ON THE POSITIVITY OF DISCRETE HARMONIC FUNCTIONS AND THE DISCRETE HARNACK INEQUALITY FOR PIECEWISE LINEAR FINITE ELEMENTS

D. LEYKEKHMAN AND M. PRUITT

ABSTRACT. The main aim of this paper is twofold. First, we investigate fine estimates of the discrete Green's function and its positivity. We establish that in two dimensions on a smooth domain the discrete Green's function with singularity in the interior of the domain must be strictly positive throughout the computational domain once the mesh is sufficiently refined. We also establish novel pointwise error estimates for the discrete Green's function that are valid up to the boundary of the domain. Then, using these estimates we establish a discrete Harnack inequality for piecewise linear discrete harmonic functions. In contrast to the discrete maximum principle the result is valid for general quasi-uniform shape regular meshes except for a condition on the layer near the boundary. Such results may prove to be useful for the analysis of discrete solutions of fully nonlinear problems.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ for $N = 2, 3$ be a convex bounded domain with sufficiently smooth boundary. Consider the Dirichlet problem for Laplace's equation:

$$
\begin{aligned}
-\Delta u &= 0, \quad \text{in } \Omega, \\
u &= b, \quad \text{on } \partial\Omega.
\end{aligned}
$$

(1)

Here we assume $b \in C(\partial\Omega)$ and $b \geq 0$. To approximate the problem we use standard piecewise linear conforming finite elements. In this paper we will investigate positivity of the finite element solution, pointwise estimates and the positivity of the discrete Green's function, and the discrete Harnack inequality.

The classical Harnack inequality states that for every fixed subdomain $\Omega_0 \Subset \Omega$, there exists a constant $C \geq 1$ depending on Ω_0 so that, for any nonnegative harmonic function u on Ω and any two points $x, y \in \Omega_0$, $u(x) \leq Cu(y)$. That is, any two values of u in the subdomain Ω_0 are comparable, with the constant independent of the particular nonnegative harmonic function. The classical Harnack inequality was extended to elliptic equations in divergence form with bounded measurable coefficients by Moser [23] using the De Giorgi-Nash-Moser iteration technique. Later, the Harnack inequality was extended to elliptic equations in nondivergence form with bounded measurable coefficients by Krylov and Safonov [19]. There is a large body of literature on the Harnack inequality in settings other than classical elliptic or parabolic partial differential equations on $\mathbb{R}^N$. For example, the Harnack

inequality appears in probability in Markov chains [22], in graph theory [3, 7], on Riemannian manifolds [25], and even for infinite dimensional operators [4].

However, there is little work when the discretization is less structured. There are almost no results on the Harnack inequality in the finite element literature, with the exception of the paper of Aguilera and Caffarelli [1]. In this work, Aguilera and Caffarelli adapted the continuous De Giorgi-Nash-Moser iteration technique to the discrete setting and established a form of the Harnack inequality valid for elliptic equations and piecewise linear finite element methods. Their technique requires the discrete maximum principle and some additional geometric constraints on the mesh. In particular, their results require that all of the off-diagonal entries of the stiffness matrix be nonpositive (essentially requiring the stiffness matrix on a bounded domain to be an M-matrix). Using a different technique, we establish the Harnack inequality for piecewise linear finite element methods on general quasi-uniform meshes, under the assumption of a mesh condition that must hold near the boundary of the domain. We believe that, as in the continuous case, the discrete Harnack inequality can be used to prove the Hölder estimates. Such Hölder estimates may be valuable in the analysis of fully nonlinear elliptic problems, for example for showing the uniform convergence of the approximate solution to the viscosity solution. A similar program was carried out for the finite differences method (cf. [20, 21]).

The mesh condition can be thought of (loosely) as requiring that the mesh be able to approximate the normal derivative of the Green's function sufficiently well. Our approach is more in the spirit of Lawler [22] and requires sharp pointwise lower and upper bounds of the corresponding continuous and discrete Green's functions and their error. The Green's function results are new and are of independent interest. For example, one consequence of the discrete Green's function estimates in this paper is that, for smooth convex domains in two dimensions, the discrete Green's function is eventually positive when the singularity is located in the strict interior of the domain. This is also valid for higher order elements and nonsmooth domains, except on a thin layer near the boundary. In [13], a quasi-uniform and shape regular mesh was constructed for which the corresponding discrete Green's function for the piecewise linear finite element method obtained persistent negative values, even as the mesh size tends to zero. Positivity of the Green's function is closely related to the maximum principle. For the continuous problem, the maximum principle can be regarded as a consequence of the nonnegativity of the Green's function. However, as the counterexample in [13] shows, the discrete Green's function need not be nonnegative, and nonnegativity of the discrete Green's function is not in general sufficient to guarantee the maximum principle (see Section 5).

In contrast to the Harnack inequality, the maximum principle is the subject of a large body of research in the finite element literature [6, 10, 12, 17, 32]. However, the maximum principle does not hold in general for discrete harmonic functions without additional restrictive hypotheses on the particular finite element method used. In fact, the classical discrete maximum principle holds essentially for piecewise linear elements only with certain mesh restrictions [16]. A sufficient (though not necessary) condition that guarantees that the maximum principle holds is to require that all of the dihedral angles in the triangulation be nonobtuse. A notable result of Schatz [26] shows that a "weak" maximum principle (also known as the Agmon-Miranda principle) holds asymptotically for general quasi-uniform meshes

in two space dimensions. When considered in perspective with the result of Schatz, our results are perhaps less unexpected.

The rest of the paper is organized as follows. In Section 2, we introduce the problem and state preliminaries, including the definitions of the various Green's functions that appear throughout. In Section 3, we review some well-known pointwise estimates of the continuous Green's function in Lemma 3.1 and a lesser-known pointwise lower bound on the Green's function in Lemma 3.3. In Section 4, we prove pointwise error estimates for the discrete Green's function which are valid up to the boundary of the domain in Theorem 4.5. At the end of Section 4, we deduce Theorem 4.6, a positivity result for the discrete Green's function in two dimensions when the singularity is located in the interior of the domain. In Section 5, we use the error estimates on the discrete Green's function to deduce a Harnack-type inequality for the discrete Green's function. Using a representation formula for discrete harmonic functions in terms of the discrete Green's function allows us to extend the Harnack-type inequality for the discrete Green's function to Theorem 5.6, a Harnack inequality for discrete harmonic functions. Finally, in Section 6 we provide some numerical examples concerning the positivity of the discrete Green's function. We show that the discrete Green's function may be negative in the interior of the domain if the mesh is not sufficiently refined.

2. Preliminaries

Throughout this paper, we adopt standard Sobolev space and finite element notation, and we use freely definitions, such as shape regularity and quasi-uniformity, and results, such as super-approximation and inverse estimates, from the finite element literature (see, for instance, [9] and [5]).

Let $0 < h < 1$ and $\{\mathcal{T}_h\}$ be a quasi-uniform and shape regular family of triangulations of size h for a polygonal computational domain $\Omega_h \subset \Omega$ approximating Ω with $dist_{x \in \partial\Omega}(x, \partial\Omega_h) \leq Ch^2$ and as a result $|\Omega \backslash \Omega_h| \leq Ch^2$. Denote by $V_h(\Omega_h)$ the set of all continuous functions on Ω_h that are linear (affine) when restricted to each triangle in $\mathcal{T}_h$, and define $V_h^0(\Omega_h) = \{v \in V_h(\Omega_h) : v|_{\partial\Omega_h} = 0\}$. After extension by zero such functions can be considered as being in $W_\infty^1(\Omega)$.

Let $\{\phi_i\}_{i=1}^{n+m}$ be a standard nodal basis for $V_h(\Omega_h)$, where the nodes x_i for $i \in \{1, \ldots, n\}$ are interior nodes, and x_j for $j \in \{n+1, \ldots, n+m\}$ are boundary nodes.

We define $u_h \in V_h(\Omega_h)$ to be the solution of the problem

$$(\nabla u_h, \nabla \chi)_{\Omega_h} = 0, \quad \forall \chi \in V_h^0(\Omega_h),$$
$$u_h = I_h b, \quad \text{on } \partial\Omega_h,$$

(2)

where the interpolant $I_h b$ is given by

$$I_h b = \sum_{j=n+1}^{n+m} b(x_j) \phi_j.$$

A function satisfying (1) is said to be a *harmonic function on* Ω, and a function satisfying (2) is said to be a *discrete harmonic function on* Ω_h.

We also require functions that are discrete harmonic on subdomains. For $D \subset \Omega_h$, let $V_h(D)$ be the set of functions on D that are the restrictions of functions in $V_h(\Omega_h)$, and define $V_h^0(D) = \{\chi \in V_h : \text{supp}(\chi) \subset D \cap \Omega_h\}$. A function

$u_h \in V_h(\Omega_h)$ is said to be *discrete harmonic on D* if

$$(3) \qquad (\nabla u_h, \nabla \chi)_D = 0, \quad \forall \chi \in V_h^0(D).$$

One way to represent the solution of (1) is by use of the Green's function. The *(continuous) Green's function* with singularity at z is the function $G^z(x)$ given by

$$(4) \qquad \begin{aligned} -\Delta G^z &= \delta^z, \quad \text{in } \Omega, \\ G^z &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where δ^z is the Dirac delta function at z. We will also use the notation $G(x, z)$ and $G^z(x)$ interchangeably (and similarly for the various other Green's functions which appear) depending on context.

The *discrete Green's function* with singularity at z is the function $G_h^z(x) \in V_h^0(\Omega_h)$ satisfying

$$(5) \qquad (\nabla G_h^z, \nabla \chi)_{\Omega_h} = \chi(z) \quad \forall \chi \in V_h^0(\Omega_h).$$

In the analysis we will also need a regularized Green's function. Let $\tilde{\delta}^z \geq 0$ denote a smooth delta function supported in an element τ_0 containing z with the property

$$(6) \qquad (\tilde{\delta}^z, \chi)_{\Omega_h} = (\tilde{\delta}^z, \chi)_{\tau_0} = \chi(z), \quad \forall \chi \in V_h(\Omega_h).$$

An explicit construction of such a function is given for example in Appendix A of [29]. In addition we also have, for C independent of z,

$$(7) \qquad \|\tilde{\delta}^z\|_{W_p^s(\tau_0)} \leq Ch^{-s-N(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty, \quad s = 0, 1.$$

Thus in particular $\|\tilde{\delta}^z\|_{L^1(\Omega)} \leq C$, $\|\tilde{\delta}^z\|_{L^2(\Omega)} \leq Ch^{-N/2}$, and $\|\tilde{\delta}^z\|_{L^\infty(\Omega)} \leq Ch^{-N}$.

Using $\tilde{\delta}^z$ we define a regularized Green's function $\tilde{G}^z(x)$ by

$$(8) \qquad \begin{aligned} -\Delta \tilde{G}^z &= \tilde{\delta}^z, \quad \text{in } \Omega, \\ \tilde{G}^z &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Notice that $G_h^z = R_h \tilde{G}^z = R_h G^z$, where $R_h u$ is the Ritz projection of a function u onto $V_h^0(\Omega_h)$ defined by

$$(\nabla R_h u, \nabla \chi)_{\Omega_h} = (\nabla u, \nabla \chi)_{\Omega_h}, \qquad \forall \chi \in V_h^0(\Omega_h).$$

3. The continuous Green's function

We will require some results for the continuous Green's function and its derivatives that are essential in our analysis. The proof of the following result for general second order elliptic equations can be found in [18].

Lemma 3.1. *Let $G(x, y)$ denote the Green's function of the Laplace equation on $\Omega \subset \mathbb{R}^N$. Then the following estimates hold:*

$$(9a) \qquad |G(x, y)| \leq \begin{cases} C(1 + |\ln|x - y||), & N = 2, \\ C|x - y|^{2-N}, & N \geq 3, \end{cases}$$

$$(9b) \qquad |\nabla_x^\alpha \nabla_y^\beta G(x, y)| \leq C|x - y|^{2-N-|\alpha|-|\beta|}, \quad |\alpha| + |\beta| \geq 1.$$

Remark 3.2. The smoothness of Ω is only required for $|\alpha| > 2$ and/or $|\beta| > 2$ in (9b). The estimates (9a) and (9b) for $|\alpha| \leq 1$ and $|\beta| \leq 1$ are known to hold for general convex domains for any $N \geq 1$ (cf. [14, 15]).

We will also need a lower bound on the continuous Green's function.

Lemma 3.3. *Let $\Omega_0 \Subset \Omega$. Then there exists a constant C so that for all $x \in \Omega_0$ and $z \in \Omega$, we have $G(x, z) \geq C\rho(z, \partial\Omega)$, where $\rho(A, B) = dist(A, B)$, the distance between sets (or points) A and B.*

Proof. For $N = 2$, by Theorem 6.23 of [8], there exists $C > 0$ such that

$$C \ln \left(1 + \frac{\rho(x, \partial\Omega)\rho(z, \partial\Omega)}{|x - z|^2} \right) \leq G(x, z).$$

Because Ω is bounded, $|x - z|^{-2}$ is bounded below by a positive constant independent of x and z, and because $\Omega_0 \Subset \Omega$, we have that $\rho(x, \partial\Omega)$ is bounded below by a positive constant independent of $x \in \Omega_0$. Therefore, we have the lower bound $C \ln (1 + C\rho(z, \partial\Omega)) \leq G(x, z)$. Because Ω is a bounded domain, $\rho(z, \partial\Omega)$ is bounded above by a constant independent of $x \in \Omega$, so that we may find $C' > 0$ such that $C'\rho(z, \partial\Omega) \leq G(x, z)$ for all $x \in \Omega_0$ and $z \in \Omega$.

For $N \geq 3$, by Theorem 1 of [33], there exists a constant $C > 0$ such that

$$G(x, z) \geq \begin{cases} C|x - z|^{2-N} & \text{if } |x - z| \leq \max\left(\rho(x, \partial\Omega), \rho(z, \partial\Omega)\right)/2, \\ C|x - z|^{-N}\rho(x, \partial\Omega)\rho(z, \partial\Omega) & \text{if } |x - z| > \max\left(\rho(x, \partial\Omega), \rho(z, \partial\Omega)\right)/2. \end{cases}$$

First, note again that because Ω is bounded, the factors $|x - z|^{2-N}$ and $|x - z|^{-N}$ are bounded below by a positive constant independent of $x, z \in \Omega$. Therefore, if $|x - z| \leq \max\left(\rho(x, \partial\Omega), \rho(z, \partial\Omega)\right)/2$, there exists a positive constant C independent of $x, z \in \Omega$ for which $G(x, z) \geq C$, and because Ω is bounded, $G(x, z) \geq C'\rho(z, \partial\Omega)$. For the case where $x \in \Omega_0$ and $z \in \Omega$ with $|x - z| > \max\left(\rho(x, \partial\Omega), \rho(z, \partial\Omega)\right)/2$, because $\rho(x, \Omega)$ is bounded below by a positive constant independent of $x \in \Omega_0$, we again obtain the lower bound $G(x, z) \geq C'\rho(z, \partial\Omega)$ for some positive constant C'. $\qquad\square$

Remark 3.4. Lemma 3.3 is the only place in the paper that requires smoothness of the domain Ω. If Ω is less smooth but the estimate

$$G(x, z) \geq C\rho^{2-\varepsilon}(z, \partial\Omega), \quad \text{for some } \varepsilon > 0,$$

still holds, then the main results of the paper are still true and the proofs require only minor modifications.

4. POINTWISE ERROR ESTIMATES FOR THE GREEN'S FUNCTIONS

To derive the desired pointwise estimates for $G_h - G$ we require several error estimates in the L^∞ norm for the error $u - u_h$ between the solution u of the elliptic problem and the Ritz projection $u_h = R_h u$ of the solution. Although we will use the results only for the piecewise linear case (i.e. $r = 2$), the results in this section are valid for $V_h(\Omega_h)$ replaced by piecewise polynomials of degree $r - 1$ for $r \geq 2$. In the results below

$$\bar{r} = \begin{cases} 1, & r = 2, \\ 0, & r > 2. \end{cases}$$

The first result is Theorem 5.1 from [28], which states that the error for $\Omega \subset \mathbb{R}^N$ smooth is almost optimal in the $L^\infty(\Omega_h)$ norm.

Theorem 4.1 (Schatz-Wahlbin 1982)**.** *For h sufficiently small there exists a constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega_h)} \le C\ell_h^{\bar{r}} \inf_{\chi \in V_h^0(\Omega_h)} \|u - \chi\|_{L^\infty(\Omega_h)},$$

where $\ell_h := |\ln h|$.

The second result is Theorem 5.1 from [27], a localized version of the above theorem on interior domains.

Theorem 4.2 (Schatz-Wahlbin 1977)**.** *Suppose* $D \Subset D_d \Subset \Omega$, *where* $D_d = \{x \in \Omega : dist(x, D) \le d\}$, *with* $d \ge ch$. *Let t be a nonnegative integer and let* $1 \le p \le \infty$. *Then there exists a constant C independent of h and d such that*

$$\|u - u_h\|_{L^\infty(D)} \le C\ell_h^{\bar{r}} \inf_{\chi \in V_h(\Omega_h)} \|u - \chi\|_{L^\infty(D_d)} + Cd^{-t-N/p}\|u - u_h\|_{W_p^{-t}(D_d)},$$

where $\ell_h := |\ln h|$.

We will also require a version of Theorem 4.2 valid up to the boundary. To establish this, first we will need Proposition 3.1 from [11], which is also valid for smooth domains.

Proposition 4.3. *Let*

$$D_4 \subset D_3 \subset D_2 \subset D_1 \subset D \subset \Omega_h$$

with $dist(D_i, \partial D_{i-1} \setminus \partial \Omega_h) \ge d$, *and similarly for* D_1 *and* D. *There is a constant C such that for each* $\chi \in V_h(D)$ *there exists an* $\eta \in V_h^0(D_1)$ *with* $\eta \equiv \chi$ *on* D_2 *and*

$$\|\nabla(\chi - \eta)\|_{L^2(D)} \le C(\|\nabla\chi\|_{L^2(D \setminus D_4)} + d^{-1}\|\chi\|_{L^2(D \setminus D_4)}).$$

The preceding three results enable us to prove the following theorem.

Theorem 4.4. *Let* $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, *be a smooth domain and let* $D \subset D_d \subset \Omega_h$, *where* $D_d = \{x \in \Omega_h : dist(x, D) \le d\}$. *Then there exists a constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(D)} \le C\ell_h^{\bar{r}} \inf_{\chi \in V_h(\Omega_h)} \|u - \chi\|_{L^\infty(D_d)} + Cd^{-N/2}\|u - u_h\|_{L^2(D_d)}.$$

Proof. The proof is an adaptation of the proof of Theorem 1 from [11]. It is sufficient to consider the case of concentric balls $B_{md}(x_0)$, $m \in \mathbb{R}^+$, intersecting Ω_h for x_0 an arbitrary point in D. By a covering argument (cf. [24], Thm. 5.1) the proof can be extended to general subdomains $D \subset D_d \subset \Omega_h$. In what follows we will use the abbreviation $mD := B_{md}(x_0) \cap \Omega_h$ and put $e := u - u_h$. Let ω be a smooth cut-off function with the properties $\omega \equiv 1$ on D, $supp(\omega) \subset 2D$, and $|\nabla\omega| \le Cd^{-1}$. Let $\tilde{u} := \omega u$. Define $\tilde{u}_h := R_h\tilde{u}$ to be the Ritz projection of $\tilde{u}$ onto $V_h^0(\Omega_h)$. Then

$$(10) \qquad |e(x_0)| \le |(\tilde{u} - \tilde{u}_h)(x_0)| + |(\tilde{u}_h - u_h)(x_0)|.$$

By Theorem 4.1 the first term on the right hand side of (10) can be estimated as

$$|(\tilde{u} - \tilde{u}_h)(x_0)| \le \|\tilde{u} - \tilde{u}_h\|_{L^\infty(\Omega_h)} \le C\ell_h^{\bar{r}}\|\tilde{u}\|_{L^\infty(\Omega_h)} \le C\ell_h^{\bar{r}}\|u\|_{L^\infty(2D)}.$$

Let $\psi_h := \tilde{u}_h - u_h$. Notice that ψ_h is discrete harmonic on D; we do not consider the properties of this function outside D. The rest of the proof is devoted to establishing that

$$|(\tilde{u}_h - u_h)(x_0)| \le C(\ell_h^{\bar{r}}\|u\|_{L^\infty(2D)} + d^{-N/2}\|u_h\|_{L^2(2D)}).$$

By Proposition 4.3 there exists $\eta_h \in V_h^0(\frac{3}{4}D)$ such that $\eta_h \equiv \psi_h$ on $\frac{1}{2}D$ and

$$(11) \qquad \|\nabla\eta_h\|_{L^2(\frac{3}{4}D)} \leq C(\|\nabla\psi_h\|_{L^2(\frac{7}{8}D)} + d^{-1}\|\psi_h\|_{L^2(\frac{7}{8}D)}).$$

Let $\tilde{G}^{x_0}$ be the regularized Green's function defined as in (8) and recall that $G_h^{x_0}$ is the Ritz projection of $\tilde{G}^{x_0}$ onto $V_h^0(\Omega_h)$. Then,

$$\psi_h(x_0) = \eta_h(x_0) = (\tilde{\delta}^{x_0}, \eta_h)_{\Omega_h} = (\nabla\tilde{G}^{x_0}, \nabla\eta_h)_{\Omega_h} = (\nabla G_h^{x_0}, \nabla\eta_h)_{\Omega_h}.$$

Also by Proposition 4.3, there exists $\zeta_h \in V_h^0(\frac{1}{2}D)$ such that $\zeta_h \equiv G_h^{x_0}$ on $\frac{1}{4}D$ and

$$\|\nabla(G_h^{x_0} - \zeta_h)\|_{L^2(\frac{3}{4}D)} \leq C(\|\nabla G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} + d^{-1}\|G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)}).$$

Recalling that η_h is supported on $\frac{3}{4}D$ and is discrete harmonic in $\frac{1}{2}D$ and using (11), we have

$$
\begin{aligned}
\psi_h(x_0) &= (\nabla(G_h^{x_0} - \zeta_h), \nabla\eta_h)_{\frac{3}{4}D} + (\nabla\zeta_h, \nabla\eta_h)_{\frac{3}{4}D} \\
&= (\nabla(G_h^{x_0} - \zeta_h), \nabla\eta_h)_{\frac{3}{4}D} \\
(12) \qquad &\leq \|\nabla(G_h^{x_0} - \zeta_h)\|_{L^2(\frac{3}{4}D)}\|\nabla\eta_h\|_{L^2(\frac{3}{4}D)} \\
&\leq C(\|\nabla G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} + d^{-1}\|G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)}) \\
&\quad \times (\|\nabla\psi_h\|_{L^2(\frac{7}{8}D)} + d^{-1}\|\psi_h\|_{L^2(\frac{7}{8}D)}).
\end{aligned}
$$

Next we need Lemma 9.2 from [30], which says that for any discrete harmonic function, i.e. for any $v_h \in V_h^0(D)$ that satisfies (5), there holds

$$(13) \qquad \|\nabla v_h\|_{L_2(D)} \leq Cd^{-1}\|v_h\|_{L_2(D_d)} \leq Cd^{-2}\|v_h\|_{H_<^{-1}(D_d)},$$

where

$$\|v_h\|_{H_<^{-1}(D_d)} = \sup_{\substack{w \in H^1(\Omega), \\ w=0 \text{ on } \Omega\backslash D_d}} \frac{(v_h, w)_{D_d}}{\|w\|_{H^1(D_d)}}.$$

Using that ψ_h is discrete harmonic on D, the triangle inequality, and the fact that $u = \tilde{u}$ on D, we have

$$
\begin{aligned}
\|\nabla\psi_h\|_{L^2(\frac{7}{8}D)} + d^{-1}\|\psi_h\|_{L^2(\frac{7}{8}D)} &\leq Cd^{-1}\|\psi_h\|_{L^2(D)} \\
(14) \qquad &\leq Cd^{-1}\left(\|u - u_h\|_{L^2(D)} + \|\tilde{u} - \tilde{u}_h\|_{L^2(D)}\right) \\
&\leq Cd^{-1}\left(\|e\|_{L^2(D)} + h\ell_h^{\bar{r}}d^{N/2}\|u\|_{L^\infty(2D)}\right),
\end{aligned}
$$

where in the last step we have used Hölder's inequality and Theorem 4.1 for $\|\tilde{u} - \tilde{u}_h\|_{L^2(D)}$, i.e.

$$
\begin{aligned}
\|\tilde{u} - \tilde{u}_h\|_{L^2(D)} &\leq Cd^{N/2}\|\tilde{u} - \tilde{u}_h\|_{L^\infty(D)} \leq Cd^{N/2}\ell_h^{\bar{r}}\|\tilde{u}\|_{L^\infty(D)} \\
&\leq Cd^{N/2}\ell_h^{\bar{r}}\|u\|_{L^\infty(2D)}.
\end{aligned}
$$

Now we turn to $\|\nabla G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} + d^{-1}\|G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)}$. Using that $G_h^{x_0}$ is discrete harmonic away from x_0, from (13) we have

$$
(15) \qquad
\begin{aligned}
\|\nabla G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} + d^{-1}\|G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} &\leq Cd^{-1}\|G_h^{x_0}\|_{L^2(D\backslash\frac{1}{8}D)} \\
&\leq Cd^{-2}\|G_h^{x_0}\|_{H_<^{-1}(D\backslash\frac{1}{8}D)}.
\end{aligned}
$$

For $N = 2$ we apply the first inequality in (15). By the Sobolev embedding theorem $(W_1^1 \hookrightarrow L^2)$,

$$\|G_h^{x_0}\|_{L^2(D\backslash\frac{1}{8}D)} \leq C\|G_h^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)}.$$

Note that the Sobolev embedding constant appearing in the inequality above is domain independent. To verify this, we may scale the domain D to a unit-sized domain $\tilde{D}$ by introducing a new variable $y = x/d$. Then it is easy to show that for any general function $V(y) = v(yd)$ we have

$$(16) \qquad \|\nabla^s V\|_{L_q(\tilde{D})} = d^{s-N/q}\|\nabla^s v\|_{L_q(D)}, \quad s = 0, 1.$$

Thus if $D\backslash\frac{1}{8}D$ is scaled to a subset of a fixed unit-sized annulus and $G_h^{x_0}$ is extended by zero in this annulus if D abuts $\partial\Omega$, by using (16) we can see that this constant is indeed independent of d.

For $N = 3$ we use the second inequality in (15). Then,

$$\|G_h^{x_0}\|_{H_<^{-1}(D\backslash\frac{1}{8}D)} = \sup_{\substack{v\in H^1(\Omega), \\ v=0 \text{ on } \Omega\backslash(D\backslash\frac{1}{8}D)}} \frac{(G_h^{x_0}, v)_{D\backslash\frac{1}{8}D}}{\|v\|_{H^1(D\backslash\frac{1}{8}D)}}.$$

Since by the Hölder's inequality and the Sobolev embedding $W_1^1 \hookrightarrow L^{3/2}$ and $H^1 \hookrightarrow L^6$, we have

$$\begin{aligned} (G_h^{x_0}, v)_{D\backslash\frac{1}{8}D} &\leq C\|G_h^{x_0}\|_{L^{3/2}(D\backslash\frac{1}{8}D)}\|v\|_{L^3(D\backslash\frac{1}{8}D)} \\ &\leq Cd^{1/2}\|G_h^{x_0}\|_{L^{3/2}(D\backslash\frac{1}{8}D)}\|v\|_{L^6(D\backslash\frac{1}{8}D)} \\ &\leq Cd^{1/2}\|G_h^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)}\|v\|_{H^1(D\backslash\frac{1}{8}D)}, \end{aligned}$$

and as a result

$$(17) \qquad \|G_h^{x_0}\|_{H_<^{-1}(D\backslash\frac{1}{8}D)} \leq Cd^{1/2}\|G_h^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)}.$$

Again the constant C in the above inequality is independent of d.

By the triangle inequality and Lemma 5.3 in [28],

$$\begin{aligned} \|G_h^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)} &\leq \|G_h^{x_0} - \tilde{G}^{x_0}\|_{W_1^1(\Omega_h)} + \|\tilde{G}^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)} \\ &\leq Ch\ell_h^{\bar{r}} + \|\tilde{G}^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)}. \end{aligned}$$

Since for some fixed $c > 0$, we have that $dist(x, \text{supp}(\tilde{\delta}^{x_0})) \geq cd$ for all $x \in D\backslash\frac{1}{8}D$, we have from Lemma 3.1 that for any such x,

$$|\nabla\tilde{G}^{x_0}(x)| = \left|\int_{\tau_0} \nabla_x G(x, y)\tilde{\delta}^{x_0}(y)dy\right| \leq Cd^{1-N}.$$

As a result,

$$\|\tilde{G}^{x_0}\|_{W_1^1(D\backslash\frac{1}{8}D)} \leq Cd^N\|\tilde{G}^{x_0}\|_{W_\infty^1(D\backslash\frac{1}{8}D)} \leq Cd.$$

Collecting the above estimates, we thus have that

$$(18) \qquad \|\nabla G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} + d^{-1}\|G_h^{x_0}\|_{L^2(\frac{7}{8}D\backslash\frac{1}{4}D)} \leq Cd^{1-N/2}.$$

Collecting (18) and (14) into (12) yields

$$|e(x_0)| \leq C\left(\ell_h^{\bar{r}}\|u\|_{L^\infty(2D)} + d^{-N/2}\|e\|_{L^2(2D)}\right).$$

We complete the proof of Theorem 4.4 by inserting $u - \chi$ and $u_h - \chi$ for u and u_h and writing D instead of $2D$. □

As an application of Theorem 4.2 and Theorem 4.4 we have the following result for the piecewise linear case, $r = 2$.

Theorem 4.5. *Let $x, y \in \Omega$ with $|x - y| \geq d$ with $B_d(x) \Subset \Omega$. Then there exists a constant C independent of h, x, y and d such that*

$$|G_h^x(y) - G^x(y)| \leq C\ell_h h^2 d^{-N}, \quad N = 2, 3,$$

where $\ell_h = |\ln h|$.

Proof. The proof follows the lecture notes of L.B. Wahlbin [31]. Since the case $y \in \Omega \backslash \Omega_h$ is trivial, we may assume that $y \in \Omega_h$. Let $B_d(x) \Subset \Omega$ and G^x be the continuous Green's function with singularity at x and G_h^x be the discrete Green's function. For any $y \in \Omega_h$, $|x - y| \geq d$, by Theorem 4.4, we have

$$|G^x(y) - G_h^x(y)| \leq C\ell_h \inf_{\chi \in V_h^0(\Omega_h)} \|G^x - \chi\|_{L^\infty(B_{d/4}(y) \cap \Omega_h)}$$
$$+ Cd^{-N/2}\|G^x - G_h^x\|_{L^2(B_{d/4}(y) \cap \Omega_h)}.$$

Since G^x is smooth away from the singularity we may take $\chi = I_h G^x$. Using the approximation theory and Green's function estimates we obtain

$$\|G^x - \chi\|_{L^\infty(B_{d/4}(y) \cap \Omega_h)} \leq Ch^2\|G^x\|_{W_\infty^2(B_{d/4}(y) \cap \Omega_h)} \leq Ch^2 d^{-N}.$$

In the last step we have used that (cf. [18])

$$|\nabla^2 G^x(z)| \leq C|x - z|^{-N} \leq Cd^{-N}, \quad \forall z \in B_{d/4}(y) \cap \Omega_h.$$

Thus we only need to estimate $d^{-N/2}\|G^x - G_h^x\|_{L^2(B_{d/4}(y) \cap \Omega_h)}$. By duality

$$\|G^x - G_h^x\|_{L^2(B_{d/4}(y) \cap \Omega_h)} = \sup_{\substack{\varphi \in C_0^\infty(B_{d/4}(y) \cap \Omega_h) \\ \|\varphi\|_{L^2} \leq 1}} (G^x - G_h^x, \varphi)_{B_{d/4}(y) \cap \Omega_h}.$$

For each such φ, let ψ solve

$$-\Delta\psi = \varphi, \quad \text{in } \Omega,$$
$$\psi = 0, \quad \text{on } \partial\Omega,$$

and $\psi_h = R_h \psi$. Then

$$(G^x - G_h^x, \varphi)_{B_{d/4}(y) \cap \Omega_h} = (G^x - G_h^x, -\Delta\psi)_\Omega = (\nabla(G^x - G_h^x), \nabla\psi)_\Omega$$
$$= (\nabla(G^x - G_h^x), \nabla(\psi - \psi_h))_\Omega$$
$$= (\nabla G^x, \nabla(\psi - \psi_h))_\Omega$$
$$= -(\Delta G^x, \psi - \psi_h)_\Omega = \psi(x) - \psi_h(x).$$

Because $B_d(x) \Subset \Omega$, we can apply Theorem 4.2 to obtain

(19) $\quad |\psi(x) - \psi_h(x)| \leq C\ell_h\|\psi - \chi\|_{L^\infty(B_{d/4}(x))} + Cd^{-N/2}\|\psi - \psi_h\|_{L^2(B_{d/4}(x))}.$

By the approximation theory

$$\|\psi - \chi\|_{L^\infty(B_{d/4}(x))} \leq Ch^2\|\psi\|_{W_\infty^2(B_{d/4}(x))}.$$

Now using the Green's function representation and properties of the Green's function we have for $z \in B_{d/4}(x)$ that

$$|\nabla^2\psi(z)| = \left|\int_{B_{d/4}(y) \cap \Omega_h} \nabla_z^2 G^z(s)\varphi(s)ds\right|$$
$$\leq C\int_{B_{d/4}(y) \cap \Omega_h} \frac{|\varphi(s)|}{|z - s|^N}ds \leq Cd^{-N}\|\varphi\|_{L^1(B_{d/4}(y) \cap \Omega)} \leq Cd^{-N/2},$$

where we have used that $dist(B_{d/4}(y), B_{d/4}(x)) \geq d/2$. Thus we conclude that

$$(20) \qquad \|\psi - \chi\|_{L^\infty(B_{d/4}(x))} \leq Ch^2 d^{-N/2}.$$

To estimate the term involving $\|\psi - \psi_h\|_{L^2(B_{d/4}(x))}$, we use a global argument. By H^2 regularity,

$$(21) \quad \|\psi - \psi_h\|_{L^2(B_{d/4}(x))} \leq \|\psi - \psi_h\|_{L^2(\Omega)} \leq Ch^2\|\psi\|_{H^2(\Omega)} \leq Ch^2\|\varphi\|_{L^2(\Omega)} \leq Ch^2.$$

Combining estimates (20) and (21), we obtain

$$d^{-N/2}\|G^x - G_h^x\|_{L^2(B_{d/4}(y)\cap\Omega_h)} \leq C\ell_h h^2 d^{-N}.$$

This concludes the proof of the theorem. □

4.1. **On the positivity of G_h in 2D for piecewise linear elements.** An example in [13] shows that on general meshes the discrete Green's function may have persistent negative values for all h. For these meshes, the singularity and the node at which a negative value is obtained are both a distance $O(h)$ from the boundary. Our next result shows that in two dimensions the values of the discrete Green's function for piecewise linear elements must eventually be positive if the singularity is a distance $O(1)$ from the boundary.

Theorem 4.6. *Suppose $D \Subset \Omega \Subset \mathbb{R}^2$ is smooth. Then there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, we have $G_h^{x_0}(x) > 0$ for all $x \in int\ \Omega_h$ and $x_0 \in D$.*

It is sufficient to consider the case when $D = B_d(x_0)$ with $d \geq ch$ and

$$dist(\partial B_d, \partial\Omega) \geq d_0,$$

for some fixed but arbitrary d_0. The case of general $D \Subset \Omega$ follows by using a covering argument. Let τ_0 be a triangle in $\mathcal{T}_h$ containing x_0. Let $\tilde{\delta}^{x_0}$ be a regularized delta function supported in τ_0, with properties (7).

Let $\tilde{G}^{x_0}$ be the regularized Green's function as in (8). The first lemma shows that near the singularity the regularized Green's function cannot be uniformly bounded in h.

Lemma 4.7. *There exists a constant C independent of h and x_0 such that*

$$(\tilde{G}^{x_0}, \tilde{\delta}^{x_0}) \geq C(|\ln h| + 1).$$

Proof. Using that $G^{x_0}(x) \geq C|\ln|x - x_0||$ for x sufficiently close to x_0, we have

$$\tilde{G}^{x_0}(x_0) = \int_{\tau_0} G(x_0, x)\tilde{\delta}^{x_0}(x)dx \geq C(|\ln h| + 1).$$

Since $\|\tilde{\delta}^{x_0}\|_{L^1(\tau_0)} = 1$ there exists a ball $B_{c_1 h}(\bar{x})$ of radius $c_1 h$ centered at $\bar{x} \in \tau_0$ (not necessarily $\bar{x} = x_0$), where $\tilde{\delta}^{x_0} \geq c_2 h^{-2}$, for some $c_1, c_2 > 0$. Using the monotonicity of the logarithm and that $diam(\tau_0) \leq h$, we have

$$|\ln|x - x_0|| \geq |\ln(|x - \bar{x}| + |\bar{x} - x_0|)| \geq |\ln(|x - \bar{x}| + h)|.$$

Switching to polar coordinates $|x - \bar{x}| = \rho$, we obtain

$$(\tilde{G}^{x_0}, \tilde{\delta}^{x_0}) \geq c_2 h^{-2} \int_{B_{c_1 h}(\bar{x})} \tilde{G}(x, x_0)dx$$

$$= Ch^{-2} \int_0^{c_1 h} \rho |\ln(\rho + h)| d\rho \geq C(|\ln h| + 1).$$

□

The next lemma is a similar estimate for the discrete Green's function.

Lemma 4.8. *There exist a constant C independent of h and x_0 and $h_0 > 0$ such that for all $h \leq h_0$,*

$$G_h^{x_0}(x_0) \geq C(|\ln h| + 1).$$

Proof.

$$
\begin{aligned}
G_h^{x_0}(x_0) &= (G_h^{x_0}, \tilde{\delta}^{x_0})_{\Omega_h} = (\nabla G_h^{x_0}, \nabla G_h^{x_0})_{\Omega_h} \\
&= \|\nabla \widetilde{G}^{x_0}\|_{L^2(\Omega_h)}^2 - \left(\|\nabla \widetilde{G}^{x_0}\|_{L^2(\Omega_h)}^2 - \|\nabla G_h^{x_0}\|_{L^2(\Omega_h)}^2 \right) \\
&= (\widetilde{G}^{x_0}, \tilde{\delta}^{x_0})_{\Omega} - (\nabla(\widetilde{G}^{x_0} - G_h^{x_0}), \nabla(\widetilde{G}^{x_0} + G_h^{x_0}))_{\Omega_h} \\
&= (\widetilde{G}^{x_0}, \tilde{\delta}^{x_0})_{\Omega} - \|\nabla(\widetilde{G}^{x_0} - G_h^{x_0})\|_{L^2(\Omega_h)}^2.
\end{aligned}
$$

From Lemma 4.7

$$(\widetilde{G}^{x_0}, \tilde{\delta}^{x_0})_{\Omega} \geq C(|\ln h| + 1).$$

On the other hand, using the best approximation properties and H^2 regularity for smooth (convex) domains we have

$$
\|\nabla(\widetilde{G}^{x_0} - G_h^{x_0})\|_{L^2(\Omega_h)} \leq \|\nabla(\widetilde{G}^{x_0} - I_h \widetilde{G}^{x_0})\|_{L^2(\Omega_h)} \leq Ch\|\nabla^2 \widetilde{G}^{x_0}\|_{L^2(\Omega_h)}
$$
$$
\leq Ch\|\tilde{\delta}\|_{L^2(\tau_0)} \leq Chh^{-1} \leq C.
$$

Thus for h_0 small enough we have the lemma. $\square$

Lemma 4.9. *There exists a constant C independent of h and x_0 such that*

$$\|\nabla G_h^{x_0}\|_{L^\infty(\Omega_h)} \leq Ch^{-1}.$$

Proof. From estimate (2.5) in [2], we have

$$\|\nabla G_h^{x_0}\|_{L^\infty(\Omega_h)} \leq C\|\nabla \tilde{G}^{x_0}\|_{L^\infty(\Omega_h)} + Ch\|\nabla \tilde{G}^{x_0}\|_{L^\infty(\Omega \backslash \Omega_h)}.$$

Using the Green's function representation and properties of $\tilde{\delta}^{x_0}$, for any $z \in \Omega$ we have

$$
\begin{aligned}
|\nabla \tilde{G}^{x_0}(z)| = \left| \int_{\tau_0} \nabla_z G(z, y) \tilde{\delta}^{x_0}(y) dy \right| &\leq Ch^{-2} \int_{\tau_0} \frac{dy}{|z - y|} \\
&\leq Ch^{-2} \int_{\tau_0} \frac{dy}{|x_0 - y|} \leq Ch^{-1}.
\end{aligned}
$$

$\square$

Now we are ready to prove Theorem 4.6.

Proof. Let K and C_K be the constants C_7 and C from estimate (6.3) in Theorem 6.1 in [27], respectively. In addition let $h_0 > 0$ be small enough such that for all $h \leq h_0$,

$$\frac{C_K \ln(K|\ln h|^{1/2})}{K^2 |\ln h|} \leq \frac{1}{2} \min_{\substack{x \in \Omega \\ dist(x, \partial\Omega) \geq d_0}} G^{x_0}(x)$$

and

$$h - C_3 h^2 |\ln h| \geq \frac{1}{2} h,$$

where C_3 is the constant from Theorem 4.5. We now consider several cases.

Case 1. $|x - x_0| \leq Kh|\ln h|^{1/2}$. Then $G_h^{x_0}(x) > 0$ in view of Lemmas 4.8 and 4.9 since a discrete function cannot go from negative values to positive values of order $|\ln h|$ in $K|\ln h|^{1/2}$ many steps of size one.

Case 2. $Kh|\ln h|^{1/2} \leq |x - x_0|$ and $dist(x, \partial\Omega) \geq d_0$. For this case we use Theorem 6.1 from [27] for $r = 2$, which states that for $x, y \in \Omega_0$, where $\Omega_0 \Subset \Omega$ and $|x - y| \geq Kh$, there holds

$$(22) \qquad |G^x(y) - G_h^x(y)| \leq \frac{C_K h^2}{|x - y|^2} \ln\left(\frac{|x - y|}{h}\right).$$

Adopting the notation $w = |x - x_0|/h$, the upper bound from (22) becomes $\frac{C_K \ln w}{w^2}$. Because $Kh|\ln h|^{1/2} \leq |x - x_0|$, we must have $w \geq K|\ln h|^{1/2}$. For h sufficiently small, the maximum of $\frac{C_K \ln w}{w^2}$ on the interval $w \geq K|\ln h|^{1/2}$ occurs at the left endpoint. Therefore, we have that

$$|G^{x_0}(x) - G_h^{x_0}(x)| \leq \frac{C_K h^2}{(Kh|\ln h|^{1/2})^2} \ln\left(K|\ln h|^{1/2}\right) = \frac{C_K \ln\left(K|\ln h|^{1/2}\right)}{K^2 |\ln h|}.$$

Thus in this case in view of the choice of h_0,

$$G_h^{x_0}(x) = G^{x_0}(x) - (G^{x_0}(x) - G_h^{x_0}(x)) \geq \frac{1}{2} G^{x_0}(x) > 0.$$

Case 3. $dist(x, \partial\Omega) \leq d_0$. Let x_j be any interior node such that $dist(x_j, \partial\Omega) \leq d_0$. Since in this case $|x_j - x_0| = O(1)$, we have by Theorem 4.5

$$G_h^{x_0}(x_j) = G^{x_0}(x_j) - (G^{x_0}(x_j) - G_h^{x_0}(x_j)) \geq Ch - C_3 h^2 |\ln h| \geq \widetilde{C} h.$$

The estimate $G^{x_0}(x_j) \geq Ch$ in the second to last step above follows from Lemma 3.3 (and the symmetry of the Green's function), as all interior nodes are a distance of at least $O(h)$ from the boundary. Combining all three cases and interpolating between nodes we have a proof of Theorem 4.6. $\qquad\square$

Remark 4.10. The order of the polynomials plays no role in the proofs of Lemmas 4.7-4.9 and cases 1 and 2 in the proof of Theorem 4.6. Thus, $G_h^{x_0}(x) > 0$ as $h \to 0$ for polynomials of all orders at nodes away from the boundary. The proof only relies on the fact that the discrete Green's function is of order $|\ln h|$ at the singularity, but its derivatives are of order h^{-1} at most. This discrepancy does not hold in three or higher dimensions. It would be interesting to see if a similar result holds in higher dimensions.

Remark 4.11. The above result can be thought of as some kind of an asymptotic interior maximum principle in 2D, although positivity of the discrete Green's function alone should not be enough to guarantee a maximum principle without an assumption on the boundary stiffness matrix H, defined in the next section.

5. DISCRETE HARNACK INEQUALITY

In this section, we prove a discrete form of the Harnack inequality for piecewise linear finite elements in two and three dimensions under the hypothesis that the mesh is well-behaved near the boundary. We must first adopt a representation for discrete harmonic functions using the discrete Green's function.

Let u_h be a discrete harmonic function (i.e. u_h solves (2)). We may expand u_h in the nodal basis as

$$u_h(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) + \sum_{j=n+1}^{n+m} \alpha_j \phi_j(x),$$

where the first sum is over the interior nodes and the second sum is over the boundary nodes.

The solution to the problem may then be represented in matrix form by

$$U = -A^{-1}HB.$$

Here U represents the solution u_h at the interior nodes, with

$$U = (u_h(x_1), \ldots, u_h(x_n))^{\top} \in \mathbb{R}^n.$$

The matrix $A \in \mathbb{R}^{n \times n}$ is the (interior) stiffness matrix, with entries given by $A_{ij} = (\nabla \phi_i, \nabla \phi_j)_{\Omega_h}$ for $i, j \in \{1, \ldots, n\}$. The matrix $H \in \mathbb{R}^{n \times m}$ is the boundary stiffness matrix, with entries given by $H_{jk} = (\nabla \phi_j, \nabla \phi_k)_{\Omega_h}$ for $j \in \{1, \ldots, n\}$ and $k \in \{n+1, \ldots, n+m\}$. The vector B contains the boundary data, with $B = (b(x_{n+1}), \ldots, b(x_{n+m}))^{\top} \in \mathbb{R}^m$.

By reinterpreting the matrix multiplication as a sum, we have the representation

$$(23) \qquad u_h(x_i) = -\sum_{j=1}^{n} \sum_{k=1}^{m} A_{ij}^{-1} H_{jk} B_k.$$

We also have that $A_{ij}^{-1} = G_h(x_j, x_i)$, because the value of the discrete Green's function at an interior node is given by the corresponding entry of the inverse stiffness matrix. Note that, by the symmetry of the stiffness matrix, the discrete Green's function is symmetric at the nodes. For more detail on this representation, see, for instance, [13].

Let $\tilde{N}(x_k)$ denote the set of all neighboring nodes to x_k, i.e. the set of all other nodes that are vertices of a triangle of which x_k is itself a vertex. Using the small support of the nodal basis functions, we can rearrange and rewrite the sum in (23) as

$$(24) \qquad u_h(x_i) = -\sum_{k=n+1}^{n+m} \sum_{x_j \in \tilde{N}(x_k)} G_h(x_j, x_i) \left(\nabla \phi_j, \nabla \phi_k\right)_{\Omega_h} b(x_k).$$

To derive the discrete Harnack inequality, we make the following assumption on the boundary stiffness matrix H:

Assumption 5.1. *For every triangulation in $\{\mathcal{T}_h\}$, the associated boundary stiffness matrix H must satisfy $H \leq 0$, i.e. $(\nabla \phi_i, \nabla \phi_j)_{\Omega_h} \leq 0$ for all $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, n+m\}$.*

Remark 5.2. This assumption can be (loosely) interpreted as requiring that the mesh be able to approximate the normal derivative of the continuous Green's function properly. This assumption implies the maximum principle if the discrete Green's function is known to be nonnegative. In two dimensions, this is equivalent to the following edge condition: for every edge in the triangulation with one node on the boundary of Ω and one node in the interior of Ω, the sum of the angles opposite the edge is at most π. For more detail and the relationship between this condition and the discrete maximum principle, see [13], where an explicit example

is constructed that produces negative values of the discrete Green's function for all $h > 0$ when this condition is violated.

As a consequence of Theorem 4.5, we obtain the following comparison between the discrete Green's function and the continuous Green's function.

Lemma 5.3. *Suppose $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Then there exist $h_0 > 0$ and a constant C_* such that for all $0 < h \leq h_0$, if $x \in \Omega_0$ and $z \in \Omega \backslash \Omega_1$, the estimate*

$$|G(x, z) - G_h(x, z)| \leq C_* h^2 |\ln h|$$

holds.

From this result, we obtain a Harnack-type inequality for the discrete Green's function.

Lemma 5.4. *Suppose $\Omega_0 \Subset \Omega$. Let $0 < c_1 < c_2$ be positive constants. Then there exist $h_0 > 0$ and a constant $C > 0$ independent of h such that, for all $0 < h \leq h_0$ and for all $x, y \in \Omega_0$ and all $z \in \Omega$ with $c_1 h \leq dist(z, \partial\Omega) \leq c_2 h$, we have*

$$G_h^z(x) \geq C G_h^z(y).$$

Proof. For h sufficiently small, by the smoothness of the boundary of Ω and the shape regularity and quasi-uniformity of $\{\mathcal{T}_h\}$, we have that there exist constants c_1' and c_2', independent of h, such that $z \in \Omega_h$ and $c_1 h \leq dist(z, \partial\Omega_h) \leq c_2 h$ implies $z \in \Omega$ and $c_1' h \leq dist(z, \partial\Omega) \leq c_2' h$.

Let c_3 be the constant C in Lemma 3.3, and let $h_0 > 0$ be so small that the conclusion of Lemma 5.3 holds, and such that $C_* h^2 |\ln h| \leq c_3 h/2$ for all $0 < h \leq h_0$. Because $G^z(x)$ is a harmonic function in x away from the singularity at z, by the Harnack inequality for harmonic functions there exists $C_0 > 0$ such that $G^z(x) \geq C_0 G^z(y)$ for all $x, y \in \Omega_0$.

Then if $0 < h \leq h_0$, for arbitrary $x \in \Omega_0$ and z satisfying $c_1 h \leq dist(z, \partial\Omega) \leq c_2 h$, by Lemma 5.3 we have

$$-C_* h^2 |\ln h| + G(x, z) \leq G_h(x, z) \leq G(x, z) + C_* h^2 |\ln h|.$$

Because $C_* h^2 |\ln h| \leq c_3 h/2 \leq G(x, z)/2$, we obtain

$$(25) \qquad \qquad \frac{1}{2} G(x, z) \leq G_h(x, z) \leq 2 G(x, z).$$

Using the classical Harnack inequality for the continuous Green's function away from the singularity, we have

$$(26) \qquad G_h(x, z) \geq \frac{1}{2} G(x, z) \geq \frac{C_0}{2} G(y, z) \geq \frac{C_0}{4} G_h(y, z),$$

where C_0 is independent of h sufficiently small. $\qquad \square$

Combining the representation in (24), the assumption that the boundary stiffness matrix satisfies $H \leq 0$, and the Harnack-type inequality for the discrete Green's function in Lemma 5.4, we obtain the nodal Harnack inequality for discrete harmonic functions.

Theorem 5.5. *Suppose $\Omega_0 \Subset \Omega$. Then there exists $h_0 > 0$ and a constant $C > 0$ such that for all $0 < h \leq h_0$, and for all discrete harmonic functions u_h satisfying $u_h(x) \geq 0$ for $x \in \partial\Omega$, and for all nodes $x_*, y_* \in \Omega_0$, we have*

$$(27) \qquad \qquad u_h(x_*) \geq C u_h(y_*).$$

Proof. By the shape regularity and quasi-uniformity of $\{\mathcal{T}_h\}$ and the smoothness of $\partial\Omega$, there exist positive constants c_1 and c_2, independent of x_k, x_j and h for h sufficiently small, so that for all nodes $x_k \in \partial\Omega_h$ and for all nodes $x_j \in \tilde{N}(x_k)$, we have $c_1 h \leq dist(x_j, \partial\Omega) \leq c_2 h$.

Therefore, by Lemma 5.4 and the symmetry of G_h, for all nodes $x_*, y_* \in \Omega_0$, we have

$$u_h(x_*) = \sum_{k=n+1}^{n+m} \sum_{x_j \in \tilde{N}(x_k)} G_h(x_j, x_*) \left(-(\nabla\phi_j, \nabla\phi_k)_{\Omega_h} \right) b(x_k)$$

$$= \sum_{k=n+1}^{n+m} \sum_{x_j \in \tilde{N}(x_k)} G_h(x_*, x_j) \left(-(\nabla\phi_j, \nabla\phi_k)_{\Omega_h} \right) b(x_k)$$

$$\geq C \sum_{k=n+1}^{n+m} \sum_{x_j \in \tilde{N}(x_k)} G_h(y_*, x_j) \left(-(\nabla\phi_j, \nabla\phi_k)_{\Omega_h} \right) b(x_k)$$

$$\geq C \sum_{k=n+1}^{n+m} \sum_{x_j \in \tilde{N}(x_k)} G_h(x_j, y_*) \left(-(\nabla\phi_j, \nabla\phi_k)_{\Omega_h} \right) b(x_k)$$

$$\geq C u_h(y_*).$$

$\square$

As a corollary, by interpolating at nodal points we obtain a Harnack inequality for piecewise linear finite elements valid for all points in Ω.

Theorem 5.6. *Suppose $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Then there exists $h_0 > 0$ and a constant $C > 0$, depending on Ω_0, Ω_1, such that for all $0 < h \leq h_0$ and for all discrete harmonic functions u_h satisfying $u_h(x) \geq 0$ for $x \in \partial\Omega$, and for all $x, y \in \Omega_0$, we have*

$$(28) \qquad\qquad u_h(x) \geq C u_h(y).$$

6. NUMERICAL RESULTS

In this section we provide some numerical examples concerning the positivity of the discrete Green's function. Since we only look at the values of the Green's functions in the interior of the domain, which corresponds to the cases 1 and 2 in the proof of Theorem 4.6, the smoothness of Ω is not required and we restrict our numerical examples to polygonal domains. These examples show that one cannot remove the asymptotic nature of Theorem 4.6, due to the fact that the discrete Green's function obtains negative values in the interior of the domain if the mesh is not sufficiently refined.

We let the domain Ω under consideration be a thin rhombus in the plane with vertices at $(-1,0)$, $(1,0)$, $(0, \tan\frac{\pi}{40})$ and $(0, -\tan\frac{\pi}{40})$, and let $\Omega_0 \Subset \Omega$ be a smaller rhombus with vertices at $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2}\tan\frac{\pi}{40})$, $(0, -\frac{1}{2}\tan\frac{\pi}{40})$. The smaller angle of each rhombus is then $\frac{\pi}{20}$. We triangulate Ω by dividing each side into 2^p segments of equal length, and use these to subdivide Ω into 2^{2p} smaller congruent rhombuses. We then split each of these smaller rhombuses along either the main or smaller diagonal. We will consider four different triangulations constructed in this fashion. In the figures, we depict these meshes for the subdivision of each side of

the original rhombus into eight segments. The first (Mesh 1), depicted in Figure 1 (left), is obtained by dividing all of the rhombuses along their smaller diagonal. Note that this triangulation is a Delaunay triangulation, unlike the other three types of meshes under consideration. The second (Mesh 2), depicted in Figure 1 (right), is obtained by dividing half the layers of rhombuses along each diagonal, with the inner layers along the main diagonal, and the outer layers along the smaller diagonal. The third (Mesh 3), depicted in Figure 2 (left), is obtained by dividing only the outermost layer along the smaller diagonal, but all of the inner layers along the main diagonal. The fourth (Mesh 4), depicted in Figure 2 (right), is obtained by dividing all of the layers along the main diagonal. Note that the small angle of the rhombus is exaggerated for visual clarity.
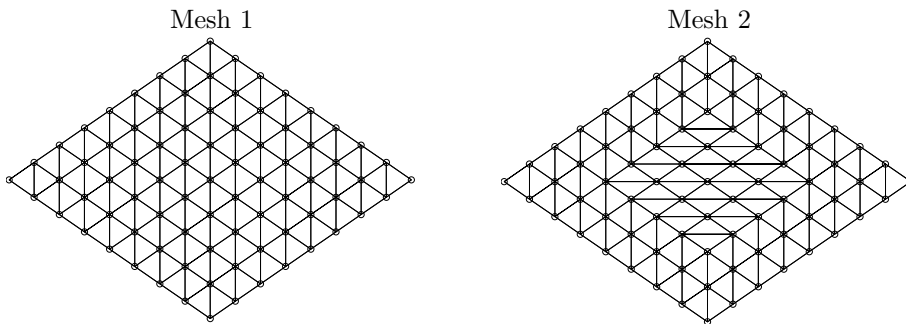

FIGURE 1. Mesh 1 with all layers divided along the smaller diagonal and Mesh 2 with outer layers divided along the smaller diagonal.
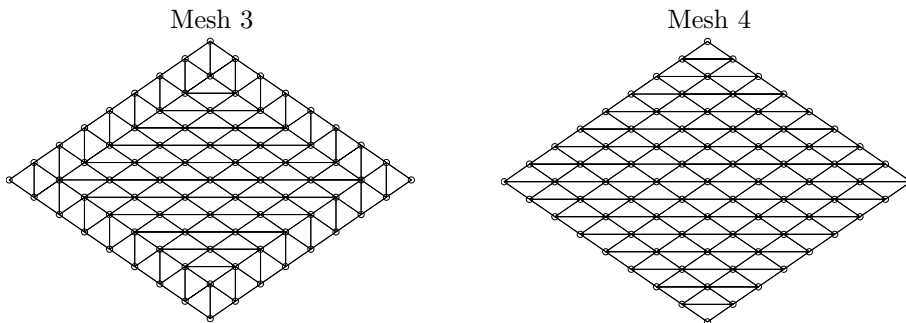

FIGURE 2. Mesh 3 with one outer layer divided along the smaller diagonal and Mesh 4 with all layers divided along the main diagonal.

For computational convenience, we place the singularity of the discrete Green's function at the origin; similar results hold for other placements of the singularity within Ω_0. Table 1 depicts $\min_{x \in \Omega_0} G_h(x, \mathbf{0})$ for the number of nodes placed along each side of Ω. (We have taken the number of nodes along each side to be one more than a power of two, so that the number of segments on each side is a power of two.)

TABLE 1. $\min\limits_{x\in\Omega_0} G_h(x,\mathbf{0})$ for each mesh.

Degrees of freedom	Mesh 1	Mesh 2	Mesh 3	Mesh 4
5^2	3.173e-05	-1.808e-03	-1.808e-03	4.559e-03
9^2	4.084e-06	4.445e-04	-7.326e-03	1.678e-03
17^2	8.626e-07	-2.020e-03	-5.455e-03	-1.642e-03
33^2	4.126e-07	-2.542e-03	-3.940e-03	-2.451e-03
65^2	3.247e-07	-2.407e-03	-2.945e-03	-2.386e-03
129^2	3.043e-07	-1.071e-03	-9.773e-04	-1.067e-03
257^2	2.993e-07	-9.374e-06	1.584e-08	-9.347e-06
513^2	2.980e-07	1.200e-07	1.989e-07	1.201e-07
1025^2	2.977e-07	2.483e-07	2.709e-07	2.483e-07
2049^2	2.976e-07	2.850e-07	2.907e-07	2.850e-07
4097^2	2.976e-07	2.944e-07	2.959e-07	2.944e-07

Because Mesh 1 is Delaunay, the discrete Green's function is nonnegative regardless of the size of h. For the three non-Delaunay meshes, for large h, the discrete Green's function with singularity at $\mathbf{0}$ may assume negative values for $x \in \Omega_0$, but upon refining the mesh, the discrete Green's function with singularity in Ω_0 becomes nonnegative for all $x \in \Omega_0$.

7. Conclusion and open problems

In this paper we have established some sharp pointwise discrete Green's function estimates. In particular, we showed in two space dimensions on a smooth domain that the discrete Green's function on any quasi-uniform shape-regular mesh is non-negative if the singularity is in the interior and the mesh is sufficiently refined. As a consequence of the discrete Green's function estimate, we establish a discrete Harnack inequality for discrete harmonic functions under some rather mild mesh restrictions. There are a number of related open questions that have not yet been addressed. These include, for instance, the qualitative behavior of the discrete Green's function, particularly in dimensions higher than two. Does Theorem 4.6 hold for the discrete Green's function in higher dimensions, or can the discrete Green's function obtain persistent negative values even if the singularity is far from the boundary? Another direction for investigation may include the extension of the Harnack inequality to the inhomogeneous case and to parabolic or more general elliptic equations on nonsmooth domains.

Acknowledgments

References

[1] N. E. Aguilera and L. A. Caffarelli, *Regularity results for discrete solutions of second order elliptic problems in the finite element method*, Calcolo **23** (1986), no. 4, 327–353 (1987), DOI 10.1007/BF02576116. MR945780 (89h:65171)

[2] N. Yu. Bakaev, V. Thomée, and L. B. Wahlbin, *Maximum-norm estimates for resolvents of elliptic finite element operators*, Math. Comp. **72** (2003), no. 244, 1597–1610 (electronic), DOI 10.1090/S0025-5718-02-01488-6. MR1986795 (2004g:65074)

[3] R. F. Bass, *A stability theorem for elliptic Harnack inequalities*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 3, 857–876, DOI 10.4171/JEMS/379. MR3085094

[4] R. F. Bass and M. Gordina, *Harnack inequalities in infinite dimensions*, J. Funct. Anal. **263** (2012), no. 11, 3707–3740, DOI 10.1016/j.jfa.2012.09.009. MR2984082

[5] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954 (2008m:65001)

[6] E. Burman and A. Ern, *Discrete maximum principle for Galerkin approximations of the Laplace operator on arbitrary meshes* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **338** (2004), no. 8, 641–646, DOI 10.1016/j.crma.2004.02.010. MR2056474

[7] M. Cerenzia and L. Saloff-Coste, *Discrete/continuous elliptic Harnack inequality and kernel estimates for functions of the Laplacian on a graph*, New York J. Math. **19** (2013), 487–509. MR3104559

[8] K. L. Chung and Z. X. Zhao, *From Brownian Motion to Schrödinger's Equation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 312, Springer-Verlag, Berlin, 1995. MR1329992 (96f:60140)

[9] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Studies in Mathematics and its Applications, Vol. 4, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. MR0520174 (58 #25001)

[10] P. G. Ciarlet and P.-A. Raviart, *Maximum principle and uniform convergence for the finite element method*, Comput. Methods Appl. Mech. Engrg. **2** (1973), 17–31. MR0375802 (51 #11992)

[11] A. Demlow, D. Leykekhman, A. H. Schatz, and L. B. Wahlbin, *Best approximation property in the W^1_∞ norm for finite element methods on graded meshes*, Math. Comp. **81** (2012), no. 278, 743–764, DOI 10.1090/S0025-5718-2011-02546-9. MR2869035 (2012m:65412)

[12] L. Diening, C. Kreuzer, and S. Schwarzacher, *Convex hull property and maximum principle for finite element minimisers of general convex functionals*, Numer. Math. **124** (2013), no. 4, 685–700, DOI 10.1007/s00211-013-0527-7. MR3073956

[13] A. Drăgănescu, T. F. Dupont, and L. R. Scott, *Failure of the discrete maximum principle for an elliptic finite element problem*, Math. Comp. **74** (2005), no. 249, 1–23 (electronic), DOI 10.1090/S0025-5718-04-01651-5. MR2085400

[14] S. J. Fromm, *Potential space estimates for Green potentials in convex domains*, Proc. Amer. Math. Soc. **119** (1993), no. 1, 225–233, DOI 10.2307/2159846. MR1156467 (93k:35076)

[15] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), no. 3, 303–342, DOI 10.1007/BF01166225. MR657523 (83h:35033)

[16] W. Höhn and H.-D. Mittelmann, *Some remarks on the discrete maximum-principle for finite elements of higher order* (English, with German summary), Computing **27** (1981), no. 2, 145–154, DOI 10.1007/BF02243548. MR632125 (83a:65109)

[17] S. Korotov, M. Křížek, and P. Neittaanmäki, *Weakened acute type condition for tetrahedral triangulations and the discrete maximum principle*, Math. Comp. **70** (2001), no. 233, 107–119 (electronic), DOI 10.1090/S0025-5718-00-01270-9. MR1803125 (2001i:65126)

[18] Ju. P. Krasovskiĭ, *Properties of Green's functions and generalized solutions of elliptic boundary value problems* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 109–137. MR0244611 (39 #5925)

[19] N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 161–175, 239. MR563790 (83c:35059)

[20] H. J. Kuo and N. S. Trudinger, *Linear elliptic difference inequalities with random coefficients*, Math. Comp. **55** (1990), no. 191, 37–53, DOI 10.2307/2008791. MR1023049 (90m:65176)

[21] H. J. Kuo and N. S. Trudinger, *Discrete methods for fully nonlinear elliptic equations*, SIAM J. Numer. Anal. **29** (1992), no. 1, 123–135, DOI 10.1137/0729008. MR1149088 (93e:65129)

[22] G. F. Lawler, *Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments*, Proc. London Math. Soc. (3) **63** (1991), no. 3, 552–568, DOI 10.1112/plms/s3-63.3.552. MR1127149 (93a:60103)

[23] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR0159138 (28 #2356)

[24] J. A. Nitsche and A. H. Schatz, *Interior estimates for Ritz-Galerkin methods*, Math. Comp. **28** (1974), 937–958. MR0373325 (51 #9525)

[25] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **2** (1992), 27–38, DOI 10.1155/S1073792892000047. MR1150597 (93d:58158)

[26] A. H. Schatz, *A weak discrete maximum principle and stability of the finite element method in L_∞ on plane polygonal domains. I*, Math. Comp. **34** (1980), no. 149, 77–91, DOI 10.2307/2006221. MR551291 (81e:65063)

[27] A. H. Schatz and L. B. Wahlbin, *Interior maximum norm estimates for finite element methods*, Math. Comp. **31** (1977), no. 138, 414–442. MR0431753 (55 #4748)

[28] A. H. Schatz and L. B. Wahlbin, *On the quasi-optimality in L_∞ of the $\dot{H}^1$-projection into finite element spaces*, Math. Comp. **38** (1982), no. 157, 1–22, DOI 10.2307/2007461. MR637283 (82m:65106)

[29] A. H. Schatz and L. B. Wahlbin, *Interior maximum-norm estimates for finite element methods. II*, Math. Comp. **64** (1995), no. 211, 907–928, DOI 10.2307/2153476. MR1297478 (95j:65143)

[30] L. B. Wahlbin, *Local behavior in finite element methods*, Handbook of Numerical Analysis, Vol. II, North-Holland, Amsterdam, 1991. MR1115238

[31] Lars B. Wahlbin, *MATH 728. Mathematical Theory of Finite Element Method*, lecture notes, unpublished, 1998.

[32] J. Wang and R. Zhang, *Maximum principles for P1-conforming finite element approximations of quasi-linear second order elliptic equations*, SIAM J. Numer. Anal. **50** (2012), no. 2, 626–642, DOI 10.1137/110833737. MR2914278

[33] Z. X. Zhao, *Green function for Schrödinger operator and conditioned Feynman-Kac gauge*, J. Math. Anal. Appl. **116** (1986), no. 2, 309–334, DOI 10.1016/S0022-247X(86)80001-4. MR842803 (88f:60142)

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269
E-mail address: dmitriy.leykekhman@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269
E-mail address: michael.dennis.pruitt@gmail.com