HOW TO SHOW A SET IS NOT ALGEBRAIC

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ABSTRACT. We revisit Akbulut and King's first example of a compact semialgebraic set which satisfies Sullivan's local Euler characteristic condition, but which is not homeomorphic to an algebraic set. A nontrivial obstruction is computed using the link operator on the ring of constructible functions.

There are many local topological conditions satisfied by real algebraic sets. The simplest of these was discovered by Sullivan [5] more than thirty years ago: The link of every point has even Euler characteristic. In low dimensions this is the only obstruction for a set to be algebraic. More precisely, if a compact semialgebraic set of dimension less than three satisfies Sullivan's condition, then it is homeomorphic to an algebraic set. Akbulut and King [1] found an example of a compact 3-dimensional semialgebraic set satisfying Sullivan's condition which is not homeomorphic to an algebraic set, and this is the example we will discuss.

The method we use to compute the obstruction is due to Parusiński and the author [2]. We have also found an enormous list of independent obstructions in dimension four [3]. The yoga of algebraically constructible functions used to prove our obstructions vanish for algebraic sets is presented in the paper of Isabelle Bonnard in this volume.

Let X be a semialgebraic set in \mathbb{R}^n , and let $p \in X$. The link L_p of X at p is the intersection of X with a sphere of small radius $\epsilon > 0$ centered at p. For sufficiently small ϵ the topological type of L_p is independent of ϵ , and a neighborhood of p in X is homeomorphic to the cone on L_p . (The link L_p is homeomorphic to the boundary of the simplicial star of p in X for a semialgebraic triangulation of X.) Sullivan's theorem is that if X is algebraic then for all $p \in X$ the Euler characteristic $\chi(L_p)$ is even.

For example, consider the Cartan umbrella $X = \{(x, y, z) \mid x^2 = zy^2\}$ in \mathbb{R}^3 (Figure 1). For each real number t, let H_t be the horizontal plane z = t. For t > 0, $X \cap H_t$ is the two lines $x = \pm(\sqrt{t})y$, z = t. For t = 0, $X \cap H_t$ is the line x = 0, z = 0. For t < 0, $X \cap H_t$ is the point x = 0, y = 0, z = t.

The space X has five strata:

- two 2-strata $f = \{x^2 = zy^2, y < 0\}, f' = \{x^2 = zy^2, y > 0\},$
- two 1-strata $d = \{x = 0, y = 0, z < 0\}, e = \{x = 0, y = 0, z > 0\},\$
- one 0-stratum $a = \{(0, 0, 0)\}.$

The local topology of X is constant along each of these strata. If $p \in X$, then

• if $p \in f \cup f'$, L_p is topologically a circle, so $\chi(L_p) = 0$;

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- if p ∈ e, L_p is a graph with two vertices and four edges joining one vertex to the other, so χ(L_p) = 2 − 4 = −2;
- if $p \in d$, L_p is two points, so $\chi(L_p) = 2$;
- if p = (0, 0, 0), L_p is a graph with two vertices and two edges which are both loops at the same vertex (L_p is the union of a figure eight and a point), so $\chi(L_p) = 2 2 = 0$.

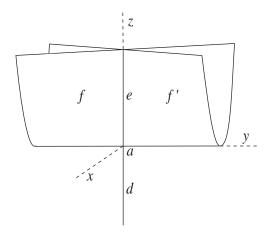


FIGURE 1. The Cartan umbrella

On the other hand, the Whitney umbrella $X = \{(x, y, z) \mid x^2 = zy^2, z \ge 0\}$ is not algebraic, because the link of the origin is a figure eight, which has Euler characteristic -1. (The Whitney umbrella is the image of the algebraic map $(u, v) \mapsto (uv, u, v^2)$.)

Now $\chi(L_p)$ is a topological invariant, since it can be computed from the local homology of X at p:

$$\chi(L_p) = 1 - \sum_k (-1)^k \operatorname{rank} H_k(X, X - \{p\}).$$

Thus the Whitney umbrella is not homeomorphic to a real algebraic set.

The known higher topological obstructions for a semialgebraic set to be algebraic are similar to Sullivan's obstruction. They are local—in fact they are mod 2 invariants of the links L_p and they are combinatorial in the sense that they depend only on the dimensions and incidence of the cells of a semialgebraic cell structure on X.

Parusiński and the author [2] have found a way to describe such obstructions using the ring of constructible functions on X and an operator on this ring generalizing the Euler characteristic of the link. Here is a brief description of our method.

Let X be a semialgebraic set. An integer-valued function $\varphi : X \to \mathbb{Z}$ is *constructible* if it is a finite combination of characteristic functions of semialgebraic subsets X_i of X with integer coefficients: $\varphi = \sum_i n_i \mathbf{1}_{X_i}$. It follows that there exists a semialgebraic cell complex \mathcal{K} on X such that φ is constant on each cell of \mathcal{K} . (A semialgebraic cell complex is a locally topologically trivial semialgebraic stratification such that each stratum is homeomorphic to a Euclidean space and has compact closure.)

The set of constructible functions on X is a ring, with $(\varphi + \psi)(p) = \varphi(p) + \psi(p)$ and $(\varphi \times \psi)(p) = \varphi(p) \times \psi(p)$. If $\varphi = \sum_i n_i \mathbf{1}_{X_i}$ with the sets X_i compact, the Euler integral of

 φ is defined by $\int_X \varphi = \sum_i n_i \chi(X_i)$, where χ is the Euler characteristic. If φ is constant on each cell of \mathcal{K} and φ has compact support, then

$$\int_X \varphi = \sum_{C \in \mathcal{K}} (-1)^{\dim C} \varphi(C),$$

where $\varphi(C)$ is the value of φ on the cell C.

The *link operator* Λ on the ring of constructible functions on X is the Euler integral of the restriction of φ to the link of X at p,

$$(\Lambda \varphi)(p) = \int_{L_p} \varphi.$$

Sullivan's theorem says that if X is an algebraic set with characteristic function $\mathbf{1}_X$, the values of the function $\Lambda \mathbf{1}_X$ are even. In other words, if $\tilde{\Lambda} = \frac{1}{2} \Lambda$ then $\tilde{\Lambda} \mathbf{1}_X$ is integer-valued (and hence $\tilde{\Lambda} \mathbf{1}_X$ is a new constructible function on X).

The main result of [2] (Theorem 2.5, p. 536) implies that if X is an algebraic set, then all the functions obtained from $\mathbf{1}_X$ using the arithmetic operations $+, -, \times$, and the operator $\tilde{\Lambda}$ are integer-valued. Furthermore this property is topological ([2] A.7, p. 550).

The following statement is a corollary: If Y is an algebraic link (the link of a point in an algebraic set), then all the functions obtained from $\mathbf{1}_Y$ using the arithmetic operations $+, -, \times$, and the operator $\tilde{\Lambda}$ are integer-valued and have even Euler integral.

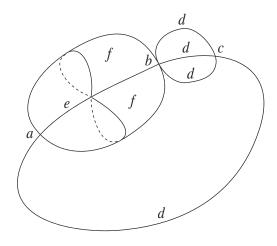


FIGURE 2. The space Y

Now we apply our method to Akbulut and King's example. Let Y be the topological space shown in Figure 2. The space Y is homeomorphic to an algebraic set in projective 3-space, the union of the projective Cartan umbrella $wx^2 = zy^2$ and the plane conic $(w+z)^2 + x^2 = z^2$, y = 0. Realized in this way, Y has a semialgebraic cell structure with three 0-cells, five 1cells, and two 2-cells. The three 0-cells are labelled a, b, c in the figure. Four of the 1-cells are labelled d, and one of the 1-cells (incident to the 2-cells) is labelled e. The two 2-cells are labelled f.

The table below summarizes the computation of a constructible function ψ obtained from $\mathbf{1}_Y$ using the arithmetic operations $+, -, \times$, and the operator $\tilde{\Lambda}$, such that $\int_Y \psi$ is odd.

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The strategy for finding a suitable function ψ is based on the following observations. We define an operator $\tilde{\Omega}$ by $\tilde{\Omega} \xi = \xi - \tilde{\Lambda} \xi$. The interesting fact $\tilde{\Lambda} \tilde{\Lambda} = \tilde{\Lambda}$ implies that $\tilde{\Omega} \tilde{\Omega} = \tilde{\Omega}$, $\tilde{\Lambda} \tilde{\Omega} = 0$, and $\tilde{\Omega} \tilde{\Lambda} = 0$. If ξ has even-dimensional support then the support of $\tilde{\Lambda} \xi$ is of lower dimension. If If ξ has odd-dimensional support then the support of $\tilde{\Omega} \xi$ is of lower dimension.

Let d be the dimension of Y. For decreasing dimensions k = d, d - 1, d - 2, ..., we search through the functions with support dimension k which are obtained from $\mathbf{1}_Y$ using $+, -, \times, \tilde{\Lambda}$, until we find one with odd Euler integral.

In the example at hand, since the dimension of Y is 2 we start with $\varphi = \tilde{\Lambda} \mathbf{1}_Y$, which has support dimension 1, and we apply $\tilde{\Omega}$ to the powers of φ to get functions with support dimension zero.

Computation of the obstruction:

	a	b	c	d	e	f	\int
1_Y	1	1	1	1	1	1	0
$arphi = ilde{\Lambda} {f 1}_Y$	0	1	2	1	-1	0	0
$arphi^2$	0	1	4	1	1	0	0
${ ilde\Lambda} arphi^2$	1	2	2	1	1	0	0
$\tilde{\Omega}\varphi^2 = \varphi^2 - \tilde{\Lambda}\varphi^2$	-1	-1	2	0	0	0	0
$\psi = \varphi \tilde{\Omega} \varphi^2$	0	-1	4	0	0	0	3

In the table the columns a, b, c, d, e, f correspond to the types of cells of Y. Two cells have the same label if they have the same local topology at an interior point. This implies that any function obtained from $\mathbf{1}_Y$ using $+, -, \times, \tilde{\Lambda}$ will be constant on all cells with the same label. The rows of the table are such functions, and the entries of the table give the values of the function on each type of cell.

The last column is the Euler integral of the function. To compute the Euler integral we take into account that there are four cells of type d and two cells of type f. Thus, for example, for the first row we have $\int \mathbf{1}_Y = (1+1+1) - (4 \cdot 1 + 1) + (2 \cdot 1) = 0$, and for the second row we have $\int \varphi = (0+1+2) - (4 \cdot 1 + (-1)) + (2 \cdot 0) = 0$.

The computation of $\varphi = \Lambda \mathbf{1}_Y$ is similar to our first example, the Cartan umbrella. The link of the point a is the union of a figure eight and a point, which has Euler characteristic 0, so $\varphi(a) = 0$. The link of b is the union of a figure eight and three points, which has Euler characteristic 2, so $\varphi(b) = 1$. The link of c is four points so $\varphi(c) = 2$. The link of a point of type d is two points so $\varphi(d) = 1$. The link of a point of type e is a graph with two vertices and four edges from one vertex to the other, which has Euler characteristic -2, so $\varphi(e) = -1$. And the link of a point of type f is a circle, which has Euler characteristic 0, so $\varphi(f) = 0$.

The computation of $\Lambda \varphi^2$ is similar. For example, the link of *a* is the union of a figure eight and a point. The point is of type *d*, at which φ^2 takes the value 1. The vertex of the figure eight is of type *e*, at which φ^2 also takes the value 1. The 1-cells of the figure eight are of type *f*, at which φ^2 takes the value 0. Thus the Euler integral of φ^2 on the link of *a* is (1+1) - 0 = 2, so $\Lambda \varphi^2(a) = 1$.

The result of our computation is

$$\int_{Y} \psi = 3, \ \psi = (\tilde{\Lambda} \mathbf{1}_{Y}) \big((\tilde{\Lambda} \mathbf{1}_{Y})^{2} - \tilde{\Lambda} (\tilde{\Lambda} \mathbf{1}_{Y})^{2} \big).$$

Thus Y is not homeomorphic to an algebraic link.

Akbulut and King's example X is the suspension of Y, the union of two cones with base Y. Now X can be realized as a semialgebraic set in a Euclidean space, with Y the link of the vertex of either cone. Since the Euler characteristic of Y is 0, X satisfies Sullivan's condition. Since Y is not homeomorphic to an algebraic link, it follows that X is not homeomorphic to an algebraic set.

A more direct way to prove that X is not homeomorphic to an algebraic set is to find a function η obtained from $\mathbf{1}_X$ using $+, -, \times, \tilde{\Lambda}$ such that η is not integer-valued. It is not hard to see that the operators $\tilde{\Lambda}$ and $\tilde{\Omega}$ are interchanged by restriction to a link (see [3] 1.3(d), p. 499). More precisely, if Y is the link of p in X, and ξ is a constructible function on X, then

$$|(\Lambda \xi)|_Y = \Omega(\xi|_Y), \ (\Omega \xi)|_Y = \Lambda(\xi|_Y).$$

We apply this to the Akbulut-King example. Since $(\mathbf{1}_X)|_Y = \mathbf{1}_Y$, we have $(\Omega \mathbf{1}_X)|_Y = \Lambda \mathbf{1}_Y$. In other words, if $\varphi' = \tilde{\Omega} \mathbf{1}_X$ then $\varphi'|_Y = \varphi$. Thus $\varphi' \tilde{\Lambda}(\varphi')^2|_Y = \varphi \tilde{\Omega} \varphi^2$. So if p is one of the two suspension vertices of X then $\tilde{\Lambda}(\varphi' \tilde{\Lambda}(\varphi')^2)(p) = \frac{1}{2} \int_Y \varphi \tilde{\Omega} \varphi^2 = \frac{3}{2}$. The function $\eta = \tilde{\Omega}(\varphi' \tilde{\Lambda}(\varphi')^2)$ is supported at the two vertices of X, and at each of these vertices $\eta(p) = -\frac{3}{2}$.

Akbulut and King [1] analyze five local topological obstructions for a 3-dimensional semialgebraic set to be algebraic—the local mod 2 Euler characteristic and four new invariants. They prove that these five obstructions are independent, and that the vanishing of these obstructions is a sufficient condition for a compact 3-dimensional semialgebraic set to be homeomorphic to an algebraic set.

The situation in dimension four is not so simple. There are at least $2^{43} - 43$ independent local topological obstructions [3], but it is not known whether the vanishing of all these obstructions is a sufficient condition for a compact 4-dimensional semialgebraic set to be homeomorphic to an algebraic set.

Significant progress toward the description of a set of sufficient conditions in dimension four has been made by Michelle Previte [4] using Akbulut and King's *resolution towers*. However, the relation between resolution tower obstructions and our constructible function invariants remains a mystery.

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