MATHEMATICS OF COMPUTATION
Volume 70, Number 234, Pages 705–717
S 0025-5718(00)01173-X
Article electronically published on November 27, 2000

SATURATION THEOREMS FOR INTERPOLATION AND THE BERNSTEIN-SCHNABL OPERATOR

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ABSTRACT. We shall study properties of box spline operators: cardinal interpolation, convolution, and the Bernstein-Schnabl operator. We prove the saturation theorem.

1. INTRODUCTION

A typical form of the saturation theorem for spline operators is given in [C2], [C3], [DM1], [Dz1], [FK]. Theorems 3.3 and 4.2 give the other forms of the saturation theorem. In the first three parts we deal with box spline operators. The main result, the saturation theorem for cardinal interpolation, is given in the third part. The proof is based on the saturation theorem for convolution operators. In the last part we deal with the saturation theorem for the Bernstein-Schnabl operator. The proof is based on Pisier's inequality.

Let us recall the definition and some properties of box splines. For more detail we refer to [BHR].

(1.1) Let $V = \{v_1, v_2, \ldots, v_n\}$ denote a set of not necessarily distinct vectors in $Z^d \backslash \{0\}$, such that

$$\operatorname{span}\{V\} = R^d.$$

We call such a set *admissible*.

(1.2) If V is admissible, then the *box spline* corresponding to V (denoted by $B(\cdot|V)$) is defined by requiring that

$$\int_{R^d} f(x)B(x|V)\,dx = \int_{[0,1]^n} f(Vu)\,du$$

holds for any continuous function f on R^d (see [BHR, page 1]).

(1.3) A family X of vectors in $Z^d \backslash \{0\}$ is called *symmetric* if it satisfies the following: if $v \in X$, then $-v \in X$.

(1.4) Let X be admissible. Then X is symmetric if and only if $X = V \cup -V$, where V is admissible and

$$-V = \{-v \colon v \in V\}.$$

Received by the editor March 17, 1998 and, in revised form, October 23, 1998 and February 4, 1999.

2000 *Mathematics Subject Classification.* 41A15, 41A35, 41A25, 41A65, 41A40, 41A05.

Key words and phrases. Box splines, cardinal interpolation, convolution operators, the Bernstein-Schnabl operator, Randon-Nikodym property, the saturation theorem.

(1.5) If X is symmetric and admissible, then the box spline corresponding to X is symmetric and it is denoted by

$$N(x) = N(x|X) = B(x|X).$$

(1.6) With the Fourier transform given by

$$\hat{f}(\xi) = \int_{R^d} f(t) e^{-2\pi i \xi \cdot t} \, dt,$$

then ([BHR, page 11])

$$\widehat{N}_X(\xi) = \prod_{v \in X} \frac{\sin(\pi \xi \cdot v)}{\pi \xi \cdot v}.$$

(1.7) The cardinality of the set V is denoted by $|V|$.

(1.8) A family V is called *unimodular* if $|\det W| \leq 1$, for all $W \subset V$, and $|W| = d$.

(1.9) Let X be admissible and symmetric. The following conditions are equivalent (see [BHR, (57) Theorem page 51, (28) Proposition page 89, see proof]):

a) X is unimodular;
b) for all $x \in R^d$

$$P(x) = \sum_{\alpha \in Z^d} N(\alpha|X) e^{2\pi i \alpha \cdot x} \neq 0.$$

(1.10) [BHR, (12) Theorem page 82, see proof] Let X be admissible, symmetric and unimodular. Let $b = \{b(\alpha), \alpha \in Z^d\}$ be the sequence of the Fourier coefficients of the periodic function $1/P(x)$. Then b decays exponentially, i.e., there are constants $C > 0$ and $0 < q < 1$ such that, for all $\alpha \in Z^d$,

$$|b(\alpha)| \leq C q^{\|\alpha\|}.$$

(1.11) For X and b as in (1.10), the *fundamental function* is defined as

$$\Phi(x) = \sum_{\alpha \in Z^d} b(\alpha) N(x - \alpha|X).$$

In particular, this definition implies that, for $\alpha \in Z^d$,

$$\Phi(\alpha) = \begin{cases} 0 & \alpha \neq 0 \\ 1 & \alpha = 0. \end{cases}$$

(1.12) Let

$$\varrho_X = \max\{r \colon \forall_{W \subset X} |W| = r, \ \operatorname{span}\{X \backslash W\} = R^d\}.$$

If $\varrho_X \geq 1$, then (cf. [BH], [BHR, (37) Proposition page 15])

$$\Phi(x), N(x|X) \in C^{\varrho_X - 1}(R^d) \backslash C^{\varrho_X}(R^d).$$

2. Convolution operators

Let $\phi\colon R^d \to R$. The following notation is used:

$$\phi_h(x) = \frac{1}{h^d}\phi\left(\frac{x}{h}\right), \quad h > 0,$$
$$\sigma_h\phi = \phi\left(\frac{x}{h}\right), \quad h > 0,$$
$$T_{\phi,h}f = \phi_h * f,$$

where

$$f * g(x) = \int_{R^d} f(y)g(x-y)\,dy.$$

As usual,

$$\|f\|_p = \left(\int_{R^d} |f(x)|^p\,dx\right)^{1/p},$$
$$|f|_{k,p} = \sum_{|\alpha|=k}\left(\int_{R^d} |D^\alpha f(x)|^p\,dx\right)^{1/p},$$
$$\alpha = (\alpha_1,\dots,\alpha_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = \alpha_1!\cdots\alpha_d!,$$
$$\alpha \geq 0 \quad \text{if} \quad \alpha_j \geq 0 \quad \text{for all} \quad j = 1,\dots,d.$$

If $\alpha \geq 0$, then

$$D^\alpha f = \frac{\partial^{\alpha_1} f}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_d} f}{\partial x_d^{\alpha_d}}$$

and

$$x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d} \quad \text{where} \quad x = (x_1,\dots,x_d).$$

For $1 \leq p < \infty$ and $k \geq 1$, let W_p^k denote the Sobolev space [S]. Let us recall the Poisson summation formula (see [SW]).

Lemma 2.1. *Let $f, \hat{f}$ be continuous functions on R^d. Suppose that there are $C > 0$ and $\delta > 0$ such that*

$$|f(x)| \leq c(1+|x|)^{-d-\delta} \quad x \in R^d$$

and

$$|\hat{f}(x)| \leq c(1+|x|)^{-d-\delta} \quad x \in R^d.$$

Then

$$\sum_{\alpha\in Z^d} f(x-\alpha) = \sum_{\alpha\in Z^d}\hat{f}(\alpha)e^{2\pi i\alpha\cdot x} \quad \textit{for all} \quad x \in R^d.$$

We study the convolution operators $T_{N,h}, T_{\Phi,h}$ corresponding to the symmetric box spline N and the fundamental function Φ. It is easy to obtain the following result on the order of approximation by these operators.

Theorem 2.2. *Let $1 \leq p < \infty$. Let X be admissible and symmetric, and let N be the box spline corresponding to X. Then there is $C_p > 0$ such that for $f \in W_p^2$*

$$\|f - T_{N,h}f\|_p \leq C_p h^2 |f|_{2,p}.$$

Moreover, assume that X is unimodular and let Φ be the fundamental function corresponding to X. Then, for $f \in W_p^{\varrho_X+1}$,

$$\|f - T_{\Phi,h} f\|_p \le C_p h^{\varrho_X+1} |f|_{\varrho_X+1,p}.$$

The following result is needed.

Theorem 2.3. *Let X be admissible and symmetric, and let N be the box spline corresponding to X. Then for $f \in W_2^2$,*

$$\frac{f - T_{N,h} f}{h^2} \to K_1 f = \frac{1}{4\pi^2} \sum_{|\alpha|=2} \frac{D^\alpha \widehat{N}(0)}{\alpha!} D^\alpha f, \quad h \to 0. \tag{2.4}$$

If $f \in L^2(R^d)$ and $\hat{f}$ is of compact support, then

$$f = T_{N,h} f + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{h^{2j}}{(2\pi)^{2j}} \sum_{|\alpha|=2j} \frac{D^\alpha \widehat{N}(0)}{\alpha!} D^\alpha f. \tag{2.5}$$

If in addition X is unimodular and Φ is the corresponding fundamental function, then, for $f \in W_2^{\varrho_X+1}$,

$$\begin{aligned} &\frac{f - T_{\Phi,h} f}{h^{\varrho_X+1}} \to K_2 f \\ &\quad = \frac{1}{(2\pi i)^{\varrho_X+1}} \sum_{|\beta|=\varrho_X+1} \frac{D^\beta f}{\beta!} \left(\sum_{\alpha \in Z^d} D^\beta \widehat{N}(\alpha) - D^\beta \widehat{N}(0) \right), \quad h \to 0, \end{aligned} \tag{2.6}$$

while, for $f \in L^2(R^d)$ with $\hat{f}$ of compact support,

$$f = T_{\Phi,h} f + \sum_{j=1}^{\infty} \frac{h^{\varrho_X+j}}{(2\pi i)^{\varrho_X+j}} \sum_{|\beta|=\varrho_X+j} \frac{D^\beta f}{\beta!} \left(\sum_{\alpha \in Z^d} D^\beta \widehat{N}(\alpha) - D^\beta \widehat{N}(0) \right). \tag{2.7}$$

In formulas (2.4)–(2.7) the convergence in the norm of $L^2(R^d)$ is considered.

Proof. Denote

$$g(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots .$$

Thus (cf. (1.6))

$$\widehat{N}(x) = \prod_{v \in X} g(\pi x \cdot v).$$

Consequently, in Maclaurin's formula

$$\widehat{N}(x) = \sum_{\alpha \ge 0} \frac{D^\alpha \widehat{N}(0)}{\alpha!} x^\alpha \tag{2.8}$$

we have only monomials of even degree. By Plancherel's formula we obtain

$$\begin{aligned} &\left\| \frac{f - T_{N,h} f}{h^2} - \frac{1}{4\pi^2} \sum_{|\alpha|=2} \frac{D^\alpha \widehat{N}(0)}{\alpha!} D^\alpha f \right\|_2 \\ &\quad = \left\| \hat{f}(x) \left(\frac{1 - \widehat{N}(hx)}{h^2} + \sum_{|\alpha|=2} \frac{D^\alpha \widehat{N}(0) x^\alpha}{\alpha!} \right) \right\|_2 . \end{aligned}$$

Since in (2.8) there are only monomials of even degree, we get

$$\frac{1-\widehat{N}(hx)}{h^2}+\sum_{|\alpha|=2}\frac{D^\alpha\widehat{N}(0)x^\alpha}{\alpha!}=-\sum_{j=2}^{\infty}\sum_{|\alpha|=2j}\frac{D^\alpha\widehat{N}(0)}{\alpha!}h^{2j}x^\alpha.$$

The last formula implies (2.4) and (2.5) for functions f such that their Fourier transforms have compact support. The boundedness of the operator $T_{N,h}$ (cf. Theorem 2.2) and a density argument gives (2.4) for $f \in W_2^2$.

Suppose now that X is unimodular. By Plancherel's formula we have

$$\left\|\frac{f-T_{\Phi,h}f}{h^{\varrho_X+1}}-K_2f\right\|_2=\left\|\hat{f}(x)\left(\frac{1-\widehat{\Phi}(hx)}{h^{\varrho_X+1}}\right)-\widehat{K_2f}(x)\right\|_2, \tag{2.9}$$

where

$$\widehat{K_2f}(x)=\hat{f}(x)\sum_{|\beta|=\varrho_X+1}\frac{x^\beta}{\beta!}\sum_{\alpha\in Z^d,\alpha\neq 0}D^\beta\widehat{N}(\alpha).$$

The definition of the fundamental function (cf. (1.11)) implies that

$$\widehat{\Phi}(hx)=\widehat{N}(hx)\sum_{\alpha\in Z^d}b(\alpha)e^{-2\pi i\alpha\cdot hx}.$$

Since X and N are symmetric, the sequence $\{b(\alpha)\}$ is also symmetric. Consequently,

$$\begin{aligned}\frac{1-\widehat{\Phi}(hx)}{h^{\varrho_X+1}}&=\frac{1-\widehat{N}(hx)\sum_{\alpha\in Z^d}b(\alpha)e^{-2\pi i\alpha\cdot hx}}{h^{\varrho_X+1}}\\&=\frac{\sum_{\alpha\in Z^d}N(\alpha|X)e^{-2\pi i\alpha\cdot hx}-\widehat{N}(hx)}{h^{\varrho_X+1}\sum_{\alpha\in Z^d}N(\alpha|X)e^{-2\pi i\alpha\cdot hx}}.\end{aligned}$$

Observe that for f, with $\hat{f}$ of compact support,

$$\sum_{\alpha\in Z^d}N(\alpha|X)e^{2\pi i\alpha\cdot hx}\to 1,\ \text{as } h\to 0,\quad \text{uniformly with respect to } x\in\operatorname{supp}\hat{f}.$$

Therefore, it is sufficient to show that

$$\begin{aligned}&\frac{\sum_{\alpha\in Z^d}N(\alpha|X)e^{-2\pi i\alpha\cdot hx}-\widehat{N}(hx)}{h^{\varrho_X+1}}\\&\quad\to\sum_{|\beta|=\varrho_X+1}\frac{x^\beta}{\beta!}\left(\sum_{\alpha\in Z^d}D^\beta\widehat{N}(\alpha)-D^\beta\widehat{N}(0)\right),\quad h\to 0.\end{aligned} \tag{2.10}$$

Since

$$e^{2\pi i\alpha\cdot hx}=\sum_{n=0}^{\infty}\frac{(2\pi i\alpha\cdot hx)^n}{n!}\quad\text{and}\quad(\alpha\cdot x)^n=\sum_{|\beta|=n}\frac{n!}{\beta!}\alpha^\beta x^\beta,$$

we obtain

$$\begin{aligned}\sum_{\alpha\in Z^d}N(\alpha|X)e^{-2\pi i\alpha\cdot hx}&=\sum_{n=0}^{\infty}\frac{(-2\pi i)^n}{n!}h^n\sum_{\alpha\in Z^d}N(\alpha)(\alpha\cdot x)^n\\&=\sum_{n=0}^{\infty}\frac{(-2\pi i)^n}{n!}h^n\sum_{|\beta|=n}\frac{n!}{\beta!}\sum_{\alpha\in Z^d}N(\alpha)\alpha^\beta x^\beta.\end{aligned} \tag{2.11}$$

Applying Poisson's formula 2.1 to the function $f(x) = x^\beta N(x)$ at the point $x = 0$, we get

$$\sum_{\alpha\in Z^d} N(\alpha)\alpha^\beta = \sum_{\alpha\in Z^d} \frac{1}{(-2\pi i)^{|\beta|}} D^\beta \widehat{N}(\alpha). \tag{2.12}$$

The conditions required by Lemma 2.1 can be checked by elementary calculations. Moreover, since N is symmetric, we get, for β with $|\beta|$ odd,

$$\sum_{\alpha\in Z^d} N(\alpha)\alpha^\beta = 0.$$

Formula (1.6) implies that, for $|\beta| \le \varrho_X$ and $\alpha \in Z^d\backslash\{0\}$,

$$D^\beta \widehat{N}(\alpha) = 0.$$

Thus, we get, for $|\beta| \le \varrho_X$,

$$\sum_{\alpha\in Z^d} N(\alpha)\alpha^\beta = \frac{1}{(-2\pi i)^{|\beta|}} D^\beta \widehat{N}(0). \tag{2.13}$$

Now, it follows from (2.11), (2.12) and (2.13) that

$$\begin{aligned}\sum_{\alpha\in Z^d} N(\alpha|X)e^{-2\pi i\alpha\cdot hx} &= \sum_{n=0}^{\varrho_X} h^n \sum_{|\beta|=n} \frac{1}{\beta!} D^\beta \widehat{N}(0)x^\beta \\ &\quad + \sum_{n=\varrho_X+1}^{\infty} h^n \sum_{|\beta|=n} \frac{1}{\beta!} \sum_{\alpha\in Z^d} D^\beta \widehat{N}(\alpha)x^\beta.\end{aligned}$$

Recall that in formula (2.8) there are only monomials of even degree, and therefore

$$\begin{aligned}&\sum_{\alpha\in Z^d} N(\alpha|X)e^{-2\pi i\alpha\cdot hx} - \widehat{N}(hx) \\ &= \sum_{n=\varrho_X+1}^{\infty} h^n \sum_{|\beta|=n} \frac{1}{\beta!} \left(\sum_{\alpha\in Z^d} D^\beta \widehat{N}(\alpha) - D^\beta \widehat{N}(0) \right) x^\beta.\end{aligned}$$

The last formula implies (2.10), which in turn gives (2.6) and (2.7). □

3. Cardinal interpolation

Let us recall the definition of *cardinal interpolation*: for a continuous and bounded function f

$$If(x) = \sum_{\alpha\in Z^d} f(\alpha)\Phi(x-\alpha), \tag{3.1}$$

and

$$I_h = \sigma_h \circ I \circ \sigma_{1/h}, \quad h > 0.$$

The following theorem can be found in [J] (cf. also [CJW]).

Theorem 3.2. *Let X be admissible, symmetric and unimodular and let Φ be the fundamental function corresponding to X. Let $\varrho_X + 1 > d/2$. Then there is a constant $C > 0$ such that, for all $f \in W_2^{\varrho_X+1}$,*

$$\|f - I_h f\|_2 \le C h^{\varrho_X+1} |f|_{\varrho_X+1,2}.$$

Denote

$$\Lambda = \{W \subset X \colon |W| = \varrho_X + 1, \quad \operatorname{span}\{X \backslash W\} \neq \mathbf{R}^d\},$$

$$D_W = \prod_{v \in W} D_v,$$

where D_v is the derivative in the direction v.

By $\beta \perp (X\backslash W)$ we mean that $v \cdot \beta = 0$ for all $v \in (X\backslash W)$. Recall that

$$K_2 f = \frac{1}{(2\pi i)^{\varrho_X+1}} \sum_{|\beta|=\varrho_X+1} \frac{D^\beta f}{\beta!} \left(\sum_{\alpha \in Z^d} D^\beta \widehat{N}(\alpha) - D^\beta \widehat{N}(0) \right).$$

The main result of this section is

Theorem 3.3. *Let X be admissible, symmetric and unimodular, and let Φ be the fundamental function corresponding to X. Let $\varrho_X + 1 > d/2$ and $f \in W_2^{\varrho_X+1}$. Then*

$$\lim_{h \to 0} \left\| \frac{f - I_h f}{h^{\varrho_X+1}} - K_2 f \right\|_2^2 = \left(\frac{1}{4\pi^2} \right)^{\varrho_X+1} \sum_{W \in \Lambda} \int_{R^d} |D_W f(u)|^2 \, du \sum_{\beta \perp (X\backslash W), \beta \neq 0} \prod_{v \in W} \frac{1}{(\beta \cdot v)^2}.$$

Proof. Let $f \in W_2^{\varrho_X+1}$ be given. Assuming that the Poisson formula can be applied for f, let us calculate the Fourier transform of If:

$$\begin{aligned} \widehat{I_h f}(x) &= \left(\sum_{\alpha \in Z^d} (\sigma_{1/h} f)(\alpha) \sigma_h(\Phi)(x - h\alpha) \right)^\wedge \\ &= h^d \widehat{\Phi}(hx) \sum_{\alpha \in Z^d} (\sigma_{1/h} f)(\alpha) e^{-2\pi i h \alpha \cdot x} \\ &= h^d \widehat{\Phi}(hx) \sum_{\beta \in Z^d} (\widehat{\sigma_{1/h} f})(hx - \beta) \\ &= \widehat{\Phi}(hx) \sum_{\alpha \in Z^d} \hat{f} \left(x - \frac{\alpha}{h} \right). \end{aligned}$$

By Plancherel's formula we have

$$\left\| \frac{f - I_h f}{h^{\varrho_X+1}} - K_2 f \right\|_2 = \left\| \frac{\hat{f} - \widehat{I_h f}}{h^{\varrho_X+1}} - \widehat{K_2 f} \right\|_2.$$

Further, let us assume that the support of $\hat{f}$ is contained in a cube $C = [-k, k]^d$. Then, for $0 < h < 1/(2k)$, we get

$$\int_C \left| \frac{\hat{f}(x) - \widehat{I_h f}(x)}{h^{\varrho_X+1}} \right|^2 dx = \int_C \left| \frac{\hat{f}(x) - \widehat{\Phi}(hx) \hat{f}(x)}{h^{\varrho_X+1}} \right|^2 dx. \tag{3.4}$$

Comparing the above formula with (2.9), we conclude that

$$\int_C \left| \frac{\hat{f}(x) - \widehat{I_h f}(x)}{h^{\varrho_X+1}} - \widehat{K_2 f}(x) \right|^2 dx \to 0 \quad \text{as} \quad h \to 0.$$

Now, let us consider the integral over C^c:

$$\begin{aligned}\int_{C^c}\left|\frac{\hat f(x)-\widehat{I_h f}(x)}{h^{\varrho_X+1}}\right|^2 dx &= \int_{C^c}\left|\frac{\widehat\Phi(hx)\sum_{\alpha\in Z^d}\hat f(x-\alpha/h)}{h^{\varrho_X+1}}\right|^2 dx\\ &= \sum_{\beta\in Z^d\setminus\{0\}}\int_{[-k,k]^d+\beta/h}\left\|\frac{\widehat\Phi(hx)\hat f(x-\beta/h)}{h^{\varrho_X+1}}\right|^2 dx\\ &= \int_{[-k,k]^d}\sum_{\beta\in Z^d\setminus\{0\}}\left|\frac{\widehat\Phi(hu+\beta)}{h^{\varrho_X+1}}\right|^2 |\hat f(u)|^2\,du.\end{aligned}\tag{3.5}$$

By the definition of the fundamental function (1.11) and by (1.6), we get

$$\begin{aligned}\widehat\Phi(hu+\beta) &= \widehat N(hu+\beta)\sum_{\alpha\in Z^d} b(\alpha)e^{2\pi i\alpha\cdot hu}\\ &= \prod_{v\in X}\frac{\sin(\pi hu\cdot v)}{\pi(hu+\beta)\cdot v}\sum_{\alpha\in Z^d} b(\alpha)e^{-2\pi i\alpha\cdot hu}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&\sum_{\beta\in Z^d\setminus\{0\}}\left|\frac{\widehat\Phi(hu+\beta)}{h^{\varrho_X+1}}\right|^2\\ &= \left(\frac{1}{h^{\varrho_X+1}}\right)^2\sum_{\beta\in Z^d\setminus\{0\}}\prod_{v\in X}\left(\frac{\sin(\pi hu\cdot v)}{\pi(hu+\beta)\cdot v}\right)^2\left|\sum_{\alpha\in Z^d} b(\alpha)e^{-2\pi i\alpha\cdot hu}\right|^2.\end{aligned}\tag{3.6}$$

Define

$$\begin{aligned}S_0 &= \{\beta\in Z^d\setminus\{0\}:\ \text{there is } v\in X, \beta\cdot v=0\},\\ S_1 &= \{\beta\in Z^d\setminus\{0\}:\ \text{for all } v\in X, \beta\cdot v\neq 0\},\end{aligned}$$

Then, it follows from (3.6) that

$$\begin{aligned}&\sum_{\beta\in Z^d\setminus\{0\}}\left|\frac{\widehat\Phi(hu+\beta)}{h^{\varrho_X+1}}\right|^2\\ &= \Bigg[\sum_{\beta\in S_0}\prod_{v\in X, v\cdot\beta=0}\left(\frac{\sin(\pi hv\cdot u)}{\pi hv\cdot u}\right)^2\\ &\qquad\times\left(\frac{\prod_{v\in X, v\cdot\beta\neq 0}\sin(\pi hv\cdot u)}{h^{\varrho_X+1}}\right)^2\left(\prod_{v\in X, v\cdot\beta\neq 0}\frac{1}{\pi(hu+\beta)\cdot v}\right)^2\\ &\qquad\qquad+\sum_{\beta\in S_1}\left(\frac{\prod_{v\in X}\sin(\pi hv\cdot u}{h^{\varrho_X+1}}\right)^2\left(\prod_{v\in X}\frac{1}{\pi(hu+\beta)\cdot v}\right)^2\Bigg]\\ &\quad\times\left|\sum_{\alpha\in Z^d} b(\alpha)e^{-2\pi i\alpha\cdot hu}\right|^2.\end{aligned}$$

For $\beta \in S_0$ denote

$$J_\beta = \{v \in X : v \cdot \beta \neq 0\}.$$

Observe that

$$|J_\beta| \geq \varrho_X + 1.$$

Moreover, there is $\beta \in S_0$ such that

$$|J_\beta| = \varrho_X + 1.$$

Let IN be the set of β satisfying the above condition, i.e.,

$$IN = \{\beta \in S_0 : |J_\beta| = \varrho_X + 1\}.$$

Observe that if $W \subset X$ and $|W| > \varrho_X + 1$, then

$$\frac{\prod_{v \in W} \sin(\pi h v \cdot u)}{h^{\varrho_X + 1}} \to 0, \quad h \to 0, \quad \text{uniformly on } C.$$

Moreover,

$$\sum_{\alpha \in Z^d} b(\alpha) = \sum_{\alpha \in Z^d} N(\alpha) = 1.$$

Thus, as $h \to 0$

$$\sum_{\beta \in Z^d \setminus \{0\}} \left| \frac{\widehat{\Phi}(hu + \beta)}{h^{\varrho_X + 1}} \right|^2 \to \sum_{\beta \in IN} \prod_{v \in J_\beta} (\pi v \cdot u)^2 \frac{1}{(\pi \beta \cdot v)^2} \quad \text{in case } d > 1, \tag{3.7}$$

$$\sum_{j \in Z \setminus \{0\}} \left| \frac{\widehat{\Phi}(hu + j)}{h^{\varrho_X + 1}} \right|^2 \to \prod_{v \in X} (\pi u v)^2 \sum_{j \in Z \setminus \{0\}} \prod_{v \in X} \frac{1}{(\pi j v)^2} \quad \text{in case } d = 1,$$

uniformly with respect to $u \in C = [-k, k]^d$.

Let us consider $d > 1$. As a consequence of (3.5) and (3.7) we get

$$\begin{aligned}
\int_{C^c} \left| \frac{\hat{f}(x) - \widehat{I_h f}(x)}{h^{\varrho_X + 1}} \right|^2 dx &\to \sum_{\beta \in IN} \prod_{v \in J_\beta} \frac{1}{(\beta \cdot v)^2} \int_C \prod_{v \in J_\beta} (v \cdot u)^2 |\hat{f}(u)|^2 \, du \\
&= \sum_{\beta \in IN} \prod_{v \in J_\beta} \frac{1}{(\beta \cdot v)^2} \int_{R^d} \prod_{v \in J_\beta} (v \cdot u)^2 |\hat{f}(u)|^2 \, du \\
&= \left(\frac{1}{4\pi^2} \right)^{\varrho_X + 1} \sum_{\beta \in IN} \prod_{v \in J_\beta} \frac{1}{(\beta \cdot v)^2} \int_{R^d} |D_{J_\beta} f(u)|^2 \, du,
\end{aligned} \tag{3.8}$$

where

$$D_{J_\beta} = \prod_{v \in J_\beta} D_v.$$

Note that if $\beta \in IN$, then

$$J_\beta = W \equiv \beta \perp (X \setminus W) \quad \text{and} \quad W \in \Lambda.$$

Moreover, for each $W \in \Lambda$ there is $\beta \in IN$ such that $W = J_\beta$. Therefore, (3.8) takes the following form

$$\sum_{\beta \in IN} \prod_{v \in J_\beta} \frac{1}{(\beta \cdot v)^2} \int_{R^d} |D_{J_\beta} f(u)|^2\, du = \sum_{W \in \Lambda} \int_{R^d} |D_W f(u)|^2\, du \sum_{\beta \perp (X \setminus W), \beta \neq 0} \prod_{v \in W} \frac{1}{(\beta \cdot v)^2}.$$

This completes the proof in the case of functions f for which the Poisson formula can be applied and $\hat{f}$ is of compact support. Note that such functions are dense in $W_2^{\varrho_X+1}$. Moreover, Theorem 3.2 implies that the operators

$$K_h = \frac{f - I_h f}{h^{\varrho_X+1}} - K_2 f$$

are bounded. More precisely there is C such that, for all $f \in W_2^{\varrho_X+1}$ and h,

$$\left\| \frac{f - I_h f}{h^{\varrho_X+1}} - K_2 f \right\|_2^2 \leq C|f|_{\varrho_X+1,2}.$$

The proof of Theorem 3.3 is now completed by the boundedness of the operators K_h and the density argument. □

Remark 3.9. The proof of Theorem 3.3 implies that $\frac{f - I_h f}{h^{\varrho_X+1}} \to K_2 f$ weakly in $L^2(R^d)$.

4. The saturation theorem for Bernstein-Schnabl operators

Let K be a convex set in a linear space B. To the set K corresponds a convex cone

$$\widetilde{K} = \{(\lambda x, \lambda)\colon x \in K, \lambda \geq 0\} \subset B \oplus R.$$

The cone defines a partial order in $B \oplus R$ by putting $x \leq y$ iff $x - y \in \widetilde{K}$. We shall consider here only bounded and closed convex sets in Banach spaces B. The set K is called a simplex if the order determined by $\widetilde{K}$ is a lattice order (on the linear subspace $\widetilde{K} - \widetilde{K}$ of $B \oplus R$).

Now let B be a real Banach space and $K \subset B$ be a separable closed bounded convex subset (with nonempty interior) of B. Assume that K has the Radon-Nikodym property, then, by Edgar's and Bourgin's theorem (see [B]), for every $x \in K$ there is a probability measure μ_x on K such that

$$x = \int_K y \mu_x(dy)$$

and $\mu_x\{ex\, K\} = 1$, where $ex\, K$ denotes the set of extreme points of K. If K is a simplex, then the measure μ_x is unique (see [B]). In this case we can define the Bernstein-Schnabl operators (see [A]) as follows. For $f \in C_u(K)$ (the space of all real valued uniformly continuous functions on K) and $n \in N$ we put

(4.1)
$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{t_1 + \cdots + t_n}{n}\right) \mu_x(dt_1) \cdots \mu_x(dt_n),\ x \in K,\ n = 1, 2, \ldots.$$

The Bernstein-Schnabl operators have a probabilistic interpretation, namely let $\xi_1, \dots, \xi_n$ be a sequence of independent identically distributed K-valued random vectors with the law μ_x, then

$$B_n(f)(x) = Ef\left(\frac{\xi_1 + \cdots + \xi_n}{n}\right), \quad f \in C_u(K),$$

where $E\eta$ means the expectation of a random variable η. By $C_u^2(K)$ we denote the space of all real functions with a uniformly continuous second derivative.

Theorem 4.2. *Let $K \subset B$ be a simplex as above (bounded by M) and*

$$\bigvee_{C>0} \bigwedge_{n} \bigwedge_{x \in K} nB_n(\| \cdot - x\|^2)(x) \le C. \tag{4.3}$$

Then for every $f \in C_u^2(K)$

$$\lim_{n\to\infty} n[B_n(f)(x) - f(x)] = \frac{1}{2}\left[\int_K D^2 f(x)(y,y)\mu_x(dy) - D^2 f(x)(x,x)\right] \tag{4.4}$$

uniformly on K.

Proof. Recall the following known inequality [P].

Lemma. *Let $\xi_1, \dots, \xi_n, \dots$ be a sequence of independent random vectors with $\|\xi_i\| \le 2M$, for each i. Then, for all $t > 0$,*

$$Pr\{\big|\|S_n\| - E\|S_n\|\big| > t\} \le 2\exp\left(-\frac{t^2}{32nM^2}\right),$$

where $S_n = \xi_1 + \cdots + \xi_n$. □

Assume that $\xi_1, \dots, \xi_n$ is a sequence of independent identically distributed K-valued random vectors with a law μ_x. For a given $t > 0$ we can choose (by a strong law of large numbers) $n_0 \in N$ such that, for $n > n_0$,

$$E\left\|\frac{S_n}{n} - x\right\| < \frac{t}{2}.$$

Thus, and by Pisier's lemma, we have for $n > n_0$

$$\begin{aligned} Pr\left\{\left\|\frac{S_n}{n} - x\right\| > t\right\} &\le Pr\left\{\left\|\frac{S_n}{n} - x\right\| > \frac{t}{2} + E\left\|\frac{S_n}{n} - x\right\|\right\} \\ &\le Pr\left\{\big|\|S_n - nx\| - E\|S_n - nx\|\big| > \frac{nt}{2}\right\} \\ &\le 2\exp\left(-\frac{nt^2}{128M^2}\right). \end{aligned}$$

So we have the estimation for $n > n_0$,

$$Pr\left\{\left\|\frac{S_n}{n} - x\right\| > t\right\} \le 2\exp\left(-\frac{nt^2}{128M^2}\right). \tag{4.5}$$

Next by Taylor's formula, we get

$$\begin{aligned}
f(y) - f(x) &= Df(y)(y-x) + \frac{1}{2}\int_0^1 (1-s)D^2 f(x+s(y-x))(y-x,y-x)\,ds \\
&= Df(x)(y-x) + \frac{1}{2}D^2 f(x)(y-x,y-x) \\
&\quad + \frac{1}{2}\int_0^1 (1-s)[D^2 f(x+s(y-x))(y-x,y-x) \\
&\qquad\qquad - D^2 f(x)(y-x,y-x)]\,ds,
\end{aligned}$$

i.e.,

$$f(y) - f(x) = Df(x)(y-x) + \frac{1}{2}D^2 f(x)(y-x,y-x) + r(x,y), \tag{4.6}$$

where $r(x,y) = o(\|y-x\|^2)$, and it is easy to check that $|r(x,y)| \le A$ for all $x, y \in K$. Thus for a fixed $\varepsilon > 0$ we can find $\delta > 0$ such that if $\|y-x\| < \delta$, then

$$|r(x,y)| \le \varepsilon \|y-x\|^2,$$

and

$$|r(x,y)| \le \varepsilon\|y-x\|^2 + A1_{\{\|y-x\|>\delta\}}. \tag{4.7}$$

Now by (4.6) we get

$$\begin{aligned}
&n[B_n(f)(x) - f(x)] \\
&\quad = \frac{1}{2}\left(\int_K D^2 f(x)(y,y)\mu_x(dy) - D^2 f(x)(x,x)\right) + nB_n(r(x,\cdot))(x).
\end{aligned} \tag{4.8}$$

To finish the proof of the theorem it is enough to demonstrate that

$$nB_n(r(x,\cdot))(x) \to 0, \quad n \to \infty \tag{4.9}$$

uniformly on K. But

$$\begin{aligned}
0 \le nB_n(|r(x,\cdot)|)(x) &\le n\varepsilon B_n(\|\cdot - x)\|^2) + nAB_n(1_{\{\|t-x\|>\delta\}})(x) \\
&= n\varepsilon B_n(\|\cdot - x)\|^2) + nAPr\left\{\left\|\frac{S_n}{n} - x\right\| > \delta\right\} \le \varepsilon C, \text{ when } n \to \infty
\end{aligned}$$

for any $\varepsilon > 0$. This completes the proof. □

Remark 4.10. It is easy to check that if a Banach space B is of type 2 (especially if B is a Hilbert space), then the condition (4.3) is satisfied.

Remark 4.11. If there exists a function $f \in C_u^2(B)$ with a bounded support, then the Banach space is of type 2 and by Remark 4.10 the condition (4.3) is satisfied.

Remark 4.12. It is not difficult to find a Banach space B (e.g., l^1) and a simplex $K \subset B$ for which the condition (4.3) is not fulfilled.

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