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DIMENSION OF MIXED SPLINES ON POLYTOPAL CELLS

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Abstract. The dimension of planar splines on polygonal subdivisions of degree at most d is known to be a degree two polynomial for $d \gg 0$. For planar C^r splines on triangulations this formula is due to Alfeld and Schumaker; the formulas for planar splines with varying smoothness conditions across edges on convex polygonal subdvisions are due to Geramita, McDonald, and Schenck. In this paper we give a bound on how large d must be for the known polynomial formulas to give the correct dimension of the spline space. Bounds are given for central polytopal complexes in three dimensions, or polytopal cells, with varying smoothness across two-dimensional faces. In the case of tetrahedral cells with uniform smoothness r we show that the known polynomials give the correct dimension for $d \geq 3r+2$; previously Hong and separately Ibrahim and Schumaker had shown that this bound holds for planar triangulations. All bounds are derived using techniques from computational commutative algebra.

1. Introduction

Let $\mathcal{P}$ be a subdivision of a region in $\mathbb{R}^n$ by convex polytopes. $C^r(\mathcal{P})$ denotes the set of piecewise polynomial functions (splines) on $\mathcal{P}$ that are continuously differentiable of order r. Splines are a fundamental tool in approximation theory and numerical analysis [8]; more recently they have also appeared in a geometric context, describing the equivariant cohomology ring of toric varieties [24]. Practical applications include surface modeling, computer-aided design, and computer graphics [8].

One of the fundamental questions in spline theory is to determine the dimension of the space $C^r_d(\mathcal{P})$ of splines of degree at most d. In the bivariate, simplicial case, these questions are studied by Alfeld and Schumaker in [2] and [3] using Bernstein-Bezier methods. A signature result in [3] is a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+1$ and $\Delta \subset \mathbb{R}^2$ is a generic simplicial complex. For $\Delta \subset \mathbb{R}^2$ simplicial and nongeneric, Hong [17] and Ibrahim and Schumaker [18] derive a formula for $\dim C^r_d(\Delta)$ when $d \geq 3r+2$ as a byproduct of constructing local bases for these spaces.

An algebraic approach to the dimension question was pioneered by Billera in [5] using homological and commutative algebra. In [6], Billera and Rose show that $C^r_d(\mathcal{P}) \cong C^r(\widehat{\mathcal{P}})_d$, the dth graded piece of the module $C^r(\widehat{\mathcal{P}})$ of splines on the cone $\widehat{\mathcal{P}}$ over $\mathcal{P}$. The function $\dim_{\mathbb{R}} C^r(\widehat{\mathcal{P}})_d$ is known as the *Hilbert function* of $C^r(\widehat{\mathcal{P}})$ in commutative algebra, and a standard result is that the values of the Hilbert function

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Table 1. Known bounds on the postulation number for planar splines

Analytic Methods

Bound	Context	Computed by
$\wp(C^r(\widehat{\Delta})) \leq 3r$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Schumaker [3]
$\wp(C^r(\widehat{\Delta})) \leq 3r+1$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Hong [17]
		Ibrahim-Schumaker [18]
$\wp(C^1(\widehat{\Delta})) \leq 3$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Alfeld-Piper-Schumaker [1]

Homological Methods

Bound	Context	Computed by
$\wp(C^r(\widehat{\Delta})) \leq 4r$	*all* simplicial $\Delta \subset \mathbb{R}^2$	Mourrain-Villamizar [23]
$\wp(C^1(\widehat{\Delta})) \leq 1$	*generic* simplicial $\Delta \subset \mathbb{R}^2$	Billera [5]

eventually agree with the *Hilbert polynomial* $HP(C^r(\widehat{\mathcal{P}}), d)$ of $C^r(\widehat{\mathcal{P}})$. An important invariant of $C^r(\widehat{\mathcal{P}})$ is the *postulation number* $\wp(C^r(\widehat{\mathcal{P}}))$, which is the largest integer d so that $HP(C^r(\widehat{\mathcal{P}}), d) \neq \dim C^r(\widehat{\mathcal{P}})_d$. In this terminology the Alfeld and Schumaker result above could be viewed as a computation of $HP(C^r(\widehat{\Delta}), d)$ plus the bound $\wp(C^r(\widehat{\Delta})) \leq 3r$.

The goal of this paper is to provide upper bounds on the postulation number $\wp(C^\alpha(\mathcal{P}))$ where $C^\alpha(\mathcal{P})$ is the module of *mixed splines* and $\mathcal{P}$ is a *polytopal cell*. By a polytopal cell we mean a collection of (three-dimensional) polytopes $\mathcal{P}$, all sharing a common vertex, so that any pair of polytopes in $\mathcal{P}$ meeting nontrivially must meet in a vertex, edge or face of both (the precise definition will be given in the next section). We denote the set of (two-dimensional) faces of polytopes in $\mathcal{P}$ by $\mathcal{P}_2$. The module of *mixed splines* $C^\alpha(\mathcal{P})$ on $\mathcal{P}$ is the module of splines in which different smoothness conditions are imposed across faces, encoded by a map $\alpha : \mathcal{P}_2 \to \mathbb{Z}_{\geq -1}$, whose values we will refer to as the *smoothness parameters* associated to $\mathcal{P}$. Note that a planar subdivision $\mathcal{P} \subset \mathbb{R}^2$ consisting of convex polygons becomes a polytopal cell via the coning operation $\mathcal{P} \to \widehat{\mathcal{P}} \subset \mathbb{R}^3$ (this is explained in detail in Section 2). Hence our results specialize to polygonal subdivisions and triangulations in $\mathbb{R}^2$.

The main reason for bounding $\wp(C^\alpha(\mathcal{P}))$, where $\mathcal{P}$ is a polytopal cell, is that the Hilbert polynomial of $C^\alpha(\mathcal{P})$ has been computed in situations where there are no known bounds on $\wp(C^\alpha(\mathcal{P}))$, rendering these formulas impractical. Currently, bounds which do not make heavy restrictions on the complex $\mathcal{P}$ are known only in the simplicial case. These bounds are recorded in Table 1. For particular types of polytopal cells, better and sometimes exact bounds are known on the postulation number. In contrast, the Hilbert polynomial $HP(C^\alpha(\mathcal{P}), d)$ has been computed for *all* polytopal cells. This is done in the simplicial case with mixed smoothness by Schenck and Geramita [14], in the polytopal case with uniform smoothness by Schenck and McDonald [20], and in the polytopal case with mixed smoothness and boundary conditions in [10]. In this paper we provide the first bound on $\wp(C^\alpha(\mathcal{P}))$ valid for any polytopal cell. Our main result is:

Theorem 6.2. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell, and let $\mathcal{P}_2^\circ$ be the interior faces of $\mathcal{P}$. Set*

$$e(\mathcal{P}) = \max_{\tau \in \mathcal{P}_2^\circ} \{ \sum_{\gamma \in (\mathrm{st}(\tau))_2} (\alpha(\gamma)+1) \},$$

where $st(\tau)$ denotes the union of two polytopes sharing the face τ and $(st(\tau))_2$ denotes the two-dimensional faces of $\mathrm{st}(\tau)$*. Then $\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P})-3$. In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_{\mathbb{R}} C^\alpha(\mathcal{P})_d$ for $d \geq e(\mathcal{P}) - 2$.*

From an algebraic perspective, another reason for bounding $\wp(C^\alpha(\mathcal{P}))$ is that almost all existing bounds, including most in Table 1, have been computed using analytic techniques. There are a few instances where algebraic techniques are applied to bound $\wp(C^\alpha(\mathcal{P}))$. In [5], Billera proves $\wp(C^1(\widehat{\Delta})) \leq 1$ for *generic* planar triangulations (this result relies on a computation of Whiteley [31]). The most general bound produced by homological techniques to date is by Mourrain and Villamizar [23]; building on work of Schenck and Stillman [27] they prove that $\wp(C^r(\widehat{\Delta})) \leq 4r$ for Δ a planar simplicial complex, recovering an earlier result of Alfeld and Schumaker [2]. Our second result is the following.

Theorem 7.2. *Let $\Delta \subset \mathbb{R}^3$ be a simplicial cell. Let $\alpha : \Delta_2 \to \mathbb{Z}_{\geq -1}$ satisfy $\alpha(\tau) = r$ for all interior faces and $\alpha(\tau) = -1$ or $\alpha(\tau) = r$ for boundary faces. Then $\wp(C^\alpha(\Delta)) \leq 3r+1$. In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \geq 3r+2$.*

This result was originally proved for planar splines as a byproduct of constructing local bases by Hong [17] and Ibrahim and Schumaker [18], and is the best bound valid for all planar triangulations recorded in Table 1. We will show that this bound also holds for simplicial cells, even if uniform vanishing of order r is imposed along some or all faces in the boundary of Δ.

A key tool we use to prove these results is the *Castelnuovo-Mumford regularity* of $C^\alpha(\mathcal{P})$, denoted $\mathrm{reg}(C^\alpha(\mathcal{P}))$. The relationship between $\mathrm{reg}(C^\alpha(\mathcal{P}))$ and $\wp(C^\alpha(\mathcal{P}))$ is discussed in detail in §3. This invariant is also used in the context of splines by Schenck and Stiller in [28]. Our particular way of using regularity is inspired by an observation used in the Gruson-Lazarsfeld-Peskine theorem, bounding the regularity of curves in projective space. In the context of splines this observation is roughly that, if we are lucky, we can bound $\mathrm{reg}(C^\alpha(\mathcal{P}))$ by the regularity of a 'bad' approximation. This statement is made precise in Proposition A.7 and Theorem 4.7. We take as our approximation the submodule $LS^{\alpha,1}(\mathcal{P}) \subset C^\alpha(\mathcal{P})$ consisting of splines $F = \sum_\tau F_\tau$ which may be written as a sum of splines F_τ which vanish outside the union two adjacent polytopes meeting along the face τ. In a sense, this is an algebraic analogue of locally-supported bases used in [17, 18].

The paper is organized as follows. In §2 we give some background on the spline module $C^\alpha(\mathcal{P})$, in particular, the algebraic approach pioneered by Billera [5] and Billera and Rose [6]. In §3 we give a brief overview of some constructions from graded commutative algebra, in particular syzygy modules and Castelnuovo-Mumford regularity. We make precise the relationship between regularity and postulation number. Then we begin the process of proving Theorem 6.2. This is done in three main reductions. First, in §4, we introduce the submodule $LS^{\alpha,1}(\mathcal{P})$ of $C^\alpha(\mathcal{P})$, which we use to approximate $C^\alpha(\mathcal{P})$. We then show that the regularity of $C^\alpha(\mathcal{P})$, $\mathcal{P}$ a polytopal cell, may be bounded by the regularity of the approximation $LS^{\alpha,1}(\mathcal{P})$ (Theorem 4.7). Second, in §5, we show that the regularity of $LS^{\alpha,1}(\mathcal{P})$ (hence of $C^\alpha(\mathcal{P})$) can be bounded by the maximum regularity of its summands (Theorem 5.5), these being the submodules $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ of splines vanishing outside the union of two polytopes sharing the face τ. Finally, in §6 we bound the regularity of $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ and deduce Theorem 6.2. We devote §7 to the proof of Theorem 7.2, which involves a tighter regularity bound for $C^\alpha_{\mathrm{st}(\tau)}(\Delta)$ in the case of

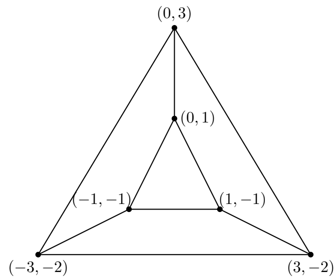


FIGURE 1. The polygonal complex in Example 1.1

uniform smoothness on simplicial cells. This proof is more technical than the proof of Theorem 6.2. We close the paper in §§8 and 9 with examples and conjectures.

1.1. **Illustrating Theorem 6.2 and Theorem 7.2.** Let $\mathcal{P} \subset \mathbb{R}^2$ be a subdivision of a simply connected polygonal domain by convex polygons. The most standard assignment of smoothness parameters is to set $\alpha(\tau) = r$ for every interior edge and $\alpha(\tau) = -1$ for every boundary edge. The module of splines on the cone $\widehat{\mathcal{P}} \subset \mathbb{R}^3$ over $\mathcal{P}$ with these smoothness parameters is denoted $C^r(\widehat{\mathcal{P}})$. By Corollary 3.14 of [20], the Hilbert polynomial of $C^r(\widehat{\mathcal{P}})$ is

$$(1) \quad HP(C^r(\widehat{\mathcal{P}}), d) = \frac{f_2}{2}d^2 + \frac{3f_2 - 2(r+1)f_1^0}{2}d + f_2 + \left(\binom{r}{2} - 1\right) f_1^0 + \sum_j c_j,$$

where

- f_0, f_1, f_2 denote, respectively, the number of vertices, edges, and polygons of $\mathcal{P}$,
- f_0°, f_1° denote, respectively, the number of interior vertices and edges of $\mathcal{P}$,
- r is the smoothness parameter, and
- the constants c_j record the dimension of certain vector spaces coming from ideals of powers of linear forms.

Example 1.1. The polygonal complex $\mathcal{Q}$ in Figure 1 has $f_2 = 4, f_1^0 = 6, f_3^0 = 3$. It is shown in §4 of [20] that there are four constants c_j in the formula (1), and they are all equal to the constant

$$\binom{r+2}{2} + \left\lceil \frac{r+1}{2} \right\rceil \left(r - \left\lceil \frac{r+1}{2} \right\rceil\right).$$

Hence by equation (1),

(2)
$$HP(C^r(\widehat{\mathcal{Q}}), d) = 2d^2 - 6rd + 6\binom{r}{2} - 2 + 4\left(\binom{r+2}{2} + \left\lceil \frac{r+1}{2} \right\rceil \left(r - \left\lceil \frac{r+1}{2} \right\rceil\right)\right).$$

By Theorem 6.2, $\wp(C^r(\widehat{\mathcal{Q}})) \le e(\mathcal{Q}) - 3$, where

$$e(\mathcal{Q}) = \max_{\tau \in \mathcal{P}_2^\circ}\{\sum_{\gamma \in (\mathrm{st}(\tau))_2} (\alpha(\gamma) + 1)\}.$$

The union of any two polygons of $\mathcal{Q}$ meeting along an interior edge have five edges which are interior. So $e(\mathcal{Q}) = 5(r + 1)$ and the Hilbert function $\dim C^r(\widehat{\mathcal{Q}})_d$ agrees with the Hilbert polynomial $HP(C^r(\widehat{\mathcal{Q}}), d)$ above for $d \ge 5(r + 1) - 2$, so

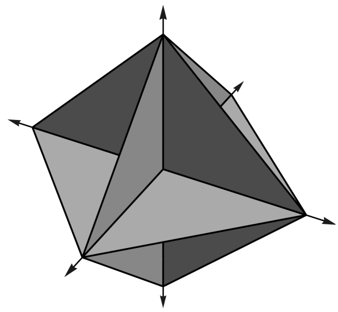


FIGURE 2. Centrally triangulated octahedron in Example 1.2

$\wp(C^r(\widehat{\mathcal{Q}})) \leq 5(r+1)-1$. Computations in Macaulay2 [15] suggest that there is agreement for $d \geq 2(r+1)$, equivalently $\wp(C^r(\widehat{\mathcal{Q}})) \leq 2r+1$. We see the same behavior in Example 8.3 in §8, indicating that there is room for improvement in Theorem 6.2.

Example 1.2. Consider a regular octahedron $\Delta \subset \mathbb{R}^3$ triangulated by placing a centrally symmetric vertex, whose interior faces are shown in Figure 2. This is the natural extension of the well-known Morgan-Scott triangulation to the context of simplicial cells. In [25, Example 5.2], Schenck shows that $C^r(\Delta)$ is free as a module over the polynomial ring $\mathbb{R}[x, y, z]$ in three variables, with one generator in degree zero, three generators in degree $r+1$, three generators in degree $2(r+1)$, and one generator in degree $3(r+1)$. It follows that

$$\dim C^r(\Delta)_d = \binom{d+2}{2} + 3\binom{d+1-r}{2} + 3\binom{d-2r}{2} + \binom{d-1-3r}{2},$$

where a binomial coefficient appearing in the above expression is interpreted as $\binom{A}{B} = 0$ if $A < 0$. Expanding these binomial coefficients as polynomials in d yields the Hilbert polynomial $HP(C^r(\Delta), d) = 4d^2 - 12rd + 12r^2 + 6r + 2$. Inspection yields that $\dim C^r(\Delta)_d = HP(C^r(\Delta), d)$ as long as $d \geq 3r+1$, so $\wp(C^r(\Delta)) = 3r$. This is only one less than the predicted bound of Theorem 7.2 (in fact, since $C^r(\widehat{\Delta})$ is free, we can adjust the bound predicted by Theorem 7.2 to be exact). Theorem 7.2 guarantees that, even if we perturb the vertices of Δ, $\dim C^r(\Delta)_d = HP(C^r(\Delta), d)$ for $d \geq 3r+2$. Computations indicate that for most perturbations, $\dim C^r(\Delta)_d = HP(C^r(\Delta), d)$ for $d \geq 2r+1$, equivalently $\wp(C^r(\Delta)) \leq 2r$. This generic behavior is akin to that of the two-dimensional Morgan-Scott triangulation. The centrally triangulated regular octahedron has additional symmetry that cannot be achieved by the Morgan-Scott triangulation, hence the jump in postulation number to $3r$ from the generic $2r$. This example shows that, even though Theorem 7.2 is tight for certain simplicial cells, there may be better bounds available for generic perturbations.

2. Preliminary material

We begin with some generalities on polytopal complexes and splines. By a *polygon* in $\mathbb{R}^2$ we mean the convex hull of a set of points in $\mathbb{R}^2$ not contained in a line. By a *polytope* in $\mathbb{R}^3$ we mean the convex hull of a set of points in $\mathbb{R}^3$ which is not contained in any plane. Denote by R the polynomial ring $\mathbb{R}[x, y]$ in two variables and by S the polynomial ring $\mathbb{R}[x,y,z]$ in three variables. Given a linear form $\alpha \in S$ and a real number a, set

$$\mathcal{H}_{\alpha=a} := \{(x,y,z) \in \mathbb{R}^3 \mid \alpha(x,y,z) = a\},$$
$$\mathcal{H}_{\alpha\geq a} := \{(x,y,z) \in \mathbb{R}^3 \mid \alpha(x,y,z) \geq a\}.$$

Let σ be a polytope and $\mathcal{H}_{\alpha\geq a}$ a half-space containing σ so that $\gamma = \mathcal{H}_{\alpha=a} \cap \sigma$ is nonempty. If γ consists of a single point, it is called a *vertex* of σ. If the smallest linear space containing γ is a line, γ is called an *edge* of σ. Otherwise the smallest linear space containing γ is $\mathcal{H}_{\alpha=a}$ and γ is called a *face* of σ. When we refer to the *boundary* of σ we will mean its set of (two-dimensional) faces, denoted $\partial\sigma$. If σ is a polygon in $\mathbb{R}^2$, define its vertices and edges the same way, and denote by $\partial\sigma$ the set of edges of σ.

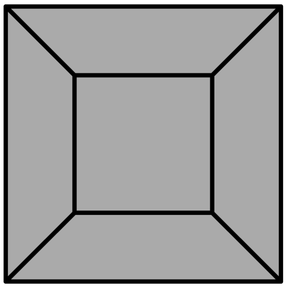
A *polytopal complex* is a finite set of polytopes $\mathcal{P} = \{\sigma_1, \ldots, \sigma_k\}$ in $\mathbb{R}^3$ satisfying the property that for any pair $\sigma_i, \sigma_j \in \mathcal{P}$ meeting nontrivially, the intersection $\sigma_i \cap \sigma_j$ is a vertex, edge, or face of both σ_i and σ_j. The book of Ziegler [32] is an excellent reference for polytopes and polytopal complexes; our terminology differs slightly from his and is more restrictive to favor geometric intuition relevant to our situation. If $\mathcal{P}$ consists entirely of tetrahedra, then $\mathcal{P}$ is a *simplicial complex* and will be denoted Δ. Likewise, a *polygonal complex* is a finite set $\mathcal{P}$ of polygons in $\mathbb{R}^2$ so that any pair of polygons in $\mathcal{P}$ intersecting nontrivially meet in a vertex or edge. If the polygons of $\mathcal{P}$ are all triangles, we call $\mathcal{P}$ a *triangulation* and denote it Δ. Whether $\mathcal{P}$ is a polygonal or polytopal complex will be evident from context.

We make the following definitions for polytopal complexes; they carry over to polygonal complexes in the obvious way. If all the polytopes of $\mathcal{P}$ share a common vertex then we say $\mathcal{P}$ is *central*. Without loss of generality, we will assume that this common vertex is the origin in $\mathbb{R}^3$. By $|\mathcal{P}|$ we denote the underlying space of $\mathcal{P}$. A vertex, edge, or face of $\mathcal{P}$ is, respectively, a vertex, edge, or face of one of the polytopes of $\mathcal{P}$. We denote the set of vertices, edges, and faces of $\mathcal{P}$ by $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$, respectively. We call a face of $\mathcal{P}$ *interior* if it is contained in two polytopes of $\mathcal{P}$ (clearly it cannot be contained in more than two); otherwise it is a *boundary* face of $\mathcal{P}$. Denote the set of boundary faces of $\mathcal{P}$ by $\partial\mathcal{P}$. A vertex or edge of $\mathcal{P}$ is interior if every face containing it is interior. We let $\mathcal{P}_0^\circ, \mathcal{P}_1^\circ, \mathcal{P}_2^\circ$ denote the set of interior vertices, edges, and faces of $\mathcal{P}$, respectively. Furthermore, we set $f_i = \#\mathcal{P}_i$ and $f_i^\circ = \#\mathcal{P}_i^\circ$ for $i = 0, 1, 2$.

Given a polytopal complex $\mathcal{P}$ and a vertex, edge, or face γ of $\mathcal{P}$, the *star* of γ in $\mathcal{P}$, denoted $\mathrm{st}_{\mathcal{P}}(\gamma)$, is the polytopal complex $\mathrm{st}_{\mathcal{P}}(\gamma) := \{\sigma \in \mathcal{P} \mid \gamma \in \sigma\}$. If the complex $\mathcal{P}$ is understood we will write $\mathrm{st}(\gamma)$.

We define an abstract graph $G(\mathcal{P})$ with a vertex corresponding to each polytope of $\mathcal{P}$; two vertices are joined by an edge if and only if the corresponding polytopes σ and σ' share a common face. We say that $\mathcal{P}$ is *hereditary* if $G(\mathrm{st}_{\mathcal{P}}(\gamma))$ is connected for every vertex and edge γ of $\mathcal{P}$. An easy example of a polytopal complex which is *not* hereditary is two polytopes meeting only at a single vertex (or along a single edge). By a *polytopal cell* we mean a central, hereditary polytopal complex.

We will frequently use the following construction of a polytopal cell. Let $i : \mathbb{R}^2 \to \mathbb{R}^3$ be the map $i(x,y) = (x,y,1)$. For a polygon σ, denote by $\widehat{\sigma}$ the convex hull of the origin in $\mathbb{R}^3$ and $i(\sigma)$. Now suppose $\mathcal{P}$ is a polygonal complex. Then define the cone $\widehat{\mathcal{P}}$ over $\mathcal{P}$ to be the set of polytopes $\{\widehat{\sigma} \mid \sigma \in \mathcal{P}\}$ (see Figure 3). This is clearly a central polytopal complex. Suppose furthermore that the polygonal complex $\mathcal{P}$



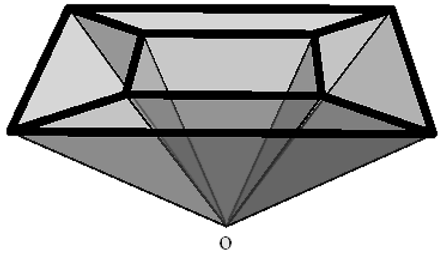


FIGURE 3. A polygonal complex $\mathcal{P}$ (left) and its cone $\widehat{\mathcal{P}}$ (right)

is hereditary (in the planar case, hereditary simply means that there is no 'pinch point'). Then clearly $\widehat{\mathcal{P}}$ is a polytopal cell.

We now recall the definition of the module of splines on a polytopal complex; again the obvious alterations are made to define splines over polygonal complexes. For a set $U \subset \mathbb{R}^3$, let $C^r(U)$ denote the set of functions $F : U \to \mathbb{R}$ continuously differentiable of order r. For a function $F : |\mathcal{P}| \to \mathbb{R}$ and a polytope $\sigma \in \mathcal{P}$, let F_σ denote the restriction of F to σ. The set $C^r(\mathcal{P})$ of piecewise polynomials continuously differentiable of order r on $\mathcal{P}$ is defined by

$$C^r(\mathcal{P}) := \{F \in C^r(|\mathcal{P}|) \mid F_\sigma \in S = \mathbb{R}[x, y, z] \text{ for every } \sigma \in \mathcal{P}\}.$$

The polynomial ring S includes in $C^r(\mathcal{P})$ as globally polynomial functions (these are the *trivial* splines); this makes $C^r(\mathcal{P})$ an S-module via pointwise multiplication. Henceforth we will refer to $C^r(\mathcal{P})$ as an S-module (or simply a module, since the S-module structure of $C^r(\mathcal{P})$ is the only structure we consider).

If, in addition, $\mathcal{P} \subset \mathbb{R}^3$ is a hereditary polytopal complex, the global C^r condition can be expressed as a differentiability condition across interior faces as follows (see [6] for a proof). For a face $\tau \in \mathcal{P}_2$, let $l_\tau \in S$ denote a choice of affine linear form (unique up to scaling) which vanishes on τ. Then a function $F : |\mathcal{P}| \to \mathbb{R}$ which restricts to a polynomial on each polytope is in $C^r(\mathcal{P})$ if and only if $l_\tau^{r+1} | (F_{\sigma_1} - F_{\sigma_2})$ for every pair of polytopes σ_1, σ_2 which intersect in the face τ.

In [14], Schenck and Geramita study the dimension of *mixed spline* spaces, in which the order of differentiability across faces varies. Specifically, let $\alpha : \mathcal{P}_2 \to \mathbb{Z}_{\geq -1}$ be a map yielding a list of *smoothness parameters* $(\alpha(\tau) \mid \tau \in \mathcal{P}_2)$. We require that $\alpha(\tau) \geq 0$ for $\tau \in \mathcal{P}_2^\circ$ and $\alpha(\tau) \geq -1$ for $\tau \in \partial\mathcal{P}$.

The module $C^\alpha(\mathcal{P})$ of mixed splines on $\mathcal{P}$ is defined as the set of splines $F \in C^0(\mathcal{P})$ satisfying

- $l_\tau^{\alpha(\tau)+1} | (F_{\sigma_1} - F_{\sigma_2})$ for $\tau \in \mathcal{P}_2^\circ$ with $\sigma_1 \cap \sigma_2 = \tau$,
- $l_\tau^{\alpha(\tau)+1} | F_\sigma$ for $\tau \in \partial\mathcal{P}$ with $\tau \in \partial\sigma$.

For hereditary polytopal complexes, the usual ring of splines $C^r(\mathcal{P})$ is recovered by setting $\alpha(\tau) = r$ for every $\tau \in \mathcal{P}_2^\circ$ and $\alpha(\tau) = -1$ for every $\tau \in \mathcal{P}_2$. The following variant of [6, Proposition 4.3] encodes the mixed spline conditions as a matrix.

Lemma 2.1. *If $\mathcal{P}$ is hereditary and $\alpha = (\alpha(\tau) \mid \tau \in \mathcal{P}_2)$, then $C^\alpha(\mathcal{P})$ fits into the exact sequence*

$$0 \to C^\alpha(\mathcal{P}) \to S^{f_3+f_2} \xrightarrow{\phi} S^{f_2} \to C \to 0,$$

$$\textit{where } \phi = \left(\begin{array}{c|ccc} & l_{\tau_1}^{\alpha(\tau_1)+1} & & \\ \delta_3 & & \ddots & \\ & & & l_{\tau_k}^{\alpha(\tau_k)+1} \end{array}\right).$$

Here $k = f_2$, $C = \operatorname{coker} \phi$ and δ_3 is the $f_2 \times f_3$ matrix of ± 1's coming from the spline conditions.

We now introduce two vector spaces of splines which will be our main objects of study. Let $S_{\leq d}$ be the vector space of polynomials $f \in S = \mathbb{R}[x, y, z]$ of total degree $\leq d$ and let S_d be the vector space of homogeneous polynomials of degree d. For $\mathcal{P} \subset \mathbb{R}^3$ we have a filtration of $C^\alpha(\mathcal{P})$ by $\mathbb{R}$-vector spaces:

$$C_d^\alpha(\mathcal{P}) := \{F \in C^\alpha(\mathcal{P}) \mid F_\sigma \in S_{\leq d} \text{ for all polytopes } \sigma \in \mathcal{P}\}.$$

For central polytopal complexes we always assign $\alpha(\tau) = -1$ for every two-dimensional face $\tau \in \mathcal{P}_2$ which does not contain the origin. Then the diagonal portion of the matrix in Lemma 2.1 consists of constants and homogeneous polynomials of degree $\alpha(\tau) + 1$. Consider the vector spaces

$$C^\alpha(\mathcal{P})_d := \{F \in C^\alpha(\mathcal{P}) \mid F_\sigma \in S_d \text{ for all polytopes } \sigma \in \mathcal{P}_2\}.$$

If $\mathcal{P}$ is central, then $C^\alpha(\mathcal{P}) \cong \bigoplus_{d \geq 0} C^\alpha(\mathcal{P})_d$. In other words, every spline $F \in C^\alpha(\mathcal{P})$ may be uniquely written as a sum $F = \sum_{d \geq 0} F_d$ where $F_d \in C^\alpha(\mathcal{P})_d$. We say $C^\alpha(\mathcal{P})$ is a *graded* S-module. We will return to this notion in §3.

Now suppose $\mathcal{P}$ is a polygonal complex in $\mathbb{R}^2$, with smoothness parameters α now attached to *edges* of $\mathcal{P}$. We extend these to smoothness parameters $\widehat{\alpha}$ on $\widehat{\mathcal{P}}$ by assigning

- $\widehat{\alpha}(\tau') = \alpha(\tau)$ for $\tau' \in \widehat{\mathcal{P}}_2$ which is a cone over $\tau \in \mathcal{P}_1$ and
- $\widehat{\alpha}(\tau') = -1$ for $\tau' \in \widehat{\mathcal{P}}_2$ so that $(0, 0, 0) \notin \tau'$.

Since this extension is natural we will abuse notation and drop the hat on α, denoting $C^{\widehat{\alpha}}(\widehat{\mathcal{P}})$ by $C^\alpha(\widehat{\mathcal{P}})$. In practice one computes the module $C^\alpha(\widehat{\mathcal{P}})$ by replacing the polynomial ring $R = \mathbb{R}[x, y]$ by $S = \mathbb{R}[x, y, z]$ in Lemma 2.1 and homogenizing the entries of the matrix ϕ with respect to z. The kernel of the homogenized matrix is $C^\alpha(\widehat{\mathcal{P}})$. The following lemma is proved the same way as Theorem 2.6 of [6].

Lemma 2.2. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a polygonal complex with smoothness parameters α. Then $C_d^\alpha(\mathcal{P}) \cong C^\alpha(\widehat{\mathcal{P}})_d$ as $\mathbb{R}$-vector spaces.*

Remark 2.3. By Lemma 2.2, if $\mathcal{P} \subset \mathbb{R}^2$ is a hereditary polygonal complex, $\dim C_d^\alpha(\mathcal{P}) = \dim C^\alpha(\widehat{\mathcal{P}})_d$, hence the study of dimension formulas for planar splines is subsumed in the study of dimension formulas for splines on polytopal cells.

For the remainder of the paper we will assume that all polytopal complexes are polytopal cells. Moreover, whenever we refer to computing splines over a (hereditary) polygonal complex $\mathcal{P}$, we assume that we are considering splines on the polytopal cell $\widehat{\mathcal{P}}$.

3. Syzygies, resolutions, and regularity

In this section we briefly summarize some commutative algebra with the aim of introducing the notion of Castelnuovo-Mumford regularity. This concept is the fundamental tool we use to study when the function $\dim C^r_d(\mathcal{P})$ starts agreeing with the known polynomial functions in d. Along the way, we introduce *syzygy modules* and their natural grading. Syzygy modules are central both to the definition of Castelnuovo-Mumford regularity and to many of our later arguments. The first chapter of [12] is an excellent introduction to the approach we take here.

Let M be a finitely generated positively graded module over the polynomial ring $S = \mathbb{R}[x, y, z]$. The module M is finitely generated if there is a finite set of elements $m_1, \dots, m_k \in M$ so that every element $m \in M$ can be written as $m = \sum_{i=1}^k f_i m_i$ for some choice of polynomials $f_1, \dots, f_k$. The module M is positively graded if there is a decomposition $M = \bigoplus_{d \geq 0} M_d$, where M_d is an $\mathbb{R}$-vector space for every d. If M is finitely generated, then M_d is finite dimensional for every d. If $\mathcal{P} \subset \mathbb{R}^3$ is a polytopal cell, the module $C^r(\mathcal{P})$ is graded by $C^r(\mathcal{P}) = \bigoplus_{d \geq 0} C^r(\mathcal{P})_d$, where $C^r(\mathcal{P})_d$ is the space of splines of degree exactly d on $\mathcal{P}$. Moreover, $C^r(\mathcal{P})$ is always finitely generated as an S-module. Theoretically, this is because $C^r(\mathcal{P})$ is a submodule of the finite rank free module S^{f_3}, so it is Noetherian. Computationally, this is a simple consequence of Schreyer's observation that Buchberger's algorithm provides a finite generating set for the kernel of any matrix like the matrix in Lemma 2.1 (see [11, §15.5]). Hence, $C^r(\mathcal{P})_d$ is finite dimensional for any $d \geq 0$.

The *Hilbert function* of a finitely generated graded S-module M is the function $\dim M_d$. A standard result from commutative algebra states that for $d \gg 0$, $\dim M_d$ agrees with a polynomial function $HP(M, d)$, called the *Hilbert polynomial* of M. As we noted in the introduction, the largest integer d for which $\dim M_d \neq HP(M, d)$ is called the *postulation number* of M, which we denote by $\wp(M)$. The *Krull dimension* of a graded module M is defined by $\dim(M) = \deg(HP(M, d)) + 1$; this is a way of measuring the size of M. For instance, the Krull dimension of the polynomial ring $S = \mathbb{R}[x, y, z]$ is three, the number of variables. The *codimension* of M, denoted $\operatorname{codim}(M)$, is defined by $\operatorname{codim}(M) = \dim(S) - \dim(M) = 3 - \dim(M)$.

We now recall the notion of a *syzygy* module. Let N be a row vector with entries $m_1, \dots, m_k \in M$, where M is a graded S-module. Take formal basis elements $e_1, \dots, e_k$ corresponding to $m_1, \dots, m_k$ and let $\bigoplus_{i=1}^k Se_i$ be the free S module on this basis. The module of syzygies (or relations) on the entries of N, denoted $\operatorname{syz}(N)$, is defined as

$$\operatorname{syz}(N) = \ker \left(\bigoplus_{i=1}^{k} Se_i \xrightarrow{\phi} M \right),$$

where $\phi(\sum_{i=1}^k f_i e_i) = \sum f_i m_i$. If the entries of N are homogeneous (that is, each $m_i \in M_{a_i}$ for some integer a_i), one can make ϕ a degree-preserving map (and thus $\operatorname{syz}(N)$ a graded S-module) by stipulating that $\deg(e_i) = \deg(m_i) = a_i$. We denote Se_i using the shorthand $S(-a_i)$ to indicate that this is a copy of S with the unit in degree a_i. In other words, $(Se_i)_d = S(-a_i)_d \cong S_{d-a_i}$.

If M is a graded S-module, a set of elements $\{m_1, \dots, m_{k_0}\}$ of M with degrees $a_{0,1}, \dots, a_{0,k_0}$ (the reason for two subscripts will soon be evident) is called a *minimal* generating set for M if no proper subset of $\{m_1, \dots, m_{k_0}\}$ generates M as an S-module. This is equivalent to requiring that no m_i can be written as a polynomial combination of $\{m_s | s \neq i\}$. If the entries of a row vector N are a set of minimal generators for the module M, then we denote $\mathrm{syz}(N)$ by $\mathrm{syz}(M)$; this is the module of syzygies on M. The S-module M is *free* if there are no relations on a minimal generating set; in other words $\mathrm{syz}(M) = 0$.

Remark 3.1. As its name suggests, different choices of a minimal generating set give rise to isomorphic syzygy modules, so $\mathrm{syz}(M)$ is well defined up to isomorphism. In particular the degrees of the generators in a minimal generating set for M is invariant, even if the generators themselves are not.

Put $\mathrm{syz}_1(M) = \mathrm{syz}(M)$. For $n > 1$, the nth syzygy module of M is inductively defined as the module $\mathrm{syz}_n(M) = \mathrm{syz}(\mathrm{syz}_{n-1}(M))$. The first index δ so that $\mathrm{syz}_\delta(M)$ is free is called the *projective dimension* of M, which we denote by $\mathrm{pd}(M)$. Since we are working over a polynomial ring in three variables, a famous result of Hilbert, known as the Hilbert syzygy theorem, states that $\mathrm{pd}(M) \leq 3$. Suppose that $\mathrm{syz}_i(M)$ is minimally generated in degrees $\{a_{i,0}, \ldots, a_{i,k_i}\}$. It is standard practice to encode all of the syzygy modules of M at once in the *minimal free resolution* of M, which is an exact complex

$$F_\bullet : \ 0 \to F_\delta \xrightarrow{\phi_\delta} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_1} F_0,$$

where $\delta = \mathrm{pd}(M)$, $\mathrm{coker}(\phi_0) = M$, $\mathrm{im}(\phi_i) = \mathrm{syz}_i(M)$, and $F_i \cong \bigoplus_j S(-a_{i,j})$ for $i = 1, \ldots, \delta$. We now come to the main concept.

Definition 3.2. Let M be a graded S-module and suppose that $\mathrm{syz}_i(M)$ is minimally generated in degrees $\{a_{i,0}, \ldots, a_{i,k_i}\}$. The *Castelnuovo-Mumford regularity* of M, denoted $\mathrm{reg}(M)$, is defined by

$$\mathrm{reg}(M) = \max_{i,j}\{a_{i,j} - i\}.$$

Remark 3.3. The degrees of the minimal generators of M are the numbers $a_{0,j}$ which appear in the summands $S(-a_{0,j})$ of F_0. According to Definition 3.2, $\mathrm{reg}(M)$ bounds the minimal degree of generators of M as an S-module. If M is a free S-module, i.e., $M \cong \bigoplus_j S(-a_{0,j})$, then $\mathrm{reg}(M) = \max_j\{a_{0,j}\}$, the maximum degree of a generator of M.

Theorem 3.4 ([12, Theorem 4.2])**.** *Let M be a finitely generated graded module over $S = \mathbb{R}[x, y, z]$. Then $\dim M_d = HP(M, d)$ for $d \geq \mathrm{reg}(M) + \mathrm{pd}(M) - 2$. Equivalently, $\wp(M) \leq \mathrm{reg}(M) + \mathrm{pd}(M) - 3$.*

Remark 3.5. Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell. In the remainder of the paper, we focus our efforts on obtaining upper bounds on the regularity of $C^\alpha(\mathcal{P})$. Due to Theorem 3.4, these bounds translate into upper bounds on the postulation number of $C^\alpha(\mathcal{P})$. We will use a number of other results concerning regularity; for the convenience of the reader these are collected in Appendix A.

4. Regularity of splines on cells: the first reduction

Now we return to the setting of splines. For $\mathcal{P}$ a polytopal cell, our goal in this section is to show that the regularity of $C^\alpha(\mathcal{P})$ is bounded by the regularity of certain 'approximations' to $C^\alpha(\mathcal{P})$, using Proposition A.7.

The approximations we use are essentially submodules of splines with 'local support.' For $\mathcal{Q}$ a polytope σ or the union $\mathrm{st}(\tau)$ of two polytopes meeting along the face τ, put

$$\Lambda(\mathcal{Q}) = \prod_{\tau \in \partial \mathcal{Q}} l_\tau^{\alpha(\tau)+1} \in S \qquad \text{and} \qquad \lambda(\mathcal{Q}) = \deg(\Lambda(\mathcal{Q})) = \sum_{\tau \in \partial \mathcal{Q}} (\alpha(\tau)+1).$$

If σ is a polytope of $\mathcal{P}$, denote by $C^\alpha_\sigma(\mathcal{P})$ the submodule of $C^\alpha(\mathcal{P})$ consisting of splines F which vanish outside of σ. Such splines are characterized by F_σ being a polynomial multiple of $\Lambda(\sigma)$. Hence $C^\alpha_\sigma(\mathcal{P}) \cong \langle \Lambda(\sigma) \rangle \subset S$, the principal ideal in S generated by the polynomial $\Lambda(\sigma)$. Likewise, if τ is an interior face of $\mathcal{P}$ denote by $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ the submodule of splines which vanish outside the union of the two polytopes which meet along τ. Define the two 'locally-supported' submodules

$$LS^{\alpha,0}(\mathcal{P}) := \sum_{\sigma \in \mathcal{P}} C^\alpha_\sigma(\mathcal{P}) \cong \bigoplus_{\sigma \in \mathcal{P}} \langle \Lambda(\sigma) \rangle \qquad \text{and} \qquad LS^{\alpha,1}(\mathcal{P}) := \sum_{\tau \in \mathcal{P}_2^\circ} C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}).$$

Remark 4.1. The isomorphism $LS^{\alpha,0}(\mathcal{P}) \cong \bigoplus \langle \Lambda(\sigma) \rangle$ follows from the isomorphisms $C^\alpha_\sigma(\mathcal{P}) \cong \langle \Lambda(\sigma) \rangle$ and the fact that the summands $C^\alpha_\sigma(\mathcal{P})$ have disjoint support inside of $C^\alpha(\mathcal{P})$.

Remark 4.2. Since $C^\alpha(\mathcal{P})$ is graded module over $S = \mathbb{R}[x, y, z]$, so are the submodules $C^\alpha_{\mathcal{Q}}(\mathcal{P})$ for $\mathcal{Q} = \sigma$ or $\mathcal{Q} = \mathrm{st}(\tau)$. Hence the submodules $LS^{\alpha,0}(\mathcal{P})$ and $LS^{\alpha,1}(\mathcal{P})$ are graded S-modules as well.

Remark 4.3. The modules $LS^{\alpha,0}(\mathcal{P})$ and $LS^{\alpha,1}(\mathcal{P})$ are introduced as part of a larger family of approximations to $C^\alpha(\mathcal{P})$ in [9]. The definition given here differs slightly from that given in [9], nevertheless it is equivalent.

Example 4.4. Consider the polygonal complex $\mathcal{Q}$ from Example 1.1. A spline $F \in LS^{\alpha,1}(\widehat{\mathcal{Q}})$ is a sum of splines $F = \sum_{\tau \in \mathcal{Q}_1^\circ} f_\tau$ so that the support of f_τ is contained in the union of two polytopes σ_1, σ_2 which meet along $\widehat{\tau}$. In other words, the support of f_τ is contained in (the cone over) one of the six shaded regions in Figure 4. The order of vanishing of f_τ along the boundary of $\mathrm{st}(\tau)$ is prescribed by the smoothness parameters α.

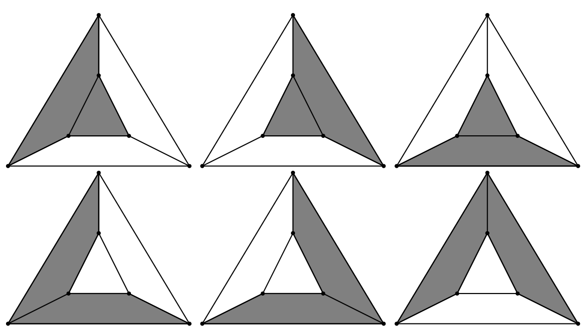


FIGURE 4. Stars of edges for the complex $\mathcal{Q}$ in Example 4.4

The crucial property of the submodules $LS^{\alpha,k}(\mathcal{P})$, explaining in what sense these approximate $C^\alpha(\mathcal{P})$, is provided by the following result. Recall that the codimension of an $S = \mathbb{R}[x, y, z]$-module M, denoted $\mathrm{codim}(M)$, is $\dim(S) - \dim(M) = 3 - \dim(M)$, where the dimension considered is the Krull dimension (see §3). We provide a proof of the following result for $k = 0, 1$.

Proposition 4.5 ([9, Theorem 4.3]). *Let $\mathcal{P}$ be a polytopal cell. Then in the tautological short exact sequence of S-modules*

$$0 \to LS^{\alpha,k}(\mathcal{P}) \xrightarrow{i} C^\alpha(\mathcal{P}) \to C \to 0,$$

where C is the cokernel of the natural inclusion i, the codimension of C is at least $k+1$.

Proof of Proposition 4.5 *for $k = 0, 1$.* The key point of the proof is that $\operatorname{codim}(C) \geq k+1$ if and only if $C_P = 0$ for all primes P of codimension at most k, where C_P is the *localization* of C at the prime P (see [11, Chapter 2]). To show this we will localize the short exact sequence

$$0 \to LS^{\alpha,k}(\mathcal{P}) \xrightarrow{i} C^\alpha(\mathcal{P}) \to C \to 0$$

at primes $P \subset S$ with codimension at most k, and prove that the localized map i_P is an isomorphism. For $k = 0$, the only prime of codimension 0 is the zero ideal. Localizing at the zero ideal is the same as tensoring with the fraction field $\mathbb{Q}(S)$ of S (this is the field of rational functions in three variables). By definition of $LS^{\alpha,0}(\mathcal{P})$, it is clear that

$$LS^{\alpha,0}(\mathcal{P}) \otimes_S \mathbb{Q}(S) \cong \left(\bigoplus_{\sigma \in \mathcal{P}} \langle \Lambda(\sigma) \rangle \right) \otimes_S \mathbb{Q}(S) \cong \mathbb{Q}(S)^{f_3}.$$

From the containment $LS^{\alpha,0}(\mathcal{P}) \subset C^\alpha(\mathcal{P}) \subset S^{f_3}$ and the fact that $S^{f_3} \otimes_S \mathbb{Q}(S) \cong \mathbb{Q}(S)^{f_3}$, we have that $C^\alpha(\mathcal{P}) \otimes_S \mathbb{Q}(S) \cong \mathbb{Q}(S)^{f_3} \cong LS^{\alpha,0}(\mathcal{P}) \otimes_S \mathbb{Q}(S)$. Hence the localized map i_P is an isomorphism in the case $k = 0$.

For $k = 1$, we must show that the localized inclusion i_P is an isomorphism for prime ideals $P \subset S$ of codimension one and zero. If P has codimension zero (i.e., $P = 0$), then the containment $LS^{\alpha,0}(\mathcal{P}) \subset LS^{\alpha,1}(\mathcal{P})$ shows $LS^{\alpha,1}(\mathcal{P}) \otimes_S \mathbb{Q}(S) \cong \mathbb{Q}(S)^{f_3} \cong C^\alpha(\mathcal{P}) \otimes_S \mathbb{Q}(S)$ as before. Now suppose P has codimension one. Codimension one ideals of S are necessarily principal, so $P = \langle f \rangle$ for some irreducible polynomial $f \in S$. If $f \neq l_\tau$ for some $\tau \in \mathcal{P}_2$, then $C^\alpha(\mathcal{P})_P = S_P^{f_n} = LS^{\alpha,1}(\mathcal{P})_P$ (all gluing conditions in the matrix from Lemma 2.1 are inverted). Otherwise, $P = \langle l_\tau \rangle$ and only gluing conditions along edges whose affine span contains τ are retained, yielding

$$C^\alpha(\mathcal{P})_P \cong S_P^h \oplus \bigoplus_{\tau \subset \operatorname{aff}(\psi)} C^\alpha_{\operatorname{st}(\psi)}(\mathcal{P})_P,$$

where in the rightmost direct sum ψ runs across faces of $\mathcal{P}$ having the same affine span as τ and h is the number of polytopes having no face whose affine span contains τ. Localizing $LS^{\alpha,1}(\mathcal{P})$ at P yields the same result, so we are done. ☐

Before proving the main result of the section, we need the following observation in order to apply Proposition A.7.

Lemma 4.6. [7, Proposition 3.4] *If $\mathcal{P} \subset \mathbb{R}^3$ is a polytopal cell, then*

(1) $\operatorname{pd}(C^\alpha(\mathcal{P})) \leq 1$,
(2) $\wp(C^\alpha(\mathcal{P})) \leq \operatorname{reg}(C^\alpha(\mathcal{P})) - 2$.

Proof. It follows from Lemma 2.1 that $C^\alpha(\mathcal{P})$ is the kernel of a map ϕ between free S-modules. This means it can be viewed as the second syzygy of $\operatorname{coker}(\phi)$. By the Hilbert syzygy theorem, the $S = \mathbb{R}[x,y,z]$-module $\operatorname{coker}(\phi)$ has projective dimension at most three. Since $C^\alpha(\mathcal{P})$ is a second syzygy module, $\mathrm{pd}(C^\alpha(\mathcal{P})) \leq 3-2=1$. Part (2) follows from (1) and Theorem 3.4. $\square$

Theorem 4.7. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell. Then*

(1) $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq \mathrm{reg}(LS^{\alpha,1}(\mathcal{P}))$,
(2) *If $C^\alpha(\mathcal{P})$ is free, then* $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq \mathrm{reg}(LS^{\alpha,0}(\mathcal{P}))$.

Proof. (1) By Lemma 4.6, $\mathrm{pd}(C^\alpha(\mathcal{P})) \leq 1$. By Proposition 4.5, the cokernel of the inclusion $LS^{\alpha,1}(\mathcal{P}) \hookrightarrow C^\alpha(\mathcal{P})$ has codimension at least two. The result follows from Proposition A.7. (2) If $C^\alpha(\mathcal{P})$ is free, then $\mathrm{pd}(C^\alpha(\mathcal{P}))=0$. By Proposition 4.5, the cokernel of the inclusion $LS^{\alpha,1}(\mathcal{P}) \hookrightarrow C^\alpha(\mathcal{P})$ has codimension at least one. The result follows again from Proposition A.7. $\square$

For the following corollary, recall $\lambda(\sigma)=\sum_{\tau\in\partial\sigma}(\alpha(\tau)+1)$.

Corollary 4.8. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell and suppose that $C^\alpha(\mathcal{P})$ is free. Set $f(\mathcal{P}) = \max\{\lambda(\sigma) \mid \sigma \in \mathcal{P}\}$. Then $C^\alpha(\mathcal{P})$ is generated in degrees at most $f(\mathcal{P})$.*

Proof. The maximum degree of a generator of $C^\alpha(\mathcal{P})$ is the regularity of $C^\alpha(\mathcal{P})$ (see Remark 3.3), so we need to show $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq f(\mathcal{P})$. The module $C^\alpha(\mathcal{P})$ is free if and only if $\mathrm{pd}(C^\alpha(\mathcal{P}))=0$. By Theorem 4.7, $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq \mathrm{reg}(LS^{\alpha,0}(\mathcal{P}))$. By definition, $LS^{\alpha,0} \cong \bigoplus_{\sigma\in\mathcal{P}}\langle\Lambda(\sigma)\rangle$. The principal ideal $\langle\Lambda(\sigma)\rangle$ is isomorphic, as a graded S-module, to $S(-\lambda(\sigma))$. Since the regularity of a direct sum of modules is the maximum of the regularity of the summands, $\mathrm{reg}(LS^{\alpha,0}(\mathcal{P})) = \max\{\mathrm{reg}(C^\alpha_\sigma(\mathcal{P})) \mid \sigma \in \mathcal{P}\} = \max\{\lambda(\sigma) \mid \sigma\in\mathcal{P}\} = f(\mathcal{P})$. $\square$

Remark 4.9. The centrally triangulated octahedron in Example 1.2 is a rare instance when Corollary 4.8 is exact. In fact, the bound in Corollary 4.8 is typically not tight. For instance, if $\Delta \subset \mathbb{R}^2$ is a triangulation so that $C^r(\widehat{\Delta})$ is free, it follows from work of Schenck and Stillman [27] that $\mathrm{reg}(C^r(\widehat{\Delta})) \leq 2(r+1)$. Corollary 4.8, on the other hand, only implies that $C^r(\widehat{\Delta})$ is generated in degrees at most $3(r+1)$.

5. Regularity of splines on cells: The second reduction

Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell. The goal in this section is to obtain a bound on the regularity of $LS^{\alpha,1}(\mathcal{P})$. To do that, we find an exact chain complex

$$0 \to C_n \to C_{n-1} \to \cdots \to C_2 \to C_1 \to LS^{\alpha,1}(\mathcal{P}) \to 0$$

of modules that will allow us to bound the regularity of M using Lemma A.4. The length of the chain complex is controlled by the maximal number of interior faces of the polytopes in $\mathcal{P}$.

We introduce a simplicial complex $\Delta(\mathcal{P})$. This complex will provide a convenient combinatorial structure that we will use to describe the chain complex of modules. The idea is that each polytope $\sigma \in \mathcal{P}$ will correspond to a $(d(\sigma)-1)$-dimensional simplex in $\Delta(\mathcal{P})$, where $d(\sigma)$ is the number of interior faces of σ.

Formally, the *vertices* of $\Delta(P)$ are given by the pairs

$$\Delta_0(\mathcal{P}) = \{(\tau,\sigma) \mid \sigma\in\mathcal{P}, \tau \text{ an interior face of } \sigma\}.$$

Fix an enumeration $\{\tau_1,\ldots,\tau_{d(\sigma)}\}$ of the interior faces of σ. For $k \geq 2$, the set of $(k-1)$-faces of $\Delta(\mathcal{P})$ is the set of k-tuples of vertices

$$\Delta_{k-1}(\mathcal{P}) = \{((\tau_{i_1},\sigma),\ldots,(\tau_{i_k},\sigma)\}.$$

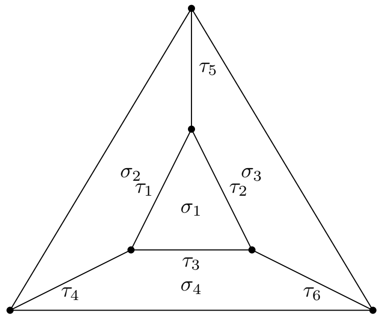


FIGURE 5. Labeling used in Examples 5.1 and 5.2 and

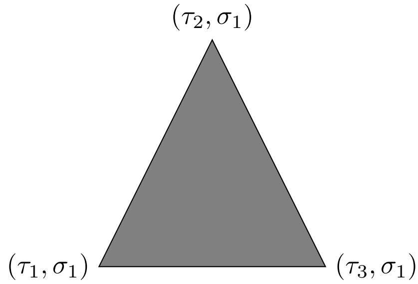


FIGURE 6. The simplex $\Delta(\sigma_1)$ in Example 5.1

Let us denote by $\Delta_k(\sigma)$ the set of k-faces whose vertices have the second coordinate σ. Finally, let

$$\Delta(\sigma) := \bigcup_{k=1}^{d(\sigma)} \Delta_k(\sigma) \text{ and } \Delta(\mathcal{P}) = \bigcup_{\sigma \in \mathcal{P}} \Delta(\sigma).$$

Note that $\Delta(\sigma)$ is a simplex for every $\sigma \in \mathcal{P}$.

Example 5.1. We describe the simplicial complex $\Delta(\mathcal{P})$ for the polytopal cell $\mathcal{P} = \widehat{\mathcal{Q}}$ in Example 1.1 and Example 4.4, using the labeling of polytopes and faces given in Figure 5. The simplicial complex $\Delta(\mathcal{P})$ is two-dimensional; it is a disjoint union of the four two-dimensional simplices $\Delta(\sigma_1), \Delta(\sigma_2), \Delta(\sigma_3), \Delta(\sigma_4)$. The simplex $\Delta(\sigma_1)$ is shown in Figure 6.

If I is an indexing set and M is the direct sum of modules $\{M_i \mid i \in I\}$ indexed by I, it is convenient to introduce formal basis elements e_i for the summand corresponding to the index i and write $M = \bigoplus_{i \in I} M_i e_i$. Elements of M are then written as $\sum_{i \in I} f_i e_i$, where $f_i \in M_i$. We are now ready to describe the chain complex of modules. For $\sigma \in \mathcal{P}$ define

$$C_0(\sigma) := C^\alpha_\sigma(\mathcal{P}) e_\sigma \qquad \text{and} \qquad C_0 := \bigoplus_{\sigma \in \mathcal{P}} C_0(\sigma) = \bigoplus_{\sigma \in \mathcal{P}} C^\alpha_\sigma(\mathcal{P}) e_\sigma,$$

where e_σ is a formal basis element corresponding to the polytope σ. For $k \geq 1$, define

$$C_k(\sigma) := \bigoplus_{s \in \Delta_{k-1}(\sigma)} C^\alpha_\sigma(\mathcal{P}) e_s \qquad \text{and} \qquad C_k := \bigoplus_{\sigma \in \mathcal{P}} C_k(\sigma) = \bigoplus_{s \in \Delta_{k-1}} C^\alpha_\sigma(\mathcal{P}) e_s,$$

where the e_s are formal basis elements corresponding to $s \in \Delta_{k-1}(\sigma)$ or $s \in \Delta_{k-1}$, respectively. Note that each of the summands $C^\alpha_\sigma(\mathcal{P})$ in the modules C_k are isomorphic (as ungraded modules) to the polynomial ring S. This will come into play in the proof of Proposition 5.4.

The map $\delta_1 : C_1 \to C_0$ is an *augmentation map.* For $F = \sum_{\tau\subset\sigma} f_{(\tau,\sigma)} e_{(\tau,\sigma)} \in C_1(\sigma)$,

$$\delta_1(F) = \left(\sum_{\tau\subset\sigma} f_{(\tau,\sigma)}\right) e_\sigma,$$

and extend to C_1 by linearity. The maps $\delta_k : C_k \to C_{k-1}$, for $k \geq 2$, are the simplicial boundary maps. For $f \in C^\alpha_\sigma(\mathcal{P})$ and $((\tau_0,\sigma),(\tau_1,\sigma),\ldots,(\tau_k,\sigma)) \in \Delta_k(\sigma)$, let

$$\delta_k(f \cdot e_{((\tau_0,\sigma),(\tau_1,\sigma),\ldots,(\tau_k,\sigma))}) := \sum_{j=0}^{k}(-1)^j f \cdot e_{((\tau_0,\sigma),(\tau_1,\sigma),\ldots,\widehat{(\tau_j,\sigma)},\ldots,(\tau_k,\sigma))},$$

and extend to C_k by linearity. We denote by $C_\bullet$ the chain complex

$$\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \to 0.$$

Example 5.2. We explicitly write out the chain complex $C_\bullet$ for the polytopal cell $\mathcal{P} = \widehat{\mathcal{Q}}$ from Example 5.1. First we describe the formal basis elements. We write e_{σ_i} for a formal basis elements in C_0, $e_{i\sigma_j}$ for the formal basis element corresponding to (τ_i,σ_j) in C_1, $e_{is\sigma_j}$ for the formal basis element corresponding to $((\tau_i,\sigma_j),(\tau_s,\sigma_j))$ in C_2, and $e_{ist\sigma_j}$ for the formal basis element corresponding to $((\tau_i,\sigma_j),(\tau_s,\sigma_j),(\tau_t,\sigma_j))$ in C_3. The chain complex $C_\bullet$ has the form

$$0 \to S^4 \xrightarrow{\delta_3} S^{12} \xrightarrow{\delta_2} S^{12} \xrightarrow{\delta_1} S^4 \to 0.$$

The matrices for the differentials (with rows and columns labeled according to the formal basis elements) are shown in Figure 7. The structure of these matrices reflect the fact that the chain complex $C_\bullet$ decomposes as a direct sum of four subcomplexes, one for each simplex $\Delta(\sigma_i)$.

Now let $C'_1 := \bigoplus_{\tau\in\mathcal{P}^\circ_2} C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})e_\tau$ and define $\delta'_1 : C'_1 \to LS^{\alpha,1}(\mathcal{P})$ by

$$\delta'_1 : \sum_\tau f_\tau e_\tau \mapsto \sum_\tau f_\tau.$$

We next note that since each interior face τ is shared by exactly two polytopes, say σ_1 and σ_2, there is a natural inclusion ι_1 of C_1 into C'_1 given by

$$\iota(f_1 \cdot e_{(\tau,\sigma_1)} + f_2 \cdot e_{(\tau,\sigma_2)}) = (f_1 + f_2) \cdot e_\tau,$$

where e_τ is the formal basis element corresponding to the summand $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ for an interior face τ. We denote by $C'_\bullet$ the chain complex

$$\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2\circ\iota_1} C'_1 \xrightarrow{\delta'_1} LS^{\alpha,1}(\mathcal{P}) \to 0.$$

There is also a natural inclusion of C_0 into $LS^{\alpha,1}(\mathcal{P})$ which we call ι_0. To see this, note that C_0 and $LS^{\alpha,1}(\mathcal{P})$ occur as submodules of $C^\alpha(\mathcal{P})$, and it is immediate that

$$\delta_3 = \begin{array}{cc} & \begin{array}{cccc} 123 & 145 & 256 & 346 \end{array} \\ \begin{array}{c} 23 \\ 13 \\ 12 \\ 45 \\ 15 \\ 14 \\ 56 \\ 26 \\ 25 \\ 46 \\ 36 \\ 34 \end{array} & \begin{pmatrix} 1 & & & \\ -1 & & & \\ 1 & & & \\ & 1 & & \\ & -1 & & \\ & 1 & & \\ & & 1 & \\ & & -1 & \\ & & 1 & \\ & & & 1 \\ & & & -1 \\ & & & 1 \end{pmatrix} \end{array},$$

$$\delta_2 = \begin{array}{cc} & \begin{array}{cccccccccccc} 23 & 13 & 12 & 45 & 15 & 14 & 56 & 26 & 25 & 46 & 36 & 34 \end{array} \\ \begin{array}{c} 1\sigma_1 \\ 2\sigma_1 \\ 3\sigma_1 \\ 1\sigma_2 \\ 4\sigma_2 \\ 5\sigma_2 \\ 2\sigma_3 \\ 5\sigma_3 \\ 6\sigma_3 \\ 3\sigma_4 \\ 4\sigma_4 \\ 6\sigma_4 \end{array} & \begin{pmatrix} & -1 & -1 & & & & & & & & & \\ -1 & & 1 & & & & & & & & & \\ 1 & 1 & & & & & & & & & & \\ & & & & -1 & -1 & & & & & & \\ & & & -1 & & 1 & & & & & & \\ & & & 1 & 1 & & & & & & & \\ & & & & & & & -1 & -1 & & & \\ & & & & & & -1 & & 1 & & & \\ & & & & & & 1 & 1 & & & & \\ & & & & & & & & & & -1 & -1 \\ & & & & & & & & & -1 & & 1 \\ & & & & & & & & & 1 & 1 & \end{pmatrix} \end{array},$$

$$\delta_1 = \begin{array}{cc} & \begin{array}{cccccccccccc} 1\sigma_1 & 2\sigma_1 & 3\sigma_1 & 1\sigma_2 & 4\sigma_2 & 5\sigma_2 & 2\sigma_3 & 5\sigma_3 & 6\sigma_3 & 3\sigma_4 & 4\sigma_4 & 6\sigma_4 \end{array} \\ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & & & & & & & & & \\ & & & 1 & 1 & 1 & & & & & & \\ & & & & & & 1 & 1 & 1 & & & \\ & & & & & & & & & 1 & 1 & 1 \end{pmatrix} \end{array}$$

FIGURE 7. Differentials of $C_\bullet$ in Example 5.2

$C_0 \subset LS^{\alpha,1}(\mathcal{P})$. It is a simple exercise to see that $\iota_0 \circ \delta_1 = \delta_1' \circ \iota_1$. We thus have an inclusion of chain complexes $\iota : C_\bullet \hookrightarrow C_\bullet'$ indicated in the following diagram:

$$\begin{tikzcd} \dots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\cong"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "\iota_1"] & C_0 \arrow[r] \arrow[d, "\iota_0"] & 0 \\ \dots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\iota_1 \circ \delta_2"] & C_1' \arrow[r, "\delta_1'"] & LS^{\alpha,1}(\mathcal{P}) \arrow[r] & 0 \end{tikzcd}$$

It is clear that the map δ_1' is surjective (this is the reason for considering the module C_1' instead of C_1), but we show below that the kernel of δ_1' is contained in the image of C_1 under ι_1.

Lemma 5.3. *For every $F \in \ker \delta_1'$, there is $G \in C_1$ such that $\iota_1(G) = F$.*

Proof. Write $F = \sum_\tau f_\tau e_\tau$, where $f_\tau \in C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$. We show that $f_\tau e_\tau$ is in the image of ι_1 for all τ. For a polytope $\sigma \in \mathcal{P}$, write $f_{(\tau,\sigma)}$ for the restriction $f_\tau|_\sigma$ (this is zero unless τ is a face of σ). It suffices to show that $f_{(\tau,\sigma)} \in C^\alpha_\sigma(\mathcal{P})$ for every pair (τ,σ). By assumption, $\sum_\tau f_\tau = 0$; restricting to a polytope σ yields $\sum_\tau f_{(\tau,\sigma)} = 0$, where the sum runs across faces of σ (no repeats). Now since $f_{(\tau,\sigma)} \in C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$, it must vanish on every face of σ other than τ. Since every summand $f_{(\tau',\sigma)}$ for $\tau' \neq \tau$ must vanish on τ, and the sum $\sum_\tau f_{(\tau,\sigma)}$ vanishes, $f_{(\tau,\sigma)}$ must also vanish along τ, hence $f_{(\tau,\sigma)} \in C^\alpha_\sigma(\mathcal{P})$, as desired. $\square$

Proposition 5.4. *The chain complex*

$$C'_\bullet = \cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2 \circ \iota_1} C'_1 \xrightarrow{\delta'_1} LS^{\alpha,1}(\mathcal{P}) \to 0$$

is exact.

Proof. The map δ'_1 is clearly surjective, so the chain complex is exact at $LS^{\alpha,1}(\mathcal{P})$. The inclusion $\iota : C_\bullet \hookrightarrow C'_\bullet$ induces a map on homologies $\iota^\#_i : H_i(C_\bullet) \to H_i(C'_\bullet)$. We claim that $\iota^\#_i$ is an isomorphism for every $i \geq 1$. This is immediate for $i \geq 2$. For $i = 1$, we need to show that

$$H_1(C_\bullet) = \frac{\ker(\delta_1)}{\mathrm{im}(\delta_2)} \cong \frac{\ker(\delta'_1)}{\mathrm{im}(\iota_1 \circ \delta_2)} = H_1(C'_\bullet).$$

Since $\iota_0 \circ \delta_1 = \iota_1 \circ \delta'_1$ and ι_0, ι_1 are both injective, $\iota_1(\ker(\delta_1)) = \iota_1(C_1) \cap \ker(\delta'_1)$. By Lemma 5.3, $\ker(\delta'_1)$ is a subset of $\iota_1(C_1)$. So we have $\iota_1(\ker(\delta_1)) = \ker(\delta'_1)$ and thus the quotient map $\bar{\iota}_1 : \ker(\delta_1) \to H_1(C'_\bullet)$ is surjective. The kernel of $\bar{\iota}_1$ is clearly $\mathrm{im}(\delta_2)$, so the isomorphism $H_1(C_\bullet) \cong H_1(C'_\bullet)$ follows from the first isomorphism theorem.

Now it suffices to show that the complex $C_\bullet$ is exact. From the definition of the differentials δ_k it is clear that the chain complex $C_\bullet$ is a direct sum of chain complexes $C_\bullet = \bigoplus_{\sigma \in \mathcal{P}} C_\bullet(\sigma)$ (see also Example 5.2), where $C_\bullet(\sigma)$ is the chain complex

$$\cdots \xrightarrow{\delta_3} C_2(\sigma) \xrightarrow{\delta_2} C_1(\sigma) \xrightarrow{\delta_1} C_0(\sigma) \to 0.$$

So it suffices to show that $H_i(C_\bullet(\sigma)) = 0$ for $i \geq 1$. The maps δ_i, $i \geq 1$, are the maps used in simplicial homology (see [16, Chapter 2]). In particular, the presence of the augmentation map $\delta_1 : C_1(\sigma) \to C_0(\sigma)$ ensures that the homology $H_\bullet(C_\bullet)$ is the *reduced simplicial homology* $\widetilde{H}_{\bullet-1}(\Delta(\sigma); S)$ of the simplex $\Delta(\sigma)$ with coefficients in S (there is a shift of -1 in dimension because simplicial homology is indexed by dimension, whereas the index i in C_i corresponds to faces of $\Delta(\mathcal{P})$ which have dimension $i-1$). It is well known that the reduced simplicial homology of a simplex vanishes in all dimensions, thus $H_i(C_\bullet) = 0$ for all $i \geq 0$. $\square$

Theorem 5.5. *Suppose $\mathcal{P} \subset \mathbb{R}^3$ is polytopal cell. Let $f(\mathcal{P}) = \max\{\lambda(\sigma) \mid \sigma \in \mathcal{P}\}$ and $T = \max_{\tau \in \mathcal{P}^\circ_2}\{\mathrm{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))\}$. Then $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq \max\{f(\mathcal{P}) - 1, T\}$.*

Proof. By Corollary 4.7, $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq \mathrm{reg}(LS^{\alpha,1}(\mathcal{P}))$. By Proposition 5.4, $LS^{\alpha,1}(\mathcal{P})$ fits into the exact sequence

$$\cdots \to C_2 \to C'_1 \xrightarrow{\delta'_1} LS^{\alpha,1}(\mathcal{P}) \to 0.$$

The summands of C_i, $i \geq 2$, have the form $C^\alpha_\sigma(\mathcal{P})$. As we saw in Section 4, $C^\alpha_\sigma(\mathcal{P}) \cong S(-\lambda(\sigma))$. Since the regularity of a direct sum is the maximum of the regularity of its summands,

$$\operatorname{reg}(C_i) \leq \max\{\lambda(\sigma) \mid \sigma \in \mathcal{P}\} = f(\mathcal{P})$$

for every $i \geq 2$. Now $C_1' = \bigoplus_{\tau \in \mathcal{P}_2^\circ} C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$, so

$$\operatorname{reg}(C_1') = \max_{\tau \in \mathcal{P}_2^\circ}\{\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))\} = T.$$

Now the conclusion follows from Lemma A.4. □

6. Regularity of splines on cells: The third reduction

Suppose $\mathcal{P} \subset \mathbb{R}^3$ is a polytopal cell. In this section we complete the proof of Theorem 6.2 by providing an explicit bound on $\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))$, where τ is an interior face of $\mathcal{P}$, and using Theorem 5.5. As in Section 4, we accomplish this by 'approximating' $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ and using Proposition A.7.

Let $\tau \in \mathcal{P}_2^\circ$ be an interior face of $\mathcal{P}$, and σ_1, σ_2 the two polytopes which meet along τ to form $\mathrm{st}(\tau)$. Recall that if $\mathcal{Q}$ is a polytope σ or the union $\mathrm{st}(\tau)$ of two polytopes sharing the face τ, then $\Lambda(\mathcal{Q}) = \prod_{\tau \in \partial \mathcal{Q}} l_\tau^{\alpha(\tau)+1}$ and $\lambda(\mathcal{Q}) = \deg(\Lambda(\mathcal{Q}))$.

Take formal basis symbols e_1, e_2 corresponding to σ_1, σ_2 and let $Se_1 + Se_2$ be the free S-module of rank two containing $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$. Let

$$F_1 = \Lambda(\sigma_1)e_1, \qquad F_2 = \Lambda(\sigma_2)e_2, \qquad F_\tau = \Lambda(\mathrm{st}(\tau))(e_1 + e_2),$$

and define $M(\tau)$ to be the submodule of $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$ generated by F_1, F_2, and F_τ. Set $L_\tau = l_\tau^{\alpha(\tau)+1}$, $L_1 = \Lambda(\sigma_1)/L_\tau$, and $L_2 = \Lambda(\sigma_2)/L_\tau$. If $\Lambda(\mathrm{st}(\tau)) \neq 1$ (in other words, if $\alpha(\tau') \neq -1$ for all $\tau' \in \partial\mathrm{st}(\tau)$), then there is a single nontrivial syzygy among F_1, F_2, F_τ given by $L_\tau F_\tau = L_2 F_1 + L_1 F_2$. So $M(\tau)$ has minimal free resolution

$$0 \longrightarrow S(-\lambda(\mathrm{st}(\tau)) - \alpha(\tau) - 1) \longrightarrow \begin{array}{c} S(-\lambda(\sigma_1)) \\ \oplus \\ S(-\lambda(\mathrm{st}(\tau))) \\ \oplus \\ S(-\lambda(\sigma_2)). \end{array}$$

It follows immediately from Definition 3.2 that, if $\Lambda(\mathrm{st}(\tau)) \neq 1$, then $\operatorname{reg}(M(\tau)) = \lambda(\mathrm{st}(\tau)) + \alpha(\tau)$. If $\Lambda(\mathrm{st}(\tau)) = 1$, then $M(\tau)$ is free, generated by F_τ and either F_1 or F_2, so it follows from Remark 3.3 that $\operatorname{reg}(M(\tau)) = \deg(F_1) = \deg(F_2) = \alpha(\tau) + 1$. In fact, it is clear in this case that $M(\tau) = C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$.

Proposition 6.1. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell and $\tau \in \mathcal{P}_2^\circ$ a face of $\mathcal{P}$. Define $\lambda(\tau) = \lambda(\mathrm{st}(\tau)) + \alpha(\tau) + 1 = \sum_{\gamma \in (\mathrm{st}(\tau))_2}(\alpha(\gamma) + 1)$. Then $\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})) \leq \lambda(\tau) - 1$ unless $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in \partial\mathrm{st}(\tau)$, when $\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})) = \alpha(\tau) + 1$.*

Proof. The remarks directly preceding Proposition 6.1 establish that $\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})) = \alpha(\tau) + 1$ if $\alpha(\gamma) = -1$ for all $\gamma \neq \tau \in \partial\mathrm{st}(\tau)$, so assume $\alpha(\gamma) \geq 0$ for at least one $\gamma \neq \tau \in \partial\mathrm{st}(\tau)$. We use the approximation $M(\tau)$ introduced above and Proposition A.7. We have already seen that $\operatorname{reg}(M(\tau)) = \lambda(\tau) - 1$. By Proposition A.7, it suffices to show that $\operatorname{codim}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})/M(\tau)) \geq 2$. As in the proof of Proposition 4.5, we accomplish this by showing equality of localizations $(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))_P = M(\tau)_P$ for every prime P of codimension at most one.

If $\mathrm{codim}(P) = 0$, then P is the zero ideal. Just as in the proof of Proposition 4.5, $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})_P \cong C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}) \otimes_S \mathbb{Q}(S) \cong \mathbb{Q}(S)^2 \cong M(\tau) \otimes_S \mathbb{Q}(S) = M(\tau)_P$.

Now suppose $\mathrm{codim}(P) = 1$. Primes of S with codimension one are principle, generated by a single irreducible polynomial. If $P \neq \langle l_\gamma \rangle$ for any $\gamma \in (\mathrm{st}(\tau))_2$, then

$$(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))_P = M(\tau)_P = S_P^2.$$

If $P = \langle l_\gamma \rangle$ for some $\gamma \neq \tau \in (\mathrm{st}(\tau))_2$, then

$$(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))_P = M(\tau)_P \cong \begin{cases} l_\gamma^{\alpha(\gamma)+1} S_P \oplus S_P & \begin{array}{l}\mathrm{aff}(\gamma) \text{ meets a face of} \\ \text{only one of } \sigma_1 \text{ and } \sigma_2,\end{array} \\ l_\gamma^{\alpha(\gamma)+1} S_P \oplus l_\gamma^{\alpha(\gamma)+1} S_P & \begin{array}{l}\mathrm{aff}(\gamma) \text{ meets a face of} \\ \text{both } \sigma_1 \text{ and } \sigma_2.\end{array} \end{cases}$$

If $P = \langle l_\tau \rangle$, then $(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P}))_P = M(\tau)_P = (C^{\alpha(\tau)}(\mathrm{st}(\tau)))_P$, where $C^{\alpha(\tau)}(\mathrm{st}(\tau))$ is the module of splines on $\mathrm{st}(\tau)$ with $\alpha(\tau') = -1$ for every $\tau' \in \partial \mathrm{st}(\tau)$.

It follows that $\mathrm{codim}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})/M(\tau)) \geq 2$. By Lemma 4.6, $\mathrm{pd}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})) \leq 1$, so $\mathrm{reg}(C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})) \leq \mathrm{reg}(M(\tau)) = \lambda(\tau) - 1$ by Proposition A.7. ☐

Theorem 6.2. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polytopal cell and set $e(\mathcal{P}) = \max\{\lambda(\tau) \mid \tau \in \mathcal{P}_2^\circ\}$, where $\lambda(\tau)$ is as in Proposition* 6.1*. Then:*

(1) $\mathrm{reg}(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 1$,
(2) $\wp(C^\alpha(\mathcal{P})) \leq e(\mathcal{P}) - 3$.

In particular, $HP(C^\alpha(\mathcal{P}), d) = \dim_\mathbb{R} C^r_d(\mathcal{P})$ for $d \geq e(\mathcal{P}) - 2$.

Proof. (1) follows by applying Theorem 5.5 to Proposition 6.1 (notice $\lambda(\tau) \geq \lambda(\sigma)$ whenever $\tau \subset \sigma$). (2) follows from (1) by Lemma 4.6. ☐

Remark 6.3. General formulas for the polynomial $HP(C^\alpha(\mathcal{P}), d)$ appearing in Theorem 6.2 may be found in [20] (uniform smoothness) and [10] (mixed smoothness). For Example 1.1 and Example 8.3 we give the polynomial $HP(C^\alpha(\mathcal{P}), d)$ explicitly.

Example 1.1 indicates that the bound given in Theorem 6.2 can be far from optimal. In the next section we bound $\mathrm{reg}(C^\alpha_{\mathrm{st}(\tau)}(\Delta))$ more precisely for $\Delta \subset \mathbb{R}^3$ a simplicial cell with uniform smoothness.

7. Regularity of splines on simplicial cells with uniform smoothness

In this section we derive our main result on the regularity of the spline module $C^\alpha(\Delta)$, where Δ is a simplicial cell with uniform smoothness r. That is, $\alpha(\tau) = r$ for every interior face $\tau \in \Delta_2^\circ$, and $\alpha(\tau) = -1$ or $\alpha(\tau) = r$ for every boundary face $\tau \in \partial\Delta$. This is slightly more general than the module $C^r(\Delta)$ since we allow vanishing of order r to be imposed along boundary faces. By Theorem 5.5, we may achieve a regularity bound on $C^\alpha(\Delta)$ by bounding the regularity of splines $C^\alpha_{\mathrm{st}(\tau)}(\Delta)$ vanishing outside the union of two tetrahedra meeting along the face τ. Our main result in this section is the following.

Theorem 7.1. *Let $\tau \in \Delta_2^\circ$ be a face of the simplicial cell Δ, and let α be a set of smoothness parameters on Δ_2 so that $\alpha(\tau) = r$ for every $\tau \in \Delta_2^\circ$ and $\alpha(\tau) = -1$ or $\alpha(\tau) = r$ for every $\tau \in \partial\Delta$. Then $\mathrm{reg}(C^\alpha_{\mathrm{st}(\tau)}(\Delta)) \leq 3r + 3$.*

Before proving Theorem 7.1 we derive the main result.

Theorem 7.2. *Let $\Delta \subset \mathbb{R}^3$ be a simplicial cell and let α be a set of smoothness parameters on Δ_2 so that $\alpha(\tau) = r$ for every $\tau \in \Delta_2^\circ$ and $\alpha(\tau) = -1$ or $\alpha(\tau) = r$ for every $\tau \in \partial\Delta$. Then:*

(1) $\operatorname{reg}(C^\alpha(\Delta)) \le 3r + 3$,
(2) $\wp(C^\alpha(\Delta)) \le 3r + 1$.

In particular, $HP(C^\alpha(\Delta), d) = \dim_{\mathbb{R}} C^r(\Delta)_d$ for $d \ge 3r + 2$.

Proof. (1) follows by applying Theorem 5.5 to Theorem 7.1 (notice that $\lambda(\sigma) \le 3r + 3$). (2) follows by applying Theorem 3.4 to (1). □

Corollary 7.3. *Let $\Delta \in \mathbb{R}^2$ be a triangulation and let α be smoothness parameters so that $\alpha(\tau) = r$ for every interior edge τ and $\alpha(\tau) = r$ or $\alpha(\tau) = -1$ for every boundary edge τ (in particular, this includes the case where vanishing of order r is imposed along the entire boundary). Then $\dim C_d^\alpha(\Delta) = HP(C^\alpha(\widehat{\Delta}), d)$ for $d \ge 3r + 2$.*

In particular, if no vanishing is imposed along the boundary, then the Alfeld-Schumaker formula for $\dim C_d^r(\Delta)$ holds for $d \ge 3r + 2$. That is, for $d \ge 3r + 2$,

$$\dim C_d^r(\Delta) = \binom{d+2}{2} + \binom{d-r+1}{2} f_1^\circ - \left(\binom{d+2}{2} - \binom{r+2}{2} \right) f_0^\circ + \sigma,$$

where

$$\sigma = \sum_{v \in \Delta_0^\circ} \sigma_v \qquad \sigma_v = \sum_{j=1}^{r} (r + j + 1 - jn(v))_+,$$

$n(v)$ is the number of distinct slopes at the interior vertex v, and

$$(r + j + 1 - jn(v))_+ = \max\{r + j + 1 - jn(v), 0\}.$$

Proof. Applying Theorem 7.2 to the simplicial cell $\widehat{\Delta}$ yields $\wp(C^\alpha(\widehat{\Delta})) \le 3r + 1$, so $\dim C_d^\alpha(\Delta) = \dim C^\alpha(\widehat{\Delta})_d = HP(C^r(\widehat{\Delta}), d)$ for $d \ge 3r + 2$, where the first equality follows from Lemma 2.2. This finishes the proof of the first statement. In the second statement, the expression for $\dim C_d^r(\Delta)$ is the well-known Schumaker lower bound [29], which was shown to be equal to $HP(C^r(\widehat{\Delta}), d)$ by Schenck [26]. □

Remark 7.4. The second statement in Corollary 7.3, that $\dim C_d^r(\Delta)$ agrees with Schumaker's lower bound for $d \ge 3r+2$, was obtained by Hong [17] and Ibrahim and Schumaker [18] (see Table 1 in the introduction). If the triangulation Δ is in addition *nondegenerate*, then Alfeld and Schumaker proved in [3] that $\dim C_d^r(\Delta)$ agrees with Schumaker's lower bound for $d \ge 3r + 1$. An expression for $HP(C^\alpha(\widehat{\Delta}), d)$ where vanishing of order r is imposed along the entire boundary of a planar triangulation Δ (relevant to the first statement of Corollary 7.3) may be found in [19].

The proof we give of Theorem 7.1 is more delicate than the proof of Proposition 6.1, so we break it up over a sequence of several lemmas proving special cases or equivalent formulations. We attempt to introduce all tools from commutative algebra in a self-contained manner.

First we set up the notation for the proof of Theorem 7.1. Figure 8 depicts our situation. We will abuse notation and write v_i both for the corresponding edge of $\operatorname{st}(\tau)$ and for the vector we obtain by taking positive real multiples of this edge.

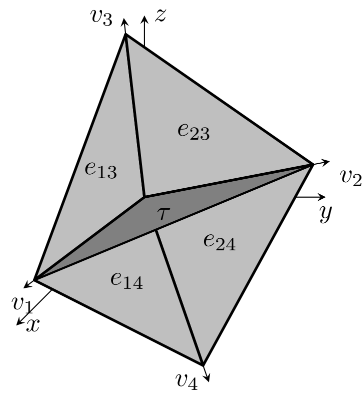


FIGURE 8. A typical configuration for $\mathrm{st}(\tau)$

Let $u_1, u_2 \in S$ be the forms vanishing on the affine spans of e_{13}, e_{23}, let w_1, w_2 be the forms vanishing on the affine spans of e_{14}, e_{24}, and l_τ be the form vanishing on the affine span of τ (for now do this without coordinates). Denote by σ_1 the tetrahedron with faces defined by u_1, u_2, l_τ and by σ_2 the tetrahedron with faces defined by w_1, w_2, l_τ. Let $\alpha_1 = \alpha(e_{13})+1, \alpha_2 = \alpha(e_{23})+1, \beta_1 = \alpha(e_{14})+1, \beta_2 = \alpha(e_{24})+1$ be the exponents to appear on u_1, u_2, w_1, w_2 corresponding to the smoothness parameters specified by α. By our assumption of uniform smoothness, $\alpha_1, \alpha_2, \beta_1$, and β_2 can only take on the values 0 and $r+1$. Since the face τ is interior, $\alpha(\tau) = r$. We first take care of a special case of Theorem 7.1 which follows immediately from Proposition 6.1.

Lemma 7.5. *Suppose that at most two of $\alpha_1, \alpha_2, \beta_1, \beta_2$ do not vanish. Then* $\mathrm{reg}(C^\alpha_{\mathrm{st}(\tau)}(\Delta)) \leq 3r+3$.

Now assume that three or more of $\alpha_1, \alpha_2, \beta_1, \beta_2$ do not vanish. Without loss of generality, we will assume that $\alpha_2 = \beta_1 = \beta_2 = r+1$, and α_1 is either 0 or $r+1$. Our next step is to identify $C^\alpha_{\mathrm{st}(\tau)}(\Delta)$ as the module of syzygies of a row vector with entries in S. Let $N(\tau)$ be the row vector

$$N(\tau) = \begin{bmatrix} l_\tau^{r+1} & u_1^{\alpha_1} u_2^{r+1} & w_1^{r+1} w_2^{r+1} \end{bmatrix}.$$

Using the convention introduced in Section 3, we take formal basis elements s_τ of degree $r+1$, s_1 of degree $\alpha_1 + r + 1$, and s_2 of degree $2(r+1)$ corresponding to the entries of $N(\tau)$ reading from left to right. By definition, a syzygy of $N(\tau)$ is a formal sum $G_\tau s_\tau + G_1 s_1 + G_2 s_2 \in S^3$ satisfying

$$G_\tau l_\tau^{r+1} + G_1 u_1^{\alpha_1} u_2^{r+1} + G_2 w_1^{r+1} w_2^{r+1} = 0.$$

Take formal basis elements e_1, e_2 (both of degree zero) corresponding to the tetrahedra σ_1, σ_2 and let $Se_1 + Se_2$ be the free S-module of rank two containing $C^\alpha_{\mathrm{st}(\tau)}(\mathcal{P})$. Consider the S-module map $\pi : Ss_\tau + Ss_1 + Ss_2 \to Se_1 + Se_2$ defined by $\pi(G_\tau s_\tau + G_1 s_1 + G_2 s_2) = G_1 u_1^{r+1} u_2^{r+1} e_1 - G_2 w_1^{r+1} w_2^{\beta_2} e_2$. Notice that the image of $\mathrm{syz}(N(\tau))$ under π is contained in $C^\alpha_{\mathrm{st}(\tau)}(\Delta)$. To see this let $F_1 = G_1 u_1^{\alpha_1} u_2^{r+1}$ and $F_2 = -G_2 w_1^{r+1} w_2^{r+1}$ and observe that the three spline conditions

$$u_1^{\alpha_1} u_2^{r+1} | F_1, w_1^{r+1} w_2^{r+1} | F_2, \text{ and } l_\tau^{r+1} | F_1 - F_2$$

are satisfied. It is straightforward to check that the map π above restricted to $\mathrm{syz}(N(\tau))$ yields an isomorphism of $\mathrm{syz}(N(\tau))$ and $C^{\alpha}_{\mathrm{st}(\tau)}(\Delta)$ as S-modules. Moreover, the grading assigned to the formal symbols $s_\tau, s_1, s_2, e_1, e_2$ guarantee that this map is a *graded* map. We have proved the following lemma.

Lemma 7.6. *The modules* $C^{\alpha}_{\mathrm{st}(\tau)}(\Delta)$ *and* $\mathrm{syz}(N(\tau))$ *are isomorphic as graded* S*-modules.*

Denote by $K(\tau)$ the ideal in S generated by the entries of $N(\tau)$. Recall from Section 3 that $K(\tau)$ is *minimally generated* by the entries of $N(\tau)$ if none of the entries of $N(\tau)$ can be written as an S-linear combination of the remaining entries.

Lemma 7.7. *Suppose* $K(\tau)$ *is not minimally generated by the entries of* $N(\tau)$*. Then* $\mathrm{reg}(\mathrm{syz}(N(\tau))) \leq 3(r+1)$*. In particular,* $\mathrm{reg}(C^{\alpha}_{\mathrm{st}(\tau)}(\Delta)) \leq 3(r+1)$*.*

Proof. By degree considerations we may assume that $w_1^{r+1}w_2^{r+1}$ can be written as a polynomial combination of l_τ^{r+1} and $u_1^{\alpha_1}u_2^{r+1}$. In other words, there is the equation

$$f_\tau l_\tau^{r+1} + f_1 u_1^{\alpha_1}u_2^{r+1} = w_1^{r+1}w_2^{r+1}.$$

Clearly this yields the syzygy $F = l_\tau^{r+1}s_\tau + f_1 s_1 - s_2 \in \mathrm{syz}(N(\tau))$ of degree $2(r+1)$. In this case the only other syzygy of interest is

$$G = u_1^{\alpha_1}u_2^{r+1}s_\tau - l_\tau^{r+1}s_1$$

of degree $2(r+1)+\alpha_1 \leq 3(r+1)$. It is not difficult to check that the two syzygies F and G generate $\mathrm{syz}(N(\tau))$ as an S-module. There are clearly no relations among F, G, so $\mathrm{syz}(N(\tau))$ is free as an S-module, generated in degrees $2(r+1)$ and $2(r+1)+\alpha_1$. Now the result follows at once from Remark 3.3 and Lemma 7.6. $\square$

From Lemma 7.7 we may now assume that $K(\tau)$ is minimally generated by the entries of $N(\tau)$ and so can identify $\mathrm{syz}(N(\tau))$ with $\mathrm{syz}(K(\tau))$. In the remainder of the cases we will fit the ideal $K(\tau)$ into exact sequences and make use of Proposition A.3. The first exact sequence we use is

$$0 \to \mathrm{syz}(K(\tau)) \to Ss_\tau + Ss_1 + Ss_2 \xrightarrow{\phi} S \to S/K(\tau) \to 0, \tag{3}$$

where $\phi(G_\tau s_\tau + G_1 s_1 + G_2 s_2) = G_\tau l_\tau^{r+1} + G_1 u_1^{\alpha_1}u_2^{r+1} + G_2 w_1^{r+1}w_2^{r+1}$. Clearly $\mathrm{im}(\phi) = K(\tau)$, so the above sequence is exact. Recalling the convention from Section 3, we will write $Ss_\tau + Ss_1 + Ss_2$ as $S(-(r+1)) \oplus S(-\alpha_1 - r - 1) \oplus S(-2(r+1))$ to encode the degrees of s_τ, s_1, s_2. The exact sequence (3), coupled with Lemma A.4 and Lemma 7.6, yields the following lemma.

Lemma 7.8. *If* $K(\tau)$ *is minimally generated by the entries of* $N(\tau)$*, then*

$$\mathrm{reg}(C^{\alpha}_{\mathrm{st}(\tau)}(\Delta)) = \mathrm{reg}(\mathrm{syz}(K(\tau))) \leq \max\{2r+2, \mathrm{reg}(S/K(\tau)) + 2\}.$$

Hence to prove Theorem 7.1 *it suffices to show that* $\mathrm{reg}(S/K(\tau)) \leq 3r+1$*.*

To handle the remainder of the cases, we introduce the *multiplication sequence* from graded commutative algebra. Recall that if $Q \subset S$ is an ideal and $f \in S$ is a polynomial, then the *colon ideal* of Q with f is the ideal $Q : f = \{g \in S \mid gf \in I\}$. For any Q, f we always have the graded short exact *multiplication sequence*

$$0 \to \frac{S(-\deg(f))}{Q : f} \xrightarrow{\cdot f} \frac{S}{Q} \to \frac{S}{Q + \langle f \rangle} \to 0.$$

We use this as follows. Let $Q = \langle l_\tau^{r+1}, u_1^{\alpha_1} u_2^{r+1} \rangle$ and $f = w_1^{r+1} w_2^{r+1}$. We then have the multiplication sequence

$$0 \to \frac{S(-2(r+1))}{Q : (w_1^{r+1} w_2^{r+1})} \xrightarrow{\cdot w_1^{r+1} w_2^{r+1}} \frac{S}{Q} \to \frac{S}{K(\tau)} \to 0. \tag{4}$$

Lemma 7.9. *With Q and $K(\tau)$ as above, the regularity of $S/K(\tau)$ satisfies*

$$\operatorname{reg}\left(\frac{S}{K(\tau)}\right) \le \max\left\{\operatorname{reg}\left(\frac{S}{Q : (w_1^{r+1} w_2^{r+1})}\right) + 2r + 1, 3r\right\}.$$

Hence, to prove Theorem 7.1 *it suffices to prove that*

$$\operatorname{reg}\left(\frac{S}{Q : (w_1^{r+1} w_2^{r+1})}\right) \le r.$$

Proof. From the multiplication sequence (4) and Proposition A.3 we conclude that

$$\operatorname{reg}\left(\frac{S}{K(\tau)}\right) \le \max\left\{\operatorname{reg}\left(\frac{S(-2(r+1))}{Q : (w_1^{r+1} w_2^{r+1})}\right) - 1, \operatorname{reg}\left(\frac{S}{Q}\right)\right\}.$$

Since the two generators of Q are relatively prime, they form a *regular sequence* (see Appendix A) with degrees $r+1$ and $r+1+\alpha_1$. Hence by Proposition A.5, $\operatorname{reg}(S/Q) = 2r + 2 + \alpha_2 - 3 \le 3r$. Now the first result follows since $S(-2(r+1))$ represents a free S-module of rank one generated in degree $2(r+1)$, so

$$\operatorname{reg}\left(\frac{S(-2(r+1))}{Q : (w_1^{r+1} w_2^{r+1})}\right) = \operatorname{reg}\left(\frac{S}{Q : (w_1^{r+1} w_2^{r+1})}\right) + 2(r+1).$$

The final statement now follows from Lemma 7.8. ☐

Next we analyze the ideal $Q : (w_1^{r+1} w_2^{r+1})$. The ideal Q can be written as $Q = \langle l_\tau^{r+1}, u_1^{\alpha_1} \rangle \cap \langle l_\tau^{r+1}, u_2^{r+1} \rangle$. Write $J_1 = \langle l_\tau^{r+1}, u_1^{\alpha_1} \rangle$ and $J_2 = \langle l_\tau^{r+1}, u_2^{r+1} \rangle$. Since colons split over intersections (symbolically, $(J_1 \cap J_2) : f = J_1 : f \cap J_2 : f$), we have $Q : (w_1^{r+1} w_2^{r+1}) = I_1 \cap I_2$, where $I_1 = J_1 : (w_1^{r+1} w_2^{r+1})$ and $I_2 = J_2 : (w_1^{r+1} w_2^{r+1})$.

Now the polynomials $l_\tau^{r+1}, u_1^{\alpha_1}, w_2^{r+1}$ form a regular sequence. In particular, any polynomial f which multiplies w_2^{r+1} into the ideal J_1 must already be in J_1. So $I_1 = J_1 : (w_1^{r+1} w_2^{r+1}) = J_1 : w_1^{r+1}$. Likewise, $I_2 = J_2 : (w_1^{r+1} w_2^{r+1}) = J_2 : w_2^{r+1}$. We prove one last special case of Theorem 7.1 before moving on to the general case. We make use of the following result of Schenck and Stillman [26] (they state the result for an arbitrary number of powers of linear forms).

Proposition 7.10 ([26, Corollary 3.4])**.** *Suppose l_1, l_2, and l_3 are distinct linear forms in $S = \mathbb{R}[x, y, z]$ all of which vanish along a common line (equivalently, up to a change of variables, $l_1, l_2, l_3 \in \mathbb{R}[x, y]$). For a positive integer r put $J = \langle l_1^{r+1}, l_2^{r+1}, l_3^{r+1} \rangle$. Then $\operatorname{reg}(S/J) = r + \lceil \frac{r+1}{2} \rceil - 1$.*

Lemma 7.11. *Suppose that $K(\tau)$ is minimally generated by the entries of $N(\tau)$ and*

(1) $\alpha_1 = 0$ *or*
(2) $\alpha_1 \neq 0$ *and* $u_1 = w_1$ *or* $u_2 = w_2$.

Then $\operatorname{reg}(C^\alpha_{\mathrm{st}(\tau)}(\Delta)) \le 3(r+1)$.

Proof. By Lemma 7.9, it suffices to prove that $\operatorname{reg}\left(\frac{S}{Q:(w_1^{r+1}w_2^{r+1})}\right) \leq r$. From the above discussion we know that $Q = I_1 \cap I_2$, where

$$I_1 = J_1 : w_1^{r+1} = \langle l_\tau^{r+1}, u_1^{\alpha_1}\rangle : w_1^{r+1}$$

and

$$I_2 = J_2 : w_2^{r+1} = \langle l_\tau^{r+1}, u_2^{r+1}\rangle : w_2^{r+1}.$$

If $\alpha_1 = 0$, then $J_1 = I_1 = S$, hence $Q : (w_1^{r+1}w_2^{r+1}) = I_2$. If $\alpha_1 \neq 0$ and $u_1 = w_1$, then $I_1 = S$ and $Q : (w_1^{r+1}w_2^{r+1}) = I_2$. Likewise if $\alpha_1 \neq 0$ and $u_2 = w_2$, then $Q : (w_1^{r+1}w_2^{r+1}) = I_1$. Since it makes no difference to the analysis, we assume that $Q : (w_1^{r+1}w_2^{r+1}) = I_2$. Hence it suffices to prove that $\operatorname{reg}(S/I_2) \leq r$. We use the multiplication sequence

$$0 \to \frac{S(-r-1)}{\langle l_\tau^{r+1}, u_2^{r+1}\rangle : w_2^{r+1}} \xrightarrow{\cdot w_2^{r+1}} \frac{S}{\langle l_\tau^{r+1}, u_2^{r+1}\rangle} \to \frac{S}{\langle l_\tau^{r+1}, u_2^{r+1}, w_2^{r+1}\rangle} \to 0.$$

The ideal $\langle l_\tau^{r+1}, u_2^{r+1}\rangle$ is generated by a regular sequence with both generators in degree $r+1$, hence by Proposition A.5 $\operatorname{reg}(S/\langle l_\tau^{r+1}, u_2^{r+1}\rangle) = 2r$.

Notice that l_τ, u_2, and w_2 all vanish along a common line, the span of the vector v_2 in Figure 8. Hence by Proposition 7.10,

$$\operatorname{reg}\left(\frac{S}{\langle l_\tau^{r+1}, u_2^{r+1}, w_2^{r+1}\rangle}\right) = r + \left\lceil \frac{r+1}{2} \right\rceil - 1 \leq 2r-1.$$

By Proposition A.3, $\operatorname{reg}(S(-r-1)/I_2) \leq 2r$, hence $\operatorname{reg}(S/I_2) \leq r-1$, completing the proof. □

7.1. **The general case.** Due to Lemma 7.11, we may assume that $\alpha_1 = r+1$, so $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha(\tau) = r$. Furthermore, referring to Figure 8, we may by Lemma 7.11 assume that each of the two sets of vectors $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$ are linearly independent.

It follows that we may make a change of coordinates so that v_1 points along the y-axis, v_2 points along the x-axis, and v_3 points along the z-axis. Applying appropriate scaling in the x, y, and positive z directions, we can assume that the vector defined by v_4 points in the direction of $\langle 1, 1, -1\rangle$. Under this change of coordinates, $\operatorname{st}(\tau)$ has four possible configurations, shown in Figure 9. The ideal $K(\tau)$ is the same for all of these. We have

$$\begin{array}{rcl} l_\tau &=& z, \\ u_1 &=& x, \\ u_2 &=& y, \\ w_1 &=& x+z, \\ w_2 &=& y+z, \end{array} \qquad \begin{array}{rcl} I_1 &=& \langle l_\tau^{r+1}, u_1^{r+1}\rangle : w_1^{r+1} = \langle x^{r+1}, z^{r+1}\rangle : (x+z)^{r+1}, \\ I_2 &=& \langle l_\tau^{r+1}, u_2^{r+1}\rangle : w_2^{r+1} = \langle y^{r+1}, z^{r+1}\rangle : (y+z)^{r+1}. \end{array}$$

Since linear changes of coordinate yield a graded isomorphism on S, the ideals I_1 and I_2 for any general configuration will be isomorphic to the above ideals.

We also make one further simplification using Proposition A.3 and the short exact sequence

$$0 \to \frac{S}{I_1 \cap I_2} \xrightarrow{q} \frac{S}{I_1} \oplus \frac{S}{I_2} \xrightarrow{\phi} \frac{S}{I_1 + I_2} \to 0, \tag{5}$$

where $q : S/(I_1 \cap I_2) \to S/I_1 \oplus S/I_2$ is the difference $q_1 - q_2$ of the two quotient maps $q_1 : S/(I_1 \cap I_2) \to S/I_1$ and $q_2 : S/(I_1 \cap I_2) \to S/I_2$, and ϕ is the sum of the two quotient maps $\phi_1 : S/I_1 \to S/(I_1 + I_2)$ and $\phi_2 : S/I_2 \to S/(I_1 + I_2)$.

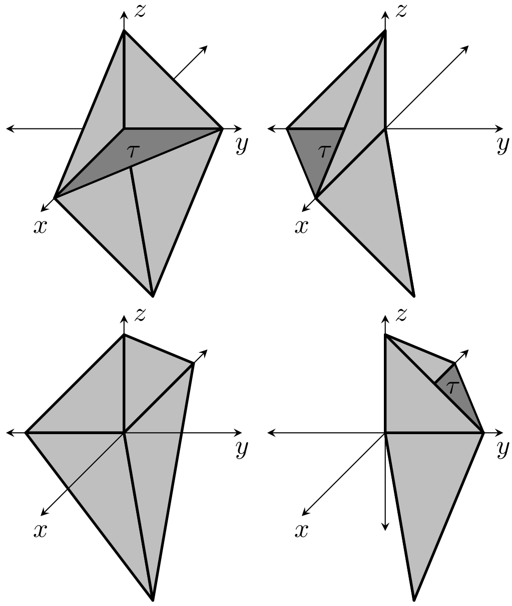


FIGURE 9. Possible configurations for $\mathrm{st}(\tau)$ in the proof of Theorem 7.2

Lemma 7.12. *Let*

$$\begin{array}{rcl} I_1 & = & \langle x^{r+1}, z^{r+1}\rangle : (x+z)^{r+1}, \\ I_2 & = & \langle y^{r+1}, z^{r+1}\rangle : (y+z)^{r+1}. \end{array}$$

To complete the proof of Theorem 7.1 *it suffices to prove that* $\mathrm{reg}(S/(I_1 \cap I_2)) \leq r$. *Equivalently, it suffices to show that* $\mathrm{reg}(S/(I_1+I_2)) \leq r-1$, *or* $S_r = (I_1+I_2)_r$.

Proof. The only thing left to prove are the two reformulations given in terms of (I_1+I_2). It follows from the proof of Lemma 7.11 that $\mathrm{reg}(S/I_1) \leq r-1$ and $\mathrm{reg}(S/I_2) \leq r-1$. Hence $\mathrm{reg}(S/I_1 \oplus S/I_2) = \max\{\mathrm{reg}(S/I_1), \mathrm{reg}(S/I_2)\} \leq r-1$. Now the exact sequence (5) and Proposition A.3 together yield the equivalent formulation $\mathrm{reg}(S/(I_1+I_2)) \leq r-1$. The module $S/(I_1+I_2)$ is a module of *finite length*. This means that in large degree d, $(S/(I_1+I_2))_d = 0$. By Corollary A.2, $\mathrm{reg}(S/(I_1+I_2))$ is the largest degree d for which $(S/(I_1+I_2))_d \neq 0$. Hence if $S_r = (I_1+I_2)_r$, then $\mathrm{reg}(S/(I_1+I_2)) \leq r-1$. ☐

Given Lemma 7.12, our goal will be to prove that every monomial $x^a y^b z^c$ in S of degree $a+b+c=r$ is contained in the ideal I_1+I_2. A main tool we use in this endeavor is the *initial ideal* with respect to the *standard graded lexicographic order* $<_{glex}$ on monomials (see [11, Chapter 15]). Recall that $x^{a_1}y^{b_1}z^{c_1} <_{glex} x^{a_2}y^{b_2}z^{c_2}$ if $(a_1+b_1+c_1) < (a_2+b_2+c_2)$ or $a_1+b_1+c_1 = a_2+b_2+c_2$ and the leftmost nonzero entry of $(a_2-a_1, b_2-b_1, c_2-c_1)$ is positive. We can pick out the initial term $\mathrm{in}_{<_{glex}}(f)$ (henceforth $\mathrm{in}(f)$) of a polynomial $f \in S$ by taking the term whose underlying monomial is largest under $<_{glex}$. For instance, if $f = 3x^2y + y^3$, then $\mathrm{in}(f) = 3x^2y$. Likewise, if I is an ideal, then $\mathrm{in}(I)$ is the monomial ideal generated by $\{\mathrm{in}(f) \mid f \in I\}$. The choice of lexicographic ordering gives a preferred basis of forms of degree d, and the point is that linear independence of this basis can be determined just from the leading terms, which are the monomials of degree d

in $\mathrm{in}(I)$. In particular, initial ideals preserve the Hilbert function of I, namely $\dim I_d = \dim \mathrm{in}(I)_d$ [11, Theorem 15.26].

Lemma 7.13. *To prove that $S_r = (I_1 + I_2)_r$, it suffices to prove that* $\mathrm{in}(I_1+I_2)_r = S_r$ *or* $(\mathrm{in}(I_1) + \mathrm{in}(I_2))_r = S_r$.

Proof. If $\mathrm{in}(I_1 + I_2)_r = S_r$, then $\dim(I_1 + I_2)_r = \dim \mathrm{in}(I_1 + I_2)_r = \dim S_r$. But this can only happen if $(I_1 + I_2)_r = S_r$. Furthermore $\mathrm{in}(I_1) + \mathrm{in}(I_2) \subset \mathrm{in}(I_1 + I_2)$, so the final statement is clear. □

With Lemma 7.13 as motivation, we study the initial ideals $\mathrm{in}(I_1)$ and $\mathrm{in}(I_2)$. Since I_1 and I_2 are the same ideal when x and y are interchanged, we simplify by working in the polynomial ring $R = \mathbb{R}[s,t]$ and studying the initial ideal of $I = I(r) = \langle s^{r+1}, t^{r+1}\rangle : (s+t)^{r+1}$. At the heart of our analysis is a matrix condition on the coefficients of a form f of degree d which distinguishes when $f \in I(r)_d$. This approach is due to Tohaneanu and Minac, who study an ideal very similar to ours in [22]. A small example illustrates how the matrix is constructed.

Example 7.14. Consider the case $r = 4$. We determine what conditions must be placed on the coefficients of a general form $f = a_{20}s^2 + a_{11}st + a_{02}t^2$ of degree two to satisfy $f \in I(4) = \langle s^5, t^5\rangle : (s+t)^5$. Since the ideal $\langle s^5, t^5\rangle$ is a monomial ideal, $f(s+t)^5 \in \langle s^5, t^5\rangle \iff$ every monomial of $f(s+t)^5$ is divisible by either s^5 or t^5. Expanding,

$$\begin{aligned} f(s+t)^5 = {} & s^7a_{20} + s^6t\,(a_{11} + 5a_{20}) + s^5t^2\,(a_{02} + 5a_{11} + 10a_{20}) \\ & + s^4t^3\,(5a_{02} + 10a_{11} + 10a_{20}) + s^3t^4\,(10a_{02} + 10a_{11} + 5a_{20}) \\ & + s^2t^5\,(10a_{02} + 5a_{11} + a_{20}) + st^6\,(5a_{02} + a_{11}) + t^7a_{02} \end{aligned}$$

is in $\langle s^5, t^5\rangle$ if and only if the coefficients in the middle row vanish, i.e.,

$$\begin{pmatrix} 5 & 10 & 10 \\ 10 & 10 & 5 \end{pmatrix} \begin{pmatrix} a_{02} \\ a_{11} \\ a_{20} \end{pmatrix} = 0.$$

Now suppose $f = \sum_{i+j=d} a_{i,j}s^it^j \in R_d$. Then, generalizing Example 7.14, we find that $f \in I(r)_d$ if and only if

$$\begin{pmatrix} \binom{r+1}{d+1} & \binom{r+1}{d} & \binom{r+1}{d-1} & \cdots & \binom{r+1}{1} \\ \binom{r+1}{d+2} & \binom{r+1}{d+1} & \binom{r+1}{d} & \cdots & \binom{r+1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{r+1}{r-1} & \binom{r+1}{r-2} & \binom{r+1}{r-3} & \cdots & \binom{r+1}{r-d-1} \\ \binom{r+1}{r} & \binom{r+1}{r-1} & \binom{r+1}{r-2} & \cdots & \binom{r+1}{r-d} \end{pmatrix} \cdot \begin{pmatrix} a_{0,d} \\ a_{1,d-1} \\ \vdots \\ a_{d-1,1} \\ a_{d,0} \end{pmatrix} = 0,$$

where we use the convention that $\binom{A}{B} = 0$ when $B < 0$ or $B > A$. Denote the $(r-d) \times (d+1)$ matrix on the left by $M(r,d)$. This matrix has entries

$$M(r,d)_{i,j} = \binom{r+1}{d+1+i-j},$$

where $i = 0, \ldots, r-d-1$ and $j = 0, \ldots, d$. With this choice of indexing, column c_j of $M(r,d)$ corresponds to the coefficient $a_{j,d-j}$.

Remark 7.15. Notice that if $f \in R_d$ and $d \geq r$, then $f \cdot (s+t)^{r+1}$ has degree $2r+1$. It is clear that any polynomial of degree $2r+1$ is in the ideal $\langle s^{r+1}, t^{r+1}\rangle$, so it makes sense that the matrix condition given by $M(r,d)$ is vacuous for $d \geq r$.

Tohaneanu and Minac [22, § 3.1] make the fundamental observation that matrices such as $M(r,d)$ appear in the representation theory of the special linear group. In particular, maximal minors of $M(r,d)$ tend to be invertible. We record this observation in the next lemma.

Lemma 7.16. *Let $M = M(r,d)$ be the $(r-d) \times (d+1)$ matrix defined above and let $\mathfrak{r}$ denote the rank of M. If $d < r$, then M has full rank and the first $\mathfrak{r}$ columns of M are linearly independent.*

Proof. Let $k = \min\{r-d-1, d\}$. Starting in the upper left-hand corner of $M(r,d)$, consider the square $(k+1) \times (k+1)$ submatrix N with entries

$$N_{i,j} = M(r,d)_{i,j} = \binom{r+1}{d+1+i-j},$$

$0 \leq i, j \leq k$. We are done if we show this matrix is invertible. Following Tohaneanu and Minac, we appeal to representation theory. Explicitly, let μ be the list with $k+1$ entries $(d+1, \ldots, d+1)$. This is a *partition.* The *conjugate partition* to μ is the list $\lambda = (k+1, \ldots, k+1)$ with $d+1$ entries. Then $\det(N)$ is the dimension of the Weyl module $\mathbb{S}_\lambda V$, which is a nontrivial irreducible representation of $SL(V)$, where V is an r-dimensional vector space. More explicitly, $\det(N) = s_\lambda(1, \ldots, 1)$, where $s_\lambda(x_1, \ldots, x_r)$ is the Schur polynomial in r variables of the conjugate partition λ of μ. Such evaluations are always positive integers. See [13, § 6.1] and Exercise A.30 in [13, Appendix A.1] for more details. ☐

Corollary 7.17. *Let $I = I(r)$ as above and set $a = \lfloor (r+1)/2 \rfloor$. We have*

$$\dim I_d = \begin{cases} 0 & 0 \leq d < a, \\ 2d+1-r & a \leq d < r, \\ d+1 & d \geq r. \end{cases}$$

Proof. If $d \geq r$, it follows from Remark 7.15 that $\dim I_d = \dim R_d = d+1$. For a fixed degree $d < r$, let $\mathfrak{r}$ be the rank of $M(r,d)$. By definition, $\dim I_d = \dim \ker M(r,d)$, hence $\dim I_d = d+1-\mathfrak{r}$. Now the result follows from the fact that $M(r,d)$ has full rank $\mathfrak{r} = \min\{r+d, d+1\}$ by Lemma 7.16. ☐

Lemma 7.18. *The vector space* $\mathrm{in}(I)_d$ *consists of the* $\dim I_d$ *monomials of R with lexicographically largest degree.*

Proof. Set $a = \lfloor (r+1)/2 \rfloor$. For fixed degree d with $a \leq d < r$, let $\mathfrak{r}$ be the rank of $M(r,d)$. We saw in the proof of Corollary 7.17 that $\dim I_d = d+1-\mathfrak{r}$. By Lemma 7.16, the first $\mathfrak{r}$ columns of M are linearly independent. It follows that for any column c_l of $M(r,d)$ with $\mathfrak{r} \leq l \leq d$, there is a unique (up to scaling) relation

$$\left(\sum_{i=0}^{\mathfrak{r}-1} a_{i,d-i} c_i\right) + a_{l,d-l} c_l = 0,$$

where $a_{l,d-l} \neq 0$. This gives rise to the polynomial $f = \sum_{i=0}^{\mathfrak{r}-1} a_{i,d-i} s^i t^{d-i} + a_{l,d-l} s^l t^{d-l} \in I$ with leading monomial $s^l t^{d-l}$. These monomials are the largest $d+1-\mathfrak{r}$ monomials of degree d with respect to lex ordering, so the result follows. ☐

Remark 7.19. Lemma 7.18 shows that $\mathrm{in}(I)$ is a so-called *lex-segment* ideal. Such ideals are of central importance to classifying Hilbert functions and providing extremal bounds on betti numbers. See [21, § 2.4] for more on these ideals.

By Lemma 7.13 and Lemma 7.12, the next proposition completes our proof of Theorem 7.1.

Proposition 7.20. *Let I_1 and I_2 be as in Lemma 7.12. Then $(I_1 + I_2)_r = S_r$.*

Proof. By Lemma 7.13, it suffices to show that $(\mathrm{in}(I_1)+\mathrm{in}(I_2))_r = S_r$. To this end, take a monomial $m = x^i y^j z^k$ of S with degree r. We claim $m \in \mathrm{in}(I_1) + \mathrm{in}(I_2)$.

Assume $i \geq j$. We then have $i + k \geq \lfloor (r-k+1)/2 \rfloor + k \geq \lfloor (r+1)/2 \rfloor$. By Corollary 7.17,

$$\begin{aligned} \dim I(r)_{i+k} &= 2(i+k) + 1 - r \\ &\geq 2(\lfloor (r-k+1)/2 \rfloor + k) + 1 - r \\ &\geq r - k + 2k + 1 - r \\ &= k + 1. \end{aligned}$$

By Lemma 7.18, $\mathrm{in}(I_1)_{i+k}$ consists of the $\dim I(r)_{i+k} \geq k+1$ largest monomials of degree $i+k$ in the variables x, z. Now notice that $x^i z^k$ is the $(k+1)$st largest monomial of degree $i+k$ in the variables x, z with respect to lex order. By Lemma 7.18, $x^i z^k \in \mathrm{in}(I_1)$, hence $m \in \mathrm{in}(I_1)$.

By the same argument, if $j \geq i$, then $m \in \mathrm{in}(I_2)$. In either case, we have shown that $m \in \mathrm{in}(I_1) + \mathrm{in}(I_2)$, completing the proof. □

8. Examples

In this section we give several examples to illustrate both how the bounds in Theorems 6.2 and 7.2 may be used and how well they approximate the actual regularity of the spline module.

Example 8.1. Our first example illustrates that the number of faces in a polytope of $\mathcal{P}$, where $\mathcal{P}$ is a polytopal cell, can indeed impact the regularity of $C^\alpha(\mathcal{P})$. In particular we show that, contrary to most of our examples, the bound of Theorem 6.2 may not be so far off in worst case scenarios. This example generalizes the construction in [9, Theorem 5.7].

For simplicity we restrict to the case of uniform smoothness. Suppose $P \subset \mathbb{R}^2$ is a polygon so that $(0,0) \in P$ and no pair of edges of P is parallel. Let $A = \widehat{P} \subset \mathbb{R}^3$. We construct a polytopal cell $\mathcal{P}(A)$ so that the regularity of $C^r(\mathcal{P}(A))$ grows in direct correlation with the number of edges of P.

We construct $\mathcal{P}(A)$ as follows. Let $\bar{A}$ be the reflection of A across the xy-plane. For a face τ of A containing $(0,0,0)$ let $\bar{\tau}$ denote the face of $\bar{A}$ obtained by reflecting τ across the xy-plane. Let $\sigma(\tau)$ denote the polytope formed by taking the convex hull of τ and $\bar{\tau}$. Now let $\mathcal{P}(A)$ be the polytopal cell with polytopes $\{A, \bar{A}\} \cup \{\sigma(\tau) \mid \tau \in \partial A, (0,0,0) \in \gamma\}$ (see Figure 10 for a depiction of the interior faces of $\mathcal{P}(A)$). We take uniform smoothness parameter r on $\mathcal{P}(A)$, so $\alpha(\tau) = r$ for $\tau \in \mathcal{P}(A)^\circ_2$ and $\alpha(\tau) = -1$ for $\tau \in \partial\mathcal{P}(A)$.

Consider the graded $S = \mathbb{R}[x,y,z]$-module $C^r(\mathcal{P}(A))$. We will show that a certain spline $G(A) \in C^r(\mathcal{P}(A))$, which is supported only on the polytope A, is a minimal generator of $C^r(\mathcal{P}(A))$. Since $\mathrm{reg}(C^r(\mathcal{P}(A)))$ bounds the degrees of generators, this will force $\mathrm{reg}(C^r(\mathcal{P}(A)))$ to be large.

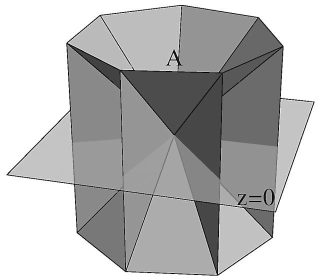


FIGURE 10. The polytopal cell $\mathcal{P}(A)$

Let $\bar{\phi} : C^r(\mathcal{P}(A)) \to S$ be the map of S-modules obtained by restricting splines $F \in C^r(\mathcal{P}(A))$ to the polytope $\bar{A}$. This is a splitting of the inclusion $S \to C^r(\mathcal{P}(A))$ as global polynomials on $\mathcal{P}(A)$. Let $NT^r(\mathcal{P}(A))$ be the kernel of $\bar{\phi}$. Then

$$C^r(\mathcal{P}(A)) \cong S \oplus NT^r(\mathcal{P}(A)).$$

Let $S' = \mathbb{R}[x,y]$ and, for $f \in S$, set $\overline{f} = f(x,y,0)$. Define an S-module map $\phi : C^r(\mathcal{P}(A)) \cong S \oplus NT^r(\mathcal{P}(A)) \to S'$ by $\phi(f,F) = \overline{F_A}$, where $f \in S$, $F \in NT^r(\mathcal{P}(A))$, and F_A is the restriction of F to the polytope A.

For a polytope σ, set $\partial^\circ(\sigma) = \partial\sigma \cap \mathcal{P}(A)_2^\circ$. We have

$$\Lambda(A) = \prod_{\tau \in \partial A} l_\tau^{\alpha(\tau)+1} = \prod_{\tau \in \partial^\circ(A)} l_\tau^{r+1}.$$

We claim that the image of ϕ is the principal ideal $I = \langle \overline{\Lambda(A)} \rangle$. Define the spline $G(A) \in NT^r(\mathcal{P}(A))$ by $G(A)_A = \Lambda(A)$ and $G(A)_{\sigma'} = 0$ for every other polytope $\sigma' \in \mathcal{P}(A)$. Clearly $\phi(G(A)) = \overline{\Lambda(A)}$, so $I \subset \operatorname{im}(\phi)$. To see that $\operatorname{im}(\phi) \subset I$, let $F \in NT^r(\mathcal{P}(A))$. Then, since $F_{\bar{A}} = 0$, $l_{\bar{\tau}}^{r+1} | F_{\sigma(\tau)}$ for every $\tau \in \partial^\circ(A)$. We also have $l_\tau^{r+1} | (F_A - F_{\sigma(\tau)})$ for every $\tau \in \partial^\circ(A)$. Hence $F_A \in \bigcap_{\tau \in \partial^\circ A} \langle l_\tau^{r+1}, l_{\bar{\tau}}^{r+1} \rangle$. But l_τ and $l_{\bar{\tau}}$ differ at most by a scalar multiple and a sign on the variable z, so $\overline{l_\tau} = \overline{l_{\bar{\tau}}}$ and

$$\phi(F) \in \bigcap_{\tau \in \partial^\circ(A)} \langle \overline{l_\tau^{r+1}} \rangle = \langle \prod_{\tau \in \partial^\circ(A)} \overline{l_\tau^{r+1}} \rangle = \langle \overline{\Lambda(A)} \rangle$$

as claimed. In the first equality we use the fact that no pair of edges of the polygon P are parallel; this guarantees all the forms $\overline{l_\tau}$ are distinct. It follows that the spline $G(A)$, which is supported only on the polytope A and generates splines supported on A, is a minimal generator of $C^r(\mathcal{P}(A))$. Since $G(A)$ has degree $|\partial^\circ(A)|(r+1)$, it follows from Remark 3.3 that $\operatorname{reg}(C^r(\mathcal{P}(A))) \geq |\partial^\circ(A)|(r+1)$.

Now we compute the bound from Theorem 6.2. Each face $\tau \in \mathcal{P}(A)_2$ either has $\lambda(\tau) = (|\partial^\circ(A)| + 3)(r+1)$ or $\lambda(\tau) = 7(r+1)$. As long as the polygon P has at least four edges, Theorem 6.2 yields $\operatorname{reg}(C^r(\mathcal{P}(A))) \leq (|\partial^\circ(A)| + 3)(r+1)$.

Remark 8.2. The construction in Example 8.1 is inherently nonsimplicial. Some other construction needs to be used to obtain high degree generators in the simplicial case. In the planar simplicial case, there is an example in [30] of a planar simplicial complex Δ with minimal generator in degree $2r+2$.

Example 8.3. In this example we apply Theorem 6.2 to bound the regularity of $C^\alpha(\mathcal{P})$ where boundary vanishing is imposed. Consider the polygonal complex $\mathcal{Q}$

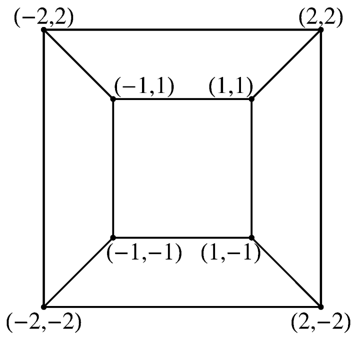


FIGURE 11. Polygonal complex $\mathcal{Q}$ in Example 8.3

in Figure 11 with five faces, eight interior edges, and four interior vertices. Impose vanishing of order r along interior edges and vanishing of order s along boundary edges. The following Hilbert polynomials are computed in [10, Example 8.5]. If $s=-1$, so $C^\alpha(\widehat{\mathcal{Q}})=C^r(\widehat{\mathcal{Q}})$, then

$$\begin{aligned}HP(C^r(\widehat{\mathcal{Q}}),d) = {} & \tfrac{5}{2}d^2+\left(-8r-\tfrac{1}{2}\right)d \\ & -4\left\lfloor\tfrac{3r}{2}\right\rfloor^2+12r\left\lfloor\tfrac{3r}{2}\right\rfloor-r^2+4r+2.\end{aligned}$$

By Theorem 6.2, $\operatorname{reg}(C^r(\widehat{\mathcal{Q}}))\le 6(r+1)-1$ and $HP(C^r(\widehat{\mathcal{Q}}),d)=\dim C^r_d(\mathcal{Q})$ for $d\ge 6(r+1)-2$. We compare the regularity bound $6(r+1)-1$ with $\operatorname{reg}(C^r(\widehat{\mathcal{Q}}))$ as computed in Macaulay2 in Table 2, where $\operatorname{reg}(C^r(\widehat{\mathcal{Q}}))$ appears to have alternating differences of 1 and 3 and grows roughly as $2(r+1)+1$. In fact, $\operatorname{reg}(C^r(\widehat{\mathcal{Q}}))$ appears to agree with the regularity of r-splines on the complex from Example 1.1.

TABLE 2

r	0	1	2	3	4	5	6	7	8	9
$6(r+1)-1$	5	11	17	23	29	35	41	47	53	59
$\operatorname{reg}(C^r(\widehat{\mathcal{Q}}))$	3	4	7	8	11	12	15	16	19	20

Now suppose that vanishing of degree $s\ge 0$ is imposed along $\partial\mathcal{Q}$. Then

$$\begin{aligned}HP(C^\alpha(\widehat{\mathcal{P}}),d) = {} & \tfrac{5}{2}d^2+\left(-8r-4s-\tfrac{9}{2}\right)d \\ & -3\left\lfloor\tfrac{2(r+s)}{3}\right\rfloor^2+4r\left\lfloor\tfrac{2(r+s)}{3}\right\rfloor+4s\left\lfloor\tfrac{2(r+s)}{3}\right\rfloor-\left\lfloor\tfrac{2(r+s)}{3}\right\rfloor \\ & -4\left\lfloor\tfrac{r}{2}\right\rfloor^2-4\left\lfloor\tfrac{3r}{2}\right\rfloor^2+4r\left\lfloor\tfrac{r}{2}\right\rfloor+12r\left\lfloor\tfrac{3r}{2}\right\rfloor \\ & -5\ r^2+4rs+8r+4s+4.\end{aligned}$$

This formula is correct when r,s are not too small; for instance if $r=3$ and $s=0$, the above formula has constant term 81 while the actual constant, according to Macaulay2, is 87. By Theorem 6.2,

$$\operatorname{reg}(C^\alpha(\widehat{\mathcal{Q}}))\le\max\{6(r+1)+(s+1),5(r+1)+2(s+1)\}-1$$

and $HP(C^\alpha(\widehat{\mathcal{P}}),d)=\dim C^\alpha_d(\mathcal{P})$ for

$$d\ge\max\{6(r+1)+(s+1),5(r+1)+2(s+1)\}-2.$$

A comparison of the bound on $\mathrm{reg}(C^\alpha(\widehat{\mathcal{Q}}))$ and its actual value computed in Macaulay2 appears in Table 3 for $r, s \leq 5$.

TABLE 3. Comparing regularity bound to regularity for Example 8.3

$\max\{6(r+1)+(s+1), 5(r+1)+2(s+1)\}-1$

	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$r=0$	7	8	10	12	14
$r=1$	13	14	15	17	19
$r=2$	19	20	21	22	24
$r=3$	25	26	27	28	29
$r=4$	31	32	33	34	35

$\mathrm{reg}(C^\alpha(\widehat{\mathcal{Q}}))$

	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$r=0$	4	4	5	6	7
$r=1$	4	5	6	8	9
$r=2$	7	8	8	9	10
$r=3$	8	8	9	10	12
$r=4$	11	11	12	12	13

Example 8.4. We now give an example which has very different behavior from Example 8.1. Consider the polygonal complex $\mathcal{Q}$ formed by placing a regular (or almost regular) n-gon inside of a scaled copy of itself and connecting corresponding vertices by edges. The complex $\mathcal{Q}$ has one polygon with n edges and n quadrilateral polygons. An example for $n=10$ is shown in Figure 12. We may or may not perturb the vertices so that the affine spans of the edges between the inner and outer n-gons do not all meet at the origin. This does not appear to have much effect on regularity, although it does change the constant term of $HP(C^r(\widehat{\mathcal{Q}}), d)$.

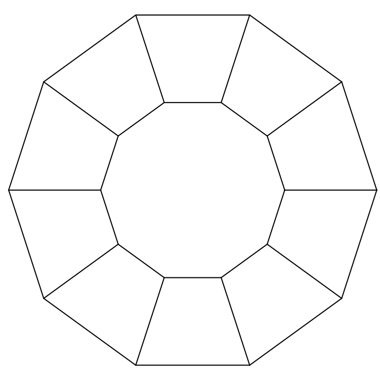


FIGURE 12. The complex $\mathcal{Q}$ in Example 8.4

According to Theorem 6.2, $\mathrm{reg}(C^r(\widehat{\mathcal{Q}})) \leq \max\{(r+1)(n+2), 5(r+1)\} \leq (r+1)(n+2)$ as long as $n \geq 3$. However, according to computations for $r \leq 3$ and $n \leq 10$ in Macaulay2, $\mathrm{reg}(C^r(\widehat{\mathcal{Q}})) \leq 3(r+1)$ regardless of what value n takes.

9. Regularity conjecture

We conclude with a conjecture in the case of uniform smoothness (see also [9, Conjecture 5.6]). As in Example 8.1, given a polytope σ of the polytopal cell $\mathcal{P}$, denote by $\partial^\circ(\sigma)$ the number of faces of σ which are interior to $\mathcal{P}$. We call the polytopal cell $\mathcal{P}$ *complete* if the origin is an interior vertex.

Conjecture 9.1. *Let $\mathcal{P} \subset \mathbb{R}^3$ be a polyhedral cell. Let $F = \max\{|\partial^\circ(\sigma)| : \sigma \in \mathcal{P}\}$.*

(1) *$\mathcal{P}$ is central and complete $\implies \operatorname{reg}(C^r(\mathcal{P})) \leq \operatorname{reg}(LS^{r,1}(\mathcal{P})) \leq F(r+1)$.*
(2) *$\mathcal{P}$ is central but not complete $\implies \operatorname{reg}(C^r(\mathcal{P})) \leq (F-1)(r+1)$.*

Remark 9.2. [9, Theorem 5.7] shows that generators can be obtained in degree $(F-1)(r+1)$ in the noncomplete central case. Example 8.1 shows generators can be obtained in degree $F(r+1)$ for the complete central case. Hence these are the lowest regularity bounds that we can conjecture which take into account the statistic $|\partial^\circ(\sigma)|$ for each polytope σ.

Remark 9.3. The generic perturbation of the octahedron in Example 1.2 suggests that Conjecture 9.1 part (1) could perhaps be made closer to the bound in part (2) for *generic* complete polytopal cells.

Remark 9.4. Conjecture 9.1 part (2) is a natural generalization of a conjecture of Schenck [28], that $\operatorname{reg}(C^r(\widehat{\Delta})) \leq 2(r+1)$ for $\Delta \subset \mathbb{R}^2$. This is a highly nontrivial conjecture in the simplicial case; it implies, for instance, that $\wp(C^1(\widehat{\Delta})) \leq 2$. To date, it is unknown whether $HP(C^1(\widehat{\Delta}), 3) = \dim C^1_3(\widehat{\Delta})$. The difficulty of this problem is in large part due to the fact that nonlocal geometry plays an increasingly important role in low degree [1,4]. Since our methods hinge on using the submodules $LS^{\alpha,1}(\mathcal{P})$, which are locally supported approximations to $C^\alpha(\mathcal{P})$, our approach will not be effective in proving Conjecture 9.1 part (2).

Remark 9.5. Example 8.4 indicates that a polytope with a large number of interior faces does not always impose large regularity. Hence there is a need to study when a large polytope actually imposes large regularity, as is the case in Example 8.1 (in practice one would want to avoid these polytopal cells).

Appendix A. Additional results concerning regularity

In this appendix we summarize some results about regularity which are used in the paper. We provide a proof for one of these results (Proposition A.7) since we are not aware of a proof of this exact statement in the literature. For this we will need an alternate characterization of regularity via local cohomology. We briefly summarize this characterization; see [12, Appendix 1] for more details. Let $S = \mathbb{R}[x_0, \ldots, x_n]$. The local cohomology modules $H^i_m(M)$ of M with respect to the homogeneous maximal ideal $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ are the right derived functors of the functor $H^0_{\mathfrak{m}}(\_)$, where

$$H^0_{\mathfrak{m}}(M) = \{m \in M \mid \mathfrak{m}^j m = 0 \text{ for some } j \geq 0\}.$$

If M is a graded module, then the local cohomology modules $H^i_{\mathfrak{m}}(M)$ are also graded. A standard fact about the modules $H^i_{\mathfrak{m}}(M)$ is that they have finite length, in other words, $H^i_{\mathfrak{m}}(M)_d = 0$ for $d \gg 0$. The module M is called d-regular if

(1) $H^0_{\mathfrak{m}}(M)_j = 0$ for every $j > d$,
(2) $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ for every $i > 0$.

Theorem A.1 (Theorem 4.3 of [12]). *Let* $\mathfrak{m} \subset S$ *be the maximal ideal of* S *and* M *a graded* S*-module. Then* $\text{reg}(M) = \min\{d \mid M \text{ is } d\text{-regular}\}$.

The next two results follow from Theorem A.1 using standard homological tools.

Corollary A.2 ([12], Corollary 4.4). *If* M *has finite length, then* $\text{reg}(M) = \max\{d \mid M_d \neq 0\}$.

Proposition A.3 ([11], Corollary 20.19). *Let* $0 \to A \to B \to C \to 0$ *be a graded exact sequence of finitely generated* S *modules. Then:*

(1) $\text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\}$,
(2) $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$,
(3) $\text{reg}(C) \leq \max\{\text{reg}(A) - 1, \text{reg}(B)\}$.

Proposition A.3 can be applied to long exact sequences by breaking them into short exact sequences. The next lemma can be proved in this way. The first statement may be found in [12, Lemma 5.9].

Lemma A.4. *Let* $m \geq 0$. *If*

$$0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to M \to 0$$

is an exact sequence of S*-modules, then* $\text{reg}(M) \leq \max_i\{\text{reg}(C_i) - i\}$. *Similarly, if*

$$0 \to M \to C_0 \to \cdots \to C_{m-1} \to C_m \to 0$$

is an exact sequence of S*-modules, then* $\text{reg}(M) \leq \max_i\{\text{reg}(C_i) + i\}$.

An additional concept we need is the *depth* of a module. The *depth* of a graded S-module M with respect to the homogeneous maximal ideal $\mathfrak{m}$, denoted $\text{depth}(M)$, is the length of a maximal sequence of homogeneous forms $\{f_1, \ldots, f_k\} \subset \mathfrak{m}$ satisfying that f_1 is a nonzerodivisor on M (there is no $m \in M$ so that $f_1 m = 0$) and f_l is a nonzerodivisor on $M/(\sum_{i=1}^{l-1} f_i M)$ for $l = 2, \ldots, k$. Such a sequence is called an M-sequence. If the module M is simply S, then an S-sequence is called a *regular sequence*. An ideal generated by a regular sequence is called a *complete intersection*. We pause to give a result concerning the regularity of complete intersections.

Proposition A.5. *If* I *is a complete intersection generated by a regular sequence of homogeneous forms* $f_1, \ldots, f_k$ *of degrees* $d_1, \ldots, d_k$, *then* $\text{reg}(S/I) = \sum_{i=1}^{k}(d_i - 1) = (\sum_{i=1}^{k} d_i) - k$.

We will use the following result of Auslander and Buchsbaum to move back and forth between the notions of depth and projective dimension.

Theorem A.6 (Auslander-Buchsbaum). *Let* M *be an* $S = \mathbb{R}[x_0, \ldots, x_n]$*-module. Then* $\text{depth}(M) + \text{pd}(M) = \text{depth}(S) = n + 1$.

In particular, Theorem A.6 implies that $\text{pd}(M) \leq n + 1$. This inequality is known as the *Hilbert syzygy theorem.*

The following proposition is one of the ingredients used in the proof of the Gruson-Lazarsfeld-Peskine theorem bounding the regularity of curves in projective space [12, Proposition 5.5].

Proposition A.7. *Let* M *be an* S*-module and* $N \subset M$ *a submodule of* M *with* $\text{codim}(M/N) > \text{pd}(M)$. *Then* $\text{reg}(M) \leq \text{reg}(N)$.

Proof. Note that the inequality $\operatorname{codim}(M/N) > \operatorname{pd}(M)$ in Proposition A.7 is equivalent, via the Auslander-Buchsbaum theorem, to $\dim(M/N) < \operatorname{depth}(M)$. We prove $\operatorname{reg}(M) \leq \operatorname{reg}(N)$ if $\dim(M/N) < \operatorname{depth}(M)$. Set $d = \operatorname{depth}(M)$. By [12, Proposition A1.16], $H^i_{\mathfrak{m}}(M) = 0$ for $i < d$ and $H^i_{\mathfrak{m}}(M/N) = 0$ for $i > \dim(M/N)$. The long exact sequence in local cohomology resulting from the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

yields a surjection $H^d_{\mathfrak{m}}(N) \twoheadrightarrow H^d_{\mathfrak{m}}(M)$ and isomorphisms $H^i_{\mathfrak{m}}(N) \cong H^i_{\mathfrak{m}}(M)$ for $i > d$. Since $H^i_{\mathfrak{m}}(M) = 0$ for $i < d$, if N is j-regular for some j, then so is M. Now Theorem A.1 yields $\operatorname{reg}(N) \geq \operatorname{reg}(M)$. $\square$

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References

[1] P. Alfeld, B. Piper, and L. L. Schumaker, *An explicit basis for C^1 quartic bivariate splines*, SIAM J. Numer. Anal. **24** (1987), no. 4, 891–911, DOI 10.1137/0724058. MR899711

[2] P. Alfeld and L. L. Schumaker, *The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r + 1$*, Constr. Approx. **3** (1987), no. 2, 189–197, DOI 10.1007/BF01890563. MR889554

[3] P. Alfeld and L. L. Schumaker, *On the dimension of bivariate spline spaces of smoothness r and degree $d = 3r + 1$*, Numer. Math. **57** (1990), no. 6-7, 651–661, DOI 10.1007/BF01386434. MR1062372

[4] P. Alfeld and L. L. Schumaker, *Nonexistence of star-supported spline bases*, SIAM J. Math. Anal. **31** (1999/00), no. 2, 455–465 (electronic), DOI 10.1137/S0036141098342349. MR1740946

[5] L. J. Billera, *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 325–340, DOI 10.2307/2001125. MR965757

[6] L. J. Billera and L. L. Rose, *A dimension series for multivariate splines*, Discrete Comput. Geom. **6** (1991), no. 2, 107–128, DOI 10.1007/BF02574678. MR1083627

[7] L. J. Billera and L. L. Rose, *Modules of piecewise polynomials and their freeness*, Math. Z. **209** (1992), no. 4, 485–497, DOI 10.1007/BF02570848. MR1156431

[8] C. de Boor, *A Practical Guide to Splines*, Revised edition, Applied Mathematical Sciences, vol. 27, Springer-Verlag, New York, 2001. MR1900298

[9] M. DiPasquale, *Lattice-supported splines on polytopal complexes*, Adv. in Appl. Math. **55** (2014), 1–21, DOI 10.1016/j.aam.2013.12.002. MR3176714

[10] M. DiPasquale, *Associated primes of spline complexes*, J. Symbolic Comput. **76** (2016), 158–199, DOI 10.1016/j.jsc.2016.01.004. MR3461264

[11] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. MR1322960

[12] D. Eisenbud, *The Geometry of Syzygies: A Second Course in Commutative Algebra and Algebraic Geometry*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005. MR2103875

[13] W. Fulton and J. Harris, *Representation Theory: A First Course*, Readings in Mathematics, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. MR1153249

[14] A. V. Geramita and H. K. Schenck, *Fat points, inverse systems, and piecewise polynomial functions*, J. Algebra **204** (1998), no. 1, 116–128, DOI 10.1006/jabr.1997.7361. MR1623949

[15] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.
[16] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002. MR1867354
[17] D. Hong, *Spaces of bivariate spline functions over triangulation*, Approx. Theory Appl. **7** (1991), no. 1, 56–75. MR1117308
[18] A. Kh. Ibrahim and L. L. Schumaker, *Super spline spaces of smoothness r and degree $d \geq 3r+2$*, Constr. Approx. **7** (1991), no. 3, 401–423, DOI 10.1007/BF01888166. MR1120412
[19] T. McDonald, *Splines with boundary conditions*, Comput. Math. Appl. **54** (2007), no. 9-10, 1234–1239, DOI 10.1016/j.camwa.2006.10.034. MR2397674
[20] T. McDonald and H. Schenck, *Piecewise polynomials on polyhedral complexes*, Adv. in Appl. Math. **42** (2009), no. 1, 82–93, DOI 10.1016/j.aam.2008.06.001. MR2475315
[21] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR2110098
[22] J. Mináč and S. O. Tohăneanu, *From spline approximation to Roth's equation and Schur functors*, Manuscripta Math. **142** (2013), no. 1-2, 101–126, DOI 10.1007/s00229-012-0594-6. MR3081001
[23] B. Mourrain and N. Villamizar, *Homological techniques for the analysis of the dimension of triangular spline spaces*, J. Symbolic Comput. **50** (2013), 564–577, DOI 10.1016/j.jsc.2012.10.002. MR2996896
[24] S. Payne, *Equivariant Chow cohomology of toric varieties*, Math. Res. Lett. **13** (2006), no. 1, 29–41, DOI 10.4310/MRL.2006.v13.n1.a3. MR2199564
[25] H. Schenck, *A spectral sequence for splines*, Adv. in Appl. Math. **19** (1997), no. 2, 183–199, DOI 10.1006/aama.1997.0534. MR1459497
[26] H. Schenck and M. Stillman, *A family of ideals of minimal regularity and the Hilbert series of $C^r(\hat{\Delta})$*, Adv. in Appl. Math. **19** (1997), no. 2, 169–182, DOI 10.1006/aama.1997.0533. MR1459496
[27] H. Schenck and M. Stillman, *Local cohomology of bivariate splines*, J. Pure Appl. Algebra **117/118** (1997), 535–548, DOI 10.1016/S0022-4049(97)00026-1. Algorithms for algebra (Eindhoven, 1996). MR1457854
[28] H. K. Schenck and P. F. Stiller, *Cohomology vanishing and a problem in approximation theory*, Manuscripta Math. **107** (2002), no. 1, 43–58, DOI 10.1007/s002290100222. MR1892771
[29] L. L. Schumaker, *On the dimension of spaces of piecewise polynomials in two variables*, Multivariate Approximation Theory (Proc. Conf., Math. Res. Inst., Oberwolfach, 1979), Internat. Ser. Numer. Math., vol. 51, Birkhäuser, Basel-Boston, Mass., 1979, pp. 396–412. MR560683
[30] Ş. O. Tohăneanu, *Smooth planar r-splines of degree $2r$*, J. Approx. Theory **132** (2005), no. 1, 72–76, DOI 10.1016/j.jat.2004.10.011. MR2110576
[31] W. Whiteley, *A matrix for splines*, Progress in Approximation Theory, Academic Press, Boston, MA, 1991, pp. 821–828. MR1114817
[32] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028

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