax \leq \mathbf{1}\}\ , \end{aligned} \tag{7.16}$$

implying (7.15).

(II) Conversely, suppose that the stable-set polytope is determined by (7.12), i.e., that the maximum in (7.13) is attained by the incidence vector of a stable set, for each $w \in \mathbb{Z}_+^V$. To show that G is perfect it suffices to show that the minimum in (7.13) also has an integer optimum solution for each $\{0,1\}$-valued w. This will be done by induction on $\sum_{v \in V} w_v$.

Let w be $\{0,1\}$-valued, and let y be a, not necessarily integral, optimum solution for the minimum in (7.13). Let K be a clique with $y_K > 0$, and let $a = \chi^K$ (we may assume $a \leq w$). Then the common value of

$$\max\{(w-a)^{\mathrm{T}}x \mid x \geq 0,\ Ax \leq \mathbf{1}\} = \min\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0,\ y^{\mathrm{T}}A \geq (w-a)^{\mathrm{T}}\} \tag{7.17}$$

is less than the common value of (7.13), since by complementary slackness, each optimum solution x in (7.13) has $a^{\mathrm{T}}x = 1$. However, the values in (7.13) and (7.17) are integers (since by assumption, the maxima have integral optimum solutions). Hence they differ by exactly 1. Moreover, by induction the minimum in (7.17) has an integral optimum solution y. Increasing component y_K of y by 1, gives an integral optimum solution of (7.13). □

Equivalent to Theorem 7.14 is:

$$G \text{ is perfect} \Leftrightarrow \mathrm{STAB}(G) = A(\mathrm{STAB}(\bar{G}))\ . \tag{7.18}$$

Note that the stable-set polytope of G is determined by (7.12) if the stable-set polytope and the clique polytope of G form an anti-blocking pair of polyhedra. Here the *clique polytope* is the convex hull of the incidence vectors of cliques.

The theory of anti-blocking polyhedra then gives directly the Perfect Graph Theorem of Lovász (1972):

Lovász's Perfect Graph Theorem 7.19. *The complement of a perfect graph is perfect.*

Proof. If G is perfect, then $\mathrm{STAB}(G) = A(\mathrm{STAB}(\bar{G}))$. Hence $\mathrm{STAB}(\bar{G}) = A(A(\mathrm{STAB}(\bar{G}))) = A(\mathrm{STAB}(G))$. Therefore, $\bar{G}$ is perfect. $\square$

By (7.14), with the ellipsoid method, a maximum-weighted stable set in a perfect graph G can be found in polynomial time if and only if a maximum-weighted clique in a perfect graph G can be found in polynomial time. Since the complement of a perfect graph is a perfect graph again, this would not give any reduction of one problem to another.

However, an alternative approach does give a polynomial-time algorithm to find a maximum-weighted stable set in a perfect graph (Grötschel et al. 1981, 1986, 1988). Let $G = (V, E)$ be a graph, with $V = \{1, \ldots, n\}$, say. Consider the collection $M(G)$ of all matrices $Y = (y_{ij})_{i,j=0}^{n}$ in $\mathbb{R}^{(n+1)\times(n+1)}$ satisfying

$$\begin{array}{ll} \text{(i)} & Y \text{ is symmetric and positive semi-definite;} \\ \text{(ii)} & y_{00} = 1\,, \qquad y_{0i} = y_{ii}\,, \quad i = 1, \ldots, n\,; \\ \text{(iii)} & y_{ij} = 0 \text{ if } i \neq j,\ \{i, j\} \in E\,. \end{array} \tag{7.20}$$

These conditions imply that $M(K)$ is a convex set (not necessarily a polytope).

Let $\mathrm{TH}(G)$ be the set of all vectors $x \in \mathbb{R}^n$ for which there exists a matrix Y in $M(G)$ so that $x_i = y_{ii}$ for $i = 1, \ldots, n$. So $\mathrm{TH}(G)$ is the projection of $M(G)$ on the diagonal coordinates [excluding the $(0, 0)$ coordinate].

Now $\mathrm{TH}(G)$ turns out to be an approximation of $\mathrm{STAB}(G)$, at least as good as $A(\mathrm{STAB}(\bar{G}))$, in the following sense:

Theorem 7.21. $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq A(\mathrm{STAB}(\bar{G}))$.

Proof. The first inclusion follows from the fact that for each stable set $S \subseteq V$, the incidence vector χ^S belongs to $\mathrm{TH}(G)$, as it is the projection of the matrix Y in $M(G)$ defined by:

$$y_{ij} = \begin{cases} 1 & \text{if } i, j \in S \cup \{0\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{7.22}$$

To see the second inclusion, first note that trivially each vector in $\mathrm{TH}(G)$ is nonnegative (since the diagonal of a positive semi-definite matrix is nonnegative). It next suffices to show: if $x \in \mathrm{TH}(G)$ and u is the incidence vector of a stable set in $\bar{G}$, then $u^{\mathrm{T}}x \leq 1$. To prove this, let x be the projection of $Y \in M(G)$. Since Y is positive semi-definite we know:

$$(1 \quad -u^{\mathrm{T}})Y\binom{1}{-u} \geq 0 . \tag{7.23}$$

As $y_{ij} = 0$ if $\{i, j\} \in E$, and as u is the incidence vector of a clique K in G, (7.23) implies

$$1 - 2\sum_{i \in K} y_{i0} + \sum_{i \in K} y_{ii} \geq 0 . \tag{7.24}$$

Since $x_i = y_{i0} = y_{ii}$, this implies $u^{\mathrm{T}}x \leq 1$. $\square$

Theorem 7.21 implies that if $\mathrm{STAB}(G) = A(\mathrm{STAB}(\bar{G}))$, i.e., if G is perfect then $\mathrm{STAB}(G) = \mathrm{TH}(G)$. Now any linear objective function $w^{\mathrm{T}}x$ can be maximized over $\mathrm{TH}(G)$ in polynomial time. This follows from the fact that any linear objective function can be maximized over $M(G)$ in polynomial time, since we can solve the separation problem over $M(G)$ in polynomial time. [The latter follows from the fact that we can test, for any given Y in $\mathbb{R}^{(n+1)\times(n+1)}$, the constraints in (7.20) in polynomial time, in such a way that we find a separating hyperplane (in the space $\mathbb{R}^{(n+1)\times(n+1)}$) if Y does not belong to $M(G)$.]

So as a consequence we have:

Theorem 7.25. *There exists a polynomial-time algorithm finding a maximum-weight stable set in any given perfect graph.*

By symmetry, the same holds for finding a maximum-weight clique in a perfect graph.

Application 7.26 (*Matchings and edge-colorings*). Let for any graph $G = (V, E)$, $P_{\mathrm{mat}}(G)$ denote the matching polytope of G. By scalar multiplication, we can normalize system (5.28) determining $P_{\mathrm{mat}}(G)$ to : $x \geq 0$, $Cx \leq \mathbf{1}$, for a certain matrix C (deleting the inequalities in (5.28) corresponding to $U \subseteq V$ with $|U| \leq 1$). The matching polytope is of type (7.4), and hence its anti-blocking polyhedron $A(P_{\mathrm{mat}}(G))$ is equal to $\{z \in \mathbb{R}_+^E \mid Dz \leq \mathbf{1}\}$, where the rows of D are the incidence vectors of all matchings in G. So by (7.8), taking $l = \mathbf{1}$:

$$\max\left\{\Delta(G), \max_{U \subseteq V, |U| \geq 2} \frac{|\langle U \rangle|}{\lfloor \frac{1}{2}|U| \rfloor}\right\} = \min\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0, y^{\mathrm{T}}D \geq \mathbf{1}^{\mathrm{T}}\} . \tag{7.27}$$

Here $\langle U \rangle$ denotes the collection of all edges contained in U.

The minimum in (7.27) can be interpreted as the *fractional edge-coloring number* $\gamma^*(G)$ of G. If the minimum is attained by an integral optimum solution y, it is equal to the edge-coloring number $\gamma(G)$ of G, since

$$\gamma(G) = \min\{y^{\mathrm{T}}\mathbf{1} \mid y \geq 0, y^{\mathrm{T}}D \geq \mathbf{1}^{\mathrm{T}}, y \text{ integral}\} . \tag{7.28}$$

By Vizing's Theorem, $\gamma(G) = \Delta(G)$ or $\gamma(G) = \Delta(G) + 1$ if G is a simple graph. If

G is the Petersen graph, then $\Delta(G) = \gamma^*(G) = 3$ while $\gamma(G) = 4$. Seymour (1979) conjectured that for each, possibly nonsimple, graph one has $\gamma(G) \leq \max\{\Delta(G) + 1, \lceil \gamma^*(G) \rceil\}$.

8. Cutting planes

For any set $P \subseteq \mathbb{R}^n$, let the *integer hull* of P, denoted by P_I, be

$$P_I := \text{conv}\{x \mid x \in P, x \text{ integral}\}.$$

Trivially, if P is bounded, then P_I is a polytope. Meyer (1974) showed that if P is a rational polyhedron, then P_I is a rational polyhedron again.

Most of the combinatorial results given above consist of a characterization of the integer hull P_I by linear inequalities for certain polyhedra P. For example, the matching polytope is the integer hull of the polyhedron determined by the inequalities (5.28) (i), (ii). For most combinatorial optimization problems it is not difficult to describe a set of linear inequalities, determining a polyhedron P, in which the integral vectors are exactly the incidence vectors corresponding to the combinatorial optimization problem. Hence, P_I is the convex hull of these incidence vectors. However, it is generally difficult to describe P_I by linear inequalities (cf. section 9).

The *cutting-plane method* was introduced by Gomory (1960) to solve integer linear programs. Chvátal (1973a) (and Schrijver 1980a, for the unbounded case) derived from it the following iterative process characterizing P_I.

Define for any polyhedron $P \subseteq \mathbb{R}^n$:

$$P' := \bigcap_{\substack{H \text{ rational affine} \\ \text{halfspace with } H \supseteq P}} H_I, \tag{8.2}$$

where a *rational affine halfspace* is a set $H := \{x \mid c^T x \leq \delta\}$, with $c \in \mathbb{Q}^n$ ($c \neq \mathbf{0}$) and $\delta \in \mathbb{Q}$. Clearly, we may assume that the components of c are relatively prime integers, which implies

$$H_I = \{x \mid c^T x \leq \lfloor \delta \rfloor\}. \tag{8.3}$$

This usually makes the set P' easy to characterize.

For instance, for any rational $m \times n$ matrix and $b \in \mathbb{Q}^m$ we have

$$\begin{aligned} \{x \mid Ax \leq b\}' &= \{x \mid (u^T A)x \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{Q}^m_+ \text{ with } u^T A \text{ integral}\}, \\ \{x \mid x \geq 0, Ax \leq b\}' &= \{x \mid x \geq 0; \lfloor u^T A \rfloor x \leq \lfloor u^T b \rfloor \text{ for all } u \in \mathbb{Q}^m_+\} \end{aligned} \tag{8.4}$$

(here $\lfloor \cdot \rfloor$ denotes component-wise lower integer parts).

The halfspaces H_I (more strictly, their bounding hyperplanes) are called *cutting planes*.

It can be shown that if P is a rational polyhedron, then P' is also a rational polyhedron. Trivially, $P \subseteq H$ implies $P_I \subseteq H_I$, and hence $P_I \subseteq P'$. Now generally $P'' \neq P'$, and repeating this operation we obtain a sequence of polyhedra

$P, P', P'', P''', \ldots$, satisfying

$$P \supseteq P' \supseteq P'' \supseteq P''' \supseteq \cdots \supseteq P_{\mathrm{I}} . \tag{8.5}$$

Denote the $(t+1)$th set in this sequence by $P^{(t)}$. Then:

Theorem 8.6. *For each rational polyhedron P there exists a number t with $P^{(t)} = P_{\mathrm{I}}$.*

A direct consequence applies to bounded, but not necessarily rational, polyhedra.

Corollary 8.7. *For each polytope P there exists a number t with $P^{(t)} = P_{\mathrm{I}}$.*

Blair and Jeroslow (1982) (cf. Cook et al. 1986b) proved the following generalization of Theorem 8.6.

Theorem 8.8. *For each rational matrix A there exists a number t such that for each column vector b one has: $\{x \mid Ax \leq b\}^{(t)} = \{x \mid Ax \leq b\}_{\mathrm{I}}$.*

Hence we can define the *Chvátal rank* of a rational matrix A as the smallest such number t. The *strong Chvátal rank* of A then is the Chvátal rank of the matrix

$$\begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix} . \tag{8.9}$$

It follows from Hoffman and Kruskal's Theorem (cf. Theorem 4.1) that an integral matrix A has a strong Chvátal rank 0 if and only if it is totally unimodular. Similar characterizations for higher Chvátal ranks are not known. In Examples 8.10 and 9.3 we shall see some classes of matrices with strong Chvátal rank 1.

For more on cutting planes, see Jeroslow (1978, 1979), and Blair and Jeroslow (1977, 1979, 1982).

Example 8.10 (*The matching polytope*). For any graph $G = (V, E)$, let P be the polytope determined by (5.28) (i), (ii). So P_{I} is the matching polytope of G. It is not difficult to show that P' is the polytope determined by (5.28) (i)–(iii). Hence Edmonds' Matching Polyhedron Theorem 5.33 is equivalent to asserting $P' = P_{\mathrm{I}}$. So the matching polytope arises from (5.28) (i), (ii) by one "round" of cutting planes.

It can be derived from Edmonds' Matching Polyhedron Theorem that each integer matrix $A=(a_{ij})$ satisfying

$$\sum_{i=1}^{m} |a_{ij}| \leq 2\,, \quad j=1,\ldots,n\,, \tag{8.11}$$

where A has order $m \times n$, has strong Chvátal rank at most 1.

9. Hard problems and the complexity of the integer hull

The integer hull P_I can be quite intractable compared with the polyhedron P. This has been shown by Karp and Papadimitriou (1982), under the generally accepted assumption NP $\neq$ co-NP.

First note that the ellipsoid method (cf. section 3) can also be used in the negative: if NP $\neq$ P, then for any NP-complete problem there is no polynomial-time algorithm for the separation problem for the corresponding polytopes. More precisely, if for each graph $G=(V,E)$ $\mathcal{F}_G$ is a subset of $\mathcal{P}(E)$, and if Optimization Problem 3.17 is NP-complete, then (if NP $\neq$ P) the Separation Problem 3.18 is not polynomially solvable.

In fact, Karp and Papadimitriou showed that for any class $(\mathcal{F}_G \mid G$ graph), if Optimization Problem 3.17 is NP-complete, and if NP $\neq$ co-NP, then the class of polytopes $\mathrm{conv}\{\chi^F \mid F \in \mathcal{F}_G\}$ has difficult facets, i.e.,

> there exists no polynomial Φ such that for each graph $G=(V,E)$ and each $c \in \mathbb{Z}^E$ and $\delta \in \mathbb{Q}$ with the property that $c^{\mathrm{T}}x \leq \delta$ defines a facet of $\mathrm{conv}\{\chi^F \mid F \in \mathcal{F}_G\}$, the fact that $c^{\mathrm{T}}x \leq \delta$ is valid for each χ^F with $F \in \mathcal{F}_G$ has a proof of length at most $\Phi(|V|+|E|+\mathrm{size}(c)+\mathrm{size}(\delta))$. (9.1)

The meaning of (9.1) might become clear by considering description (5.28) of the matching polytope: although (5.28) consists of exponentially many inequalities, each facet-defining inequality is of form (5.28), and for them it is easy to show that they are valid for the matching polytope.

Another negative result was given by Boyd and Pulleyblank (1984): let, for a given class $(\mathcal{F}_G \mid G$ graph), for each graph $G=(V,E)$ the polytope P_G in $\mathbb{R}^E$ satisfy $(P_G)_I = \mathrm{conv}\{\chi^F \mid F \in \mathcal{F}_G\}$ and have the property that

$$\text{given } G=(V,E) \text{ and } c \in \mathbb{R}^E, \text{ find } \max\{c^{\mathrm{T}}x \mid x \in P_G\} \tag{9.2}$$

is polynomially solvable. Then if Optimization Problem 3.17 is NP-complete and if NP $\neq$ co-NP, then there is no fixed t such that for each graph G, $(P_G)^{(t)} = \mathrm{conv}\{\chi^F \mid F \in \mathcal{F}_G\}$.

Similar results hold for subcollections $\mathcal{F}_G$ of $\mathcal{P}(V)$ and for directed graphs. See also Papadimitriou (1984) and Papadimitriou and Yannakakis (1982) for the complexity of facets.

Example 9.3 (*The stable-set polytope*). Let $G=(V,E)$ be a graph, and let

STAB(G) be the stable-set polytope of G. Let $P(G)$ be the polytope in $\mathbb{R}^V$ determined by

$$\begin{aligned} &\text{(i)} \quad x_v \geq 0\,, \quad v \in V\,, \\ &\text{(ii)} \quad \sum_{v\in K} x_v \leq 1\,, \quad K \subseteq V,\ K \text{ clique}\,. \end{aligned} \tag{9.4}$$

So $P(G) = A(\mathrm{STAB}(\bar{G}))$ (cf. Section 7).

Clearly, $\mathrm{STAB}(G) \subseteq P(G)$. In fact, since the integral solutions to (9.4) are exactly the incidence vectors of stable sets, we have

$$\mathrm{STAB}(G) = P(G)_{\mathrm{I}}\,. \tag{9.5}$$

Chvátal (1973a, 1984) showed that there is no fixed t such that $P(G)^{(t)} = P(G)_{\mathrm{I}}$ for all graphs G (if NP $\neq$ co-NP, this follows from Boyd and Pulleyblank's result mentioned above), even if we restrict G to graphs with $\alpha(G) = 2$.

By Chvátal's Theorem 7.14, the class of graphs with $P(G)_{\mathrm{I}} = P(G)$ is exactly the class of perfect graphs. In Example 8.10 we mentioned Edmonds' result that if G is the line graph of some graph H, then $P(G)' = P(G)_{\mathrm{I}}$, which is the matching polytope of H.

The smallest t for which $P(G)^{(t)} = P(G)_{\mathrm{I}}$ is an indication of the computational complexity of the stability number $\alpha(G)$. Chvátal (1973) raised the question of whether there exists, for each fixed t, a polynomial-time algorithm determining $\alpha(G)$ for graphs G with $P(G)^{(t)} = P(G)_{\mathrm{I}}$. This is true for $t = 0$, i.e., for perfect graphs (Grötschel et al. 1981).

Minty (1980) and Sbihi (1978, 1980) extended Edmonds' result of the polynomial solvability of the maximum-weighted matching problem, by describing polynomial-time algorithms for finding a maximum-weighted stable set in $K_{1,3}$-free graphs (i.e., graphs with no $K_{1,3}$ as induced subgraph). Hence, by (3.9), the separation problem for stable-set polytopes of $K_{1,3}$-free graphs is polynomially solvable. Yet no explicit description of a linear inequality system defining STAB(G) for $K_{1,3}$-free graphs has been found. This would extend Edmonds' description of the matching polytope. It follows from Chvátal's result mentioned above that there is no fixed t such that $P(G)^{(t)} = P(G)_{\mathrm{I}}$ for all $K_{1,3}$-free graphs (see Giles and Trotter 1981).

Perhaps the most natural "relaxation" of the stable-set polytope of $G = (V, E)$ is the polytope $Q(G)$ determined by

$$\begin{aligned} &\text{(i)} \quad x_v \geq 0\,, \quad v \in V\,, \\ &\text{(ii)} \quad x_v + x_w \leq 1\,, \quad \{v, w\} \in E\,. \end{aligned} \tag{9.6}$$

Again, $Q(G)_{\mathrm{I}} = \mathrm{STAB}(G)$. Since $Q(G) \supseteq P(G)$, there is no t with $Q(G)^{(t)} = Q(G)_{\mathrm{I}}$ for all G. It is not difficult to see that $Q(G)'$ is the polytope determined by (9.6) together with

$$\sum_{v\in C} x_v \leq \frac{|C| - 1}{2}\,, \quad C \text{ is the vertex set of an odd circuit}\,. \tag{9.7}$$

It was shown by Gerards and Schrijver (1986) that if G has no subgraph H which arises from K_4 by replacing edges by paths such that each triangle in K_4 becomes an odd circuit in H, then $Q(G)' = \mathrm{STAB}(G)$. Graphs G with $Q(G)' = \mathrm{STAB}(G)$ are called by Chvátal (1975) *t-perfect*.

Gerards and Schrijver showed more generally the following. Let $A = (a_{ij})$ be an integral $m \times n$ matrix satisfying

$$\sum_{j=1}^{n} |a_{ij}| \leq 2 , \quad i = 1, \ldots, m . \tag{9.8}$$

Then A has strong Chvátal rank at most 1 if and only if A cannot be transformed to the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{9.9}$$

by a series of the following operations: deleting or permuting rows or columns, or multiplying them by -1; replacing $\left[\begin{smallmatrix} 1 & c^{\mathrm{T}} \\ b & D \end{smallmatrix}\right]$ by $D - bc^{\mathrm{T}}$, where D is a matrix and b and c are column vectors.

Chvátal (1973a) showed that for $G = K_n$ the smallest t with $Q(G)^{(t)} = \mathrm{STAB}(G)$ is about $\log n$.

Chvátal (1975) observed that the incidence vectors of two stable sets C, C' are adjacent on the stable-set polytope if and only if $C \,\Delta\, C'$ induces a connected graph. For more on the stable-set polytope, see Fulkerson (1971), Chvátal (1973a, 1975, 1984, 1985), Padberg (1973, 1974, 1977, 1979), Nemhauser and Trotter (1974, 1975), Trotter (1975), Wolsey (1976b), Balas and Zemel (1977), Ikura and Nemhauser (1985), Grötschel et al. (1986), and Lovász and Schrijver (1991).

Example 9.10 (*The traveling-salesman polytope*). For any graph $G = (V, E)$, the *traveling-salesman polytope* is equal to $\mathrm{conv}\{\chi^H \mid H \subseteq E,\ H \text{ Hamiltonian circuit}\}$. As the traveling salesman problem is NP-complete, by Karp and Papadimitriou's result, the traveling-salesman polytope will have "difficult" facets [cf. (9.1)] if NP $\neq$ co-NP.

Define the polyhedron $P \subseteq \mathbb{R}^E$ by:

$$\begin{aligned} &\text{(i)} \quad 0 \leq x_e \leq 1 , \quad e \in E , \\ &\text{(ii)} \quad \sum_{e \ni v} x_e = 2 , \quad v \in V , \\ &\text{(iii)} \quad \sum_{e \in \delta(U)} x_e \geq 2 , \quad U \subseteq V,\ 3 \leq |U| \leq |V| - 3 . \end{aligned} \tag{9.11}$$

Since the integral solutions of (9.11) are exactly the incidence vectors of Hamiltonian circuits, P_{I} is equal to the traveling-salesman polytope. Note that the problem of minimizing a linear function $c^{\mathrm{T}}x$ over P is polynomially solvable, with the ellipsoid method, since system (9.11) can be checked in polynomial time [(iii) can be checked by reduction to a minimum-cut problem]. So if NP $\neq$ co-NP, by Boyd and Pulleyblank's result, there is no fixed t such that $P^{(t)} = P_{\mathrm{I}}$ for each graph G.

The system (9.11), however, has been useful in solving large-scale instances of the traveling salesman problem: for any $c \in \mathbb{Q}^E$, the minimum of $c^{\mathrm{T}}x$ over (9.11) is a lower bound for the traveling salesman problem, which can be computed with the simplex method using a row-generating technique. This lower bound can be used in a "branch-and-bound" procedure for the traveling salesman problem.

This approach was initiated by Dantzig et al. (1954, 1959), and developed and sharpened by Miliotis (1978), Grötschel and Padberg (1979a,b), Grötschel (1980), Crowder and Padberg (1980), and Padberg and Hong (1980) (see Grötschel and Padberg 1985, and Padberg and Grötschel 1985 for a survey).

Grötschel and Padberg (1979a) showed that the diameter of the traveling-salesman polytope for $G = K_n$ is equal to $\frac{1}{2}n(n-3)$. They also proved that for complete graphs all inequalities in (9.11) are facet-defining.

For more about facets of the traveling-salesman polytope, see Held and Karp (1970, 1971), Chvátal (1973b), Grötschel and Padberg (1975, 1977, 1979a,b), Maurras (1975), Grötschel (1977, 1980), Grötschel and Pulleyblank (1986), Grötschel and Wakabayashi (1981a,b), and Cornuéjols and Pulleyblank (1982).

Papadimitriou and Yannakakis (1982) showed that it is co-NP-complete to decide if a given vector belongs to the traveling-salesman polytope. Moreover, Papadimitriou (1978) showed that it is co-NP-complete to check if two Hamiltonian circuits H, H' yield adjacent incidence vectors (see also Rao 1976).

On the other hand, Padberg and Rao (1974) showed that the diameter of the "asymmetric" traveling-salesman polytope (i.e., convex hull of incidence vectors of Hamiltonian cycles in a directed graph) is equal to 2, for the complete directed graph with at least six vertices. Grötschel and Padberg (1985) conjecture that also the "undirected" traveling-salesman polytope has diameter 2.

Other hard problems 9.12. The following references deal with polyhedra associated with further difficult problems. *Set-packing problem*: Fulkerson (1971), Padberg (1973, 1977, 1979), Balas and Zemel (1977), and Ikura and Nemhauser (1985). *Set-covering problem*: Padberg (1979), Balas (1980), and Balas and Ho (1980). *Set-partitioning problem*: Balas and Padberg (1972), Balas (1977), Padberg (1979), and Johnson (1980). *Linear ordering and acyclic subgraph problem*: Grötschel et al. (1984, 1985a,b), and Jünger (1985). *Knapsack problem and* 0,1-*programming*: Balas (1975), Hammer et al. (1975), Wolsey (1975, 1976a, 1977), Johnson (1980), Zemel (1978), and Crowder et al. (1983). *Bipartite subgraph and maximum-cut problem*: Grötschel and Pulleyblank (1981), Barahona (1983a,b), and Barahona et al. (1985).

For more background information on hard problems, see Grötschel (1977, 1982).

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