

A PRIMAL-DUAL WEAK GALERKIN FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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ABSTRACT. This article proposes a new numerical algorithm for second order elliptic equations in non-divergence form. The new method is based on a discrete weak Hessian operator locally constructed by following the weak Galerkin strategy. The numerical solution is characterized as a minimization of a non-negative quadratic functional with constraints that mimic the second order elliptic equation by using the discrete weak Hessian. The resulting Euler-Lagrange equation offers a symmetric finite element scheme involving both the primal and a dual variable known as the Lagrange multiplier, and thus the name of *primal-dual weak Galerkin finite element method*. Error estimates of optimal order are derived for the corresponding finite element approximations in a discrete H^2 -norm, as well as the usual H^1 - and L^2 -norms. The convergence theory is based on the assumption that the solution of the model problem is H^2 -regular, and that the coefficient tensor in the PDE is piecewise continuous and uniformly positive definite in the domain. Some numerical results are presented for smooth and non-smooth coefficients on convex and non-convex domains, which not only confirm the developed convergence theory but also a superconvergence result.

1. INTRODUCTION

This paper is concerned with development of numerical methods for second order elliptic problems in non-divergence form. For simplicity, we consider the model problem that seeks an unknown function $u = u(x)$ satisfying

$$(1.1) \quad \begin{cases} \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded domain in \mathbb{R}^d ($d = 2, 3$) with Lipschitz continuous boundary $\partial\Omega$, $\mathcal{L} := \sum_{i,j=1}^d a_{ij} \partial_{ij}^2$ is the second order partial differential operator with coefficients $a_{ij} \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$ is a given function.

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Assume that the coefficient tensor $a(x) = (a_{ij}(x))_{d \times d}$ is symmetric, uniformly bounded and positive definite. Namely, there exist positive constants α and β such that

$$(1.2) \quad \alpha \xi^T \xi \leq \xi^T a(x) \xi \leq \beta \xi^T \xi \quad \forall \xi \in \mathbb{R}^d, x \in \Omega.$$

If the coefficient tensor $a(x)$ is smooth in the domain Ω , then the operator \mathcal{L} can be written in a divergence form,

$$\mathcal{L}u = \sum_{i,j=1}^d \partial_j (a_{ij} \partial_i u) - \sum_{i,j=1}^d (\partial_j a_{ij}) \partial_i u,$$

so that the existing finite element methods (see [3, 6] for example) can be employed for an accurate approximation of the problem (1.1). In this paper, we assume that the coefficient tensor $a(x) \in L^\infty(\Omega)$ is non-smooth so that a variational formulation using integration by parts is not possible.

Problems in the form of (1.1) arise in many applications from applied areas such as probability and stochastic processes [10]. They also appear in the study of fully non-linear partial differential equations in conjunction with linearization techniques such as Newton's iterative method [2, 18]. In many such applications, the coefficient tensor $a(x)$ is hardly smooth nor even continuous. For example, the coefficient $a(x)$ is merely essentially bounded in the application to Hamilton-Jacobi-Bellman equations [10]. For fully non-linear PDEs discretized by discontinuous finite elements, their linearization involves at most piecewise smooth coefficients. Therefore, it is important and crucial to develop efficient numerical methods for problem (1.1) with rough coefficient tensor.

Several numerical methods were recently designed and studied for PDEs in non-divergence form by using finite element approaches based on ad hoc variational forms. In [14], a Galerkin type method was introduced by using C^0 -conforming finite elements via a *finite element Hessian* computed in the same finite element space without boundary constraints. This finite element scheme was further modified and analyzed for its convergence in [18]. In [9], a non-standard primal finite element method, which uses finite-dimensional subspaces consisting globally continuous piecewise polynomial functions, was proposed and analyzed when the coefficient tensor $a(x)$ is continuous over the domain. The main idea of [9] is to augment a non-symmetric piecewise defined bilinear form by using an interior penalty term for the jump of the flux across the interior element edges/faces. In [21], an hp -version discontinuous Galerkin finite element method was designed and analyzed for a class of such problems that satisfy the Cordès condition. The numerical scheme of [21] was based on a variational form arising from testing the original PDE against the Laplacian of sufficiently smooth functions (e.g., twice differentiable functions in L^2). It was proved in [21] that the method exhibits a convergence rate that is optimal with respect to the meshsize h and suboptimal with respect to the polynomial degree p by half an order. The result in [21] is limited to PDEs satisfying the Cordès condition, but is applicable to the case of discontinuous coefficient tensor $a(x)$. In [20], a two-scale finite element method was presented and analyzed for the model problem (1.1), in which the fine scale is given by the meshsize h whereas the coarse scale is dictated by an integro-differential approximation of the PDE. It was shown that the corresponding numerical solution satisfies the discrete maximum principle

provided that the mesh is weakly acute. Furthermore, a convergence of the numerical solution to the viscosity solution was established under some assumptions, including $a(x) \in C(\Omega)^{d \times d}$, $f \in C(\Omega)$, and the restriction of $\epsilon \geq Ch|\ln(h)|$ for the coarse scale.

The goal of this paper is to develop a new finite element method for the model problem (1.1) by using the weak Galerkin strategy recently introduced in [16, 24–26] for partial differential equations. One of the two basic principles for weak Galerkin is the use of locally constructed differential operators, called *discrete weak differential operators*, in the space of discontinuous functions including necessary boundary information. The discrete weak differential operators form the critical building block in discretization of the underlying PDEs. For the model problem (1.1), the Hessian is the primary differential operator which shall be locally reconstructed by using the weak Galerkin approach. The resulting discrete weak Hessian, denoted by $\{\partial_{ij,d}^2 v\}_{d \times d}$ to be detailed in Sections 3 and 4, is then employed to approximate (1.1) as

$$(1.3) \quad \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 u_h, w) = (f, w), \quad \forall w \in W_{h,k},$$

where $W_{h,k}$ is a test space and u_h is sought from a trial space $V_{h,k}$. The discrete problem (1.3), however, is not well-posed unless an *inf-sup* condition of Babuška [1] and Brezzi [4] is satisfied. To overcome this difficulty, this paper proposes a constraint optimization algorithm which seeks $u_h \in V_{h,k}$ as a minimization of a prescribed non-negative quadratic functional $J(v) = \frac{1}{2}s(v, v)$ with constraint given by equation (1.3). The functional $J(v)$ measures the “continuity” of $v \in V_{h,k}$ in the sense that $v \in V_{h,k}$ is a classical C^1 -conforming element if and only if $s(v, v) = 0$. The weak continuity of the finite element approximation u_h as characterized by the functional $J(v)$ forms the second basic principle of weak Galerkin. The resulting Euler-Lagrange equation for the constraint optimization problem gives rise to a symmetric numerical algorithm involving not only the primal variable u_h but also a dual variable λ_h known as the Lagrange multiplier. This numerical scheme, called *primal-dual weak Galerkin finite element method* (PD-WG), is the main contribution of the present paper.

Our theory for the primal-dual weak Galerkin finite element method is based on the assumption that the solution of (1.1) is H^2 -regular, and that the coefficient tensor $a(x)$ is piecewise continuous and satisfies the uniform ellipticity condition (1.2). Under those assumptions, an optimal order error estimate is derived in a discrete H^2 -norm for the primal variable and in the L^2 -norm for the dual variable. We shall also establish a convergence theory and a superconvergence result for the primal variable in the H^1 - and L^2 -norms under some smoothness assumptions for the coefficient tensor. Numerical experiments are presented to illustrate the accuracy and to confirm the theory developed for the primal-dual weak Galerkin finite element method.

The primal-dual weak Galerkin method has the following advantages over the schemes in [9, 14, 21]: (1) PD-WG offers a symmetric numerical scheme while the other three are all non-symmetric; (2) PD-WG works for finite element partitions consisting of arbitrary polygons or polyhedra, while [9, 14] are limited to those on which C^0 elements are possible; (3) PD-WG covers a wider class of PDE problems than those in [9, 14, 21], as globally continuous coefficient tensor $a(x)$ is required in

[9] and the Cordès condition is essential in [21]. On the other hand, the numerical schemes in [9, 14, 21] involve a fewer number of degrees of freedom than PD-WG in general.

The paper is organized as follows. In Section 2, we present some preliminary results on strong solutions for the model problem (1.1). Section 3 is devoted to a discussion of weak Hessian and its discretizations. In Section 4, we describe the primal-dual weak Galerkin finite element method for the model problem (1.1). Section 5 is devoted to a stability analysis for the new finite element method. In Section 6, we derive an optimal order error estimate for the numerical method in a discrete H^2 -norm for piecewise continuous coefficient tensors. Section 7 continues the error analysis by establishing some error estimates in the usual H^1 - and L^2 -norms for the primal variable under some smoothness assumptions on the coefficient tensor. Finally in Section 8, we conduct some numerical experiments for the model problem (1.1) with smooth and non-smooth coefficients $a(x)$ on convex and non-convex domains.

2. PRELIMINARIES

Let $D \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz continuous boundary. We use the standard definition for the Sobolev space $H^s(D)$ and the associated inner product $(\cdot, \cdot)_{s,D}$, norm $\|\cdot\|_{s,D}$, and seminorm $|\cdot|_{s,D}$ for any $s \geq 0$ [3, 6]. We also use $\langle \cdot, \cdot \rangle_{\partial D}$ to denote the usual inner products in $L^2(\partial D)$. For simplicity, we shall drop the subscript D in the norm and inner product notation when $D = \Omega$. In addition, $\|\cdot\|_{0,D}$ and $\|\cdot\|_{0,\partial D}$ are simplified as $\|\cdot\|_D$ and $\|\cdot\|_{\partial D}$, respectively.

The classical Schauder's theory [11] states that if the coefficient matrix $a = a(x)$ is of $C^{0,\alpha}(\Omega)$ and $\partial\Omega \in C^{2,\alpha}$, then there exists a unique solution $u \in C^{2,\alpha}(\Omega)$ satisfying the model problem (1.1). The Calderón-Zygmund theory states that if $a = a(x)$ is of $C^0(\bar{\Omega})$ and $\partial\Omega \in C^{1,1}$, then there exists a unique solution $u \in W^{2,p}(\Omega)$ satisfying (1.1); see Theorem 9.15 in [11] for details. Furthermore, one has the following a priori estimate:

$$(2.1) \quad \|u\|_{2,p} \leq C\|f\|_{0,p}.$$

Here $p \in (1, \infty)$ is any given real number and $\|\cdot\|_{m,p}$ stands for the standard norm in the Sobolev space $W^{m,p}(\Omega)$ for any $m \geq 0$.

The solution uniqueness may break down when $d \geq 3$ for coefficients $a(x)$ that are not continuous. One such example is given by

$$(2.2) \quad a(x) = I_{d \times d} + \frac{(d + \lambda - 2)xx^T}{(1 - \lambda)|x|^2}.$$

With $\Omega = B_1(0)$, $d > 2(2 - \lambda)$, it can be verified that $u = |x|^\lambda \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the partial differential equation in (1.1) with $f = 0$. For this reason, in the case $a(x)$ is discontinuous, we assume the following Cordès condition is satisfied: There exists an $\varepsilon \in (0, 1]$ such that

$$(2.3) \quad \frac{\sum_{i,j=1}^d a_{ij}^2}{(\sum_{i=1}^d a_{ii})^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{in } \Omega.$$

Theorem 2.1 ([21]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain, and let the differential operator defined in (1.1) satisfy $a \in [L^\infty(\Omega)]^{d \times d}$, the ellipticity condition (1.2), and the Cordès condition (2.3). Then, for any given $f \in L^2(\Omega)$, there exists*

a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ that is a strong solution of (1.1), and this strong solution satisfies

$$(2.4) \quad \|u\|_2 \leq C\|f\|_0,$$

where C is a constant depending only on d , the diameter of Ω , α , β , and ε .

For problems in two dimensions, the uniform ellipticity assumption (1.2) implies the validity of the Cordès condition (2.3); see [21] and the references cited therein. In fact, let λ_1 and λ_2 be the smallest and the largest eigenvalues of $a(x)$. Using the identities

$$(2.5) \quad \sum_{i,j=1}^2 a_{ij}^2 = \lambda_1^2 + \lambda_2^2, \quad \sum_{i=1}^2 a_{ii} = \lambda_1 + \lambda_2,$$

we arrive at

$$\frac{\sum_{i,j=1}^2 a_{ij}^2}{(\sum_{i=1}^2 a_{ii})^2} = \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 + \lambda_2)^2} = \frac{1}{1 + 2\kappa/(1 + \kappa^2)},$$

where $\kappa = \lambda_2/\lambda_1$ is the condition number of the matrix $a(x)$. It follows that the Cordès condition is satisfied with $\varepsilon = 2\kappa/(1 + \kappa^2)$. Note that the uniform ellipticity (1.2) implies $\kappa \in [1, \beta/\alpha]$. Since the minimum value of $2\kappa/(1 + \kappa^2)$ in the interval $[1, \beta/\alpha]$ is given by $\varepsilon_0 = 2\alpha\beta/(\alpha^2 + \beta^2)$, then the Cordès condition (2.3) is satisfied with $\varepsilon = 2\alpha\beta/(\alpha^2 + \beta^2)$ under the condition (1.2) for two-dimensional problems.

Throughout this paper, we assume that the problem (1.1) has a unique strong solution in $H^2(\Omega) \cap H_0^1(\Omega)$ with the a priori estimate

$$(2.6) \quad \|u\|_2 \leq C\|f\|_0,$$

where C is a generic constant which represents different values at different appearances.

Let $X = H^2(\Omega) \cap H_0^1(\Omega)$ and $Y = L^2(\Omega)$. Introduce the following bilinear form in $X \times Y$:

$$(2.7) \quad b(v, \sigma) := (\mathcal{L}v, \sigma), \quad v \in X, \sigma \in Y.$$

Then, the strong solution of the problem (1.1) satisfies the following variational equation: Find $u \in X$ such that

$$(2.8) \quad b(u, w) = (f, w) \quad \forall w \in Y.$$

It follows from the regularity assumption (2.6) that the bilinear form $b(\cdot, \cdot)$ satisfies the following *inf-sup* condition

$$\sup_{v \in X, v \neq 0} \frac{b(v, \sigma)}{\|v\|_X} \geq \Lambda \|\sigma\|_Y$$

for all $\sigma \in Y$, where Λ is a generic constant related to the constant C in the H^2 regularity estimate (2.6). Here $\|\cdot\|_X$ stands for the $H^2(\Omega)$ -norm, and $\|\cdot\|_Y$ is the standard $L^2(\Omega)$ -norm.

Remark 2.1. If the problem (1.1) has the $W^{2,p}$ -regularity (2.1) instead of (2.6), then the variational equation (2.8) still holds true with $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $Y = L^q(\Omega)$, where q is the conjugate of $p \in (1, \infty)$ so that $p^{-1} + q^{-1} = 1$.

3. WEAK HESSIAN AND DISCRETE WEAK HESSIAN

For classical functions, the Hessian is a square matrix of second order partial derivatives if they all exist. Note that Hessian is the primary differential operator in the composition of the second order elliptic problem (1.1) in the non-divergence form. It is therefore necessary to develop numerical techniques targeted at the Hessian operator. The objective of this section is to review the discrete weak Hessian operator introduced in [23].

Let K be a polygonal or polyhedral domain with boundary ∂K . By a weak function on K we mean a triplet $v = \{v_0, v_b, \mathbf{v}_g\}$ such that $v_0 \in L^2(K)$, $v_b \in L^2(\partial K)$ and $\mathbf{v}_g \in [L^2(\partial K)]^d$. The first and second components, namely v_0 and v_b , represent the value of v in the interior and on the boundary of K . The third one, $\mathbf{v}_g = (v_{g1}, \dots, v_{gd}) \in \mathbb{R}^d$, intends to represent the gradient vector ∇v on the boundary of K . Note that v_b and \mathbf{v}_g may or may not be related to the trace of v_0 and ∇v_0 on ∂K . In the case that traces are used (if they exist), the weak function v is uniquely determined by its first component v_0 , and it becomes a classical function. It is also possible to take v_b as the trace of v_0 and leave \mathbf{v}_g completely free or vice versa. Denote by $W(K)$ the space of all weak functions on K :

$$(3.1) \quad W(K) = \{v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in L^2(K), v_b \in L^2(\partial K), \mathbf{v}_g \in [L^2(\partial K)]^d\}.$$

For any $v \in W(K)$, the generalized weak second order partial derivative is defined as a bounded linear functional $\partial_{ij,w}^2 v$ on the Sobolev space $H^2(K)$ so that its action on each $\varphi \in H^2(K)$ is given by

$$(3.2) \quad \langle \partial_{ij,w}^2 v, \varphi \rangle_K := (v_0, \partial_{ji}^2 \varphi)_K - \langle v_b n_i, \partial_j \varphi \rangle_{\partial K} + \langle v_{gi}, \varphi n_j \rangle_{\partial K}.$$

Here, $\mathbf{n} = (n_1, \dots, n_d)$ is the unit outward normal direction on ∂K . The weak Hessian of $v \in W(K)$ is defined as $\nabla_{w,K}^2 v = \{\partial_{ij,w}^2 v\}_{d \times d}$.

Let $S_r(K)$ be a finite-dimensional linear space consisting of polynomials on K . A discrete analogy of $\partial_{ij,w}^2$, denoted by $\partial_{ij,w,r,K}^2$, is defined as the unique polynomial $\partial_{ij,w,r,K}^2 v \in S_r(K)$ such that

$$(3.3) \quad (\partial_{ij,w,r,K}^2 v, \varphi)_K = (v_0, \partial_{ji}^2 \varphi)_K - \langle v_b n_i, \partial_j \varphi \rangle_{\partial K} + \langle v_{gi}, \varphi n_j \rangle_{\partial K}, \quad \forall \varphi \in S_r(K).$$

Analogously, for any $v \in W(K)$, its discrete weak Hessian is given by

$$\nabla_{w,r,K}^2 v = \{\partial_{ij,w,r,K}^2 v\}_{d \times d}.$$

If $v \in W(K)$ has a smooth component $v_0 \in H^2(K)$, then the usual integration by parts can be applied to the first term on the right-hand side of (3.3), yielding

$$(3.4) \quad (\partial_{ij,w,r,K}^2 v, \varphi)_K = (\partial_{ij}^2 v_0, \varphi)_K - \langle (v_b - v_0) n_i, \partial_j \varphi \rangle_{\partial K} + \langle v_{gi} - \partial_i v_0, \varphi n_j \rangle_{\partial K},$$

for all $\varphi \in S_r(K)$.

4. PRIMAL-DUAL WEAK GALERKIN

Let \mathcal{T}_h be a finite element partition of the domain Ω into polygons in 2D or polyhedra in 3D. Denote by \mathcal{E}_h the set of all edges or flat faces in \mathcal{T}_h and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or flat faces. Assume that \mathcal{T}_h satisfies the shape regularity conditions described as in [25]. Denote by h_T the diameter of $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize of the partition \mathcal{T}_h . For any integer $m \geq 0$, denote by $P_m(T)$ the set of all polynomials of total degree m or less.

For any given integer $k \geq 2$, let $W_k(T) \subset W(T)$ be a subspace consisting of (piecewise) polynomials in the following form:

$$(4.1) \quad W_k(T) := \{v = \{v_0, v_b, \mathbf{v}_g\} \in P_k(T) \times P_k(e) \times [P_{k-1}(e)]^d, e \in \partial T \cap \mathcal{E}_h\}.$$

By patching $W_k(T)$ over all $T \in \mathcal{T}_h$ through a common value on the interface \mathcal{E}_h^0 for v_b and \mathbf{v}_g , we arrive at the following weak finite element space

$$W_{h,k} := \{\{v_0, v_b, \mathbf{v}_g\} : \{v_0, v_b, \mathbf{v}_g\}|_T \in W_k(T), T \in \mathcal{T}_h\}.$$

Denote by $W_{h,k}^0$ the subspace of $W_{h,k}$ with vanishing boundary value for v_b on $\partial\Omega$:

$$(4.2) \quad W_{h,k}^0 = \{\{v_0, v_b, \mathbf{v}_g\} \in W_{h,k}, v_b|_e = 0, e \subset \partial\Omega\}.$$

Next, let $S_k(T)$ be a linear space of polynomials satisfying

$$(4.3) \quad P_{k-2}(T) \subseteq S_k(T) \subseteq P_{k-1}(T).$$

Correspondingly, we have the following finite element space

$$(4.4) \quad S_{h,k} = \left\{ \sigma : \sigma|_T \in S_k(T), T \in \mathcal{T}_h \right\}.$$

For simplicity of notation, we denote by $\partial_{ij,d}^2$ the discrete weak second order partial differential operator defined by (3.3) with $S_r(T) = S_k(T)$ on each element T ; i.e.,

$$(\partial_{ij,d}^2 v)|_T = \partial_{ij,w,r,T}^2(v|_T), \quad v \in W_{h,k}.$$

On each element T , we introduce

$$(4.5) \quad b_T(v, \sigma) = \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 v, \sigma)_T,$$

$$(4.6) \quad s_T(u, v) = h_T^{-3} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} + h_T^{-1} \langle \nabla u_0 - \mathbf{u}_g, \nabla v_0 - \mathbf{v}_g \rangle_{\partial T},$$

for $u, v \in W_k(T)$ and $\sigma \in S_k(T)$. Summing up over $T \in \mathcal{T}_h$ gives the following two bilinear forms:

$$(4.7) \quad b_h(v, \sigma) = \sum_{T \in \mathcal{T}_h} b_T(v, \sigma), \quad v \in W_{h,k}, \sigma \in S_{h,k},$$

$$(4.8) \quad s_h(u, v) = \sum_{T \in \mathcal{T}_h} s_T(u, v), \quad u, v \in W_{h,k}.$$

Using the bilinear forms defined in (4.7) and (4.8), the second order elliptic problem (1.1) can be discretized as a constrained optimization problem read as follows: Find $u_h \in W_{h,k}^0$ such that

$$(4.9) \quad u_h = \arg \min_{v \in W_{h,k}^0, b_h(v, \sigma) = (f, \sigma), \forall \sigma \in S_{h,k}} \left(\frac{1}{2} s_h(v, v) \right).$$

The Euler-Lagrange equation for the constrained minimization problem (4.9) gives rise to the following numerical scheme.

Algorithm 4.1 (Primal-dual weak Galerkin FEM). *For a numerical approximation of the second order elliptic problem (1.1) in the non-divergence form, find $(u_h; \lambda_h) \in W_{h,k}^0 \times S_{h,k}$ satisfying*

$$(4.10) \quad s_h(u_h, v) + b_h(v, \lambda_h) = 0, \quad \forall v \in W_{h,k}^0,$$

$$(4.11) \quad b_h(u_h, \sigma) = (f, \sigma), \quad \forall \sigma \in S_{h,k}.$$

Remark 4.1. The primal-dual weak Galerkin finite element scheme (4.10)-(4.11) is symmetric with respect to the unknown variables u_h and λ_h . Note that the primal variable u_h contributes the majority of the degrees of freedom so that the dual variable λ_h does not add much computational complexity to the linear system.

From (4.3), the finite element space $S_k(T)$ for the Lagrange multiplier can be chosen as any linear space between $P_{k-2}(T)$ and $P_{k-1}(T)$. The choice of $S_k(T) = P_{k-2}(T)$ has the least degrees of freedom, but the resulting numerical solution may not be as accurate as the case of $S_k(T) = P_{k-1}(T)$. Numerical results will be presented in Section 8 for a comparison on the approximation accuracies and their order of convergence.

Of interest to us is the special case of the weak finite element space $W_{h,k}$ in which $v_b = v_0|_{\partial T}$ for each weak function $v = \{v_0, v_b, \mathbf{v}_g\}$ on each element $T \in \mathcal{T}_h$. The corresponding finite element method shall be called C^0 -type, and will be numerically investigated in Section 8. Analogously, C^{-1} -type primal-dual WG methods refer to the general case of v_b being totally independent of v_0 . It is clear that C^0 -type finite element schemes involve less number of degrees of freedom than C^{-1} -type, as the boundary component v_b has already been represented by v_0 . However, for C^{-1} -type primal-dual WG schemes, it is possible to devise a hybridized formulation that involves only the degrees of freedom corresponding to the dual of v_b and \mathbf{v}_g . Results on the hybridized primal-dual weak Galerkin will be reported in forthcoming papers.

5. STABILITY AND SOLVABILITY

In this section, we first derive an *inf-sup* condition for the bilinear form $b_h(\cdot, \cdot)$, and then show the existence and uniqueness for the solution of the Algorithm 4.1 defined by the equations (4.10)-(4.11).

For each element T , denote by Q_0 the L^2 -projection onto $P_k(T)$, $k \geq 2$. For each edge or face $e \subset \partial T$, denote by Q_b and $\mathbf{Q}_g = (Q_{g1}, Q_{g2}, \dots, Q_{gd})$ the L^2 -projections onto $P_k(e)$ and $[P_{k-1}(e)]^d$, respectively. For any $w \in H^2(\Omega)$, denote by $Q_h w$ the L^2 -projection onto the weak finite element space $W_{h,k}$ such that on each element T ,

$$(5.1) \quad Q_h w = \{Q_0 w, Q_b w, \mathbf{Q}_g(\nabla w)\}.$$

Next, denote by \mathcal{Q}_h the L^2 -projection onto the space $S_{h,k}$, which is clearly a composition of local L^2 -projections into $S_k(T)$.

Lemma 5.1 ([23]). *The projection operators Q_h and \mathcal{Q}_h satisfy the commutative property*

$$(5.2) \quad \partial_{ij,d}^2(Q_h w) = \mathcal{Q}_h(\partial_{ij}^2 w), \quad i, j = 1, \dots, d,$$

for all $w \in H^2(T)$.

Proof. For any $\varphi \in S_k(T)$ and $w \in H^2(T)$, from (3.3) and the usual integration by parts we have

$$\begin{aligned} (\partial_{ij,d}^2(Q_h w), \varphi)_T &= (Q_0 w, \partial_{ji}^2 \varphi)_T - \langle Q_b w, \partial_j \varphi n_i \rangle_{\partial T} + \langle Q_{gi}(\partial_i w), \varphi n_j \rangle_{\partial T} \\ &= (w, \partial_{ji}^2 \varphi)_T - \langle w, \partial_j \varphi n_i \rangle_{\partial T} + \langle \partial_i w, \varphi n_j \rangle_{\partial T} \\ &= (\partial_{ij}^2 w, \varphi)_T \\ &= (\mathcal{Q}_h \partial_{ij}^2 w, \varphi)_T. \end{aligned}$$

It follows that (5.2) holds true. This completes the proof of the lemma. \square

In the weak finite element space $W_{h,k}$, let us introduce the following seminorm:

$$(5.3) \quad \|v\|_{2,h}^2 = \sum_{T \in \mathcal{T}_h} \left\| \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij}^2 v_0) \right\|_T^2 + s_h(v, v).$$

The following lemma shows that $\|\cdot\|_{2,h}$ is indeed a norm in the subspace $W_{h,k}^0$ when the meshsize h is sufficiently small.

Lemma 5.2. *Assume that the coefficient functions a_{ij} are uniformly piecewise continuous in Ω with respect to the finite element partition \mathcal{T}_h . There exists a fixed $h_0 > 0$ such that if $v = \{v_0, v_b, \mathbf{v}_g\} \in W_{h,k}^0$ satisfies $\|v\|_{2,h} = 0$, then one must have $v \equiv 0$ when $h \leq h_0$.*

Proof. Assume that $v = \{v_0, v_b, \mathbf{v}_g\} \in W_{h,k}^0$ satisfies $\|v\|_{2,h} = 0$. It follows from (5.3) and (4.8) that

$$(5.4) \quad \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij}^2 v_0) = 0, \quad v_0|_{\partial T} = v_b, \quad \nabla v_0|_{\partial T} = \mathbf{v}_g$$

for all $T \in \mathcal{T}_h$. Thus, $v_0 \in C_0^1(\Omega)$ and satisfies

$$(5.5) \quad \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij}^2 v_0) = 0.$$

Hence,

$$(5.6) \quad \begin{aligned} \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 v_0 &= \sum_{i,j=1}^d (I - \mathcal{Q}_h)(a_{ij} \partial_{ij}^2 v_0) \\ &= \sum_{i,j=1}^d (I - \mathcal{Q}_h)((a_{ij} - \bar{a}_{ij}) \partial_{ij}^2 v_0) =: F, \end{aligned}$$

where \bar{a}_{ij} is the average of a_{ij} on $T \in \mathcal{T}_h$. Using the H^2 -regularity assumption (2.6), there exists a constant C such that

$$(5.7) \quad \|v_0\|_2 \leq C \|F\|_0.$$

Note that a_{ij} is uniformly piecewise continuous in Ω with respect to \mathcal{T}_h . Thus, for any $\varepsilon > 0$, there exists a $h_0 > 0$ such that $\|a_{ij} - \bar{a}_{ij}\|_{L^\infty} \leq \varepsilon$. Using the stability of the L^2 -projection \mathcal{Q}_h , we arrive at

$$\|F\|_0 \leq C\varepsilon \|v_0\|_2.$$

Substituting the above into (5.7) yields

$$(5.8) \quad \|v_0\|_2 \leq C\varepsilon \|v_0\|_2.$$

This implies that $v_0 = 0$ if ε is so small that it satisfies $C\varepsilon < 1$, which can be easily achieved by adjusting the parameter h_0 . \square

For convenience, in the weak finite element space $W_{h,k}$, we introduce another seminorm

$$(5.9) \quad \|v\|_2^2 = \sum_{T \in \mathcal{T}_h} \left\| \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij,d}^2 v) \right\|_T^2 + s_h(v, v).$$

Observe that the only difference between $\|v\|_{2,h}$ and $\|v\|_2$ lies in the first term of (5.3) and (5.9) where the strong second order partial derivatives are replaced by the discrete weak second order partial derivatives. The following lemma shows that they are indeed equivalent.

Lemma 5.3. *Assume that the coefficient functions a_{ij} are uniformly piecewise continuous in Ω with respect to the finite element partition \mathcal{T}_h . There exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that*

$$(5.10) \quad \alpha_1 \|v\|_{2,h} \leq \|v\|_2 \leq \alpha_2 \|v\|_{2,h}$$

for all $v \in W_{h,k}$.

Proof. Note that, for any $\phi \in S_k(T)$, we have

$$(\mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v), \phi)_T = (\partial_{ij,d}^2 v, \mathcal{Q}_h(a_{ij}\phi))_T.$$

With $\varphi = \mathcal{Q}_h(a_{ij}\phi)$, we have

$$(\partial_{ij}^2 v_0, \varphi)_T = (\partial_{ij}^2 v_0, \mathcal{Q}_h(a_{ij}\phi))_T = (\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0), \phi)_T.$$

Thus, using (3.4) we arrive at

$$(5.11) \quad \begin{aligned} (\mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v), \phi)_T &= (\partial_{ij,d}^2 v, a_{ij}\phi)_T = (\partial_{ij,d}^2 v, \varphi)_T \\ &= (\partial_{ij}^2 v_0, \varphi)_T - \langle (v_b - v_0)n_i, \partial_j \varphi \rangle_{\partial T} \\ &\quad + \langle v_{gi} - \partial_i v_0, n_j \varphi \rangle_{\partial T} \\ &= (\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0), \phi)_T - \langle (v_b - v_0)n_i, \partial_j \varphi \rangle_{\partial T} \\ &\quad + \langle v_{gi} - \partial_i v_0, n_j \varphi \rangle_{\partial T}. \end{aligned}$$

It now follows from the Cauchy-Schwarz and the trace inequality (6.7) that

$$(5.12) \quad \begin{aligned} |(\mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v), \phi)_T| &\leq \|\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0)\|_T \|\phi\|_T + \|v_b - v_0\|_{\partial T} \|\partial_j \varphi\|_{\partial T} \\ &\quad + \|v_{gi} - \partial_i v_0\|_{\partial T} \|\varphi\|_{\partial T} \\ &\leq \|\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0)\|_T \|\phi\|_T + Ch_T^{-\frac{3}{2}} \|v_b - v_0\|_{\partial T} \|\varphi\|_T \\ &\quad + Ch_T^{-\frac{1}{2}} \|v_{gi} - \partial_i v_0\|_{\partial T} \|\varphi\|_T. \end{aligned}$$

It is easy to see that $\|\varphi\|_T \leq C\|\phi\|_T$. Thus, by choosing $\phi = \mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v)$ in (5.12) we obtain

$$\|\mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v)\|_T^2 \leq C (\|\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0)\|_T^2 + h_T^{-3} \|v_b - v_0\|_{\partial T}^2 + h_T^{-1} \|v_{gi} - \partial_i v_0\|_{\partial T}^2),$$

which, after summing over all $T \in \mathcal{T}_h$, gives the upper-bound estimate of $\|v\|_2$ in (5.10). The lower-bound estimate of $\|v\|_2$ can be established in a similar manner by representing $(\mathcal{Q}_h(a_{ij}\partial_{ij}^2 v_0), \phi)_T$ in terms of $(\mathcal{Q}_h(a_{ij}\partial_{ij,d}^2 v), \phi)_T$ and the other two boundary integrals in (5.11). This completes the proof of the lemma. \square

Lemma 5.4 (inf-sup condition). *Assume that the coefficient matrix $a = \{a_{ij}\}_{d \times d}$ is uniformly piecewise continuous in Ω with respect to the finite element partition \mathcal{T}_h . For any $\sigma \in S_{h,k}$, there exists $v_\sigma \in W_{h,k}^0$ satisfying*

$$(5.13) \quad b_h(v_\sigma, \sigma) \geq \frac{1}{2} \|\sigma\|_0^2,$$

$$(5.14) \quad \|v_\sigma\|_{2,h}^2 \leq C \|\sigma\|_0^2,$$

provided that the meshsize $h < h_0$ for a sufficiently small, but fixed parameter $h_0 > 0$.

Proof. Consider the following second order elliptic problem:

$$(5.15) \quad \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 w = \sigma, \quad \text{in } \Omega,$$

$$(5.16) \quad w = 0, \quad \text{on } \partial\Omega.$$

By the H^2 -regularity assumption (2.6), the problem (5.15)-(5.16) has a unique solution in $H^2(\Omega)$ satisfying

$$(5.17) \quad \|w\|_2 \leq C \|\sigma\|_0.$$

We claim that $v_\sigma = Q_h w$ satisfies (5.13)-(5.14). In fact, by setting $v = v_\sigma = Q_h w$ in $b_h(v, \sigma)$, we have from the commutative property (5.2), the equation (5.15), and the a priori estimate (5.17) that

$$\begin{aligned} b_h(v_\sigma, \sigma) &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij,d}^2 Q_h w, \sigma \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \mathcal{Q}_h \partial_{ij}^2 w, \sigma \right)_T \\ (5.18) \quad &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij}^2 w, \sigma \right)_T + \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} (\mathcal{Q}_h - I) \partial_{ij}^2 w, \sigma \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \|\sigma\|_T^2 + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d ((\mathcal{Q}_h - I) \partial_{ij}^2 w, (a_{ij} - \bar{a}_{ij}) \sigma)_T \\ &\geq \|\sigma\|_0^2 - \varepsilon(h) \|w\|_2 \|\sigma\|_0 \\ &\geq (1 - C\varepsilon(h)) \|\sigma\|_0^2, \end{aligned}$$

where $\varepsilon(h)$ is given by $\|a_{ij} - \bar{a}_{ij}\|_{L^\infty(\Omega)}$. Since a_{ij} is uniformly piecewise continuous, there exists a small, but fixed h_0 , such that $1 - C\varepsilon(h) \geq \frac{1}{2}$ when $h < h_0$. It follows that

$$b_h(v_\sigma, \sigma) \geq \frac{1}{2} \|\sigma\|_0^2,$$

which verifies the inequality (5.13).

Next, for the same $v_\sigma = Q_h w$, from the commutative property (5.2) and the stability of the L^2 -projection \mathcal{Q}_h , we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \|\mathcal{Q}_h \left(\sum_{i,j=1}^d a_{ij} \partial_{ij,d}^2 v_\sigma \right)\|_T^2 &\leq C \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \|a_{ij} \partial_{ij,d}^2 Q_h w\|_T^2 \\ (5.19) \quad &= C \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \|a_{ij} \mathcal{Q}_h \partial_{ij}^2 w\|_T^2 \\ &\leq C \|w\|_2^2 \leq C \|\sigma\|_0^2. \end{aligned}$$

For $v = Q_h w$, by the trace inequality (6.6) and (5.17), the estimate (6.8) with $m = 1$, we have

$$\begin{aligned}
 \sum_{T \in \mathcal{T}_h} h_T^{-3} \|v_0 - v_b\|_{\partial T}^2 &= \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 w - Q_b w\|_{\partial T}^2 \\
 &\leq \sum_{T \in \mathcal{T}_h} h_T^{-3} \|Q_0 w - w\|_{\partial T}^2 \\
 (5.20) \quad &\leq C \sum_{T \in \mathcal{T}_h} h_T^{-4} (\|Q_0 w - w\|_T^2 + h_T^2 \|\nabla Q_0 w - \nabla w\|_T^2) \\
 &\leq C \|w\|_2^2 \leq C \|\sigma\|_0^2.
 \end{aligned}$$

A similar argument can be applied to yield the following estimate:

$$(5.21) \quad \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla v_0 - \mathbf{v}_g\|_{\partial T}^2 \leq C \|\sigma\|_0^2.$$

Now combining (5.19) with (5.20) and (5.21) gives $\|v\|_2^2 \leq C \|\sigma\|_0^2$, and hence from (5.10) we obtain

$$\|v_\sigma\|_{2,h}^2 \leq C \|\sigma\|_0^2,$$

which, together with (5.18), completes the proof of the lemma. \square

Lemma 5.5 (Boundedness). *The following inequalities hold true:*

$$(5.22) \quad |s_h(u, v)| \leq \|u\|_{2,h} \|v\|_{2,h}, \quad \forall u, v \in W_{h,k}^0,$$

$$(5.23) \quad |b_h(v, \sigma)| \leq C \|v\|_{2,h} \|\sigma\|_0, \quad \forall v \in W_{h,k}^0, \sigma \in S_{h,k}.$$

Proof. To derive (5.22), we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
 |s_h(u, v)| &= \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} + h_T^{-1} \langle \nabla u_0 - \mathbf{u}_g, \nabla v_0 - \mathbf{v}_g \rangle_{\partial T} \right| \\
 &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|u_0 - u_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
 &\quad + \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla u_0 - \mathbf{u}_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla v_0 - \mathbf{v}_g\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
 &\leq \|u\|_{2,h} \|v\|_{2,h}.
 \end{aligned}$$

As to (5.23), by the definition of \mathcal{Q}_h and the Cauchy-Schwarz inequality, for any $v \in W_{h,k}^0$ and $\sigma \in S_h$, we have

$$\begin{aligned}
 b_h(v, \sigma) &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij,d}^2 v, \sigma \right)_T \\
 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\mathcal{Q}_h(a_{ij} \partial_{ij,d}^2 v), \sigma)_T \\
 &\leq \left(\sum_{T \in \mathcal{T}_h} \left\| \sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij,d}^2 v) \right\|_T^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\sigma\|_T^2 \right)^{\frac{1}{2}} \\
 &\leq \|v\|_2 \|\sigma\|_0.
 \end{aligned}$$

This, along with (5.10), completes the proof. \square

Introduce the following subspace of $W_{h,k}^0$:

$$Z_h = \{v \in W_{h,k}^0 : b_h(v, \sigma) = 0, \forall \sigma \in S_{h,k}\}.$$

Lemma 5.6 (Coercivity). *There exists a constant $\alpha > 0$ such that*

$$(5.24) \quad s_h(v, v) \geq \alpha \|v\|_{2,h}^2, \quad \forall v \in Z_h.$$

Proof. Given any $v \in Z_h$, we have $b_h(v, \sigma) = 0$ for all $\sigma \in S_{h,k}$. Using (4.7) and (4.5) we obtain

$$\begin{aligned} 0 &= b_h(v, \sigma) \\ &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij,d}^2 v, \sigma \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij,d}^2 v), \sigma \right)_T \end{aligned}$$

for all $\sigma \in S_{h,k}$. Thus, on each element $T \in \mathcal{T}_h$ we have

$$\sum_{i,j=1}^d \mathcal{Q}_h(a_{ij} \partial_{ij,d}^2 v) = 0.$$

It follows that $\|v\|_2^2 = s_h(v, v)$, which, together with (5.10), implies the desired coercivity (5.24) for some $\alpha > 0$. \square

Using the abstract theory for saddle-point problems developed by Babuška [1] and Brezzi [4], we arrive at the following result.

Theorem 5.7. *Assume that the coefficient functions a_{ij} are uniformly piecewise continuous in Ω with respect to the finite element partition \mathcal{T}_h . The primal-dual weak Galerkin finite element scheme (4.10)-(4.11) has a unique solution $(u_h; \lambda_h) \in W_{h,k}^0 \times S_{h,k}$, provided that the meshsize $h < h_0$ holds true for a sufficiently small, but fixed parameter value $h_0 > 0$. Moreover, there exists a constant C such that the solution u_h and λ_h satisfies*

$$\|u_h\|_{2,h} + \|\lambda_h\|_0 \leq C \|f\|_0.$$

6. ERROR ESTIMATES

Let $(u_h; \lambda_h) \in W_{h,k}^0 \times S_{h,k}$ be the approximate solution of problem (1.1) arising from the primal-dual weak Galerkin finite element scheme (4.10)-(4.11). Note that $\lambda = 0$ is the solution of the trivial dual problem of $b(v, \lambda) = 0$ for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Define the error functions by

$$(6.1) \quad e_h = u_h - Q_h u, \quad \gamma_h = \lambda_h - \mathcal{Q}_h \lambda,$$

where Q_h and \mathcal{Q}_h are the corresponding L^2 -projection operators.

Lemma 6.1. *The error functions e_h and γ_h given by (6.1) satisfy the equations*

$$(6.2) \quad s_h(e_h, v) + b_h(v, \gamma_h) = -s_h(Q_h u, v), \quad \forall v \in W_{h,k}^0,$$

$$(6.3) \quad b_h(e_h, \sigma) = \ell_u(\sigma), \quad \forall \sigma \in S_{h,k},$$

where

$$(6.4) \quad \ell_u(\sigma) = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d ((I - \mathcal{Q}_h) \partial_{ij}^2 u, a_{ij} \sigma)_T.$$

Proof. First, by subtracting $s_h(Q_h u, v)$ from both sides of (4.10) we obtain

$$s_h(u_h - Q_h u, v) + b_h(v, \lambda_h) = -s_h(Q_h u, v), \quad \forall v \in W_{h,k}^0.$$

It follows from $\lambda = 0$ that $\gamma_h = \lambda_h$. Thus, the above equation can be rewritten as

$$(6.5) \quad s_h(e_h, v) + b_h(v, \gamma_h) = -s_h(Q_h u, v), \quad \forall v \in W_{h,k}^0,$$

which is the first error equation (6.2).

To derive (6.3), we use (1.1) and (5.2) in Lemma 5.1 to obtain

$$\begin{aligned} b_h(Q_h u, \sigma) &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij,d}^2 Q_h u, \sigma \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \mathcal{Q}_h \partial_{ij}^2 u, \sigma \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} \partial_{ij}^2 u, \sigma \right)_T + \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d a_{ij} (\mathcal{Q}_h - I) \partial_{ij}^2 u, \sigma \right)_T \\ &= (f, \sigma) + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d ((\mathcal{Q}_h - I) \partial_{ij}^2 u, a_{ij} \sigma)_T, \end{aligned}$$

for all $\sigma \in S_{h,k}$. Now subtracting the above equation from (4.11) yields the desired equation (6.3). This completes the proof of the lemma. \square

The equations (6.2) and (6.3) are called *error equations* for the primal-dual WG finite element scheme (4.10)-(4.11). This is a saddle point system for which Brezzi's Theorem [4] can be applied for a stability analysis.

Recall that \mathcal{T}_h is a shape-regular finite element partition of the domain Ω . For any $T \in \mathcal{T}_h$ and $\varphi \in H^1(T)$, the following trace inequality holds true [25]:

$$(6.6) \quad \|\varphi\|_{\partial T}^2 \leq C(h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2).$$

If φ is a polynomial on the element $T \in \mathcal{T}_h$, then from the inverse inequality (see also [25]) we have

$$(6.7) \quad \|\varphi\|_{\partial T}^2 \leq C h_T^{-1} \|\varphi\|_T^2.$$

The following estimates for the L^2 -projections are extremely useful in the forthcoming error analysis.

Lemma 6.2 ([25]). *Let \mathcal{T}_h be a finite element partition of Ω satisfying the shape regularity assumptions given in [25]. Then, for any $0 \leq s \leq 2$ and $1 \leq m \leq k$, one*

has

$$(6.8) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|u - Q_0 u\|_{s,T}^2 \leq Ch^{2(m+1)} \|u\|_{m+1}^2,$$

$$(6.9) \quad \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h_T^{2s} \|u - Q_h u\|_{s,T}^2 \leq Ch^{2(m-1)} \|u\|_{m-1}^2,$$

$$(6.10) \quad \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h_T^{2s} \|\partial_{ij}^2 u - Q_h \partial_{ij}^2 u\|_{s,T}^2 \leq Ch^{2(m-1)} \|u\|_{m+1}^2.$$

Theorem 6.3. Assume that the coefficient functions a_{ij} are uniformly piecewise continuous in Ω with respect to the finite element partition \mathcal{T}_h . Let u and $(u_h; \lambda_h) \in W_{h,k}^0 \times S_{h,k}$ be the solutions of (1.1) and (4.10)-(4.11), respectively. Assume that the exact solution u of (1.1) is sufficiently regular such that $u \in H^{k+1}(\Omega)$. There exists a constant C such that

$$(6.11) \quad \|u_h - Q_h u\|_{2,h} + \|\lambda_h - Q_h \lambda\|_0 \leq Ch^{k-1} \|u\|_{k+1},$$

provided that the meshsize $h < h_0$ holds true for a sufficiently small, but fixed $h_0 > 0$.

Proof. It follows from Lemma 5.4, Lemma 5.5, and Lemma 5.6 that Brezzi's stability conditions are satisfied for the saddle-point system (6.2)-(6.3). Thus, there exists a constant C such that

$$(6.12) \quad \|e_h\|_{2,h} + \|\gamma_h\|_0 \leq C \left(\sup_{v \in W_{h,k}^0, v \neq 0} \frac{|s_h(Q_h u, v)|}{\|v\|_{2,h}} + \sup_{\sigma \in S_h, \sigma \neq 0} \frac{|\ell_u(\sigma)|}{\|\sigma\|_0} \right).$$

Recall that

$$(6.13) \quad \begin{aligned} s_h(Q_h u, v) &= \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, v_0 - v_b \rangle_{\partial T} \\ &+ \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \nabla Q_0 u - \mathbf{Q}_g(\nabla u), \nabla v_0 - \mathbf{v}_g \rangle_{\partial T}. \end{aligned}$$

The first term on the right-hand side of (6.13) can be estimated by using the Cauchy-Schwarz inequality, the trace inequality (6.6), and the estimate (6.8) with $m = k$ as follows:

$$(6.14) \quad \begin{aligned} &\left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - Q_b u, v_0 - v_b \rangle_{\partial T} \right| \\ &= \left| \sum_{T \in \mathcal{T}_h} h_T^{-3} \langle Q_0 u - u, v_0 - v_b \rangle_{\partial T} \right| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|u - Q_0 u\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-3} \|v_0 - v_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{-4} (\|u - Q_0 u\|_T^2 + h_T^2 \|u - Q_0 u\|_{1,T}^2) \right)^{\frac{1}{2}} \|v\|_{2,h} \\ &\leq Ch^{k-1} \|u\|_{k+1} \|v\|_{2,h}. \end{aligned}$$

Similarly, the second term on the right-hand side of (6.13) has the following estimate

$$(6.15) \quad \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \nabla Q_0 u - \mathbf{Q}_g(\nabla u), \nabla v_0 - \mathbf{v}_g \rangle_{\partial T} \right| \leq Ch^{k-1} \|u\|_{k+1} \|v\|_{2,h}.$$

Combining (6.13) with (6.14) and (6.15) gives

$$(6.16) \quad |s_h(Q_h u, v)| \leq Ch^{k-1} \|u\|_{k+1} \|v\|_{2,h}.$$

As to the second term on the right-hand side of (6.12), using (6.4) and the estimate (6.10) with $m = k$ we have

$$(6.17) \quad \begin{aligned} |\ell_u(\sigma)| &= \left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (I - \mathcal{Q}_h) \partial_{ij}^2 u, a_{ij} \sigma \right|_T \\ &\leq \sum_{i,j=1}^d \|a_{ij}\|_{L^\infty} \|(I - \mathcal{Q}_h) \partial_{ij}^2 u\|_0 \|\sigma\|_0 \\ &\leq Ch^{k-1} \|u\|_{k+1} \|\sigma\|_0. \end{aligned}$$

Substituting (6.16) and (6.17) into (6.12) gives the error estimate (6.11). \square

7. ERROR ESTIMATES IN H^1 AND L^2

We first establish an estimate for the discrete weak second order partial derivatives.

Lemma 7.1. *There exists a constant C such that for any $v \in W_k(T)$, we have*

$$(7.1) \quad \|\partial_{ij,d}^2 v\|_T^2 \leq C (\|\partial_{ij}^2 v_0\|_T^2 + s_T(v, v)),$$

where C is a generic constant independent of $T \in \mathcal{T}_h$.

Proof. From (3.4), for any $\varphi \in S_k(T)$, we have

$$(\partial_{ij,d}^2 v, \varphi)_T = (\partial_{ij}^2 v_0, \varphi)_T - \langle v_b - v_0, \partial_j \varphi n_i \rangle_{\partial T} + \langle v_{gi} - \partial_i v_0, \varphi n_j \rangle_{\partial T}.$$

Using the Cauchy-Schwarz inequality, the trace inequality (6.7), and the inverse inequality we arrive at

$$\begin{aligned} |(\partial_{ij,d}^2 v, \varphi)_T| &\leq \|\partial_{ij}^2 v_0\|_T \|\varphi\|_T + \|v_b - v_0\|_{\partial T} \|\partial_j \varphi\|_{\partial T} + \|v_{gi} - \partial_i v_0\|_{\partial T} \|\varphi\|_{\partial T} \\ &\leq \left(\|\partial_{ij}^2 v_0\|_T + Ch_T^{-\frac{3}{2}} \|v_b - v_0\|_{\partial T} + Ch_T^{-\frac{1}{2}} \|v_{gi} - \partial_i v_0\|_{\partial T} \right) \|\varphi\|_T. \end{aligned}$$

Thus,

$$\|\partial_{ij,d}^2 v\|_T^2 \leq C (\|\partial_{ij}^2 v_0\|_T^2 + h_T^{-3} \|v_b - v_0\|_{\partial T}^2 + h_T^{-1} \|v_{gi} - \partial_i v_0\|_{\partial T}^2),$$

which verifies the inequality (7.1). This completes the proof of the lemma. \square

Consider the problem of solving an unknown function w such that

$$(7.2) \quad \sum_{i,j=1}^d \partial_{ji}^2 (a_{ij} w) = \theta, \quad \text{in } \Omega,$$

$$(7.3) \quad w = 0, \quad \text{on } \partial\Omega,$$

where θ is a given function. With the bilinear form $b(\cdot, \cdot)$ given by (2.7), a variational formulation for (7.2)-(7.3) reads as follows: Find $w \in L^2(\Omega)$ such that

$$(7.4) \quad b(v, w) = (\theta, v) \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

The problem (7.2)-(7.3) is said to be H^{1+s} -regular, $s \in [0, 1]$, if for any $\theta \in H^{s-1}(\Omega)$, there exists a unique $w \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ satisfying (7.4) and the following a priori estimate:

$$(7.5) \quad \|w\|_{1+s} \leq C \|\theta\|_{s-1}.$$

Lemma 7.2. *Assume that the coefficients a_{ij} are in $C^1(\Omega)$. Then, for any $v = \{v_0, v_b, \mathbf{v}_g\} \in W_{h,k}^0$, the following identity holds true:*

$$(7.6) \quad \begin{aligned} (v_0, \theta) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 v, w)_T - \langle (v_{gi} - \partial_i v_0) n_j, (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \\ &\quad + \langle (v_b - v_0) n_i, \partial_j (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T}. \end{aligned}$$

Proof. By testing (7.2) with v_0 on each element $T \in \mathcal{T}_h$, we obtain from the usual integration by parts,

$$(7.7) \quad \begin{aligned} (\theta, v_0) &= \sum_{T \in \mathcal{T}_h} \left(\sum_{i,j=1}^d \partial_{ij}^2 (a_{ij} w), v_0 \right)_T \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} w, \partial_{ij}^2 v_0)_T - \langle a_{ij} w n_j, \partial_i v_0 \rangle_{\partial T} + \langle \partial_j (a_{ij} w), v_0 n_i \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} w, \partial_{ij}^2 v_0)_T - \langle a_{ij} w n_j, \partial_i v_0 - v_{gi} \rangle_{\partial T} \\ &\quad + \langle \partial_j (a_{ij} w) n_i, v_0 - v_b \rangle_{\partial T}, \end{aligned}$$

where we have used the homogeneous boundary condition (7.3) in the third line and the fact that $a_{ij} \in C^1(\Omega)$ and $v_b = 0$ on $\partial\Omega$ in the fourth line.

From (3.4) with $\varphi = \mathcal{Q}_h(a_{ij} w)$, we have

$$\begin{aligned} (\partial_{ij,d}^2 v, \mathcal{Q}_h(a_{ij} w))_T &= (\partial_{ij}^2 v_0, \mathcal{Q}_h(a_{ij} w))_T - \langle v_b - v_0, n_i \partial_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T} \\ &\quad + \langle v_{gi} - \partial_i v_0, n_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T} \\ &= (\partial_{ij}^2 v_0, a_{ij} w)_T - \langle v_b - v_0, n_i \partial_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T} \\ &\quad + \langle v_{gi} - \partial_i v_0, n_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T}, \end{aligned}$$

which leads to

$$(7.8) \quad \begin{aligned} (\partial_{ij}^2 v_0, a_{ij} w)_T &= (\partial_{ij,d}^2 v, \mathcal{Q}_h(a_{ij} w))_T \\ &\quad - \langle v_{gi} - \partial_i v_0, n_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T} + \langle v_b - v_0, n_i \partial_j \mathcal{Q}_h(a_{ij} w) \rangle_{\partial T}. \end{aligned}$$

Using (7.8), we can rewrite (7.7) as

$$\begin{aligned}
 (v_0, \theta) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (\partial_{ij,d}^2 v, \mathcal{Q}_h(a_{ij}w))_T - \langle v_{gi} - \partial_i v_0, n_j \mathcal{Q}_h(a_{ij}w) \rangle_{\partial T} \\
 &\quad + \langle v_b - v_0, n_i \partial_j \mathcal{Q}_h(a_{ij}w) \rangle_{\partial T} - \langle a_{ij}w n_j, \partial_i v_0 - v_{gi} \rangle_{\partial T} \\
 &\quad + \langle n_i \partial_j(a_{ij}w), v_0 - v_b \rangle_{\partial T} \\
 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 v, w)_T - \langle (v_{gi} - \partial_i v_0) n_j, (\mathcal{Q}_h - I)(a_{ij}w) \rangle_{\partial T} \\
 &\quad + \langle (v_b - v_0) n_i, \partial_j (\mathcal{Q}_h - I)(a_{ij}w) \rangle_{\partial T},
 \end{aligned} \tag{7.9}$$

which is the desired identity (7.6). \square

The following lemma is developed for an estimate of the last two terms on the right-hand side of (7.6) with the H^1 -regularity assumption for the dual problem (7.4).

Lemma 7.3. *Assume that the coefficient matrix $\{a_{ij}\}_{d \times d}$ is regular so that $a_{ij} \in \Pi_{T \in \mathcal{T}_h} W^{1,\infty}(T)$. Then, there exists a constant C such that for any $v \in W_{h,k}^0$, we have*

$$(7.10) \quad \left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle (v_{gi} - \partial_i v_0) n_j, (\mathcal{Q}_h - I)(a_{ij}w) \rangle_{\partial T} \right| \leq Ch \|v\|_{2,h} \|\theta\|_{-1},$$

provided that the dual problem (7.4) has the H^1 -regularity estimate (7.5) with $s = 0$.

Proof. From the Cauchy-Schwarz inequality, the trace inequality (6.6), and the estimates in Lemma 6.2 we have

$$\begin{aligned}
 &\left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle (v_{gi} - \partial_i v_0) n_j, (\mathcal{Q}_h - I)(a_{ij}w) \rangle_{\partial T} \right| \\
 &\leq \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \|v_{gi} - \partial_i v_0\|_{\partial T} \|(\mathcal{Q}_h - I)(a_{ij}w)\|_{\partial T} \\
 &\leq C \left(\sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d h_T \|(\mathcal{Q}_h - I)(a_{ij}w)\|_{\partial T}^2 \right)^{\frac{1}{2}} \|v\|_{2,h} \\
 &\leq Ch \|w\|_1 \|v\|_{2,h} \leq Ch \|\theta\|_{-1} \|v\|_{2,h},
 \end{aligned} \tag{7.11}$$

where we have used the H^1 -regularity assumption in the last line. This completes the proof of the lemma. \square

Note that if $P_1(T) \subseteq S_k(T)$ for all $T \in \mathcal{T}_h$ and $a_{ij} \in \Pi_{T \in \mathcal{T}_h} W^{2,\infty}(T)$, then from the trace inequality (6.6) and the standard error estimate for the L^2 -projection \mathcal{Q}_h we have

$$\begin{aligned}
 \|(\mathcal{Q}_h - I)(a_{ij}w)\|_{\partial T}^2 &\leq Ch_T^{-1} (\|(\mathcal{Q}_h - I)(a_{ij}w)\|_T^2 + h_T^2 \|(\mathcal{Q}_h - I)(a_{ij}w)\|_{1,T}^2) \\
 &\leq Ch_T^3 \|a_{ij}\|_{2,\infty,T}^2 \|w\|_{2,T}^2.
 \end{aligned} \tag{7.12}$$

By substituting the above inequality into the third line of (7.11) and then assuming the H^2 -regularity (7.5) we obtain the following result.

Lemma 7.4. *Assume that the coefficients a_{ij} are sufficiently smooth on each element such that $a_{ij} \in \Pi_{T \in \mathcal{T}_h} W^{2,\infty}(T)$. In addition, assume $P_1(T) \subset S_k(T)$ for each element $T \in \mathcal{T}_h$. Then, there exists a constant C such that for any $v \in W_{h,k}^0$, we have*

$$(7.13) \quad \left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle (v_{gi} - \partial_i v_0) n_j, (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \right| \leq Ch^2 \|v\|_{2,h} \|\theta\|_0,$$

$$(7.14) \quad \left| \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d \langle (v_b - v_0) n_i, \partial_j (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \right| \leq Ch^2 \|v\|_{2,h} \|\theta\|_0,$$

provided that the regularity estimate (7.5) holds true with $s = 1$.

Theorem 7.5. *Let $u_h = \{u_0, u_b, \mathbf{u}_g\} \in W_{h,k}^0$ be the approximate solution of (1.1) arising from the primal-dual weak Galerkin finite element algorithm (4.10)-(4.11) with C^0 -type elements. Assume that $a_{ij} \in C^1(\Omega)$ and the exact solution u of (1.1) satisfies $u \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$(7.15) \quad \left(\sum_{T \in \mathcal{T}_h} \|\nabla u_0 - \nabla u\|_T^2 \right)^{\frac{1}{2}} \leq Ch^k \|u\|_{k+1},$$

provided that the meshsize h is sufficiently small and the dual problem (7.2)-(7.3) has the H^1 -regularity estimate (7.5) with $s = 0$.

Proof. For any $\eta \in [C^1(\Omega)]^d$ with $\eta = 0$ on \mathcal{E}_h , let w be the solution of the dual problem (7.2)-(7.3) with $\theta = -\nabla \cdot \eta$. Thus, from Lemma (7.2) with $v = e_h$ given as in (6.1) we obtain

$$\begin{aligned} -(e_0, \nabla \cdot \eta) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,a}^2 e_h, w)_T - \langle (e_{gi} - \partial_i e_0) n_j, (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \\ &= I_1 - I_2, \end{aligned}$$

where I_j are defined in the obvious way. Since η vanishes on the wired basket \mathcal{E}_h , then from the integration by parts we have

$$(7.16) \quad (\nabla e_0, \eta) = I_1 - I_2.$$

Using the two estimates in Lemma 7.3, we can bound the term I_2 as follows:

$$(7.17) \quad |I_2| \leq Ch \|\theta\|_{-1} \|e_h\|_{2,h} \leq Ch \|\eta\|_0 \|e_h\|_{2,h}.$$

As to the term I_1 , we use the error equation (6.3) to obtain

$$\begin{aligned}
 I_1 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, w)_T \\
 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, \mathcal{Q}_h w)_T + (a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h)w)_T \\
 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d ((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, a_{ij} \mathcal{Q}_h w)_T + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h)w)_T.
 \end{aligned}$$

Note that

$$\begin{aligned}
 |((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, a_{ij} \mathcal{Q}_h w)_T| &= |((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, (I - \mathcal{Q}_h) a_{ij} \mathcal{Q}_h w)_T| \\
 &\leq \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_T \|(I - \mathcal{Q}_h) a_{ij} \mathcal{Q}_h w\|_T \\
 &\leq Ch_T \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_T \|w\|_{1,T}
 \end{aligned}
 \tag{7.19}$$

and by (7.1)

$$\begin{aligned}
 |(a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h)w)_T| &= |((a_{ij} - \bar{a}_{ij}) \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h)w)_T| \\
 &\leq \|a_{ij} - \bar{a}_{ij}\|_{L^\infty(T)} \|\partial_{ij,d}^2 e_h\|_T \|(I - \mathcal{Q}_h)w\|_T \\
 &\leq \varepsilon(h_T) h_T \|w\|_{1,T} (\|\partial_{ij,d}^2 e_0\|_T^2 + s_T(e_h, e_h))^{\frac{1}{2}},
 \end{aligned}
 \tag{7.20}$$

where $\varepsilon(h_T) \rightarrow 0$ as $h \rightarrow 0$. Using (7.19) and (7.20), we obtain the following estimate for the term I_1 :

$$\begin{aligned}
 |I_1| &\leq Ch \left(\varepsilon(h) \|\nabla^2 e_0\|_0 + \|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_0 \right) \|w\|_1 \\
 &\leq C \left(\varepsilon(h) \|\nabla e_0\|_0 + h \|e_h\|_{2,h} + h \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_0 \right) \|\eta\|_0,
 \end{aligned}
 \tag{7.21}$$

where we have used the inverse inequality and the estimate $\|w\|_1 \leq C \|\eta\|_{-1} \leq C \|\eta\|_0$. Substituting (7.21) and (7.17) into (7.16) yields

$$|(\nabla e_0, \eta)| \leq C \left(\varepsilon(h) \|\nabla e_0\|_0 + h \|e_h\|_{2,h} + h \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_0 \right) \|\eta\|_0.$$

Since the set of all such η is dense in $L^2(\Omega)$, then the above inequality implies

$$\|\nabla e_0\|_0 \leq C \left(\varepsilon(h) \|\nabla e_0\|_0 + h \|e_h\|_{2,h} + h \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_0 \right),$$

which leads to

$$\|\nabla e_0\|_0 \leq Ch \left(\|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_0 \right)
 \tag{7.22}$$

for sufficiently small meshsize h . The inequality (7.22), together with the error estimate (6.11) and the usual triangle inequality, verifies the estimate (7.15). \square

The following is an error estimate for the primal variable u_h in the usual L^2 -norm.

Theorem 7.6. *Assume that each entry of the coefficient matrix $\{a_{ij}\}_{d \times d}$ is in $C^1(\Omega) \cap [\Pi_{T \in \mathcal{T}_h} W^{2,\infty}(T)]$. In addition, assume that the dual problem (7.2)-(7.3) has H^2 -regularity with the a priori estimate (7.5) (i.e., $s = 1$), and $P_1(T) \subset S_k(T)$ for all $T \in \mathcal{T}_h$. Then, there exists a constant C such that*

$$(7.23) \quad \|u_0 - u\|_0 \leq Ch^{k+1} \|u\|_{k+1},$$

provided that the meshsize h is sufficiently small.

Proof. Let w be the solution of the dual problem (7.2)-(7.3) with $\theta \in L^2(\Omega)$. From Lemma 7.2 with $v = e_h$ given by (6.1), we have

$$(7.24) \quad \begin{aligned} (e_0, \theta) &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, w)_T - \langle (e_{gi} - \partial_i e_0) n_j, (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \\ &\quad + \langle (e_b - e_0) n_i, \partial_j (\mathcal{Q}_h - I)(a_{ij} w) \rangle_{\partial T} \\ &= J_1 - J_2 + J_3, \end{aligned}$$

where J_m are defined accordingly. Using the two estimates in Lemma 7.4 we obtain the following estimates:

$$(7.25) \quad |J_2| + |J_3| \leq Ch^2 \|\theta\|_0 \|e_h\|_{2,h}.$$

For the term J_1 , we use the error equation (6.3) to obtain

$$(7.26) \quad \begin{aligned} J_1 &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, w)_T \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, \mathcal{Q}_h w)_T + (a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h) w)_T \\ &= \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d ((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, a_{ij} \mathcal{Q}_h w)_T + \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^d (a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h) w)_T. \end{aligned}$$

Since $P_1(T) \subseteq S_k(T)$ and \mathcal{Q}_h is the L^2 -projection onto $S_k(T)$, then

$$(7.27) \quad \begin{aligned} |((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, a_{ij} \mathcal{Q}_h w)_T| &= |((I - \mathcal{Q}_h) \partial_{ij,d}^2 u, (I - \mathcal{Q}_h) a_{ij} \mathcal{Q}_h w)_T| \\ &\leq \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_T \|(I - \mathcal{Q}_h) a_{ij} \mathcal{Q}_h w\|_T \\ &\leq Ch_T^2 \|(I - \mathcal{Q}_h) \partial_{ij,d}^2 u\|_T \|w\|_{2,T} \end{aligned}$$

and by (7.1) we arrive at

$$(7.28) \quad \begin{aligned} &|(a_{ij} \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h) w)_T| \\ &= |((a_{ij} - \bar{a}_{ij}) \partial_{ij,d}^2 e_h, (I - \mathcal{Q}_h) w)_T| \\ &\leq \|a_{ij} - \bar{a}_{ij}\|_{L^\infty(T)} \|\partial_{ij,d}^2 e_h\|_T \|(I - \mathcal{Q}_h) w\|_T \\ &\leq Ch_T^3 \|w\|_{2,T} (\|\partial_{ij,d}^2 e_0\|_T^2 + s_T(e_h, e_h))^{\frac{1}{2}}. \end{aligned}$$

It follows from (7.27) and (7.28) that

$$\begin{aligned}
 |J_1| &\leq C \left(h^3 \|\nabla^2 e_0\|_0 + h^3 \|e_h\|_{2,h} + h^2 \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij}^2 u\|_0 \right) \|w\|_2 \\
 (7.29) \quad &\leq C \left(h \|e_0\|_0 + h^3 \|e_h\|_{2,h} + h^2 \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij}^2 u\|_0 \right) \|\theta\|_0,
 \end{aligned}$$

where we have used the inverse inequality and the regularity assumption (7.5) with $s = 1$. Substituting (7.29) and (7.25) into (7.24) yields

$$|(e_0, \theta)| \leq Ch^2 \left(h^{-1} \|e_0\|_0 + h \|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij}^2 u\|_0 \right) \|\theta\|_0.$$

Thus, we have

$$\|e_0\|_0 \leq Ch^2 \left(h^{-1} \|e_0\|_0 + h \|e_h\|_{2,h} + \sum_{i,j=1}^d \|(I - \mathcal{Q}_h) \partial_{ij}^2 u\|_0 \right),$$

which, together with the error estimate (6.11) and the usual triangle inequality, gives rise to the L^2 -error estimate (7.23) when the meshsize h is sufficiently small. This completes the proof of the theorem. \square

Remark 7.1. The optimal order error estimate (7.23) is based on the assumption that $P_1(T) \subseteq S_2(T)$. This assumption was used in the derivation of the inequalities (7.25), (7.27), and (7.28). In the case of $P_1(T) \not\subseteq S_2(T)$, those inequalities need to be modified by replacing $\|w\|_{2,T}$ by $h_T^{-1} \|w\|_{1,T}$. As a result, the following sub-optimal order error estimate holds true

$$(7.30) \quad \|u_0 - u\|_0 \leq Ch^k \|u\|_{k+1}$$

provided that (1) the coefficient matrix $\{a_{ij}\}_{d \times d}$ satisfies $a_{ij} \in C^1(\Omega)$, (2) the meshsize h is sufficiently small, and (3) the dual problem (7.2)-(7.3) has the H^1 -regularity with $s = 0$ in the a priori estimate (7.5). On the other hand, the last inequality in the proof of Theorem 7.6 implies the superconvergence estimate of $\|e_0\|_0 \leq Ch^{k+2} \|u\|_{k+2}$ if $S_k(T) = P_{k-1}(T)$ is used in the construction of the finite element space $S_{h,k}$.

To establish some error estimates for the two boundary components u_b and \mathbf{u}_g , we introduce the following norms

$$(7.31) \quad \|e_b\|_{L^2} := \left(\sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 \right)^{\frac{1}{2}}, \quad \|\mathbf{e}_g\|_{L^2} := \left(\sum_{T \in \mathcal{T}_h} h_T \|\mathbf{e}_g\|_{\partial T}^2 \right)^{\frac{1}{2}}.$$

Theorem 7.7. *Under the assumptions of Theorem 7.6, there exists a constant C such that*

$$(7.32) \quad \|u_b - Q_b u\|_{L^2} \leq Ch^{k+1} \|u\|_{k+1},$$

$$(7.33) \quad \|\mathbf{u}_g - \mathbf{Q}_b \nabla u\|_{L^2} \leq Ch^k \|u\|_{k+1}.$$

Proof. On each element $T \in \mathcal{T}_h$, we have from the triangle inequality that

$$\|e_b\|_{\partial T} \leq \|e_0\|_{\partial T} + \|e_b - e_0\|_{\partial T}.$$

Thus, by the trace inequality (6.7) we obtain

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T \|e_b\|_{\partial T}^2 &\leq 2 \sum_{T \in \mathcal{T}_h} h_T \|e_0\|_{\partial T}^2 + Ch^4 \sum_{T \in \mathcal{T}_h} h_T^{-3} \|e_b - e_0\|_{\partial T}^2 \\ &\leq C(\|e_0\|_0^2 + h^4 \|e_h\|_{2,h}^2), \end{aligned}$$

which, together with the error estimates (6.11) and (7.23), gives rise to (7.32).

To derive (7.33), we apply the same approach to the error component $\mathbf{e}_g = \mathbf{u}_g - \mathbf{Q}_b \nabla u$ as follows:

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T \|\mathbf{e}_g\|_{\partial T}^2 &\leq 2 \sum_{T \in \mathcal{T}_h} h_T \|\nabla e_0\|_{\partial T}^2 + Ch^2 \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{e}_g - \nabla e_0\|_{\partial T}^2 \\ &\leq C \left(\sum_{T \in \mathcal{T}_h} h^{-2} \|e_0\|_T^2 + h^2 \|e_h\|_{2,h}^2 \right). \end{aligned}$$

It then follows from the error estimates (6.11) and (7.24) that (7.33) holds true. \square

8. NUMERICAL RESULTS

In this section, we present some numerical results for the primal-dual WG finite element method proposed and analyzed in the previous sections. The test problems are defined in 2D polygonal domains in the following form: Find $u \in H^2(\Omega)$ such that

$$\begin{aligned} (8.1) \quad &\sum_{i,j=1}^2 a_{ij} \partial_{ij}^2 u = f, \quad \text{in } \Omega, \\ &u = g, \quad \text{on } \partial\Omega. \end{aligned}$$

For simplicity, in the numerical scheme (4.10)-(4.11), we shall make use of the lowest order WG element on triangular partitions; i.e., $k = 2$ in $W_k(T)$ on triangles $T \in \mathcal{T}_h$ given by (4.1). The goal is to illustrate the efficiency and confirm the convergence theory established in the previous sections through numerical experiments.

For the lowest order WG element with $k = 2$, the corresponding finite element spaces are given by

$$W_{h,2} = \{v = \{v_0, v_b, \mathbf{v}_g\} : v_0 \in P_2(T), v_b \in P_2(e), \mathbf{v}_g \in [P_1(e)]^2, \forall T \in \mathcal{T}_h, e \in \mathcal{E}_h\}$$

and

$$S_{h,2} = \{\sigma : \sigma|_T \in S_2(T), \forall T \in \mathcal{T}_h\}.$$

A finite element function $v \in W_{h,2}$ is said to be of C^0 -type if $v_b = v_0|_{\partial T}$ for each element T . For C^0 -type WG elements, the boundary component v_b can be merged with v_0 in all the formulations since it coincides with the trace of v_0 on the element boundary. This clearly results in a linear system that has less computational complexity than fully discontinuous type WG elements. But the C^0 continuity limits the pool of availability of polygonal elements due to the obvious constraints.

The local finite element space $S_2(T)$ is chosen such that $P_0(T) \subseteq S_2(T) \subseteq P_1(T)$. Our numerical experiments are conducted for the case of both $S_2(T) = P_1(T)$ and $S_2(T) = P_0(T)$ with C^0 -type $W_{h,2}$. For convenience, the C^0 -type WG element with $S_2(T) = P_1(T)$ shall be called the $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element. Analogously, the C^0 -type WG element with $S_2(T) = P_0(T)$ is called the $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element.

It should be pointed out that all of the theoretical results developed in the previous sections can be extended to C^0 -type triangular elements without any difficulty.

For C^0 -type elements, the discrete weak second order partial derivative $\partial_{ij,d}^2 v$ should be computed as a polynomial in $S_2(T)$ on each element T by solving the following equation:

$$(8.2) \quad (\partial_{ij,d}^2 v, \varphi)_T = -(\partial_i v_0, \partial_j \varphi)_T + \langle v_{gi}, \varphi n_j \rangle_{\partial T}, \quad \forall \varphi \in S_2(T).$$

For the convergence theory to work, one needs to establish the commutative property (5.2) for a properly defined projection operator Q_h given as in (5.1). It turns out that \mathbf{Q}_g should remain to be the usual L^2 -projection into the space of polynomials of degree $k-1$ on each side and Q_0 can be chosen as the interpolation operator \tilde{I}_k given as in Lemma A.3 of [12]. Note that for any $v \in H^2(T)$, the interpolant polynomial $\tilde{I}_k v \in P_k(T)$ satisfies

$$(8.3) \quad \int_e (v - \tilde{I}_k v) \phi ds = 0 \quad \forall \phi \in P_{k-2}(e), \quad \forall \text{ sides } e \text{ of } T,$$

$$(8.4) \quad \int_T (v - \tilde{I}_k v) \phi ds = 0 \quad \forall \phi \in P_{k-3}(T).$$

Thus, from the integration by parts, (8.3), and (8.4), we have

$$(8.5) \quad \begin{aligned} (\partial_i \tilde{I}_k v, \partial_j \varphi)_T &= -(\tilde{I}_k v, \partial_{ji}^2 \varphi)_T + \langle \tilde{I}_k v, \partial_j \varphi n_i \rangle_{\partial T} \\ &= -(v, \partial_{ji}^2 \varphi)_T + \langle v, \partial_j \varphi n_i \rangle_{\partial T} \\ &= (\partial_i v, \partial_j \varphi)_T \end{aligned}$$

for all $\varphi \in P_{k-1}(T)$. It follows from (8.2) and (8.5) that

$$\begin{aligned} (\partial_{ij,d}^2 Q_h v, \varphi)_T &= -(\partial_i \tilde{I}_k v, \partial_j \varphi)_T + \langle (\mathbf{Q}_g \nabla v)_i, \varphi n_j \rangle_{\partial T} \\ &= -(\partial_i v, \partial_j \varphi)_T + \langle (\nabla v)_i, \varphi n_j \rangle_{\partial T} \\ &= (\partial_{ij}^2 v, \varphi)_T \\ &= (Q_h \partial_{ij}^2 v, \varphi)_T \end{aligned}$$

for all $\varphi \in P_{k-1}(T)$, which gives rise to the commutative property (5.2). Readers are referred to [17] for a detailed discussion on the use of C^0 -type elements in the context of weak Galerkin approach.

Three domains are used in our numerical experiments: the unit square $\Omega = (0, 1)^2$, the reference domain $\Omega = (-1, 1)^2$, and the L-shaped domain with vertices $A_0 = (0, 0)$, $A_1 = (2, 0)$, $A_2 = (1, 1)$, $A_3 = (1, 2)$, and $A_4 = (0, 2)$. Given an initial coarse triangulation of the domain, a sequence of triangular partitions are obtained successively through a uniform refinement procedure that divides each coarse level triangle into four congruent sub-triangles by connecting the three mid-points on the edges of each triangle.

We use $u_h = \{u_0, \mathbf{u}_g\} \in W_{h,2}$ and $\lambda_h \in S_{h,2}$ to denote the primal-dual WG-FEM solution arising from (4.10)-(4.11). These numerical solutions are compared with some interpolants of the exact solution in various norms. Specifically, the numerical component u_0 is compared with the standard Lagrange interpolation of the exact solution u on each triangular element by using three vertices and three mid-points on the edge, which is denoted as $I_h u$. The vector component \mathbf{u}_g is compared with the linear interpolant of ∇u , denoted as $\mathbf{I}_g(\nabla u)$, on each edge $e \in \mathcal{E}_h$. The Lagrange multiplier λ_h is compared with $\lambda = 0$, as it is the trivial solution of the dual problem. Denote their differences by

$$e_h = \{e_0, \mathbf{e}_g\} := \{u_0 - I_h u, \mathbf{u}_g - \mathbf{I}_g(\nabla u)\}, \quad \gamma_h = \lambda_h - 0.$$

The following norms are used to measure the magnitude of the error:

$$\begin{aligned}
 L^2\text{-norm:} \quad & \|e_0\|_0 = \left(\sum_{T \in \mathcal{T}_h} \int_T |e_0|^2 dT \right)^{\frac{1}{2}}, \\
 H^1\text{-seminorm:} \quad & \|\mathbf{e}_g\|_{L^2} = \left(\sum_{T \in \mathcal{T}_h} h_T \int_{\partial T} |\mathbf{e}_g|^2 ds \right)^{\frac{1}{2}}, \\
 L^2\text{-norm:} \quad & \|\gamma_h\|_0 = \left(\sum_{T \in \mathcal{T}_h} \int_T |\gamma_h|^2 dT \right)^{\frac{1}{2}}.
 \end{aligned}$$

8.1. Numerical experiments with continuous coefficients. Tables 1–2 illustrate the performance of the primal-dual WG finite element method for the test problem (8.1) with exact solution given by $u = \sin(x_1)\sin(x_2)$ on the unit square domain and the L-shaped domain. The right-hand side function and the Dirichlet boundary condition are chosen to match the exact solution. The results indicate that the convergence rates for the solution of the weak Galerkin algorithm (4.10)–(4.11) is of order $r = 4.0$ and $r = 3.5$ in the discrete L^2 -norm for u_0 on the unit square domain and the L-shaped domain, respectively. For the discrete H^1 -seminorm (i.e., the L^2 -norm for \mathbf{e}_g), the numerical order of convergence is $r = 2.0$ on both domains. For the Lagrange multiplier λ_h , the numerical order of convergence is $r = 1.0$ in the L^2 -norm on the square and the L-shaped domain. In comparison, the theoretical order of convergence for u_0 in the L^2 -norm is $r = 3.0$, and that for \mathbf{u}_g and λ_h are $r = 2.0$ and $r = 1.0$, respectively, for the unit square domain. For the L-shaped domain, the theoretical rate of convergence for u_0 in the L^2 -norm should be between $r = 2$ and $r = 3$ due to the lack of needed H^2 -regularity for the dual problem (7.2)–(7.3). However, the theoretical rates of convergence for \mathbf{u}_g and λ_h remain to be of order $r = 2.0$ and $r = 1.0$, respectively. It is clear that the numerical results are in good consistency with the theory for \mathbf{u}_g and λ_h , but greatly outperform the theory for u_0 in the discrete L^2 -norm. Note that the primal-dual weak Galerkin finite element method has a superconvergence of order $r = k + 2$ for smooth solutions with smooth data on convex domains.

Table 3 contains some numerical results for problem (8.1) in $\Omega = (-1, 1)^2$ with exact solution $u = \sin(x_1)\sin(x_2)$ with varying coefficients. Observe that the coefficient function $a_{12} = 0.5|x_1|^{\frac{1}{3}}|x_2|^{\frac{1}{3}}$ is continuous in the domain, but its derivative

TABLE 1. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.1) with exact solution $u = \sin(x_1)\sin(x_2)$ on $\Omega = (0, 1)^2$. The coefficient matrix is $a_{11} = 3$, $a_{12} = a_{21} = 1$, and $a_{22} = 2$.

$1/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.00624		0.126		0.0335	
2	0.00147	2.09	0.0448	1.50	0.0650	-0.96
4	1.39e-004	3.40	0.0116	1.95	0.0284	1.20
8	1.03e-005	3.75	0.00284	2.03	0.0132	1.10
16	6.95e-007	3.89	7.02e-004	2.02	0.00643	1.04
32	4.52e-008	3.94	1.75e-004	2.01	0.00317	1.02

TABLE 2. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.1) with exact solution $u = \sin(x_1)\sin(x_2)$ on the L-shaped domain. The coefficient matrix is $a_{11} = 3$, $a_{12} = a_{21} = 1$, and $a_{22} = 2$.

$1/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.0168		0.481		0.448	
2	0.00248	2.76	0.125	1.95	0.195	1.20
4	2.30e-004	3.43	0.0310	2.01	0.0875	1.16
8	1.93e-005	3.57	0.00767	2.01	0.0413	1.08
16	1.61e-006	3.59	0.00191	2.01	0.0202	1.03
3.2	1.37e-007	3.56	4.75e-004	2.00	0.00999	1.01

TABLE 3. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.1) with exact solution $u = \sin(x_1)\sin(x_2)$ on the domain $(-1, 1)^2$. The coefficient matrix is $a_{11} = 1 + |x_1|$, $a_{12} = a_{21} = 0.5|x_1|^{\frac{1}{3}}|x_2|^{\frac{1}{3}}$, $a_{22} = 1 + |x_2|$.

$2/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.1763728		1.2455105		0.0038959	
2	0.0356693	2.31	0.4859078	1.36	0.0082045	-1.07
4	0.0036026	3.31	0.1304043	1.90	0.0032424	1.34
8	2.78e-004	3.70	0.0318454	2.03	0.0015142	1.10
16	2.02e-005	3.78	0.0078262	2.02	7.42e-004	1.03
32	2.37e-006	3.09	0.00194	2.01	3.68e-004	1.01

has a singularity at the origin so that the corresponding second order elliptic equation cannot be written in a divergence form. The performance of the primal-dual WG finite element method is similar to the case of constant coefficient matrix, except that the superconvergence seems to be weakened in the convergence order.

In Table 4, we present some numerical results for the test problem (8.1) with exact solution $u = \sin(x_1)\sin(x_2)$ in $\Omega = (-1, 1)^2$ when the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is employed in the primal-dual WG finite element scheme (4.10)-(4.11). Note that the Lagrange multiplier λ is now approximated by piecewise constant functions; i.e., $S_2(T) = P_0(T)$. The results indicate that the numerical solution \mathbf{u}_g converges to the exact solution ∇u at the rate of $r = 2.0$ in the usual L^2 -norm. The same rate of convergence is also observed for $u_h - u$ in the L^2 -norm. The Lagrange multiplier has a convergence rate slightly higher than $r = 1.0$ to the exact solution of $\lambda = 0$. The numerical convergence for the primal variable u is in great consistency with the theory developed in this paper, while the convergence for the dual variable λ outperforms the theory of $r = 1.0$.

8.2. Numerical experiments with discontinuous coefficients. In the second part of the numerical experiment, we consider problems with discontinuous coef-

TABLE 4. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element applied to problem (8.1) with exact solution $u = \sin(x_1)\sin(x_2)$ on the domain $(-1, 1)^2$. The coefficient matrix is $a_{11} = 1 + |x_1|$, $a_{12} = a_{21} = 0.5|x_1|^{\frac{1}{3}}|x_2|^{\frac{1}{3}}$, $a_{22} = 1 + |x_2|$.

$2/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	2.80e-006		1.7557720		2.10e-006	
2	0.1756863	-15.94	0.6755226	1.38	0.0894908	-15.38
4	0.0395431	2.15	0.1637125	2.04	0.0517686	0.79
8	0.0089637	2.14	0.0386493	2.08	0.0190018	1.45
16	0.0021665	2.05	0.0093809	2.04	0.0068545	1.47
32	5.37e-004	2.01	0.00231	2.02	0.00288	1.25

ficients that satisfy the Cordès condition (2.3). The first such problem is given as

$$(8.6) \quad \sum_{i,j=1}^2 (1 + \delta_{ij}) \frac{x_i}{|x_i|} \frac{x_j}{|x_j|} \partial_{ij}^2 u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = (-1, 1)^2$ is the reference square domain and the function f is chosen so that the exact solution of (8.6) is

$$(8.7) \quad u = x_1 x_2 \left(1 - e^{1-|x_1|}\right) \left(1 - e^{1-|x_2|}\right).$$

It is not hard to see that the Cordès condition (2.3) is satisfied for the problem (8.6) with $\varepsilon = 3/5$ and the coefficients matrix is discontinuous across the x_1 - and x_2 -axis. This is a test problem suggested in [21].

Table 5 contains some numerical results for the test problem (8.6) when the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element is employed in the WG finite element scheme (4.10)-(4.11). Note that the Lagrange multiplier λ is approximated by piecewise linear functions; i.e., $S_2(T) = P_1(T)$. The results indicate that the numerical solution \mathbf{u}_g converges to the exact solution ∇u at the rate of $r = 2.0$ in the usual L^2 -norm, which is consistent with the theoretical rate of convergence. The Lagrange multiplier has a convergence rate that seems to be higher than the theory-predicted rate of $r = 1.0$. For the approximation of u , the convergence rate in the usual L^2 -norm seems to exceed $r = 2$. It should be pointed out that there is no theoretical result on optimal order of error estimates for $u - u_h$ in the L^2 -norm, as it is not clear if the dual problem (7.2)-(7.3) has the required regularity necessary for carrying out the convergence analysis. Table 5 shows that the numerical performance of the primal-dual WG finite element method is typically better than what theory predicts.

In Table 6, we present some numerical results for the test problem (8.6) when the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element is employed in the WG finite element scheme (4.10)-(4.11). It is interesting to note that the absolute error for each numerical approximation is smaller than those arising from the use of C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$

TABLE 5. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.6) with exact solution given by (8.7).

$2/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.094005		0.765566		0.337760	
2	0.248887	-1.40	1.346963	-0.82	0.642055	-0.93
4	0.106414	1.23	0.538155	1.32	1.284597	-1.0
8	0.030602	1.80	0.137486	1.97	0.537170	1.26
16	0.007488	2.03	0.032750	2.07	0.212136	1.34
32	0.001736	2.11	0.007848	2.06	0.092301	1.20

TABLE 6. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element applied to problem (8.6) with exact solution given by (8.7).

$2/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.0393		0.672		0.137	
2	0.0322	0.28	0.322	1.06	0.104	0.40
4	0.00750	2.10	0.0791	2.03	0.0532	0.96
8	0.00161	2.22	0.0180	2.13	0.0204	1.39
16	3.85e-004	2.07	0.00427	2.08	0.00818	1.32
32	9.52e-005	2.02	0.00104	2.04	0.00371	1.14

element in Table 5, while the rate of convergence remains to be comparable. Readers are invited to draw their own conclusions for the results illustrated in this table.

The final test equation is given by

$$(8.8) \quad \sum_{i,j=1}^2 \left(\delta_{ij} + \frac{x_i x_j}{|x|^2} \right) \partial_{ij}^2 u = f \quad \text{in } \Omega,$$

where $|x| = \sqrt{x_1^2 + x_2^2}$ is the length of x . Note that the coefficient $a_{ij} = \frac{x_i x_j}{|x|^2}$ fails to be continuous at the origin for $i \neq j$. For $\alpha > 1$, it can be seen that $u = |x|^\alpha \in H^2(\Omega)$ satisfies (8.8) with $f = (2\alpha^2 - \alpha)|x|^{\alpha-2}$. The linear operator in (8.8) satisfies the Cordès condition with $\varepsilon = 4/5$. The solution $u = |x|^\alpha$ has the regularity of $H^{1+\alpha-\tau}(\Omega)$ for arbitrarily small $\tau > 0$. In the numerical experiments, we take $\alpha = 1.6$ with problem (8.8) defined on two square domains: $(0, 1)^2$ and $(-1, 1)^2$. The case of $\Omega = (0, 1)^2$ was tested in [21].

Tables 7 and 8 illustrate the performance of the primal-dual WG scheme for the domain $\Omega = (0, 1)^2$. Note that the coefficient matrix $\{a_{ij}\}_{2 \times 2}$ is continuous in the interior of the domain, but it fails to be continuous at the corner point $A = (0, 0)$. The numerical approximation suggests a convergence rate of $r = 1.6$ in the H^1 -seminorm (i.e., L^2 for \mathbf{e}_g) and $r = 0.6$ in L^2 for the Lagrange multiplier λ_h . These are in great consistency with theory developed in earlier sections, as the solution $u = |x|^{1.6}$ has the regularity of $H^{2.6-\tau}(\Omega)$ for any small $\tau > 0$. It seems that the L^2 norm for $u - u_h$ has a numerical convergence rate of $r = 2$, for which no theory was available to apply or compare with.

TABLE 7. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.8) on $\Omega = (0, 1)^2$ with exact solution $u = |x|^{1.6}$.

$1/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.020		0.315		0.304	
2	0.00629	1.68	0.126	1.32	0.248	0.296
4	0.00174	1.86	0.0446	1.50	0.182	0.445
8	4.43e-004	1.97	0.0152	1.56	0.126	0.537
16	1.08e-004	2.03	0.00508	1.58	0.0846	0.570
32	2.60e-005	2.05	0.00169	1.59	0.0564	0.584

TABLE 8. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element applied to problem (8.8) on $\Omega = (0, 1)^2$ with exact solution $u = |x|^{1.6}$.

$1/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.00405		0.489		0.0623	
2	0.00803	-0.988	0.177	1.46	0.0616	0.0156
4	0.00263	1.61	0.0616	1.53	0.0476	0.372
8	7.90e-004	1.74	0.0210	1.55	0.0327	0.544
16	2.20e-004	1.85	0.00705	1.57	0.0218	0.582
32	5.85e-005	1.91	0.00235	1.59	0.0145	0.593

TABLE 9. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_1(T)$ element applied to problem (8.8) on $\Omega = (-1, 1)^2$ with exact solution $u = |x|^{1.6}$.

$2/h$	$\ e_0\ _0$	order	$\ \mathbf{e}_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.532		0.511		0.280	
2	0.266	1.00	0.403	0.344	0.623	-1.15
4	0.117	1.19	0.211	0.933	0.562	0.149
8	0.0563	1.05	0.111	0.927	0.405	0.471
16	0.0271	1.06	0.0576	0.945	0.277	0.547
32	0.0129	1.07	0.0290	0.987	0.187	0.572

Tables 9 and 10 illustrate the performance of the primal-dual WG finite element scheme (4.10)-(4.11) for the equation (8.8) in the domain $\Omega = (-1, 1)^2$. For this test problem, the coefficient matrix $\{a_{ij}\}_{2 \times 2}$ is discontinuous at the center of the domain so that the duality argument in the convergence theory is not applicable. Consequently, the corresponding numerical results are less accurate than the case of $\Omega = (0, 1)^2$ as shown in Tables 7 and 8. However, the numerical approximation suggests a convergence rate of $r = 0.6$ in L^2 for the Lagrange multiplier λ_h which is consistent with the theory. The convergence in H^1 - and L^2 -norms seems to have a rate of $r = 1.0$ or slightly higher.

TABLE 10. Convergence rates for the C^0 - $P_2(T)/[P_1(\partial T)]^2/P_0(T)$ element applied to problem (8.8) on $\Omega = (-1, 1)^2$ with exact solution $u = |x|^{1.6}$.

$2/h$	$\ e_0\ _0$	order	$\ e_g\ _{L^2}$	order	$\ \gamma_h\ _0$	order
1	0.647		0.487		0.0862	
2	0.611	0.08	0.697	-0.517	0.0769	0.165
4	0.254	1.26	0.407	0.774	0.0500	0.619
8	0.113	1.18	0.218	0.903	0.0417	0.264
16	0.0512	1.14	0.110	0.984	0.0297	0.490
32	0.0235	1.12	0.0540	1.03	0.0201	0.561

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REFERENCES

- [1] I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math. **20** (1972/73), 179–192. MR0359352
- [2] S. C. Brenner, T. Gudi, M. Neilan, and L. Sung, C^0 penalty methods for the fully nonlinear Monge-Ampère equation, Math. Comp. **80** (2011), no. 276, 1979–1995, DOI 10.1090/S0025-5718-2011-02487-7. MR2813346
- [3] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954
- [4] F. Brezzi, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8** (1974), no. R-2, 129–151. MR0365287
- [5] M. Chicco, *Dirichlet problem for a class of linear second order elliptic partial differential equations with discontinuous coefficients*, Ann. Mat. Pura Appl. (4) **92** (1972), 13–22. MR0328316
- [6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. Studies in Mathematics and its Applications, Vol. 4. MR0520174
- [7] M. Crouzeix and V. Thomée, *The stability in L_p and W_p^1 of the L_2 -projection onto finite element function spaces*, Math. Comp. **48** (1987), no. 178, 521–532, DOI 10.2307/2007825. MR878688
- [8] A. Dedner and T. Pryer, *Discontinuous Galerkin methods for nonvariational problems*, arXiv:1304.2265v1.
- [9] X. Feng, L. Hennings, and M. Neilan, *C^0 discontinuous Galerkin finite element methods for second order linear elliptic partial differential equations in non-divergence form*, arXiv:1505.02842.
- [10] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., Stochastic Modelling and Applied Probability, vol. 25, Springer, New York, 2006. MR2179357
- [11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR737190
- [12] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986. MR851383

- [13] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683]; With a foreword by Susanne C. Brenner. MR3396210
- [14] O. Lakkis and T. Pryer, *A finite element method for second order nonvariational elliptic problems*, SIAM J. Sci. Comput. **33** (2011), no. 2, 786–801, DOI 10.1137/100787672. MR2801189
- [15] A. Maugeri, D. K. Palagachev, and L. G. Softova, *Elliptic and parabolic equations with discontinuous coefficients*, Mathematical Research, vol. 109, Wiley-VCH Verlag Berlin GmbH, Berlin, 2000. MR2260015
- [16] L. Mu, J. Wang, and X. Ye, *Weak Galerkin finite element methods on polytopal meshes*, Int. J. Numer. Anal. Model. **12** (2015), no. 1, 31–53. MR3286455
- [17] L. Mu, J. Wang, X. Ye, and S. Zhang, *A C^0 -weak Galerkin finite element method for the biharmonic equation*, J. Sci. Comput. **59** (2014), no. 2, 473–495, DOI 10.1007/s10915-013-9770-4. MR3188449
- [18] M. Neilan, *Convergence analysis of a finite element method for second order non-variational elliptic problems*, J. Numer. Math., DOI: 10.1515/jnma-2016-1017.
- [19] M. Neilan, *Quadratic finite element approximations of the Monge-Ampère equation*, J. Sci. Comput. **54** (2013), no. 1, 200–226, DOI 10.1007/s10915-012-9617-4. MR3002620
- [20] R. H. Nochetto and W. Zhang, *Discrete ABP estimate and convergence rates for linear elliptic equations in non-divergence form*, arXiv:1411.6036.
- [21] I. Smears and E. Süli, *Discontinuous Galerkin finite element approximation of nondivergence form elliptic equations with Cordès coefficients*, SIAM J. Numer. Anal. **51** (2013), no. 4, 2088–2106, DOI 10.1137/120899613. MR3077903
- [22] I. Smears and E. Süli, *Discontinuous Galerkin finite element approximation of Hamilton-Jacobi-Bellman equations with Cordes coefficients*, SIAM J. Numer. Anal. **52** (2014), no. 2, 993–1016, DOI 10.1137/130909536. MR3196952
- [23] C. Wang and J. Wang, *An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes*, Comput. Math. Appl. **68** (2014), no. 12, 2314–2330, DOI 10.1016/j.camwa.2014.03.021. MR3284276
- [24] J. Wang and X. Ye, *A weak Galerkin finite element method for second-order elliptic problems*, J. Comput. Appl. Math. **241** (2013), 103–115, DOI 10.1016/j.cam.2012.10.003. MR2994424
- [25] J. Wang and X. Ye, *A weak Galerkin mixed finite element method for second order elliptic problems*, Math. Comp. **83** (2014), no. 289, 2101–2126, DOI 10.1090/S0025-5718-2014-02852-4. MR3223326
- [26] J. Wang and X. Ye, *A weak Galerkin finite element method for the stokes equations*, Adv. Comput. Math. **42** (2016), no. 1, 155–174, DOI 10.1007/s10444-015-9415-2. MR3452926

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