# The smallest 5-chromatic tournament \*

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#### **Abstract**

A coloring of a digraph is a partition of its vertex set such that each class induces a digraph with no directed cycles. A digraph is k-chromatic if k is the minimum number of classes in such partition, and a digraph is oriented if there is at most one arc between each pair of vertices. Clearly, the smallest k-chromatic digraph is the complete digraph on k vertices, but determining the order of the smallest k-chromatic oriented graphs is a challenging problem. It is known that the smallest 2-, 3- and 4-chromatic oriented graphs have 3, 7 and 11 vertices, respectively. In 1994, Neumann-Lara conjectured that a smallest 5-chromatic oriented graph has 17 vertices. We solve this conjecture and show that the correct order is 19.

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# 1 Introduction

Finding proper colorings of graphs lies among the most studied problems in graph theory. The goal consists in coloring vertices so that adjacent ones receive distinct colors. In [Neu82], Neumann-Lara introduced a generalization of this problem to digraphs. A digraph consists of a vertex set V plus a set of (ordered) pairs of vertices called arcs. Graphs can be seen as a special case of digraphs where, for every arc uv, there also exists an arc vu (such digraphs are called symmetric and these pairs of arcs are called symmetric and these pairs of arcs are called symmetric and symmetric and symmetric and symmetric and symmetric arc symmetric arc symmetric and symmetric and symmetric arc symmetric and symmetric and symmetric arc symmetric and symmetric and symmetric arc symmetric arc symmetric and symmetric arc symmetric and symmetric arc symmetric arc symmetric and symmetric arc symmetric arc symmetric arc symmetric and symmetric arc symmetric arc symmetric and symmetric arc symmetric

Neumann-Lara defines a proper coloring of a digraph as a partition of the vertex set into acyclic sets (i.e., subsets of vertices which do not contain any oriented cycle). Note that when all the arcs come in digons, this notion indeed reduces to the usual graph coloring definition of unoriented graphs. The smallest number of colors required to color properly a digraph D is called the *chromatic number of* D and will be denoted by  $\chi(D)$  in the rest of the paper<sup>1</sup>.

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<sup>&</sup>lt;sup>1</sup>Note that it is sometimes denoted  $\overrightarrow{\chi}(D)$ , especially when confusion is possible with chromatic number of unoriented graphs. Since we consider only digraphs we prefer keeping the notation as simple as possible.

While there exist other generalizations of coloring to digraphs for instance based on graph homomorphisms, see e.g. [Cou94], Neumann-Lara's is the most classical one and the one that received ever-growing attention since its introduction.

An oriented graph is a digraph that does not contain any digon. Understanding the behavior and structure of graphs of small order and large chromatic number becomes much harder on oriented graphs. Indeed, for any integer k, the smallest digraph of chromatic number k is the complete graph on k vertices (i.e. there is a digon between every pair of vertices). On the contrary, determining the order of the smallest oriented graph of chromatic number k was already raised by Neumann-Lara in 1982 in [Neu94]. The goal of this paper is to tackle that problem for k=5.

Observe that adding arcs to an oriented graph cannot decrease its chromatic number. Therefore, if there exists an oriented graph of chromatic number k there exists a tournament of chromatic number k; a tournament being an orientation of the undirected complete graph. Then, the quest of the smallest oriented graph of chromatic number k can be restricted to tournaments. A tournament (or by abuse of notations a set of vertices) is transitive if for every pair of arcs uv and vw, the arc between uw also exists. One can easily observe that a tournament can be colored with k colors if and only if it can be partitioned in k transitive subtournaments. Thus, Neumann-Lara's question can be rephrased as follows:

Question 1.1. For every k, what is the smallest value of  $n_k$  for which there exists a tournament D on  $n_k$  vertices which cannot be partitioned into (k-1) transitive subtournaments?

This formulation connects this problem to questions raised by Erdős and Moser about 20 years before Neumann-Lara's question [EM64]. The question was asymptotically solved since the maximum chromatic number of a tournament on n vertices is  $\Theta(\frac{n}{\log n})$  [Erd79, EM64]. However, the question of determining the exact values even for small values of k is still widely open.

The smallest tournament of chromatic number 2 (*i.e.* non-transitive) is the directed cycle of length 3. The constructions for k=4,5 rely on the so-called Paley tournaments. For every prime integer n of the form 4k+3, the *Paley tournaments* on n vertices  $\operatorname{Pal}_n$  is the tournament whose vertex set is  $\{0,...,n-1\}$  and containing the arc ij if and only if i-j is a square modulo n. In [Neu94], Neumann-Lara proved that the smallest tournament of chromatic number 3 has order 7 and that there exist four such tournaments, including  $\operatorname{Pal}_7$ . He also proved that the smallest tournament of chromatic number 4 has order 11, is unique and is actually  $\operatorname{Pal}_{11}$ . In the conclusion of [Neu94], Neumann-Lara discussed the possible order of the smallest 5-chromatic tournament. He claimed to know that the answer is between 17 and 19 and conjectured that it is 17. Note that the next "natural" candidate, namely the Paley tournament  $\operatorname{Pal}_{19}$  is actually 4-colorable. Neumann-Lara actually published his construction of a 5-chromatic tournament on 19 vertices six years later in [NL00].

Even if the question received a considerable attention and was mentioned often as an open problem in the literature in the last 30 years (see [BBSS20, KS20] for recent examples), determining the exact value of  $n_5$  is still open today.

The goal of this paper is to answer this question and provides a definitive answer to Neumann-Lara's question for k = 5. Namely,

### **Theorem 1.2.** The smallest order of a 5-chromatic tournament is 19.

After presenting some tools in Sections 2 and 3, we disprove Neumann-Lara's conjecture in Section 4 by showing that every tournament on 17 vertices is 4-colorable. The proof relies on a surprising intermediate result (Theorem 3.1) of independent interest proved by a computer

analysis. Namely all the 4-chromatic tournaments on 12 vertices contain  $Pal_{11}$  as a subtournament. We derive from it that all the tournaments on 17 vertices have chromatic number 4 with a short and human-readable proof. We leave as an open problem a human-readable proof that all the 4-tournaments on 12 vertices contain  $Pal_{11}$  as a subtournament. So  $n_5 \geqslant 18$ .

We then exhibit an example of a 5-chromatic tournament on 19 vertices in Section 5, which ensures that  $n_5 \le 19$ .

We finally present in Section 6 a computer-assisted proof showing that all the tournaments on 18 vertices are 4-colorable, which ensures that  $n_5 = 19$  and settles the case k = 5.

Note that the number of non-isomorphic tournaments on 17, 18 and 19 vertices have respectively 27, 31 and 35 digits [SI], generating them up to isomorphism is already a very challenging task and the problem of 5-colorability that we need to solve on each of them is NP-complete. Therefore, it is definitely out of reach to solve the problem by bruteforce. Instead, we use the approach summarized in the following sketch. We observe that any 5-chromatic 18-vertex tournament must contain two or three (vertex-)disjoint copies of  $TT_5$ , the transitive tournament of order 5. In the latter case, we can thus decompose its vertex set as  $A_1, A_2, A_3, B$  such that each  $A_i$  induces  $TT_5$  and B induces a directed triangle. We may observe that  $A_i \cup B$  induces a 3-chromatic tournament on 8 vertices and that  $A_i \cup A_j \cup B$  induces a 4-chromatic tournament on 13 vertices. We proceed as follows:

- 1. We generate the so-called 8-completions, that are the non-isomorphic ways to orient the arcs between a  $TT_5$  and a directed triangle so that the resulting tournament is 3-chromatic. To this end, we use a branching algorithm involving a trimming operation when we detect that the branch will not generate any 3-chromatic tournament. There are 256 such 8-completions.
- 2. For each pair of 8-completions, we identify their distinguished directed triangle and we generate the possible orientations of the arcs between their respective  $TT_5$  so that the result is a 4-chromatic 13-vertex tournament (with two distinguished copies of  $TT_5$ ). These are called 13-completions. For each pair of 8-completions, the maximum amount of 13-completions is 2072. However only 4508 pairs have at least one 13-completion (with an average of 47.6 completions for each pair), and a quarter of them has precisely one 13-completion.
- 3. We then consider the triples of 8-completions  $(C_1, C_2, C_3)$  where for each  $1 \le i < j \le 3$ ,  $(C_i, C_j)$  lies among these 4508 pairs. We generate all the 18-vertex tournaments obtained by identifying their distinguished directed triangle, and adding the arcs of each 13-completion between  $C_i$  and  $C_j$  for  $i, j \in \{1, 2, 3\}$ . We finally check whether one of these tournaments is 5-chromatic.

When our candidate has exactly two disjoint copies of  $TT_5$ , note that the remaining 8-vertex tournament  $X_8$  must be 3-chromatic and without  $TT_5$ , hence lies among a list of only 94 elements. For each such tournament  $X_8$ , we proceed as follows:

- 1. We re-use our branching algorithm to generate all orientations of arcs between  $X_8$  and  $TT_5$  that yield a 4-chromatic 13-vertex tournament (and adapt the trimming to cut the branch when we detect two disjoint  $TT_5$ ).
- 2. For each pair of such orientations, we generate a 13-vertex oriented graph by identifying their common 8-vertex tournament.

- 3. We discard *incompatible pairs*, that are pairs with a 4-coloring where colors 1, 2 are used only on one  $TT_5$  and 3, 4 on the other. Indeed, for these pairs, every orientation of the remaining arcs will stay 4-colorable.
- 4. For each remaining pair, we try all possible orientations of the remaining 25 arcs and check whether the resulting tournament is 5-chromatic (re-using our branching algorithm).

For 17-vertex tournaments, we could follow roughly the same approach as in the two  $TT_5$  case (since we can show each candidate must contain two copies, but cannot contain three of them). However, we can show by hand that all pairs are incompatible in Step 3, which directly concludes without using Step 4. To this end, we do not need to consider the full output of Step 2, but only the intermediate result stated in Section 3.

The case k=6. It was already known to Neumann-Lara that  $n_6 \leqslant 26$  since there is a  $TT_6$ -free tournament on 26-vertices, which is thus not 5-colorable. Our result actually implies that  $n_6 \geqslant 24$  since every tournament on 23 vertices must contain  $TT_5$ , and the remaining vertices induce a 4-colorable tournament. Besides these easy observations, nothing seems to be known about the exact value of  $n_6$ .

**Related work.** In [BBSS20], Bang-Jensen *et al.* establish some structural results about k-critical digraphs, i.e. digraphs with chromatic number k that are minimal by inclusion. The average degree of such digraphs was also source of attention in recent years. In [HK15, KS20], the authors provide some bounds on the smallest possible value of this parameter among all k-critical digraphs on n vertices. Note that the question is easily answered without the dependency in n since each vertex of a k-critical digraph needs to have in- and out-degree at least k-1 and this value is reached by complete digraphs on k vertices. But here again, the question becomes much more difficult when digons are forbidden: the smallest average degree of oriented k-critical graphs is still open even for k=3 [ABHR22].

These works are also reminiscent of numerous works in the undirected case that look for the smallest graph of chromatic number k that does not contain any complete subgraph of order c. The problem has been especially well-studied for triangle-free graphs (the case c=3), since this is the smallest value of c that makes the problem non-trivial. In [Chv70], Chvátal proved that the smallest triangle-free k-chromatic graph has order 11 for k=4 and Jensen and Royle proved in [JR95] through a computer search that it has order 22 for k=5. The question is still open for k=6 where Goedgebeur proved in [Goe20] that it is between 32 and 40. For digraphs, forbidding cliques of size c=2 corresponds to considering oriented graphs, and actually yields again Neumann-Lara's question we study in this paper.

## 2 Tools

In this section, we introduce structural results that are used throughout the paper. The first of these results answers the question on the smallest 3-chromatic tournaments.

**Theorem 2.1** ([Neu94]). Every tournament on 6 vertices is 2-colorable. Moreover, there are exactly four 3-chromatic tournaments on 7 vertices; namely, the tournament  $\operatorname{Pal}_7, W, W_0, W_1$  depicted in Figure 1.

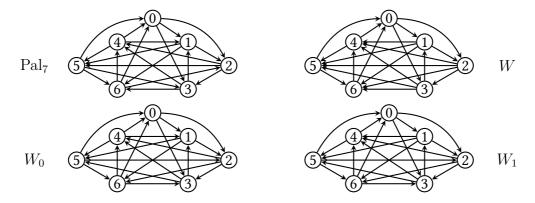


Figure 1: The 3-chromatic tournaments on 7 vertices.

We then investigate tournaments which contain no transitive subtournaments of prescribed order. We let  $TT_k$  denote the transitive tournament on k vertices, and we say that an oriented graph is  $TT_k$ -free if it does not contain  $TT_k$  as a subgraph. A simple inductive argument (using that the in-neighborhood or the out-neighborhood of each vertex contains at last half of the other vertices of a tournament) yields the following.

**Lemma 2.2** ([Ste59]). Every tournament on  $2^{k-1}$  vertices contains  $TT_k$ .

The bound of Lemma 2.2 is not tight, and determining the order of smallest  $TT_k$ -free tournaments is an open question. A precise answer is known only for  $k \le 6$  [SF98], and we need the case k = 5.

**Theorem 2.3** ([RP70]). Every tournament on 14 vertices contains  $TT_5$ .

Interestingly, there is precisely one  $TT_5$ -free tournament on 12 vertices, and precisely one  $TT_5$ -free tournament on 13 vertices as shown by the following.

**Theorem 2.4** ([SF98]). There is a unique  $TT_5$ -free tournament on 12 vertices, and its chromatic number is 3.

**Theorem 2.5** ([RP70]). There is a unique  $TT_5$ -free tournament on 13 vertices, and it can be represented so that the vertices are integers  $0, \ldots, 12$  and ij is an arc if and only if  $j - i \in \{1, 2, 3, 5, 6, 9\}$  modulo 13.

We let  $X_{13}$  be the unique  $TT_5$ -free tournament on 13 vertices, and we conclude this section with two propositions on the properties of  $X_{13}$ . It is well-known that this tournament is *vertex-transitive*, which means that its automorphism group acts transitively on its vertices. The following stronger result actually holds.

**Proposition 2.6** ([RP70]). The tournament  $X_{13}$  is vertex-transitive and for every arc ij, there exists an automorphism of  $X_{13}$  mapping ij to either 01 or 02.

Using Proposition 2.6, we determine the structure of the copies of  $TT_4$  in  $X_{13}$  as follows.

**Proposition 2.7.** Let A be a set of vertices inducing  $TT_4$  in  $X_{13}$  whose vertices of highest out-degree are either  $\{0,1\}$  or  $\{0,2\}$ . Then A is either  $\{0,1,2,3\}$ ,  $\{0,1,3,6\}$ ,  $\{0,1,6,2\}$  or  $\{0,2,3,5\}$ . Moreover, the four possible tournaments obtained by removing A from  $X_{13}$  are pairwise non-isomorphic, and each of them has no non-trivial automorphism.

*Proof.* Let a, b, c and d be four vertices in transitive order in  $X_{13}$ , that is, ab, ac, ad and bc, bd and cd are arcs of  $X_{13}$ . By hypothesis, we have ab = 01 or ab = 02. Now we use that c and d are out-neighbors of both a and b. If the arc ab is 01, then we can only complete 01 into a  $TT_4$  by choosing cd as 23, 36 or 62. If ab is 02, then the only way to complete 02 into a  $TT_4$  is with cd chosen as 35. This concludes the first part of the statement.

We let  $A_1 = \{0, 1, 2, 3\}$ ,  $A_2 = \{0, 1, 3, 6\}$ ,  $A_3 = \{0, 1, 6, 2\}$  and  $A_4 = \{0, 2, 3, 5\}$ , and let  $T_i = X_{13} \setminus A_i$  for every i of  $\{0, 1, 2, 3\}$ .

In order to show that the tournaments  $T_1, T_2, T_3$  and  $T_4$  are pairwise non-isomorphic, we consider the subtournament  $T_i'$  of  $T_i$  induced by the set of all vertices whose in-degree in  $T_i$  is 4 (see tournaments  $T_1', \ldots, T_4'$  depicted in Figure 2). We note that  $T_1', \ldots, T_4'$  are pairwise non-isomorphic, and thus  $T_1, \ldots, T_4$  are pairwise non-isomorphic.

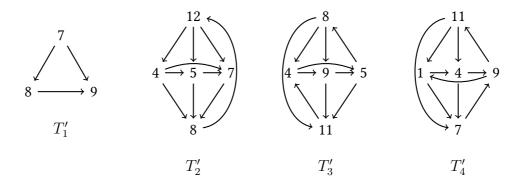


Figure 2: Tournaments  $T'_1, \ldots, T'_4$ .

Finally, we show that each of  $T_1, \ldots, T_4$  has no non-trivial automorphism. For  $T_1$ , we observe that vertices 4, 5, 6 have in-degree 3, vertices 7, 8, 9 have in-degree 4, and vertices 10, 11, 12 have in-degree 5 in  $T_1$ . Furthermore, each of the sets  $\{4, 5, 6\}, \{7, 8, 9\}$ , and  $\{10, 11, 12\}$  induces a  $TT_3$ . Since automorphisms preserve in-degrees and  $TT_3$  has no non-trivial automorphism, we conclude that each vertex of  $T_1$  has to be mapped to itself in every automorphism of  $T_1$ . Thus,  $T_1$  has no non-trivial automorphism.

For every i of  $\{2, 3, 4\}$ , we note that  $T_i$  has precisely two vertices of in-degree 3 and precisely two vertices of in-degree 5 in  $T_i$ . In particular, each of these vertices has to be mapped to itself in every automorphism of  $T_i$ . We recall that the remaining vertices induce  $T'_i$ , and it remains to show that  $T'_i$  has no non-trivial automorphism.

For every i of  $\{2,3,4\}$ , we note that  $T_i'$  contains precisely one vertex of in-degree 1 and precisely one vertex of in-degree 3 in  $T_i'$ , and each of these vertices has to be mapped to itself in every automorphism of  $T_i'$ . For each of  $T_2'$  and  $T_3'$ , we note that the set of all vertices of in-degree 2 induces a  $TT_3$ . It follows that each of  $T_2'$  and  $T_3'$  has no non-trivial automorphism. For  $T_4'$ , we observe that vertex 9 has to be mapped to itself in every automorphism of  $T_4'$  (since there is an arc from 9 to the unique vertex of in-degree 1 in  $T_4'$ ). The desired conclusion for  $T_4'$  follows.

# 3 The 4-chromatic tournaments on 12 vertices

Our disproof of Neumann-Lara's conjecture heavily relies on the following result, that is interesting by itself and has already been useful for other projects.

**Theorem 3.1.** Every 4-chromatic tournament on 12 vertices contains  $Pal_{11}$ .

This result has already been proven of interest; the first author used it for another project with other co-authors in [ABHR22]. The authors show that for every  $k \ge 2$ , there exist k-critical oriented graphs of any possible order larger than some threshold  $p_k$ . They then used Theorem 3.1 to prove that there is no 4-critical oriented graphs on 12 vertices (while  $Pal_{11}$  is one on 11 vertices). In particular, this implies that  $p_k$  is not necessarily the order  $n_k$  of a smallest k-critical oriented graph (which is actually true for k = 2 and 3).

Our proof of Theorem 3.1 relies on a computer program, that basically went through an extensive case analysis, that would be too long to do by hand. In this section, we introduce the ideas behind the program. These ideas will then be reused and developed to prove Theorem 6.1 in Section 6. All our programs can be found at https://github.com/tpierron/5chromatictournaments/.

Let T be a 4-chromatic tournament on 12 vertices. By Theorem 2.4, T contains a  $TT_5$ . Since T is 4-chromatic, the remaining 7 vertices induce a 3-chromatic tournament. Using Theorem 2.1, it follows that the vertices of T can be partitioned into a copy of  $TT_5$  and a copy of a tournament X among  $\{\operatorname{Pal}_7, W, W_0, W_1\}$ . In particular, we say that T is a gluing of  $TT_5$  and X.

A naive way to prove Theorem 3.1 is then to try all the  $4 \times 2^{35}$  possible gluings, keep the 4-chromatic ones and check whether they all contain  $Pal_{11}$ . While this is almost doable, we explain here how to make this process faster, so that Theorem 3.1 can be checked in a matter of hours on a standard computer.

Fix a tournament  $X \in \{\operatorname{Pal}_7, W, W_0, W_1\}$ . Instead of generating the  $2^{35}$  gluings of X with  $TT_5$  directly, and then filter out the 3-colorable ones, we generate them using a branching algorithm (see Algorithm 1) in such a way that we will be able to cut branches.

```
Algorithm 1: completions(T)

Input: An oriented graph T.

Output: All arc-extensions of T to 4-chromatic tournaments on V(T).

1 if V(T) cannot be partitioned into 3 transitive tournaments then

2 | if T is a tournament then

3 | return [T]

4 | else

5 | Choose two non-adjacent vertices a, b in T.

6 | return completions(T + ab) + completions(T + ba)
```

We start from disjoint union of X and  $TT_5$ , and apply completions. At each step, we choose a pair of non-adjacent vertices and add an arc joining them which gives two branches of the computation (one branch for each possible direction of the new arc). The main observation is that, if at some point V(T) can be partitioned into three transitive tournaments, then we can immediately cut the branch since all the tournaments we could obtain from this point onwards will be 3-colorable.

In order to prove Theorem 3.1, we just run completions four times (once for each choice of X), and then check for a  $\operatorname{Pal}_{11}$  in each of the resulting tournaments. One can check that when  $X \neq W_1$ , this step is not needed since the output is always empty, i.e. all the gluings are 3-colorable. This yields the following by-product of our proof, which can actually be deduced from Theorem 3.1 (while not being necessary for proving it).

**Corollary 3.2.** Every 4-chromatic tournament on 12 vertices is a gluing of  $W_1$  and  $TT_5$ .

*Proof.* Let T be a 4-chromatic tournament on 12 vertices. By Theorem 3.1, there is a vertex x such that T-x is  $\operatorname{Pal}_{11}$ . Moreover, by Lemma 2.2, T also contains a set S of 5 vertices inducing  $TT_5$ . Since  $\operatorname{Pal}_{11}$  is  $TT_5$ -free, S must contain x.

Now, the four remaining vertices of  $S \setminus \{x\}$  induce  $TT_4$  in  $Pal_{11}$ . Observe that  $Pal_{11}$  is arc-transitive, hence up to renaming vertices, we may assume that 0 and 1 are the first and second vertices in the  $TT_4$ . One can then easily check that  $S = \{x, 0, 1, 4, 5\}$ . Now observe that the remaining vertices of  $Pal_{11}$  induce  $W_1$ .

We conclude this section by outlining the implementation of the 3-colorability test. The full implementation can be found in section3.ml.

If T is k-colorable, one can choose a k-coloring such that the size of the first color class is maximized. Therefore, with the list L of sets of vertices inducing maximal transitive subtournaments of T we can test k-colorability recurrently as follows: for each V from L, we check if T-V is (k-1)-colorable. The list of maximal transitive subtournaments of T-V can be obtained by removing the vertices of V from the elements of L.

In particular, when running completions, we do not recompute the list L from scratch at each call. Instead, we just update it when adding an arc.

# 4 Disproving Neumann-Lara's conjecture

**Theorem 4.1.** Every 17-vertex tournament is 4-colorable.

This section is devoted to the proof of Theorem 4.1. By contradiction, we consider a 17-vertex tournament  $T_{17}$  which is not 4-colorable. We show that, due to this assumption,  $T_{17}$  has a very rigid structure, which allows us reach a contradiction by constructing a 4-coloring of  $T_{17}$ . The first structural result is summarized in the following lemma.

**Lemma 4.2.** The vertices of  $T_{17}$  can be partitioned in three sets  $A_1, A_2, B$  such that  $A_1$  and  $A_2$  both induce  $TT_5$  and B induces  $W_1$ .

*Proof.* By Lemma 2.2, every tournament on 17 vertices contains a set  $A_1$  of five vertices inducing a transitive tournament. Removing  $A_1$  from  $T_{17}$  gives a tournament  $T_{12}$  on 12 vertices, which is not 3-colorable (otherwise  $T_{17}$  would be 4-colorable). The result now follows by applying Corollary 3.2 to  $T_{12}$ .

The contradiction then follows directly from the next lemma.

**Lemma 4.3.** One can split B as  $B_1 \cup B_2$  such that each of the subtournaments of  $T_{17}$  induced by  $A_1 \cup B_1$  and  $A_2 \cup B_2$  is 2-colorable.

The rest of the proof is devoted to prove Lemma 4.3. To prove this lemma, we consider the tournament  $T_{12}$  induced by  $A_1 \cup B$ , and we identify some subsets  $B_1$  of B such that  $A_1 \cup B_1$  induces a 2-colorable tournament. Up to renaming, we can assume that B = [0, 6] (with the labeling depicted in Figure 1). Let us first state four claims whose proofs are postponed to the end of this section. For readability, we write  $\chi(X)$  to denote the chromatic number of the oriented graph induced in  $T_{12}$  by a set X of vertices.

Claim 4.4.  $\chi(A_1 \cup \{0, 1, 4\}) = 2$ .

**Claim 4.5.**  $\chi(A_1 \cup \{0,1,2,3\}) = 2 \text{ or } \chi(A_1 \cup \{0,4,5,6\}) = 2.$ 

Claim 4.6. If  $\chi(A_1 \cup \{4,5,6\}) > 2$  and  $\chi(A_1 \cup \{2,3,5,6\}) > 2$ , then  $\chi(A_1 \cup \{0,2,4\}) = \chi(A_1 \cup \{1,3,5,6\}) = 2$ .

Claim 4.7. If  $\chi(A_1 \cup \{1,2,3\}) > 2$  and  $\chi(A_1 \cup \{2,3,5,6\}) > 2$ , then  $\chi(A_1 \cup \{0,1,6\}) = \chi(A_1 \cup \{2,3,4,5\}) = 2$ .

Let us now explain how we can derive Lemma 4.3 now follows from these claims. First note that, by symmetry, all these claims hold with  $A_1$  replaced by  $A_2$ . By Claim 4.4, Lemma 4.3 holds if  $\chi(A_1 \cup \{2,3,5,6\}) = 2$  or  $\chi(A_2 \cup \{2,3,5,6\}) = 2$ . So from now on we assume that  $\chi(A_1 \cup \{2,3,5,6\}) > 2$  and  $\chi(A_2 \cup \{2,3,5,6\}) > 2$ . Observe that if  $\chi(A_1 \cup \{0,1,2,3\}) = \chi(A_2 \cup \{0,4,5,6\}) = 2$ , then Lemma 4.3 holds with  $B_1 = \{1,2,3\}$ . Therefore, by symmetry, Claim 4.5 leads to two cases:

- $\chi(A_1 \cup \{0,1,2,3\}) = \chi(A_2 \cup \{0,1,2,3\}) = 2$ . In that case, Lemma 4.3 holds with  $B_i = \{0,1,2,3\}$  unless  $\chi(A_i \cup \{4,5,6\}) > 2$  for all  $i \in \{1,2\}$ . In that case, we can apply Claim 4.6 to both  $A_1$  and  $A_2$ , and Lemma 4.3 holds with  $B_1 = \{0,2,4\}$ .
- $\chi(A_1 \cup \{0,4,5,6\}) = \chi(A_2 \cup \{0,4,5,6\}) = 2$ . In that case, Lemma 4.3 holds with  $B_i = \{1,2,3\}$  unless  $\chi(A_i \cup \{1,2,3\}) > 2$  for all  $i \in \{1,2\}$ . In that case, we can apply Claim 4.7 to both  $A_1$  and  $A_2$ , and Lemma 4.3 holds with  $B_1 = \{0,1,6\}$ .

It remains to prove the four claims. First recall that  $A_1 \cup B$  contains a copy of  $\operatorname{Pal}_{11}$ , and the missing vertex lies in  $A_1$ . We can thus write  $A_1 = \{a, b, c, d, x\}$  where  $B \cup \{a, b, c, d\}$  induces a copy of  $\operatorname{Pal}_{11}$  and a, b, c, d are in transitive order (see Figure 3). Observe also that (up to automorphism),  $\operatorname{Pal}_{11}$  contains a unique copy of  $TT_4$ . Moreover,  $W_1$  has no automorphism. Therefore, there is a unique way to put the arcs between B and  $\{a, b, c, d\}$ , depicted in Figure 3). Each claim thus boils down to show that we can find a 2-coloring of the right subgraph regardless of the neighborhoods of x.

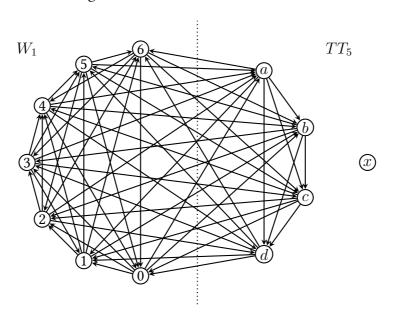


Figure 3: The known arcs in  $A_1 \cup B$ .

*Proof of Claim* 4.4. We separate five cases depending on the rank of x in the transitive order among  $\{a, b, c, d, x\}$ .

• If x < a, then  $\{x, 4, a, b, d\}$  induces a  $TT_5$ , with either x or 4 as source depending on the orientation of the arc 4x. To improve readability, we will present the vertices of the upcoming transitive tournaments in order within their set, using parenthesis when

some vertices might be flipped depending on the orientation of the arc between them. In particular, the above copy of  $TT_5$  will be written  $\{(x,4),a,b,d\}$ . Together with  $\{c,0,1\}$  it gives a 2-coloring of  $A_1 \cup \{0,1,4\}$ .

- If a < x < b, then  $(\{c, 4, d, 0\}, \{a, (1, x), b\})$  is a 2-coloring of  $A_1 \cup \{0, 1, 4\}$ .
- If b < x < c, then either  $(\{4, a, b, x, d\}, \{c, 0, 1\})$  or  $(\{0, a, 1, b\}, \{x, c, 4, d\})$  is a 2-coloring of  $A_1 \cup \{0, 1, 4\}$  (depending on the arc between 4 and x).
- If c < x < d, then  $(\{0, a, 1, b\}, \{c, (4, x), d\})$  is a 2-coloring of  $A_1 \cup \{0, 1, 4\}$ .
- If d < x, then  $(\{a, c, d, (1, x)\}, \{4, 0, b\})$  is a 2-coloring of  $A_1 \cup \{0, 1, 4\}$ .

Proof of Claim 4.5. Observe that  $\gamma_1 = (\{a,c,d,1\},\{0,b,2,3\})$  and  $\gamma_2 = (\{b,c,2,d\},\{0,3,a,1\})$  are two 2-colorings of  $\{a,b,c,d,0,1,2,3\}$ , and that  $\gamma_3 = (\{a,6,b,c\},\{4,d,5,0\})$  and  $\gamma_4 = (\{4,a,b,d\},\{5,6,c,0\})$  are two 2-colorings of  $\{a,b,c,d,0,4,5,6\}$ . We separate five cases depending on the rank of x in the transitive order among  $\{a,b,c,d,x\}$ . In each case, we look for an extension of  $\gamma_1$  or  $\gamma_2$  into a 2-coloring of  $A_1 \cup \{0,1,2,3\}$  or an extension of  $\gamma_3,\gamma_4$  to  $A_2 \cup \{0,4,5,6\}$ .

- If x < a, then we can extend  $\gamma_4$ .
- If a < x < b, then we can extend  $\gamma_3$ .
- If b < x < c, then we must have the arc 2x (resp. x6) for otherwise we can extend  $\gamma_2$  (resp.  $\gamma_3$ ). Now if there is an arc 0x, we can extend  $\gamma_1$ , otherwise there is an arc x0 and we can extend  $\gamma_4$ .
- If c < x < d, then we can extend  $\gamma_2$ .
- If d < x, then we can extend  $\gamma_1$ .

*Proof of Claim* **4.6**. We know that b < x and x6 is an arc otherwise  $(\{x,4,a,b,d\},\{5,6,c\})$  or  $(\{a,6,x,b,c\},\{4,d,5\})$  is a 2-coloring of  $A_1 \cup \{4,5,6\}$ , which is not possible by hypothesis. Moreover, x2 is an arc otherwise  $(\{2,a,x,d,6\},\{b,3,5,c\})$  is a 2-coloring of  $A_1 \cup \{2,3,5,6\}$ . Finally, we have d < x, otherwise either  $(\{5,a,6,c\},\{b,x,2,d,3\})$  or  $(\{2,a,d,6\},\{b,3,x,5,6\})$  is a 2-coloring of  $A_1 \cup \{2,3,5,6\}$  (depending on the arc x3).

Now  $(\{4, a, b, d\}, \{c, 0, x, 2\})$  is a 2-coloring of  $A_1 \cup \{0, 2, 4\}$  and  $(\{a, d, 1, x, 6\}, \{b, 3, 5, c\})$  is a 2-coloring of  $A_1 \cup \{1, 3, 5, 6\}$ .

*Proof of Claim 4.7.* We know that x < b and 1x, 2x are arcs otherwise  $(\{a, c, d, 1, x\}, \{b, 2, 3\})$  or  $(\{3, a, c, 1\}, \{b, 2, x, d\})$  is a 2-coloring of  $A_1 \cup \{1, 2, 3\})$  (depending on the orientation of xd). Moreover, 6x is an arc otherwise  $(\{2, a, x, d, 6\}, \{b, 3, 5, c\})$  is a 2-coloring of  $A_1 \cup \{2, 3, 5, 6\}$ .

Assume that a < x < b. Then we have the arc x3 otherwise  $(\{b, c, 2, d\}, \{3, a, 1, x\})$  is a 2-coloring of  $A_1 \cup \{1, 2, 3\}$ . But this is impossible since  $(\{2, a, d, 6\}, \{x, b, 3, 5, c\})$  or  $(\{5, a, 6, x, c\}, \{b, 2, d, 3\})$  is a 2-coloring of  $A_1 \cup \{2, 3, 5, 6\}$  (depending on the arc x5).

Therefore we have x < a, so  $(\{a, c, d, 1\}, \{6, 0, x, b\})$  is a 2-coloring of  $A_1 \cup \{0, 1, 6\}$  and  $(\{2, 4, x, a, d\}, \{b, 3, 5, c\})$  is a 2-coloring of  $A_1 \cup \{2, 3, 4, 5\}$ .

## 5 A 5-chromatic tournament on 19 vertices

In his seminal paper [Neu94], Neumann-Lara said that there exists a 5-chromatic tournament on 19 vertices but gave no details on the structure of this tournament or the proof of this fact. He actually explained how to construct such a tournament six years later in [NL00], as an illustration of his results on the Zykov sums of digraphs. Before we knew about this paper, we looked for a 5-chromatic tournament and found independently the same tournament. For the sake of completeness, we present this tournament in this section and prove its 5-chromaticity (the construction is outlined in Figure 4). We leave the existence of another 5-chromatic tournament on 19 vertices as an important open question.

#### **Theorem 5.1.** There is a 5-chromatic tournament on 19 vertices.

*Proof.* We consider the tournament  $\operatorname{Pal}_7$  with vertices labeled as in Figure 1. Let D be the 19-vertex tournament obtained from  $\operatorname{Pal}_7$  by blowing-up every vertex besides 0 into a triangle (see Figure 4). More precisely, every vertex  $i \in \{1, \ldots, 6\}$  is replaced by three vertices  $i_1, i_2, i_3$  inducing a triangle and  $i_j k_\ell$  is an arc of D if and only if ik is an arc of  $Pal_7$  and  $i_j 0$  is an arc of D if and only if ik is an arc of  $Pal_7$ .

For the sake of a contradiction, suppose that D admits a 4-coloring. In particular, the vertices of the each triangle  $i_1, i_2, i_3$  receive at least two different colors. We consider the multicoloring of  $\operatorname{Pal}_7$  naturally associated with the coloring of D (that is, each vertex  $i \in \{1, \ldots, 6\}$  is given the colors of  $i_1, i_2, i_3$ ) and note the following.

- Vertex 0 is colored with one color.
- Every vertex  $1, \ldots, 6$  is colored with at least two colors.
- Every color class induces a transitive tournament.

It follows that there are at least  $1+6\times 2=13$  associations of colors to vertices of  $Pal_7$ . Therefore, some color appears on at least  $\lceil \frac{13}{4} \rceil = 4$  vertices of  $Pal_7$ , which is a contradiction since  $Pal_7$  is  $TT_4$ -free.

For n up to 4, we actually know all the smallest n-chromatic tournaments, and this knowledge has been crucial for determining  $n_5$ . Thus, we believe it could be important to determine how many 5-chromatic tournaments there are on 19 vertices. We remark that for every tournament on 7 vertices distinct from  $\operatorname{Pal}_7$ , the construction of blowing-up all vertices into triangles results in a 4-colorable tournament (on 21 vertices). Also, note that any small modification of D such as reverting or removing an arbitrary arc would make it 4-colorable too.

If D is the unique 5-chromatic tournament on 19 vertices, it would imply that removing any  $TT_5$  from a 6-chromatic tournament on 24 vertices yields D, and that any 6-chromatic tournament on 25 vertices is either  $TT_6$ -free or is obtained by gluing  $TT_6$  with D. However, while such a result would be a nice step forward, the methods from Sections 4 and 6 still do not seem powerful enough in their current state to tackle the case of 6-chromatic tournaments.

# 6 The answer to Neumann-Lara's question

In Sections 4 and 5, we showed that all tournaments of order 17 are 4-colorable and that there is a 5-chromatic tournament of order 19. In this section, we consider tournaments on 18 vertices and we answer the question on the order of a smallest 5-chromatic tournament. We present a computer proof of the following.

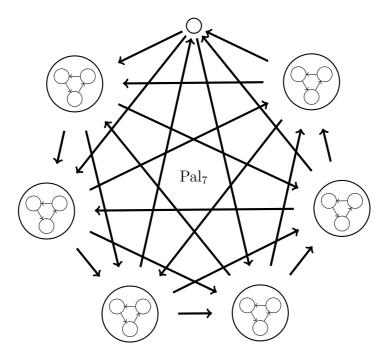


Figure 4: The construction of a 5-chromatic tournament on 19 vertices. Each thick arrow represents a set of arcs with the same orientation. Thick arrows outline the structure of Pal<sub>7</sub>

### **Theorem 6.1.** Every tournament on 18 vertices is 4-colorable.

We cannot simply check every tournament on 18 vertices since there are more than  $10^{30}$  such tournaments [SI]. We first need to reduce the problem to be able to solve it even with the help of a computer. To this end, we consider a hypothetical counterexample to Theorem 6.1 and investigate its structural properties.

We let T be a 5-chromatic tournament of order 18, and we apply a similar but more involved reasoning as in Sections 3 and 4. First, we show that T contains at least two disjoint copies of  $TT_5$  in Subsection 6.1. In Subsections 6.2 and 6.3, we then discuss two cases based on the number of disjoint copies of  $TT_5$  in T. For each case, we present an algorithm which leads to a contradiction with the choice of T.

# **6.1** T must contain two disjoint $TT_5$

This section is devoted to proving the following generalization of Lemma 4.2.

### **Lemma 6.2.** The tournament T contains at least two disjoint copies of $TT_5$ .

Using Lemma 2.2, T must contain a copy of  $TT_5$ . Fix an arbitrary such copy and consider the tournament T' induced by the remaining 13 vertices. If T' is not  $TT_5$ -free, then the lemma follows. So we can assume that T' is  $TT_5$ -free. In particular, T' is isomorphic to  $X_{13}$  by Theorem 2.5.

Let  $a_0 < \cdots < a_4$  be the vertices of  $T \setminus T'$  in transitive order. Note that  $T' + a_0$  has 14 vertices, hence contains a  $TT_5$  by Theorem 2.3. Since T' is  $TT_5$ -free, this  $TT_5$  contains  $a_0$ . Let us denote by  $\{a_0, a_1', a_2', a_3', a_4'\}$  the vertices of this  $TT_5$ , where  $a_1' < \cdots < a_4'$ . If  $T' \setminus \{a_1', a_2', a_3', a_4'\} \cup \{a_1, a_2, a_3, a_4\}$  contains  $TT_5$  then we are done. Otherwise, this tournament is also isomorphic to  $X_{13}$ .

By Proposition 2.6, there is an isomorphism  $f: T\setminus \{a'_1, a'_2, a'_3, a'_4\} \to X_{13}$  such that  $f(a_1)=0$  and  $f(a_2)\in \{1,2\}$ . Moreover, by Proposition 2.7, we may even assume that the quadruple  $(f(a_1), f(a_2), f(a_3), f(a_4))$  is either (0,1,2,3), (0,1,3,6), (0,1,6,2) or (0,2,3,5). Similarly, we may define an isomorphism  $f': T' \to X_{13}$  where  $(f'(a'_1), f'(a'_2), f'(a'_3), f'(a'_4))$  is one of these four quadruples.

Using again Proposition 2.7, we first get that  $T'\setminus\{a_1',a_2',a_3',a_4'\}$  has no non-trivial automorphism, hence f and f' coincide on these 9 vertices, and moreover that  $f(a_i)=f'(a_i')$  for each  $i\in[1,4]$ . Let  $j\in[1,4]$  such that  $f(a_j)=\max\{f(a_1),f(a_2),f(a_3),f(a_4)\}$ . Then  $\{f^{-1}(10),f^{-1}(11),f^{-1}(12),a_1,a_1'\}$  and  $\{a_j,a_j',a_j'+1,a_j'+2,a_j'+3\}$  are two disjoint copies of  $TT_5$  in T. This concludes the proof of Lemma 6.2.

# 6.2 Description of the program

Let  $A_1, A_2$  be two disjoint copies of  $TT_5$  in T and B be the subtournament induced by the 8 remaining vertices. Observe that B is a 3-chromatic tournament (otherwise T would be 4-colorable). Therefore, B lies among a list of 258 tournaments<sup>2</sup>.

Note that the direction of 105 arcs remains unfixed (the 25 arcs between  $A_1$  and  $A_2$  and then 80 arcs between B and  $A_1 \cup A_2$ ), hence an exhaustive search of all these tournaments is still unreasonable. Our method consists in using an approach similar to the one used in Section 4. More precisely, we would like to prove adapt Lemma 4.3 and prove that, for each choice of B, it is possible to split B into  $B_1 \cup B_2$  such that  $A_1 \cup B_1$  and  $A_2 \cup B_2$  are both 2-colorable. Unfortunately this method will fail for some choices of B but will permit to restrict the number of cases to consider.

To explain how our program works, we need some terminology. Fix a 3-chromatic  $TT_5$ -free tournament B on 8 vertices. Let A be a copy of  $TT_5$  and let C be a gluing of A and B. We say that C is a 13-completion of B if C is 4-chromatic. Observe that, in T,  $A_1 \cup B$  and  $A_2 \cup B$  are two 13-completions of B.

The type of C is the set of subtournaments B' of B such that  $3 \leq |B'| \leq 5$  and  $A \cup B'$  induce a 2-colorable tournament. We say that two types  $\mathcal{T}_1, \mathcal{T}_2$  are compatible if the vertices of B can be partitioned as  $B_1 \cup B_2$  where  $B_1 \in \mathcal{T}_1$  and  $B_2 \in \mathcal{T}_2$ . Since T is 5-chromatic, the completions  $A_1 \cup B$  and  $A_2 \cup B$  are not compatible.

Our program works as follows. For each possible choice of B, we generate all of its 13-completions using Algorithm 1. Then, we consider each pair of completions with incompatible types. For each such pair, we construct an 18-vertex oriented graph by identifying the two copies of B. We then apply a slightly modified version of Algorithm 1 to check that orienting the 25 missing arcs only yields 4-colorable tournaments.

For each of the 258 choices for B, this algorithm's running time may take up to roughly ten days on a standard computer. This directly yields a parallel algorithm (using one core per choice of B), which concludes in several years of total computation time. In the following, we present a deeper analysis of the tournaments, which allows us to design a faster algorithm.

To this end, we separate two new cases: either T contains precisely two disjoint copies of  $TT_5$ , or it contains three of them. We handle the former case with the current approach, and the latter in the next subsection. This allows for a faster treatment since we only need to consider the 94 cases where B is  $TT_5$ -free. Moreover, it makes also Algorithm 1 run faster since we can cut branches as soon as we find two disjoint copies of  $TT_5$ .

We provide a sequential implementation of this procedure in the file section62.ml. For each oriented graph, our program takes between a few hours and a few days on a standard

<sup>&</sup>lt;sup>2</sup>Generated using nauty [MP14], see the file 3-chromatic tournaments on 8 vertices.ipynb.

computer and outputs no tournament. Note that this part can again be easily parallelized. Therefore, we get the following.

**Lemma 6.3.** If there is a tournament T on 18 vertices that is 5-chromatic then T contains three pairwise disjoint copies of  $TT_5$ .

## **6.3** T has three disjoint $TT_5$

Let  $A_1, A_2, A_3$  be three disjoint copies of  $TT_5$  in T, and B be the set of the three remaining vertices (that must induce a directed triangle). Note that for each i,  $A_i \cup B$  induces a 3-chromatic tournament on 8 vertices that contains a  $TT_5$ . Moreover, for every  $i \neq j$ ,  $A_i \cup A_j \cup B$  induces a 4-chromatic tournament on 13-vertices.

Similarly to the previous section, our goal is to generate the candidates for  $A_i \cup B$ , then for  $A_i \cup A_j \cup B$ , and finally for T. We again rely on the notion of completion. An 8-completion is a 3-chromatic tournament on 8 vertices together with a fixed copy of  $TT_5$  in it. Two 8-completions are isomorphic if there is an isomorphism between the tournaments that fixes the distinguished copies of  $TT_5$ . Equivalently, this means that one can be obtained from the other by a circular permutation of the three vertices that are not in the distinguished  $TT_5$  (which is a triangle).

There are 256 non-isomorphic 8-completions. Note that it is not surprising that this number is larger than the number of 3-chromatic tournaments on 8 vertices containing  $TT_5$  since such a tournament may actually contain several copies of  $TT_5$ .

For every pair (C,C') of 8-completions, we consider the 13-vertex oriented graphs obtained by identifying the three non-distinguished vertices of C with the three non-distinguished vertices of C'. More precisely, the set of the non-distinguished vertices induces a directed triangle in C and in C', this yields three possible ways for making the identification, and hence we obtain three 13-vertex oriented graphs. For each of these 13-vertex oriented graphs, we use Algorithm 1 which outputs all arc-extensions to 4-chromatic tournaments (each tournament is obtained by adding 25 arcs). The resulting tournaments are candidates for  $A_i \cup A_j \cup B$ , and we call them 13-completions of (C,C').

We may now generate all candidates for T. We consider three 8-completions  $C_1$ ,  $C_2$ ,  $C_3$ , then generate all 13-completions  $C_{12}$  (resp.  $C_{13}$ ,  $C_{23}$ ) of  $(C_1, C_2)$  (resp.  $(C_1, C_3)$ ,  $(C_2, C_3)$ ) again with Algorithm 1. We finally construct a 18-vertex tournament by identifying the vertices inducing  $C_1$  in  $C_{12}$  and  $C_{13}$ , those inducing  $C_2$  in  $C_{12}$  and  $C_{23}$  and those inducing  $C_3$  in  $C_{13}$  and  $C_{23}$  (see Figure 5). We then compute the chromatic number of each such 18-vertex tournament. Each time, this chromatic number is 4, which concludes the proof of Theorem 6.1. The corresponding program can be found in section63.ml. We provide here a sequential implementation of this procedure, which runs in roughly six months on a standard computer. However, note that the computations of the 13-completions can be parallelized (and so can be the 4-colorability check for the 18-vertex candidates), so the program could run much faster if several cores are available.

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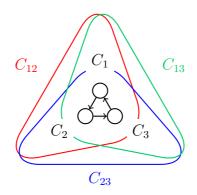


Figure 5: The structure of T.

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