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ON GENERALIZED AVERAGED GAUSSIAN FORMULAS. II

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ABSTRACT. Recently, by following the results on characterization of positive quadrature formulae by Peherstorfer, we proposed a new $(2\ell+1)$-point quadrature rule $\widehat{G}_{2\ell+1}$, referred to as a generalized averaged Gaussian quadrature rule. This rule has $2\ell+1$ nodes and the nodes of the corresponding Gauss rule G_ℓ with ℓ nodes form a subset. This is similar to the situation for the $(2\ell+1)$-point Gauss-Kronrod rule $H_{2\ell+1}$ associated with G_ℓ. An attractive feature of $\widehat{G}_{2\ell+1}$ is that it exists also when $H_{2\ell+1}$ does not. The numerical construction, on the basis of recently proposed effective numerical procedures, of $\widehat{G}_{2\ell+1}$ is simpler than the construction of $H_{2\ell+1}$. A disadvantage might be that the algebraic degree of precision of $\widehat{G}_{2\ell+1}$ is $2\ell+2$, while the one of $H_{2\ell+1}$ is $3\ell+1$. Consider a (nonnegative) measure $d\sigma$ with support in the bounded interval $[a,b]$ such that the respective orthogonal polynomials, above a specific index r, satisfy a three-term recurrence relation with constant coefficients. For $\ell \geq 2r-1$, we show that $\widehat{G}_{2\ell+1}$ has algebraic degree of precision at least $3\ell+1$, and therefore it is in fact $H_{2\ell+1}$ associated with G_ℓ. We derive some interesting equalities for the corresponding orthogonal polynomials.

1. Gauss quadratures and their Kronrod extensions

Let $d\sigma$ be a given nonnegative measure on a bounded or an unbounded interval $[a,b] = \operatorname{supp}(d\sigma)$. If σ is an absolutely continuous function on $[a,b]$, then $d\sigma(t) = \omega(t)\,dt$, where ω is a weight function. We call an *interpolatory quadrature formula* (abbreviated q.f.) of the form

$$I(f) = \int_a^b f(t)\,d\sigma(t) = Q_n[f] + R_n[f], \quad Q_n[f] = \sum_{j=1}^{n} \omega_j f(x_j), \tag{1.1}$$

where $x_1 < x_2 < \cdots < x_n$, $\omega_j \in \mathbb{R}$ $(j = 1,\ldots,n)$ and $R_n[f] = 0$ for $f \in \mathbb{P}_{2n-m-1}$ ($\mathbb{P}_n$ denotes as usual the set of polynomials of degree at most n), $0 \leq m \leq n$, a $(2n-m-1, n, d\sigma)$ q.f. If in addition all quadrature weights ω_j, $j = 1,\ldots,n$, are positive, then it is called a *positive* $(2n-m-1, n, d\sigma)$ q.f. Furthermore, we say that a polynomial $t_n \in \mathbb{P}_n$ generates a $(2n-m-1, n, w)$ q.f. if t_n has n simple zeros $x_1 < x_2 < \cdots < x_n$, $t_n(t) = \prod_{j=1}^{n}(t - x_j)$, and if the interpolatory q.f. based on the nodes x_j, $j = 1,\ldots,n$, is a $(2n-m-1, n, d\sigma)$ q.f. A $(2n-m-1, n, d\sigma)$ q.f. is *internal* if all its nodes belong to the closed interval $[a,b]$. A node not belonging to the interval $[a,b]$ is called *external*.

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Let us denote by p_k the monic polynomial of degree k, which is orthogonal to $\mathbb{P}_{k-1}$ with respect to $d\sigma$, i.e.,

$$\int_a^b t^j p_k(t) d\sigma(t) = 0, \quad j = 0, 1, \ldots, k-1,$$

and let us recall that (p_k) satisfies a three-term recurrence relation of the form

$$p_{k+1}(t) = (t - \alpha_k) p_k(t) - \beta_k p_{k-1}(t), \quad k = 0, 1, \ldots, \tag{1.2}$$

where $p_{-1}(t) = 0$, $p_0(t) = 1$, $\alpha_k \in \mathbb{R}$, and the $\beta_k > 0$ for all k; see, e.g., Gautschi [5] for details.

The unique interpolatory q.f. $(2\ell - 1, \ell, d\sigma)$ with ℓ nodes and the highest possible degree of exactness $2\ell - 1$ is the Gaussian formula with respect to the weight $d\sigma$,

$$G_\ell[f] = \sum_{j=1}^{\ell} \omega_j^G f\left(x_j^G\right).$$

Gauss (see [4]), in an attempt to improve upon the Newton-Cotes formula, discovered this quadrature rule G_ℓ named after him in the simplest case $d\sigma(t) = dt$.

As discussed, e.g., by Wilf [26], the nodes of the q.f. G_ℓ are the eigenvalues, and the weights are proportional to the squares of the first components of the eigenvectors, of the symmetric Jacobi tridiagonal matrix

$$J_\ell^G(d\sigma) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & \mathbf{0} \\ \sqrt{\beta_1} & \alpha_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\beta_{\ell-1}} \\ \mathbf{0} & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-1} \end{bmatrix} \in \mathbb{R}^{\ell \times \ell}.$$

Both the nodes and weights of the q.f. G_ℓ can be conveniently computed by the Golub-Welsch algorithm [11], which is based on the observation by Wilf [26].

In practice the common problem is to find some other q.f. for estimating the error of $G_\ell[f]$. Typically, if it exists, the Gauss-Kronrod q.f. $H_{2\ell+1}$, with $2\ell + 1$ points and degree of exactness at least $3\ell + 1$, is used under the assumption that the ℓ nodes of G_ℓ are part of the $2\ell + 1$ nodes,

$$\begin{aligned} I(f) = \int_a^b f(t)\, d\sigma(t) &= H_{2\ell+1}[f] + R_{2\ell+1}^{GK}[f], \\ H_{2\ell+1}[f] &= \sum_{j=1}^{\ell} \omega_j^{GK} f\left(x_j^G\right) + \sum_{k=1}^{\ell+1} \widetilde{\omega}_k^{GK} f\left(x_k^S\right). \end{aligned} \tag{1.3}$$

This is an idea that was first put forward and implemented in the case $d\sigma(t) = dt$ by Kronrod [14]. Moreover, it seems that this problem has a longer tradition; see Gautschi [7].

The polynomial of degree $\ell + 1$, which vanishes at the $\ell + 1$ additional nodes x_k^S $(k = 1, \ldots, \ell+1)$, the so-called Stieltjes polynomial, usually denoted by $E_{\ell+1}$, is characterized by an orthogonality relation with respect to a sign-changing weight.

The efficient numerical methods for calculating the positive Gauss-Kronrod q.f. are proposed by Laurie [15], and Calvetti et al. [1] (see also Monegato [17] and Gautschi [5], jointly with [6]).

The existence of the positive Gauss-Kronrod q.f. depends on $d\sigma$, and there are several known cases of nonexistence, e.g. for the Gauss-Laguerre and Gauss-Hermite cases [13]. Recently, for the Gegenbauer weight $\omega^{(\alpha,\alpha)}(t) = (1-t^2)^\alpha$, Peherstorfer and Petras [20] have shown nonexistence of Gauss-Kronrod formulas for ℓ sufficiently large and $\alpha > 5/2$. Analogous results for the Jacobi weight function $\omega^{(\alpha,\beta)}(t) = (1-t)^\alpha(1+t)^\beta$ can be found in their paper [21], particularly nonexistence for large ℓ of Gauss-Kronrod formulas when $\min(\alpha,\beta) \geq 0$ and $\max(\alpha,\beta) > 5/2$. In such cases it is of interest to find an adequate alternative to the corresponding Gauss-Kronrod q.f.

2. Generalized averaged Gaussian quadrature formulas

Recently, by following the results on characterization of the positive quadrature formulae by Peherstorfer [18] (see also [19]), we proposed in [24] a new $(2\ell+1)$-point quadrature rule, referred to as a generalized averaged Gaussian quadrature rule and denoted by $\widehat{G}_{2\ell+1}$ below, for estimating the error $|(I - G_\ell)(f)|$ by the difference $|(\widehat{G}_{2\ell+1} - G_\ell)(f)|$. This rule has $2\ell+1$ nodes and the nodes of the corresponding Gauss rule G_ℓ with ℓ nodes form a subset. This is similar to the situation for the $(2\ell+1)$-point Gauss-Kronrod rule $H_{2\ell+1}$ associated with G_ℓ, and estimating the error $|(I - G_\ell)(f)|$ by the difference $|(H_{2\ell+1} - G_\ell)(f)|$. An attractive feature of $\widehat{G}_{2\ell+1}$ is that it exists also when $H_{2\ell+1}$ does not. The numerical construction, on the basis of recently proposed effective numerical procedures, of $\widehat{G}_{2\ell+1}$ in [24] is simpler than the construction of $H_{2\ell+1}$ in [15] (see also [5], jointly with [6]) and [1]. A disadvantage might be that the algebraic degree of precision of $\widehat{G}_{2\ell+1}$ is $2\ell+2$, while the one of $H_{2\ell+1}$ is $3\ell+1$. In the next section we will show that, for one class of measures $d\sigma$, $\widehat{G}_{2\ell+1}$ has algebraic degree of precision at least $3\ell+1$, and therefore it coincides with $H_{2\ell+1}$ associated with G_ℓ. In this case the best way to compute $\widehat{G}_{2\ell+1} = H_{2\ell+1}$ is to use the simple numerical procedure for $\widehat{G}_{2\ell+1}$ proposed in [24].

Generalized averaged Gaussian quadrature formulas may yield higher accuracy than Gauss quadrature formulas that use the same moment information. This makes them attractive to use when moments or modified moments are expensive or difficult to evaluate. More details on the generalized averaged Gaussian quadrature rules $\widehat{G}_{2\ell+1}$ and their applications can be found in the recent papers [23], [22], [12], [3], besides [24], [25]. The rule $\widehat{G}_{2\ell+1}$ is the optimal stratified extension of the Gauss rule G_ℓ (see [24]). This rule is of particular interest since it covers nested and stratified formulas; see Peherstorfer [19, p. 2245] for a discussion. The rule $\widehat{G}_{2\ell+1}$ can be expected to give a more accurate approximation of $I(f)$ than the Gauss rule $G_{\ell+1}$ for essentially the same computational effort. The fact that $\widehat{G}_{2\ell+1}$ may give a smaller quadrature error than $G_{\ell+1}$ has been shown in [23].

Peherstorfer [18] showed that a polynomial t_n generates a positive $(2n-1-m, n, d\sigma)$ q.f. $(0 \leq m \leq n)$ if and only if t_n can be generated by a three-term recurrence relation of the form

$$t_{j+1}(x) = (x - \tilde{\alpha}_j)t_j(x) - \tilde{\beta}_j t_{j-1}(x), \quad j = 0, 1, \dots, n-1,$$

where $t_{-1}(x) \equiv 0,\ t_0(x) \equiv 1$, $\tilde{\alpha}_j \in \mathbb{R}$, $\tilde{\beta}_j > 0$, and

$$\begin{aligned} \tilde{\alpha}_j &= \alpha_j \text{ for } j = 0, 1, \ldots, n-1-\left[\frac{m+1}{2}\right], \\ \tilde{\beta}_j &= \beta_j \text{ for } j = 0, 1, \ldots, n-1-\left[\frac{m}{2}\right], \end{aligned} \tag{2.1}$$

such that

$$\operatorname{sgn} t_j(a) = (-1)^j,\ t_j(b) > 0, \quad j = 1, 2, \ldots, n.$$

These properties of the polynomials t_j are equivalent to that t_n can be represented in the form ($\ell := [(m+1)/2],\ n \geq 2\ell$, i.e., $n - \ell \geq \ell$)

$$t_n = g_\ell p_{n-\ell} - \tilde{\beta}_{n-\ell} g_{\ell-1} p_{n-\ell-1}, \tag{2.2}$$

where $g_{\ell-1}$ and g_ℓ are generated by a three-term recurrence relation

$$g_{j+1}(x) = (x - \tilde{\alpha}_{n-1-j}) g_j(x) - \tilde{\beta}_{n-j} g_{j-1}(x), \quad j = 0, 1, \ldots, \ell - 1,$$

and $g_{-1}(x) \equiv 0$, $g_0(x) \equiv 1$, with $\tilde{\alpha}_{n-1-j} \in \mathbb{R}$ and $\tilde{\beta}_{n-j} > 0$ for $j = 0, 1, \ldots, \ell - 1$; $\tilde{\beta}_{n-\ell} > 0$, $\tilde{\beta}_{n-\ell} = \beta_{n-\ell}$ if $m = 2\ell - 1$, such that

$$\operatorname{sgn} g_j(a) = (-1)^j,\ g_j(b) > 0, \quad j = 1, 2, \ldots, \ell;$$

see the proof of [18, Theorem 3.2], in particular $(d) \Longrightarrow (a)$.

We may define quadrature formulas of the kind discussed as follows. Let $d\mu$ be a nonnegative measure on the same bounded or unbounded interval $[a, b] = \operatorname{supp}(d\mu)$. Let $\tilde{p}_k$ denote the monic polynomial of degree k that is orthogonal to $\mathbb{P}_{k-1}$ with respect to $d\mu$, i.e.,

$$\int_a^b x^j \tilde{p}_k(x) d\mu(x) = 0, \quad j = 0, 1, \ldots, k-1.$$

Then the polynomials $\{\tilde{p}_k\}_{k=0}^{\infty}$ satisfy a three-term recurrence relation of the form

$$\tilde{p}_{k+1}(x) = (x - \gamma_k) \tilde{p}_k(x) - \lambda_k \tilde{p}_{k-1}(x), \quad k = 0, 1, \ldots,$$

where $\tilde{p}_{-1}(x) \equiv 0$, $\tilde{p}_0(x) \equiv 1$, $\gamma_k \in \mathbb{R}$ and $\lambda_k > 0$.

Consider the positive quadrature formula $(2n-1-m, n, d\sigma, d\mu)$, in which

$$\begin{aligned} \tilde{\alpha}_{n-1-j} &= \gamma_j \quad \text{and} \quad \tilde{\beta}_{n-j} = \lambda_j \quad \text{for } j = 0, 1, \ldots, \ell - 1, \\ \tilde{\beta}_{n-\ell} &= \beta_{n-\ell}\ (m = 2\ell - 1), \quad \text{i.e.,} \quad \tilde{\beta}_{n-\ell} = \lambda_\ell\ (m = 2\ell). \end{aligned} \tag{2.3}$$

We then obtain

$$g_j \equiv \tilde{p}_j, \quad j = 1, 2, \ldots, \ell.$$

Conversely, letting

$$g_\ell \equiv \tilde{p}_\ell \quad \text{and} \quad g_{\ell-1} \equiv \tilde{p}_{\ell-1}, \tag{2.4}$$

we obtain the relations (2.3). Hence, if (2.4) or (2.3) holds, then (2.2) is reduced to

$$t_n = \tilde{p}_\ell \cdot p_{n-\ell} - \tilde{\beta}_{n-\ell}\, \tilde{p}_{\ell-1} \cdot p_{n-\ell-1}, \tag{2.5}$$

and t_n generates a positive quadrature formula, which we denote by $(2n - m - 1, n, d\sigma, d\mu)$. The associated symmetric tridiagonal matrix $J_{n,\ell}(d\sigma, d\mu) \in \mathbb{R}^{n \times n}$ is given by

(2.6)
$$\begin{bmatrix}
\alpha_0 & \sqrt{\beta_1} & & & & & & & & 0 \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
 & & \sqrt{\beta_{n-\ell-2}} & \alpha_{n-\ell-2} & \sqrt{\beta_{n-\ell-1}} & & & & & \\
 & & & \sqrt{\beta_{n-\ell-1}} & \alpha_{n-\ell-1} & \boxed{\sqrt{\tilde{\beta}_{n-\ell}}} & & & & \\
 & & & & \boxed{\sqrt{\tilde{\beta}_{n-\ell}}} & \gamma_{\ell-1} & \sqrt{\lambda_{\ell-1}} & & & \\
 & & & & & \sqrt{\lambda_{\ell-1}} & \gamma_{\ell-2} & \ddots & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \gamma_1 & \sqrt{\lambda_1} \\
0 & & & & & & & & \sqrt{\lambda_1} & \gamma_0
\end{bmatrix},$$

where we circumscribe the last entries determined by the measure $d\sigma$ by rectangles.

3. The main result

The special case $d\mu = d\sigma$ is analyzed in [24, 25]. In that case, with $\tilde{\beta}_{n-\ell} = \beta_{n-\ell}$, it holds that

$$t_n = p_\ell \cdot p_{n-\ell} - \beta_{n-\ell}\, p_{\ell-1} \cdot p_{n-\ell-1}, \tag{3.1}$$

and t_n generates a positive quadrature formula, which we denote by $(2n-2\ell, n, d\sigma)$. The associated symmetric tridiagonal matrix (2.6) reduces to $J_{n,\ell}(d\sigma) \in \mathbb{R}^{n\times n}$ in the form

(3.2)
$$\begin{bmatrix}
\alpha_0 & \sqrt{\beta_1} & & & & & & & & 0 \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
 & & \sqrt{\beta_{n-\ell-2}} & \alpha_{n-\ell-2} & \sqrt{\beta_{n-\ell-1}} & & & & & \\
 & & & \sqrt{\beta_{n-\ell-1}} & \alpha_{n-\ell-1} & \boxed{\sqrt{\beta_{n-\ell}}} & & & & \\
 & & & & \boxed{\sqrt{\beta_{n-\ell}}} & \alpha_{\ell-1} & \sqrt{\beta_{\ell-1}} & & & \\
 & & & & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-2} & \ddots & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \alpha_1 & \sqrt{\beta_1} \\
0 & & & & & & & & \sqrt{\beta_1} & \alpha_0
\end{bmatrix}.$$

Finally, it seems the most interesting and useful subcase $n = 2\ell + 1$ has been proposed and analyzed in [24]. In this case it holds that

$$t_n \equiv t_{2\ell+1} = p_\ell \cdot F_{\ell+1}, \tag{3.3}$$

where

$$F_{\ell+1} = p_{\ell+1} - \beta_{\ell+1} \cdot p_{\ell-1}, \tag{3.4}$$

and $t_{2\ell+1}$ generates a positive quadrature formula $(2\ell+2, 2\ell+1, d\sigma)$, i.e., $\widehat{G}_{2\ell+1}$, which we name the generalized averaged Gaussian quadrature rule. The associated symmetric tridiagonal matrix (3.2) reduces to $J_{2\ell+1,\ell}(d\sigma) \in \mathbb{R}^{2\ell+1\times 2\ell+1}$ in the form

$$
\tag{3.5}
\begin{bmatrix}
\alpha_0 & \sqrt{\beta_1} & & & & & & & & 0 \\
\sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & \\
 & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-1} & \sqrt{\beta_\ell} & & & & & \\
 & & & \sqrt{\beta_\ell} & \alpha_\ell & \boxed{\sqrt{\beta_{\ell+1}}} & & & & \\
 & & & & \boxed{\sqrt{\beta_{\ell+1}}} & \alpha_{\ell-1} & \sqrt{\beta_{\ell-1}} & & & \\
 & & & & & \sqrt{\beta_{\ell-1}} & \alpha_{\ell-2} & \ddots & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \alpha_1 & \sqrt{\beta_1} \\
0 & & & & & & & & \sqrt{\beta_1} & \alpha_0
\end{bmatrix}.
$$

The nodes of Gauss rule G_ℓ form a subset of the nodes of the rule $\widehat{G}_{2\ell+1}$ (cf. (3.1); see also [23, proof of Proposition 2.1]). The leading and trailing principal $\ell\times\ell$ submatrices of $J_{2\ell+1,\ell}(d\sigma)$ given by (3.5) have the same eigenvalues. By considering the characteristic polynomial for (3.5) and expanding the determinant along row $\ell+1$, one can show that the spectrum of the leading principal submatrix is a subset of the spectrum of the latter matrix; see [15] for details.

Now $\widehat{G}_{2\ell+1}$ is constructed as the Gaussian q.f. by the associated symmetric tridiagonal matrix $J_{2\ell+1,\ell}(d\sigma) \in \mathbb{R}^{2\ell+1\times 2\ell+1}$ of the form (3.5), and therefore by the simpler manner than the corresponding positive Gauss-Kronrod q.f. $H_{2\ell+1}$ (cf. [1], [15]), if the latter exists. $\widehat{G}_{2\ell+1}$ can be constructed whenever G_ℓ is constructed. The weights of $\widehat{G}_{2\ell+1}$ are all positive, and the interlacing property holds for the nodes of p_ℓ and $F_{\ell+1}$ (cf. [24]). A disadvantage might be that the algebraic degree of precision of $\widehat{G}_{2\ell+1}$ is $2\ell+2$, while the one of $H_{2\ell+1}$ is $3\ell+1$. An interesting heuristic analysis concerning this question has been done in [25], where we try to answer whether the averaged Gaussian formula $\widehat{G}_{2\ell+1}$ is an adequate alternative to the corresponding Gauss-Kronrod quadrature formula $H_{2\ell+1}$, to estimate the remainder term of the Gaussian rule G_ℓ.

Assume now that the monic orthogonal polynomials relative to $d\sigma$ satisfy a three-term recurrence relation of the following kind:

$$
\tag{3.6}
\begin{aligned}
p_{\ell+1}(t) &= (t-\alpha_\ell)p_\ell(t) - \beta_\ell p_{\ell-1}(t), \quad \ell = 0,1,\ldots,\\
\alpha_\ell &= \alpha,\ \beta_\ell = \beta \quad \text{for} \quad \ell \ge r,
\end{aligned}
$$

where $\alpha_\ell \in \mathbb{R},\ \beta_\ell > 0,\ r \in \mathbb{N}$, and $p_0(t) = 1,\ p_{-1}(t) = 0$. Thus, the coefficients α_ℓ and β_ℓ are constant equal, respectively, to some $\alpha \in \mathbb{R}$ and $\beta > 0$ for $\ell \ge r$. Any such measure $d\sigma$ is known to be supported on a finite interval [16, Theorem 10] (see also Chihara [2, Theorem 2.2 on p. 109]), say $[a,b]$, together with the property (3.6), has been denoted by $d\sigma \in \mathcal{M}_r^{\alpha,\beta}[a,b]$ (see [9]).

Theorem 3.1. *Consider a measure $d\sigma \in \mathcal{M}_r^{\alpha,\beta}[a,b]$. Then for $\ell \ge 2r-1$ the generalized averaged Gaussian q.f. $\widehat{G}_{2\ell+1}$ has the algebraic degree of precision at least $3\ell+1$. Therefore it coincides with the corresponding Gauss-Kronrod q.f. and the monic polynomials $F_{\ell+1}$ coincide with the corresponding monic Stieltjes polynomials given by*

$$
\tag{3.7}
F_{\ell+1}(t) \equiv E_{\ell+1}(t) = p_{l+1}(t) - \beta p_{\ell-1}(t) \quad \textit{for} \quad \ell \ge 2r-1.
$$

Proof. Let $d\sigma \in \mathcal{M}_r^{\alpha,\beta}[a,b]$.

It is well known that the algebraic degree of precision $2n-1-m$ of a positive $(2n-1-m, n, d\sigma)$ q.f. $(0 \le m \le n)$ is equal to the number of the entries in the corresponding Jacobi matrix, i.e., the coefficients of the three-term recurrence relation in (1.2), without β_0 (cf. (2.1); see also Characterization theorems in [18], [19]).

If $\ell \in \{1,2,\dots,r-1\}$, then $\widehat{G}_{2\ell+1}$ has the algebraic degree of precision $2\ell+2$. (For $\ell = 1$ this means that $2\ell+2 = 3\ell+1$ and $\widehat{G}_{2\ell+1} = H_{2\ell+1}$, i.e., $\widehat{G}_3 = H_3$.)

Let $\ell \ge r$. Then the algebraic degree of precision of the corresponding positive q.f. based on the construction by the Jacobi matrix of the type (3.2), with $n = 2\ell+1$, is $2n-1-(2r-1) = 2(2\ell+1)-1-(2r-1)$ and it is greater than or equal to $3\ell+1$ if and only if $4\ell+1-(2r-1) \ge 3\ell+1$, i.e.,

$$\ell \ge 2r-1.$$

The given matrix (3.2) coincides with (3.5) for construction of the generalized averaged Gaussian q.f. $\widehat{G}_{2\ell+1}$, relative to (3.6). Therefore, if $\ell \ge 2r-1$, then $\widehat{G}_{2\ell+1}$ has the algebraic degree of precision at least $3\ell+1$ and it coincides with the corresponding Gauss-Kronrod q.f. $H_{2\ell+1}$, which is uniquely determined in this way. On the basis of (3.4), relative to (3.6), the monic polynomials $F_{\ell+1}$ coincide with the corresponding monic Stieltjes polynomials $E_{\ell+1}$ given by (3.7). $\square$

A statement similar to Theorem 3.1 has been proved by Gautschi and Notaris (see [9, Theorem 2.3]) in a different manner, from the viewpoint of Gauss-Kronrod quadrature formulae theory. The generalized averaged Gaussian q.f. are introduced much later (see [24], [25]).

Among the many orthogonal polynomials satisfying (3.6) we mention the four Chebyshev-type polynomials and their modifications discussed by Allaway in his thesis, as well as those associated with the Bernstein-Szegő measures; see [10], [8], and [9] with references therein.

We end with an interesting representation for orthogonal polynomials with respect to the measure $d\sigma \in \mathcal{M}_r^{\alpha,\beta}[-1,1]$. As noted by Peherstorfer (cf. [19, p. 2245]) for the recurrence coefficients of (p_ℓ) by Rakhmanov's theorem it holds that

$$\lim_{\ell\to 0} \alpha_\ell = 0, \quad \lim_{\ell\to 0} \beta_\ell = \frac{1}{4}$$

if $\sigma' > 0$ a.e. on $[-1,1]$, i.e., $d\sigma(t) = \omega(t)\,dt$, where ω is a weight function on $[-1,1]$. Therefore, in this case it has to be $\mathcal{M}_r^{\alpha,\beta}[-1,1] = \mathcal{M}_r^{0,1/4}[-1,1]$, and for any orthogonal polynomial p_n with respect to $d\sigma \in \mathcal{M}_r^{0,1/4}[-1,1]$, on the basis of (2.5), (2.6), it holds that

$$p_n = \hat{U}_\ell \cdot p_{n-\ell} - \frac{1}{4}\hat{U}_{\ell-1} \cdot p_{n-\ell-1} \qquad (n \ge 2\ell,\ \ell \ge r), \tag{3.8}$$

where $\hat{U}_\ell$ is the ℓ-degree monic Chebyshev polynomial of the second kind. Recall that for the monic Chebyshev polynomials of the second kind $\hat{U}_\ell$ it holds that

$$\gamma_k = 0\ (k = 0,1,2,\dots); \quad \lambda_k = \frac{1}{4}\ (k = 1,2,\dots) \quad [\lambda_0 = \pi/2].$$

For example, as a consequence of (3.8) we have

$$\hat{T}_n = \hat{U}_\ell \cdot \hat{T}_{n-\ell} - \frac{1}{4}\hat{U}_{\ell-1} \cdot \hat{T}_{n-\ell-1} \qquad (n \ge 2\ell,\ \ell \ge 2),$$

i.e.,

$$T_n = U_\ell \cdot T_{n-\ell} - U_{\ell-1} \cdot T_{n-\ell-1} \qquad (n \geq 2\ell,\ \ell \geq 2),$$

since for the monic Chebyshev polynomials of the first kind $\hat{T}_\ell$ it holds that

$$\alpha_k = 0\ (k = 0, 1, 2, \dots); \quad \beta_1 = \frac{1}{2},\ \beta_k = \frac{1}{4}\ (k = 2, \dots) \quad [\lambda_0 = \pi].$$

Further,

$$U_n = U_\ell \cdot U_{n-\ell} - U_{\ell-1} \cdot U_{n-\ell-1} \qquad (n \geq 2\ell,\ \ell \geq 1).$$

In a similar manner we derive the given representation for the Chebyshev polynomials of the third and fourth kind, monic orthogonal polynomials relative to the Bernstain-Szegő weight functions introduced in [8], etc.

4. Conclusion

The $(2\ell + 1)$-point generalized averaged Gaussian quadrature rule $\widehat{G}_{2\ell+1}$ for the class of nonnegative measures $d\sigma \in \mathcal{M}_r^{\alpha,\beta}[a, b]$ has algebraic degree of precision at least $3\ell + 1$ and coincides in this case with the corresponding Gauss-Kronrod quadrature rule $H_{2\ell+1}$. We proposed its construction by using the method from [24]. We derived some interesting equalities for the orthogonal polynomials relative to $d\sigma \in \mathcal{M}_r^{0,1/4}[-1, 1]$.

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